

A closed formula for linear recurrences with constant coefficients

Glenn Bruda, Bruce Fang, Pico Gilman, Raul Marquez, Steven J. Miller, Beni Prapashtica, Daeyoung Son, Saad Waheed, Janine Wang

ARTICLE HISTORY

Compiled January 8, 2026

Article type: Research

ABSTRACT

Given a linear recurrence of the form $c_n = a_1c_{n-1} + \cdots + a_jc_{n-j}$, it is well-known that $c_n = \sum_r p_r(n)r^n$, where the sum is taken over the set of characteristic roots and each $p_r(n)$ is some polynomial. We give a closed formula for the coefficients of each polynomial $p_r(n)$ for any linear recurrence of this form.

KEYWORDS

Linear recurrence; Binet formula; closed formula

1. Introduction

Let $\{a_n\}_{1 \leq n \leq j}$ be a finite sequence of complex numbers and $\{c_n\}_{n \geq 0}$ be the sequence defined by the recurrence $c_n = a_1c_{n-1} + \cdots + a_jc_{n-j}$ for all $n \geq j$, where the initial conditions $\{c_k\}_{0 \leq k \leq j-1}$ are fixed. Such a sequence $\{c_n\}$ is called a *linear recurrence with constant coefficients*. The *characteristic polynomial* of $\{c_n\}$ is the polynomial $p(z) = z^j - a_1z^{j-1} - a_2z^{j-2} - \cdots - a_{j-1}z - a_j$. A *characteristic root* of the recurrence satisfied by $\{c_n\}$ is defined to be a root of p .

A classical result (e.g., [9, Theorem IV.9]) in the theory of recurrence relations states that $\{c_n\}$ admits a *generalized Binet formula*: for all $n \geq 0$, we have that $c_n = \sum_r p_r(n)r^n$, where the sum is taken over the set of characteristic roots and each $p_r(n)$ is a polynomial depending on the initial conditions. We construct closed formulas for the coefficients of each $p_r(n)$.

Theorem 1.1. *Define the sequence $\{c_n\} \subseteq \mathbb{C}$ by the recurrence relation*

$$c_n = a_1c_{n-1} + \cdots + a_jc_{n-j} \tag{1.1}$$

with $c_0, c_1, \dots, c_{j-1} \in \mathbb{C}$. For each characteristic root r and integer $0 \leq k \leq j-1$, let

(i) m_r be the multiplicity of r ,

Email addresses: glenn.bruda@ufl.edu (Glenn Bruda), fangbaojun2002@gmail.com (Bruce Fang), pico@mit.edu (Pico Gilman), raul.marquez02@utrgv.edu (Raul Marquez), sjm1@williams.edu (Steven J. Miller), beniprap@gmail.com (Beni Prapashtica), ds15@williams.edu (Daeyoung Son), sw21@uchicago.edu (Saad Waheed), jjw3@williams.edu (Janine Wang)

- (ii) $s_{r,k,h} := \sum_{\ell \geq j-k} \binom{k+\ell}{h} a_\ell r^{-k-\ell}$, and
 (iii) $N_r := (-1)^{m_r-1} \left(\sum_{\ell \geq m_r} \binom{\ell}{m_r} a_\ell r^{-\ell} \right)^{-1}$.

Then $c_n = \sum_r \sum_{i=1}^{m_r} \kappa_{r,i} n^{m_r-i} r^n$, where the outer sum is taken over the set of characteristic roots and

$$\kappa_{r,i} = \frac{(-1)^{i-1}}{(m_r - i)!} \sum_{\substack{\emptyset \neq \{d_1, \dots, d_p\} \subseteq \{0, 1, \dots, i-1\} \\ d_1 < \dots < d_p = i-1}} N_r^p \sum_{k_1=0}^{j-1} c_{k_1} s_{r,k_1,d_1} \prod_{u=1}^{p-1} \sum_{k_2=0}^{j-1} \frac{k_2^{m_r-1-d_u} r^{k_2} s_{r,k_2,d_{u+1}}}{(-1)^{d_u-1} (m_r - 1 - d_u)!}. \quad (1.2)$$

Computing $p_r(n)$ is traditionally done by algorithmically computing the generating function for $\{c_n\}$ and determining its partial fraction decomposition. Algorithms for computing partial fraction decompositions are well-studied, some of which can be found in [10, 11, 16].

From Theorem 1.1, one may obtain a closed formula for the partial fraction decomposition of a proper rational function, circumventing any algorithmic computations. A closed formula for such a partial fraction decomposition was also recently given by Chargeishvili, Fekésházy, Somogyi, and Van Thurenhout in [6]. Of course, from [6, Equation 11], a closed formula for the linear recurrence may be obtained by applying Newton's binomial expansion to each term in the composition. However, our formula contrasts favorably with this approach in two ways:

- (i) the formula given for each $\kappa_{r,i}$ is not a function of any of the other characteristic roots, and
- (ii) our formula provides an exact formula for each coefficient of $p_r(n)$ rather than representing $p_r(n)$ as a sum of binomial coefficients.

Thus, if one wished to compute a particular $p_r(n)$, none of the roots other than r would need to be computed.

