Dynamics of the Fibonacci Order of Appearance Map

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ABSTRACT
The *order of appearance* $z(n)$ of a positive integer n in the Fibonacci sequence is defined as the smallest positive integer j such that n divides the j^{th} Fibonacci number. We prove that for every $k \geq 0$ there exist infinitely many integers that reach a fixed point of z after applying exactly k iterations of z. In addition, we show that, if x is a fixed point of z greater than 5, then there exist infinitely many integers whose orbits under z reach x. We also give a new proof of the theorem stating that every positive integer reaches a fixed point of z after a finite number of iterations.

This paper began when the first and fourth named authors were looking through papers of the Fibonacci Quarterly in a search for a research project for the 2023 SMALL REU at Williams College. We gratefully dedicate this article to Curtis Cooper for his quarter century of excellent stewardship as Editor-in-Chief of the Fibonacci Quarterly.

1. Introduction

The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined recursively as $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with initial terms $F_0 = 0$ and $F_1 = 1$. The arithmetic properties of the Fibonacci sequence modulo integers have been studied by many. Perhaps Lagrange was the first who noticed the periodicity of the Fibonacci sequence modulo integers by realizing that the last digit of F_n repeats with period 60 [Liv, p. 105]. Jarden showed that for $d \geq 3$ the last d digits of the Fibonacci sequence repeat with period $15 \cdot 10^{d-1}$ [Jar, Th. 1].

It's straightforward to see, using the pigeonhole principle and the recurrence relation above, that the Fibonacci sequence modulo any integer is periodic [Luc]. The smallest period of the Fibonacci sequence modulo n is called the *Pisano* period, and is denoted by $\pi(n)$. Clearly, $F_{\pi(n)} = F_0 = 0 \pmod{n}$, hence $F_{\pi(n)}$ is divisible by n. However, it's possible that an earlier term in the Fibonacci sequence is divisible by n.

Definition 1.1. The *rank of apparition* or the *order of appearance* of n is the smallest

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positive integer $z(n)$ such that $n|F_{z(n)}$.

We note the first few output values of the order of appearance map are

$$z(1) = 1, \quad z(2) = 3, \quad z(3) = 4, \quad z(4) = 6, \quad z(5) = 5, \quad z(6) = 12.$$

The diagrams in Figure 1 illustrate the behavior of the order of appearance map

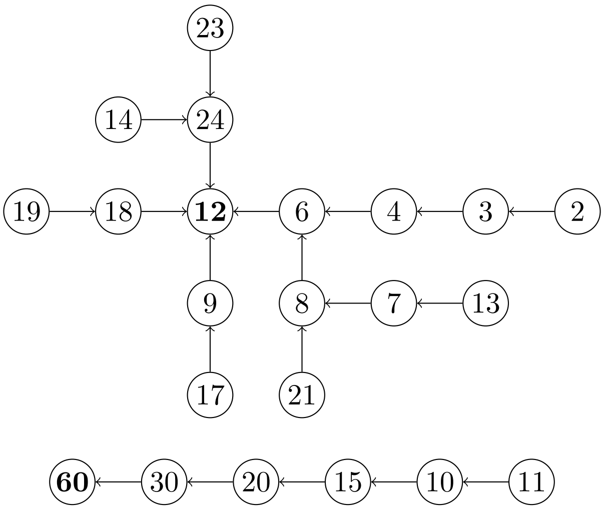

Figure 1. Illustrative example of the paths from various starting points to the fixed points 12 and 60.