The primary application of Theorem 1.1 is toward an implementation to symbolically compute the generalized Binet formula of any linear recurrence with constant coefficients, allowing us to quantifiably determine how varying a_1, \dots, a_j and the initial conditions impact the recurrence. Most of the difficulty lies in accounting for multiple roots, since each leading coefficient $\kappa_{r,1}$ reduces to

$$\kappa_{r,1} = \frac{N_r}{(m_r - 1)!} \sum_{k=0}^{j-1} c_k s_{r,k,0}. \quad (1.3)$$

Thus, Theorem 1.1 simplifies dramatically if all of the characteristic roots of the recurrence are of multiplicity one. A prototypical class of recurrences with this property are the j -nacci numbers, which are defined by the recurrence $F(n, j) = \sum_{\ell=1}^j F(n-\ell, j)$ and initial conditions $F(0, j) = F(1, j) = \dots = F(j-2, j) = 0$ and $F(j-1, j) = 1$. Letting $j \geq 2$ and $n \geq 0$, from Theorem 1.1 we have the formula

$$F(n, j) = \sum_r \frac{r^{n+1}(1-r^{-j})}{r^{j+1}-j}, \quad (1.4)$$

where the sum is taken over the set of roots of the characteristic polynomial $x^j -$

$x^{j-1} - \dots - x - 1$. This follows from Theorem 1.1 by a simplification of geometric series and the identity $\sum_{k=0}^{j-1} r^k = r^j$. Of course, this result is equivalent to the various closed forms for the j -nacci numbers given in the literature (e.g., [5, Lemma 3.2]).

However, our formula permitting multiple roots allows us to consider more complicated recurrences. Exposed by Ahlgren in [1], Ramanujan stated in his last notebook [15, pg.54] that the circular summation of the sixth powers of the symmetric theta function decomposes into a product of Ramanujan's general theta function and another function F_6 . By a formula for $F_6(q)$ given in [1, Theorem 1], Zabolotskiy notes in [14] that the exponents in the expansion of $F_6(q^2)$ are precisely the nonnegative integers congruent to 0, 5, 8, 9 mod 12, which satisfy the recurrence $c_n = 2c_{n-1} - 2c_{n-2} + 2c_{n-3} - c_{n-4}$. By definition, $c_0 = 0$, $c_1 = 5$, $c_2 = 8$, and $c_3 = 9$. The characteristic roots are 1, i , $-i$, which have multiplicities 2, 1, 1 respectively. Applying Theorem 1.1, we find that $\kappa_{1,1} = 3$, $\kappa_{1,2} = 1$, $\kappa_{i,1} = (-1 - i)/2$, $\kappa_{-i,1} = (-1 + i)/2$, thus obtaining

$$c_n = 3n + 1 + \frac{1}{2}(-1 - i)i^n + \frac{1}{2}(-1 + i)(-i)^n = 3n + 1 + \sin\left(\frac{\pi n}{2}\right) - \cos\left(\frac{\pi n}{2}\right). \quad (1.5)$$

We now consider another example demonstrating the power of allowing for multiple roots: the *generalized pentagonal numbers* are the exponents in the expansion given by the celebrated pentagonal number theorem [4], which asserts the q -series identity

$$\prod_{n \geq 1} (1 - q^n) = \sum_{k \in \mathbb{Z}} (-1)^k q^{k(3k-1)/2}. \quad (1.6)$$

That is, n is a generalized pentagonal number if and only if $n = k(3k - 1)/2$ for some $k \in \mathbb{Z}$. We aim to prove a closed formula for the generalized pentagonal numbers in increasing order, which in turn would provide a singly infinite series expansion of monomials. By [13], this sequence satisfies the recurrence $c_n = c_{n-1} + 2c_{n-2} - 2c_{n-3} - c_{n-4} + c_{n-5}$ with $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, $c_3 = 5$, and $c_4 = 7$. The characteristic roots are 1 and -1 , which have multiplicities 3 and 2 respectively. Applying Theorem 1.1, we find that $\kappa_{1,1} = 3/8$, $\kappa_{1,2} = 3/8$, $\kappa_{1,3} = 1/16$, $\kappa_{-1,1} = -1/8$, $\kappa_{-1,2} = -1/16$, thus obtaining

$$c_n = \frac{3n^2}{8} + \frac{3n}{8} + \frac{1}{16} + \left(-\frac{n}{8} - \frac{1}{16}\right) (-1)^n, \quad (1.7)$$

which matches the formula given by Barry in [3, Section 7].

Most of the remaining part of the paper is dedicated to proving Theorem 1.1, followed by a discussion on directions for future work.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we appeal to the fact that the coefficient of the asymptotically dominant term in the expansion formula for $\{c_n\}$ is straightforward to compute by [9, Note IV.26]. In particular, to compute the coefficient of $n^{m_r-i} r^n$, we subtract all terms which asymptotically dominate $n^{m_r-i} r^n$, obtaining a new linear recurrence $\{c'_n\}$ satisfying $c'_n = \kappa_{r,i} n^{m_r-i} r^n (1 + O(1/n))$. To leverage [9, Note IV.26], we first must

compute the generating function of $\{c_n\}$ in closed form. To do this, we break $\{c_n\}$ into a linear combination of a simpler sequence, which we refer to henceforth as the (a_1, \dots, a_j) -nacci sequence.

Definition 2.1 ((a_1, \dots, a_j) -nacci sequence). *Let $\{a_n\}_{1 \leq n \leq j}$ be a finite sequence of complex numbers. Define the sequence $\{F_n^{(a_1, \dots, a_j)}\}_{n \geq 0}$, which we call the (a_1, \dots, a_j) -nacci sequence, by the recurrence relation*

$$F_n^{(a_1, \dots, a_j)} = \sum_{\ell \geq 1} a_\ell F_{n-\ell}^{(a_1, \dots, a_j)}, \quad (2.1)$$

with $F_0^{(a_1, \dots, a_j)} = F_1^{(a_1, \dots, a_j)} = \dots = F_{j-2}^{(a_1, \dots, a_j)} = 0$ and $F_{j-1}^{(a_1, \dots, a_j)} = 1$.

The upside of considering the simpler (a_1, \dots, a_j) -nacci sequence is that its generating function is trivial to compute in closed form. Thus, once we precisely describe the linear combination yielding $\{c_n\}$, the desired generating function formula for $\{c_n\}$ quickly follows.

Lemma 2.2. *Let $\{a_n\}_{1 \leq n \leq j}$ be a finite sequence of complex numbers. Then*

$$\sum_{n \geq 0} F_n^{(a_1, \dots, a_j)} z^n = \frac{z^{j-1}}{1 - \sum_{\ell \geq 1} a_\ell z^\ell}. \quad (2.2)$$

Proof. To determine the ordinary generating function of $\{F_n^{(a_1, \dots, a_j)}\}$, we use the standard ansatz that this function is rational with denominator $1 - \sum_{\ell \geq 1} a_\ell z^\ell$. Indeed, we have

$$\begin{aligned} \left(1 - \sum_{\ell \geq 1} a_\ell z^\ell\right) \sum_{k \geq 0} F_k^{(a_1, \dots, a_j)} z^k &= \sum_{k \geq 0} F_k^{(a_1, \dots, a_j)} z^k - \sum_{\ell \geq 1} \sum_{k \geq 0} a_\ell F_k^{(a_1, \dots, a_j)} z^{k+\ell} \\ &= \sum_{k \geq 0} F_k^{(a_1, \dots, a_j)} z^k - \sum_{\ell \geq 1} \sum_{k \geq \ell} a_\ell F_{k-\ell}^{(a_1, \dots, a_j)} z^k \\ &\stackrel{*}{=} F_{j-1}^{(a_1, \dots, a_j)} z^{j-1} + \sum_{k \geq j} \left(F_k^{(a_1, \dots, a_j)} - \sum_{\ell \geq 1} a_\ell F_{k-\ell}^{(a_1, \dots, a_j)} \right) z^k \\ &= F_{j-1}^{(a_1, \dots, a_j)} z^{j-1} = z^{j-1}, \end{aligned} \quad (2.3)$$

where the starred equality follows since $F_k^{(a_1, \dots, a_j)} = 0$ if $k \leq j-2$. \square

Lemma 2.3. *Define the sequence $\{c_n\} \subseteq \mathbb{C}$ by the recurrence relation $c_n = a_1 c_{n-1} + a_2 c_{n-2} + \dots + a_j c_{n-j}$ with $c_0, c_1, \dots, c_{j-1} \in \mathbb{C}$. Then for all $n \geq j$,*

$$c_n = \sum_{k=0}^{j-1} c_k \sum_{\ell \geq j-k} a_\ell F_{n+j-1-k-\ell}^{(a_1, \dots, a_j)}. \quad (2.4)$$

In [12, pg.6], Liu proves a result similar to Lemma 2.3, using instead an explicit formula for $F_n^{(a_1, \dots, a_j)}$. As we are ultimately only concerned with the generating function

of $F_n^{(a_1, \dots, a_j)}$, we need not consider its explicit form.

We now recall the definition of the Kronecker delta for use in the proof of Lemma 2.3. For two numbers k, k' , the Kronecker delta is defined by

$$\delta_{k,k'} := \begin{cases} 1 & \text{if } k = k', \\ 0 & \text{if } k \neq k'. \end{cases} \quad (2.5)$$

Proof of Lemma 2.3. We have that

$$Q(a_1, \dots, a_j) := \begin{bmatrix} a_1 & a_2 & \cdots & a_{j-1} & a_j \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad (2.6)$$

is the $j \times j$ companion matrix of the (a_1, \dots, a_j) -nacci sequence. This matrix encodes the recurrence relation for the (a_1, \dots, a_j) -nacci sequence in that

$$\begin{bmatrix} F_{n+1}^{(a_1, \dots, a_j)} \\ F_n^{(a_1, \dots, a_j)} \\ \vdots \\ F_{n-j+2}^{(a_1, \dots, a_j)} \end{bmatrix} = Q(a_1, \dots, a_j) \begin{bmatrix} F_n^{(a_1, \dots, a_j)} \\ F_{n-1}^{(a_1, \dots, a_j)} \\ \vdots \\ F_{n-j+1}^{(a_1, \dots, a_j)} \end{bmatrix}. \quad (2.7)$$

For each $0 \leq k \leq j-1$, let $e_{k,n}$ be such that $c_n = \sum_{k=0}^{j-1} c_k e_{k,n}$. From the linear recurrence satisfied by $\{c_n\}$, we deduce $e_{k,n} = \sum_{\ell \geq 1} a_\ell e_{k,n-\ell}$. As this is the same recurrence satisfied by the (a_1, \dots, a_j) -nacci sequence,

$$\begin{bmatrix} e_{k,n+1} \\ e_{k,n} \\ \vdots \\ e_{k,n-j+2} \end{bmatrix} = Q(a_1, \dots, a_j) \begin{bmatrix} e_{k,n} \\ e_{k,n-1} \\ \vdots \\ e_{k,n-j+1} \end{bmatrix}. \quad (2.8)$$

Thus,

$$\begin{bmatrix} e_{k,n+j-1} \\ e_{k,n+j-2} \\ \vdots \\ e_{k,n} \end{bmatrix} = (Q(a_1, \dots, a_j))^n \begin{bmatrix} e_{k,j-1} \\ e_{k,j-2} \\ \vdots \\ e_{k,0} \end{bmatrix} = (Q(a_1, \dots, a_j))^n \begin{bmatrix} \delta_{k,j-1} \\ \delta_{k,j-2} \\ \vdots \\ \delta_{k,0} \end{bmatrix}. \quad (2.9)$$