Wall showed that $n|F_k$ if and only if $z(n)|k$. In particular, $z(n)|\pi(n)$ for all integers n [W]. A more detailed relationship between $\pi(n)$ and $z(n)$ was given by Vinson. He showed that, for $n > 2$, one has $\pi(n)/z(n) \in \{1, 2\}$ iff $z(n)$ is even and $\pi(n)/z(n) = 4$ iff $z(n)$ is odd [V]. Lucas showed that $z(p)|(p-1)$ if $p = \pm 1 \pmod{10}$ and $z(p)|(p+1)$ if $p = \pm 3 \pmod{10}$ for odd primes $p \neq 5$ [Luc]. Since $z(2) = 3$ and $z(5) = 5$, for any prime p we have $z(p) \leq p + 1$. In general, one has $z(n) \leq 2n$ for arbitrary integers n [Sal]; this bound is sharp, since the equality $z(n) = 2n$ occurs for $n = 6 \cdot 5^d$, where $d \geq 0$. Other results on $z(n)$ include explicit formulas for the order of appearance of some integers related to the sums and products of Fibonacci numbers [M1, M2)], and upper bounds in Lucas sequences [SK].

We are interested in the dynamical properties of the order of appearance map z.

Definition 1.2. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, where $\mathbb{N}$ is the set of positive integers. The f-orbit of an integer m is the smallest set containing m that is closed under the mapping of f. In other words, the f-orbit of m is the set

$$\mathcal{O}_f(m) = \left\{m, f(m), f^2(m), f^3(m), \ldots\right\},$$

where f^k denotes the composition of f with itself k times, $k \geq 0$ (with $f^0(m) = m$). We say the integer m is a *periodic* point of f if there exists a positive integer t such that $f^{k+t}(m) = f^k(m)$ for all $k \geq 0$. Finally, an integer m is called a *fixed point* of f if $f(m) = m$.

Example 1.3. The z-orbit of $m = 4273$ is

$$4273, \quad 2137, \quad 1069, \quad 89, \quad 11, \quad 10, \quad 15, \quad 20, \quad 30, \quad 60,$$

as 60 is a fixed point of z and thus further iterations just return 60.

n	$z(n)$	$z^2(n)$	$z^3(n)$	$z^4(n)$	$z^5(n)$
1	**1**				
2	3	4	6	**12**	
3	4	6	**12**		
4	6	**12**			
5	**5**				
6	**12**				
7	8	6	**12**		
8	6	**12**			
9	**12**				
10	15	20	30	**60**	
11	10	15	20	30	**60**
12	**12**				

Table 1. Iterations of z on n; numbers in bold are fixed points.

Table 1 shows the z-orbits of the first 12 positive integers, which all end in a fixed point. Every orbit of an integer map is either infinite or it reaches a periodic point. The orbits of the order of appearance map have been well studied. The following results on the fixed points of z and the end-behavior of z-orbits are known.

i) The fixed points of z are of the form 5^d or $12 \cdot 5^d$, where $d \geq 0$ [Jar, M4].
ii) Given an integer $n > 0$, there exists an integer $k \geq 0$ such that $z^k(n)$ is a fixed point of z [LT]. In other words, all z-orbits reach a fixed point, and so there are no infinite orbits and no periodic points of period greater than one.

We give a new proof of (ii) in Theorem 4.3. In the study of dynamical properties of an integer map, in addition to studying the fixed points, periodic points, and the end-behavior of orbits, one is interested in studying the fixed point order (defined below) as well as the set of integers whose orbits reach a given fixed point.

Definition 1.4. The *(Fibonacci) fixed point order* of a natural number n is the smallest positive integer $k = \alpha(n)$ such that $z^k(n)$ is a fixed point of z. If n itself is a fixed point of z, we let $\alpha(n) = 0$.

The fixed point order map measures how far a number n is from being a fixed point of z: how many iterations of z are required to reach a fixed point starting from n? It follows from (ii) above that the fixed point order of n is one less than the number of elements in the z-orbit of n; i.e.,

$$\alpha(n) = |\mathcal{O}_z(n)| - 1,$$

where $|A|$ denotes the number of elements of the set A.