We claim that $(Q(a_1, \dots, a_j))^n$ equals

$$\begin{bmatrix} \sum_{\ell \geq 1} a_\ell F_{n+j-1-\ell}^{(a_1, \dots, a_j)} & \sum_{\ell \geq 2} a_\ell F_{n+j-\ell}^{(a_1, \dots, a_j)} & \cdots & \sum_{\ell \geq j-1} a_\ell F_{n+2j-3-\ell}^{(a_1, \dots, a_j)} & \sum_{\ell \geq j} a_\ell F_{n+2j-2-\ell}^{(a_1, \dots, a_j)} \\ \sum_{\ell \geq 1} a_\ell F_{n+j-2-\ell}^{(a_1, \dots, a_j)} & \sum_{\ell \geq 2} a_\ell F_{n+j-1-\ell}^{(a_1, \dots, a_j)} & \cdots & \sum_{\ell \geq j-1} a_\ell F_{n+2j-4-\ell}^{(a_1, \dots, a_j)} & \sum_{\ell \geq j} a_\ell F_{n+2j-3-\ell}^{(a_1, \dots, a_j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sum_{\ell \geq 1} a_\ell F_{n+1-\ell}^{(a_1, \dots, a_j)} & \sum_{\ell \geq 2} a_\ell F_{n+2-\ell}^{(a_1, \dots, a_j)} & \cdots & \sum_{\ell \geq j-1} a_\ell F_{n+j-1-\ell}^{(a_1, \dots, a_j)} & \sum_{\ell \geq j} a_\ell F_{n+j-\ell}^{(a_1, \dots, a_j)} \\ \sum_{\ell \geq 1} a_\ell F_{n-\ell}^{(a_1, \dots, a_j)} & \sum_{\ell \geq 2} a_\ell F_{n+1-\ell}^{(a_1, \dots, a_j)} & \cdots & \sum_{\ell \geq j-1} a_\ell F_{n+j-2-\ell}^{(a_1, \dots, a_j)} & \sum_{\ell \geq j} a_\ell F_{n+j-1-\ell}^{(a_1, \dots, a_j)} \end{bmatrix}$$

for any integer $n \geq j$. Indeed, let $q_{a,b}^{(n)}$ denote the (a, b) th entry of $(Q(a_1, \dots, a_j))^n$. By [7, Theorem 3.2],

$$\sum_{n \geq 0} q_{a,b}^{(n)} z^n = \frac{z^{a-b} \left(1 - \sum_{\ell=1}^{b-1} a_\ell z^\ell\right)}{1 - \sum_{\ell \geq 1} a_\ell z^\ell} \quad (2.10)$$

if $a \geq b$, and

$$\sum_{n \geq 0} q_{a,b}^{(n)} z^n = \frac{\sum_{\ell \geq b} a_\ell z^{a-b+\ell}}{1 - \sum_{\ell \geq 1} a_\ell z^\ell} \quad (2.11)$$

if $a < b$. Lemma 2.2 gives that

$$\sum_{n \geq 0} F_n^{(a_1, \dots, a_j)} z^n = \frac{z^{j-1}}{1 - \sum_{\ell \geq 1} a_\ell z^\ell}. \quad (2.12)$$

From these formulas, we see that

$$q_{a,b}^{(n)} = F_{n+j-1+b-a}^{(a_1, \dots, a_j)} - \sum_{\ell=1}^{b-1} a_\ell F_{n+j-1+b-a-\ell}^{(a_1, \dots, a_j)} = \sum_{\ell \geq b} F_{n+j-1+b-a-\ell}^{(a_1, \dots, a_j)} \quad (2.13)$$

for any a, b , verifying the formula for $(Q(a_1, \dots, a_j))^n$.

The last entry $e_{k,n}$, of the vector in (2.9) is obtained by taking the $(j-k)$ th entry (with entries numbered from left to right starting from 1) of the last row of $(Q(a_1, \dots, a_j))^n$. Thus,

$$e_{k,n} = \sum_{\ell \geq j-k} a_\ell F_{n+j-1-k-\ell}^{(a_1, \dots, a_j)}, \quad (2.14)$$

completing the proof. \square

From Lemmas 2.2 and 2.3, we obtain a closed formula for the generating function of an arbitrary linear recurrence with constant coefficients. We note that this formula is equivalent to other such generating function formulas in the literature, such as [17, Theorem 5]; however, writing the formula in this particular manner is conducive to ultimately simplifying the formula for $\kappa_{r,i}$.

Proposition 2.4. Define the sequence $\{c_n\} \subseteq \mathbb{C}$ by the recurrence relation

$$c_n = a_1 c_{n-1} + a_2 c_{n-2} + \cdots + a_j c_{n-j} \quad (2.15)$$

with $c_0, c_1, \dots, c_{j-1} \in \mathbb{C}$. Then

$$\sum_{n \geq 0} c_n z^n = \frac{\sum_{k=0}^{j-1} c_k \sum_{\ell \geq j-k} a_\ell z^{k+\ell} + (1 - a_1 z - \cdots - a_j z^j) \sum_{k=0}^{j-1} c_k z^k}{1 - a_1 z - \cdots - a_j z^j}. \quad (2.16)$$