To study the fixed point order map, we define two related sets of integers as follows. Let Λ_k denote the set of numbers whose fixed point order is k:

$$\Lambda_k \ := \ \{n \in \mathbb{N} : \alpha(n) = k\} .$$

Also, given a fixed point x, let Ω_x denote the set of integers whose z-orbit reaches x:

$$\Omega_x \ := \ \left\{n \in \mathbb{N} : \exists k \ z^k(n) = x\right\}.$$

We prove the following new results regarding Λ_k and Ω_x.

i) We prove in Theorem 2.4 that $|\Lambda_k| = \infty$ for all $k \geq 0$. Equivalently, there are infinitely many numbers whose z-orbits have a given arbitrary positive length.
ii) We prove in Theorem 3.3 that $|\Omega_x| = \infty$ for all fixed points $x > 5$. In other words, there are infinitely many numbers whose z-orbits reach x (for $x \leq 5$, one notes that $\Omega_1 = \{1\}$ and $\Omega_5 = \{5\}$).

Several questions regarding Λ_k and Ω_x remain unanswered. To prove that $|\Lambda_k| = \infty$, we show that $5^b \cdot 2^{2k+2} \in \Lambda_k$ for all $b \geq 0$ and $k \geq 1$. Is it true that for every $k \geq 1$ there exist infinitely many relatively prime integers whose fixed point order is k?

It is also interesting to learn about the frequency of a fixed point appearing in a z-orbit. To be more precise, if x is a fixed point of z, set

$$\sigma(x) \ := \ \limsup_{n\to\infty} \frac{|\Omega_x \cap [1,n]|}{n}.$$

Are some fixed points (or some types of fixed points) more likely to appear in a z-orbit?

Finally, since $2^{2k+2} \in \Lambda_k$, one has $\min \Lambda_k \leq 2^{2k+2}$ (see also Table 2). Are there more efficient lower and upper bounds on the least element of Λ_k?

2. Infinitude of integers with a given fixed point order

We prove that, given an integer $k \geq 0$, there exist infinitely many integers whose fixed point order is k (Theorem 2.4). Table 2 lists the smallest integer n that takes exactly k iterations of z to reach a fixed point, $1 \leq k \leq 10$.

k	Least n with $\alpha(n) = k$	Ending Fixed Point
0	1	1
1	6	12
2	4	12
3	3	12
4	2	12
5	11	60
6	89	60
7	1069	60
8	2137	60
9	4273	60
10	59833	60

Table 2. First n that takes k iterations to reach a fixed point.

We need the following result by Vinson in the computation of $z(n)$ from the values of z on the prime power factors of n [V, Lemma 2].

Lemma 2.1 ([V]). *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1, \dots, p_r$ are distinct prime numbers and $\alpha_i \geq 1$ for all $1 \leq i \leq r$. Then*

$$z(n) \;=\; \operatorname{lcm}\left(z(p_1^{\alpha_1}), z(p_2^{\alpha_2}), \dots, z(p_r^{\alpha_r})\right). \tag{1}$$

Given nonzero integers n_1, n_2, we write $n_1 \sim n_2$ iff $n_1/n_2 = 5^t$ for some integer t. It is readily checked that the relation $\sim$ is an equivalence relation on the set of positive integers. Moreover, a positive integer n is a fixed point of z if and only if $n \sim 1, 12$ [M4].

Lemma 2.2. *If $n_1 \sim n_2$, then $z^k(n_1) \sim z^k(n_2)$ for all $k \geq 1$.*

Proof. Since every congruence class of $\sim$ contains a number relatively prime with 5, it is sufficient to prove the claim when $n_1 = n$ and $n_2 = 5^a \cdot n$, where $a \geq 0$ and $5 \nmid n$. First, we prove the claim for $k = 1$. Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factorization of n, where $\alpha_i \geq 1$ for all $1 \leq i \leq r$. By Lemma 2.1, we have

$$\begin{aligned} z\left(5^a \cdot n\right) &= \operatorname{lcm}\left(z\left(5^a\right), z(p_1^{\alpha_1}), \dots, z(p_r^{\alpha_r})\right) \\ &= \operatorname{lcm}\left(5^a, \operatorname{lcm}\left(z(p_1^{\alpha_1}), \dots, z(p_r^{\alpha_r})\right)\right) \\ &= \operatorname{lcm}\left(5^a, z(n)\right), \end{aligned}$$

and so $z(5^a \cdot n) \sim z(n)$, and the claim follows for $k = 1$.