Proof. By Lemmas 2.2 and 2.3,

$$\begin{aligned} \sum_{n \geq 0} c_n z^n &= \sum_{k=0}^{j-1} c_k z^k + \sum_{n \geq j} z^n \sum_{k=0}^{j-1} c_k \sum_{\ell \geq j-k} a_\ell F_{n+j-1-k-\ell}^{(a_1, \dots, a_j)} \\ &= \sum_{k=0}^{j-1} c_k z^k + \sum_{k=0}^{j-1} c_k \sum_{\ell \geq j-k} a_\ell \sum_{n \geq j} F_{n+j-1-k-\ell}^{(a_1, \dots, a_j)} z^n \\ &= \sum_{k=0}^{j-1} c_k z^k + \sum_{k=0}^{j-1} c_k \sum_{\ell \geq j-k} a_\ell \sum_{n \geq 2j-1-k-\ell} F_n^{(a_1, \dots, a_j)} z^{n-j+1+k+\ell} \\ &= \sum_{k=0}^{j-1} c_k z^k + \sum_{k=0}^{j-1} c_k \sum_{\ell \geq j-k} a_\ell \sum_{n \geq j-1} F_n^{(a_1, \dots, a_j)} z^{n-j+1+k+\ell} \\ &= \sum_{k=0}^{j-1} c_k z^k + \sum_{k=0}^{j-1} c_k \sum_{\ell \geq j-k} a_\ell \frac{z^{k+\ell}}{1 - a_1 z - \cdots - a_j z^j} \\ &= \frac{\sum_{k=0}^{j-1} c_k \sum_{\ell \geq j-k} a_\ell z^{k+\ell} + (1 - a_1 z - \cdots - a_j z^j) \sum_{k=0}^{j-1} c_k z^k}{1 - a_1 z - \cdots - a_j z^j}. \end{aligned} \quad (2.17)$$

□

The final lemma before the proof of Theorem 1.1, Lemma 2.5, is crucial toward simplifying our formula for $\kappa_{r,i}$. In particular, it is due to this result that our formula for $\kappa_{r,i}$ does not depend on any characteristic root other than r .

Lemma 2.5. Let $p(z) = z^j - a_1 z^{j-1} - \cdots - a_j$. Then for any nonzero distinct roots r_1, r_2 of p ,

$$\sum_{k=0}^{j-1} k^{M_1} r_1^k \sum_{\ell \geq j-k} \binom{k+\ell}{M_2} a_\ell r_2^{-k-\ell} = 0 \quad (2.18)$$

for any nonnegative integers $M_1 \leq m_{r_1} - 1$ and $M_2 \leq m_{r_2} - 1$, where m_{r_1} and m_{r_2} are the multiplicities of r_1 and r_2 respectively.

Proof. Rewriting the sum as

$$\sum_{k=0}^{j-1} k^{M_1} r_1^k \sum_{\ell \geq j-k} \binom{k+\ell}{M_2} a_\ell r_2^{-k-\ell} = \sum_{\ell=1}^j a_\ell r_2^{-\ell} \sum_{k=j-\ell}^{j-1} \left(\frac{r_1}{r_2}\right)^k \binom{k+\ell}{M_2} k^{M_1}$$

$$= \sum_{\ell=1}^j a_\ell r_2^{-j} r_1^{-\ell} \sum_{k=0}^{\ell-1} \left(\frac{r_1}{r_2}\right)^k \binom{j+k}{M_2} (j+k-\ell)^{M_1}, \quad (2.19)$$

we see that it suffices to show that $b_\ell := r_2^{-j} r_1^{-\ell} \sum_{k=0}^{\ell-1} (r_1/r_2)^k \binom{j+k}{M_2} (j+k-\ell)^{M_1}$ satisfies the linear recurrence with characteristic polynomial $z^j p(1/z) = a_j z^j + a_{j-1} z^{j-1} + \dots + a_1 z - 1$ for all $\ell \geq j$. We proceed by determining the generating function of $\{b_\ell\}_{\ell \geq 0}$. Splitting into two series, we observe that

$$\begin{aligned} \sum_{\ell \geq 0} b_\ell z^\ell &= \sum_{\ell \geq 0} z^\ell \left(r_2^{-j} r_1^{-\ell} \sum_{k=0}^{\ell} \left(\frac{r_1}{r_2}\right)^k \binom{j+k}{M_2} (j+k-\ell)^{M_1} \right) - \sum_{\ell \geq 0} z^\ell \left(r_2^{-j} r_1^{-\ell} \left(\frac{r_1}{r_2}\right)^\ell \binom{j+\ell}{M_2} j^{M_1} \right) \\ &= r_2^{-j} \sum_{\ell \geq 0} \left(\frac{z}{r_1}\right)^\ell \sum_{k=0}^{\ell} \left(\frac{r_1}{r_2}\right)^k \binom{j+k}{M_2} (j+k-\ell)^{M_1} - \frac{D(z)}{(1-z/r_2)^{M_2+1}} \end{aligned} \quad (2.20)$$

for some polynomial $D(z)$ of degree at most M_2 . Recognizing that $\sum_{k=0}^{\ell} (r_1/r_2)^k \binom{j+k}{M_2} (j+k-\ell)^{M_1}$ is a Cauchy product, we see that

$$r_2^{-j} \sum_{\ell \geq 0} \left(\frac{z}{r_1}\right)^\ell \sum_{k=0}^{\ell} \left(\frac{r_1}{r_2}\right)^k \binom{j+k}{M_2} (j+k-\ell)^{M_1} = \frac{A(z)}{(1-z/r_1)^{M_1+1} (1-z/r_2)^{M_2+1}} \quad (2.21)$$

for some polynomial $A(z)$ of degree at most $M_1 + M_2$. Let $L(z) = (1-z/r_1)^{M_1+1} (1-z/r_2)^{M_2+1}$. Since r_1 and r_2 are distinct, $\deg(A) \leq M_1 + M_2 < j$. So (2.20) and (2.21) show that $\{b_\ell\}_{\ell \geq 0}$ is a linear combination of sequences which, by [9, pg.255], all satisfy the linear recurrence with characteristic polynomial $z^{M_1+M_2+2} L(1/z)$ for all $\ell \geq j$. As r_1, r_2 are distinct, $z^{M_1+M_2+2} L(1/z)$ divides $z^j p(1/z)$, whence it follows that b_ℓ satisfies the linear recurrence with characteristic polynomial $z^j p(1/z)$ for all $\ell \geq j$. \square

Definition 2.6. Let $\mathbf{u} = \{c_n\}$ be as in Theorem 1.1. For each characteristic root r , let $p_r^{\mathbf{u}}(n)$ be the polynomial such that $c_n = \sum_r p_r^{\mathbf{u}}(n) r^n$ for all $n \geq 0$.

Proof of Theorem 1.1. By a standard result in the theory of recurrence relations (e.g., [9, Theorem IV.9]), we know that $c_n = \sum_r \sum_{i=1}^{m_r} \kappa_{r,i} n^{m_r-i} r^n$ for some constants $\kappa_{r,i}$. So it suffices to show that formula (1.2) holds for all characteristic roots r and $1 \leq i \leq m_r$.

Let $\mathbf{u} = \{c_n\}_{n \geq 0}$. Let r be a characteristic root, let $1 \leq i \leq m_r$, and let $S \not\ni r$ be the set of characteristic roots greater than or equal to r in modulus. Define

$$\mathbf{u}' = \left\{ c_n - \sum_{R \in S} p_R^{\mathbf{u}}(n) R^n - \sum_{v=1}^{i-1} \kappa_{r,v} n^{m_r-v} r^n \right\}_{n \geq 0}. \quad (2.22)$$

Noting that $\kappa_{r,i} = [z^{m_r-i}] p_r^{\mathbf{u}}(z) = [z^{m_r-i}] p_r^{\mathbf{u}'}(z)$, we proceed by computing $[z^{m_r-i}] p_r^{\mathbf{u}'}(z)$. As $n^M R^n$ satisfies recurrence (1.1) for any characteristic root R and $M \leq m_R - 1$, it follows that \mathbf{u}' also satisfies recurrence (1.1) with initial conditions

$\{c_k - \sum_{R \in S} p_R^{\mathbf{u}}(k) R^k - \sum_{v=1}^{i-1} \kappa_{r,v} k^{m_r-v} r^k\}_{0 \leq k \leq j-1}$. Let $p(z)$ be the characteristic polynomial of (1.1). Then by Proposition 2.4, the generating function of \mathbf{u}' is

$$f(z) := \frac{1}{p(z)} \left(\sum_{k=0}^{j-1} \left(c_k - \sum_{R \in S} p_R^{\mathbf{u}}(k) R^k - \sum_{v=1}^{i-1} \kappa_{r,v} k^{m_r-v} r^k \right) \sum_{\ell \geq j-k} a_{\ell} z^{k+\ell} + p(z) \sum_{k=0}^{j-1} c_k z^k \right). \quad (2.23)$$

Note that f has a single dominant pole r^{-1} with multiplicity $m_r - i + 1$ since the asymptotically dominant term of \mathbf{u}' is $\kappa_{r,i} n^{m_r-i} r^n$. So by [9, Note IV.26], $\kappa_{r,i} = C / (m_r - i)!$, where $C = \lim_{z \rightarrow r^{-1}} (1 - rz)^{m_r-i+1} f(z)$. Since the numerator of f is divisible by $(1 - rz)^{i-1}$, it follows that $C = C_1 / C_2$, where $C_2 = \lim_{z \rightarrow r^{-1}} p(z) / (1 - rz)^{m_r}$ and C_1 equals

$$\lim_{z \rightarrow r^{-1}} \frac{1}{(1 - rz)^{i-1}} \left(\sum_{k=0}^{j-1} \left(c_k - \sum_{R \in S} p_R^{\mathbf{u}}(k) R^k - \sum_{v=1}^{i-1} \kappa_{r,v} k^{m_r-v} r^k \right) \sum_{\ell \geq j-k} a_{\ell} z^{k+\ell} + p(z) \sum_{k=0}^{j-1} c_k z^k \right). \quad (2.24)$$

As $i - 1 < m_r$,

$$\lim_{z \rightarrow r^{-1}} \frac{p(z)}{(1 - rz)^{i-1}} \sum_{k=0}^{j-1} c_k z^k = 0; \quad (2.25)$$

hence, (2.24) reduces to

$$\lim_{z \rightarrow r^{-1}} \frac{1}{(1 - rz)^{i-1}} \left(\sum_{k=0}^{j-1} \left(c_k - \sum_{R \in S} p_R^{\mathbf{u}}(k) R^k - \sum_{v=1}^{i-1} \kappa_{r,v} k^{m_r-v} r^k \right) \sum_{\ell \geq j-k} a_{\ell} z^{k+\ell} \right). \quad (2.26)$$

Thus,

$$\begin{aligned} C_1 &= \frac{(-1)^{i-1}}{r^{i-1}(i-1)!} \sum_{k=0}^{j-1} \left(c_k - \sum_{R \in S} p_R^{\mathbf{u}}(k) R^k - \sum_{v=1}^{i-1} \kappa_{r,v} k^{m_r-v} r^k \right) \sum_{\ell \geq j-k} (i-1)! \binom{k+\ell}{i-1} a_{\ell} r^{-k-\ell+i-1} \\ &= (-1)^{i-1} \sum_{k=0}^{j-1} \left(c_k - \sum_{R \in S} p_R^{\mathbf{u}}(k) R^k - \sum_{v=1}^{i-1} \kappa_{r,v} k^{m_r-v} r^k \right) \sum_{\ell \geq j-k} \binom{k+\ell}{i-1} a_{\ell} r^{-k-\ell} \\ &= (-1)^{i-1} \sum_{k=0}^{j-1} \left(c_k - \sum_{R \in S} p_R^{\mathbf{u}}(k) R^k - \sum_{v=1}^{i-1} \kappa_{r,v} k^{m_r-v} r^k \right) s_{r,k,i-1} \\ &\stackrel{*}{=} (-1)^{i-1} \sum_{k=0}^{j-1} \left(c_k - \sum_{v=1}^{i-1} \kappa_{r,v} k^{m_r-v} r^k \right) s_{r,k,i-1}, \end{aligned} \quad (2.27)$$

where the starred equality follows from Lemma 2.5. Computing C_2 , we obtain

$$C_2 = \frac{-\sum_{\ell \geq m_r} m_r! \binom{\ell}{m_r} a_\ell r^{-\ell+m_r}}{(-1)^{m_r} r^{m_r} m_r!} = (-1)^{m_r-1} \sum_{\ell \geq m_r} \binom{\ell}{m_r} a_\ell r^{-\ell} = N_r^{-1}. \quad (2.28)$$

Therefore, we have the recurrence

$$\kappa_{r,i} = \frac{(-1)^{i-1} N_r}{(m_r - i)!} \sum_{k=0}^{j-1} \left(c_k - \sum_{v=1}^{i-1} \kappa_{r,v} k^{m_r-v} r^k \right) s_{r,k,i-1}. \quad (2.29)$$

It now only remains to show that recurrence (2.29) admits the explicit formula

$$\kappa_{r,i} = \frac{(-1)^{i-1}}{(m_r - i)!} \sum_{\substack{\emptyset \neq \{d_1, \dots, d_p\} \subseteq \{0, 1, \dots, i-1\} \\ d_1 < \dots < d_p = i-1}} N_r^p \sum_{k_1=0}^{j-1} c_{k_1} s_{r,k_1,d_1} \prod_{u=1}^{p-1} \sum_{k_2=0}^{j-1} \frac{k_2^{m_r-1-d_u} r^{k_2} s_{r,k_2,d_{u+1}}}{(-1)^{d_u-1} (m_r - 1 - d_u)!}. \quad (2.30)$$

Let $\{\kappa'_{r,i}\}_{1 \leq i \leq m_r}$ be the finite sequence defined by the explicit formula given in (2.30). For abbreviation, let

$$\Pi(\{d_1, \dots, d_p\}) = \sum_{k_1=0}^{j-1} c_{k_1} s_{r,k_1,d_1} \prod_{u=1}^{p-1} \sum_{k_2=0}^{j-1} \frac{k_2^{m_r-1-d_u} r^{k_2} s_{r,k_2,d_{u+1}}}{(-1)^{d_u-1} (m_r - 1 - d_u)!}. \quad (2.31)$$

Towards proving that $\kappa_{r,i} \equiv \kappa'_{r,i}$, it suffices to show that $\{\kappa'_{r,i}\}_{1 \leq i \leq m_r}$ satisfies the recurrence since $\{\kappa_{r,i}\}_{1 \leq i \leq m_r}$ is completely determined by (2.29). Note

$$\begin{aligned} \sum_{v=1}^{i-1} \kappa'_{r,v} \sum_{k=0}^{j-1} k^{m_r-v} r^k s_{r,k,i-1} &= \sum_{v=1}^{i-1} \sum_{\substack{\emptyset \neq \{d_1, \dots, d_p\} \subseteq \{0, 1, \dots, v-1\} \\ d_1 < \dots < d_p = v-1}} N_r^p \Pi(\{d_1, \dots, d_p\}) \sum_{k=0}^{j-1} \frac{k^{m_r-v} r^k s_{r,k,i-1}}{(-1)^{v-1} (m_r - v)!} \\ &= - \sum_{v=1}^{i-1} \sum_{\substack{\emptyset \neq \{d_1, \dots, d_{p+1}\} \subseteq \{0, 1, \dots, i-1\} \\ d_1 < \dots < d_p = v-1, d_{p+1} = i-1}} N_r^p \Pi(\{d_1, \dots, d_{p+1}\}). \end{aligned} \quad (2.32)$$

Now with the aid of (2.32), we have

$$\begin{aligned} &\frac{(-1)^{i-1} N_r}{(m_r - i)!} \sum_{k=0}^{j-1} \left(c_k - \sum_{v=1}^{i-1} \kappa'_{r,v} k^{m_r-v} r^k \right) s_{r,k,i-1} \\ &= \frac{(-1)^{i-1}}{(m_r - i)!} \left(N_r \sum_{k=0}^{j-1} c_k s_{r,k,i-1} - N_r \sum_{v=1}^{i-1} \kappa'_{r,v} \sum_{k=0}^{j-1} k^{m_r-v} r^k s_{r,k,i-1} \right) \\ &= \frac{(-1)^{i-1}}{(m_r - i)!} \left(N_r \Pi(\{i-1\}) + \sum_{v=1}^{i-1} \sum_{\substack{\emptyset \neq \{d_1, \dots, d_{p+1}\} \subseteq \{0, 1, \dots, i-1\} \\ d_1 < \dots < d_p = v-1, d_{p+1} = i-1}} N_r^{p+1} \Pi(\{d_1, \dots, d_{p+1}\}) \right) \end{aligned}$$

$$= \frac{(-1)^{i-1}}{(m_r - i)!} \sum_{\substack{\emptyset \neq \{d_1, \dots, d_{p+1}\} \subseteq \{0, 1, \dots, i-1\} \\ d_1 < \dots < d_{p+1} = i-1}} N_r^{p+1} \Pi(\{d_1, \dots, d_{p+1}\}) = \kappa'_{r,i}. \quad (2.33)$$

Thus, $\kappa_{r,i} \equiv \kappa'_{r,i}$. □

3. Directions for Future Work

A natural continuation of our work is to generalize Theorem 1.1 to a broader class of recurrences. A particularly natural generalization is to consider linear recurrences of the form $c_n = a_1 c_{n-1} + \dots + a_j a_{n-j} + d(n)$, where $d(n)$ is some polynomial. These recurrences are classified as non-homogeneous linear recurrences: a linear recurrence $c_n - a_1 c_{n-1} - \dots - a_j a_{n-j} = f(n)$ is *homogeneous* if $f(n) \equiv 0$, and *non-homogeneous* otherwise. This proposed generalization only encapsulates the case when f is a polynomial; however, this induces the generating function of $\{c_n\}$ to remain rational, thereby permitting the techniques presented in this work to be readily generalized.

We note that this direction has already been explored for specific cases. For example, in 1987, Asveld [2] provided a formula analogous to Theorem 1.1 for the case where $j = 2$ and $c_0 = c_1 = 1$ (see [8] for exposition on this case and related work). This potential direction goes much further, seeking to provide a general formula echoing Theorem 1.1 for arbitrary j and arbitrary initial conditions $\{c_k\}_{0 \leq k \leq j-1}$.

Acknowledgements

This work was supported by the National Science Foundation grant *DMS-2241623*, Williams College, the Finnerty Fund, and the Winston Churchill Foundation. We are grateful to the anonymous referee for their feedback, which prompted a vast improvement of a previous version of this work. We thank Jay Pantone, Robin Pemantle, Richard Stanley, Vincent Vatter, and Doron Zeilberger for their consultation on this project. We acknowledge Joe Cooper for helpful conversations and for highlighting errors and typos in an earlier draft. We also thank participants of the 21st International Fibonacci Conference for comments on an earlier version of this work.

Disclosure statement

No conflict of interest has been reported by the authors.

References

- [1] Scott Ahlgren, *The Sixth, Eighth, Ninth, and Tenth Powers of Ramanujan's Theta Function*, Proceedings of the American Mathematical Society **128** (2000), no. 5, 1333–1338.
- [2] Peter R.J. Asveld, *A Family of Fibonacci-Like Sequences*, The Fibonacci Quarterly **25** (1987), 81–83.
- [3] Paul Barry, *On sequences with $\{-1, 0, 1\}$ Hankel transforms*, 2012. <https://arxiv.org/abs/1205.2565>.
- [4] Jordan Bell, *A summary of Euler's work on the pentagonal number theorem*, Archive for History of Exact Sciences **64** (2010), no. 3, 301–373.
- [5] N.B. Boer and A.E. Sterk, *Generalized Fibonacci numbers and extreme value laws for the Rényi map*, Indagationes Mathematicae **32** (2021), no. 3, 704–718.

- [6] B. Chargeishvili, L. Fekésházy, G. Somogyi, and S. Van Thurenhout, *Linapart: Optimizing the univariate partial fraction decomposition*, *Computer Physics Communications* **307** (2025), 109395.
- [7] William Y.C. Chen and James D. Louck, *The Combinatorial Power of the Companion Matrix*, *Linear Algebra and its Applications* **232** (1996), 261–278.
- [8] Carlos M da Fonseca, Can Kızılateş, Paulo Saraiva, and Anthony G Shannon, *Generalised Leonardo numbers*, *Logic Journal of the IGPL* **33** (2025), no. 6.
- [9] Philippe Flajolet and Robert Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.
- [10] Ellis Horowitz, *Algorithms for partial fraction decomposition and rational function integration*, SYMSAC '71: Proceedings of the Second ACM Symposium on Symbolic and Algebraic Manipulation (1971), 441–457.
- [11] H.T. Kung and D.M. Tong, *Fast Algorithms for Partial Fraction Decomposition*, *SIAM Journal on Computing* **6** (1977), no. 3, 582–593.
- [12] Bolian Liu, *A Matrix Method to Solve Linear Recurrences with Constant Coefficients*, *The Fibonacci Quarterly* **30** (1992), no. 1, 2–8.
- [13] *Generalized pentagonal numbers*, 2025. Entry A001318 in The On-Line Encyclopedia of Integer Sequences, <https://oeis.org/A001318>.
- [14] *Numbers that are congruent to 0, 5, 8, 9 mod 12*, 2025. Entry A072833 in The On-Line Encyclopedia of Integer Sequences, <https://oeis.org/A072833>.
- [15] Srinivasa Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa Publishing House, 1988.
- [16] Guoce Xin, *A Fast Algorithm for Partial Fraction Decompositions*, 2004. <https://arxiv.org/abs/math/0408189>.
- [17] Roman Zatorsky and Taras Goy, *Parapermanents of Triangular Matrices and Some General Theorems on Number Sequences*, *Journal of Integer Sequences* **19** (2016).

MSC2020: 11B37, 11B39