We proceed by induction on $k \geq 1$. Suppose the claim is true for k. Now, if $n_1 \sim n_2$, then $z^k(n_1) \sim z^k(n_2)$ by the inductive hypothesis, and so by the base case $k = 1$, we have

$$z^{k+1}\left(n_1\right) \;=\; z\left(z^k\left(n_1\right)\right) \;\sim\; z\left(z^k\left(n_2\right)\right),$$

which implies that $z^{k+1}\left(n_1\right) \;\sim\; z^{k+1}\left(n_2\right)$. □

In the next lemma, we find an explicit formula for $z^k\left(2^\alpha\right)$. Jarden, in his computation of $z(10^d)$, showed that if $\alpha \geq 3$, then $z(2^\alpha) = 3 \cdot 2^{\alpha-2}$ [Jar, Th. 5], a result that we use next.

Lemma 2.3. *For all $k \geq 1$ and $\alpha \geq 2k + 2$, we have $z^k\left(2^\alpha\right) = 3 \cdot 2^{\alpha-2k}$.*

Proof. If $k = 1$, then $\alpha \geq 3$, and so $z(2^\alpha) = 3 \cdot 2^{\alpha-2}$ [Jar]. We proceed by induction on $k \geq 1$. Suppose the claim is true for k, and suppose that $\alpha \geq 2(k+1)+2$. It follows that $\alpha - 2k \geq 3$, and so $z(2^{\alpha-2k}) = 3 \cdot 2^{\alpha-2k-2}$. Therefore, by Lemma 2.1, one has

$$\begin{aligned} z^{k+1}(2^\alpha) \;=\; z\left(z^k\left(2^\alpha\right)\right) &= z\left(3 \cdot 2^{\alpha-2k}\right) \\ &= \operatorname{lcm}\left(z\left(3\right), z(2^{\alpha-2k})\right) \\ &= \operatorname{lcm}\left(4, 3 \cdot 2^{\alpha-2k-2}\right) \;=\; 3 \cdot 2^{\alpha-2(k+1)}, \end{aligned} \tag{2}$$

since $\alpha - 2k - 2 \geq 2$. □

We can now prove that $|\Lambda_k| = \infty$ for all $k \geq 0$.

Theorem 2.4. *For every integer $k \geq 0$, there exist infinitely many integers whose fixed point order is k.*

Proof. If $k = 0$, then the claim follows from the fact that z has infinitely many fixed points, since $\Lambda_0 = \{5^d, 12 \cdot 5^d : d \geq 0\}$. Thus, suppose that $k \geq 1$. By Lemma 2.3, we have

$$z^k \left(2^{2k+2}\right) \;=\; 3 \cdot 2^{(2k+2)-2k} \;=\; 12, \tag{3}$$

which is a fixed point of z. If $k = 1$, then $z^{k-1}\left(2^{2k+2}\right) = 2^{2k+2}$, which is not a fixed point of z. If $k \geq 2$, then

$$z^{k-1} \left(2^{2k+2}\right) \;=\; 3 \cdot 2^{(2k+2)-2(k-1)} \;=\; 48, \tag{4}$$

which is again not a fixed point of z. Therefore, $2^{2k+2} \in \Lambda_k$. Next, we show that $5^b \cdot 2^{2k+2} \in \Lambda_k$ for all $b \geq 0$ (hence proving that $|\Lambda_k| = \infty$). Since $5^b \cdot 2^{2k+2} \sim 2^{2k+2}$, by Lemma 2.2, we have $z^k(5^b \cdot 2^{2k+2}) \sim z^k(2^{2k+2}) \sim 1, 12$ and $z^{k-1}(5^b \cdot 2^{2k+2}) \sim z^{k-1}(2^{2k+2}) \nsim 1, 12$, and so $5^b \cdot 2^{2k+2} \in \Lambda_k$ for all $b \geq 0$. □

Theorem 2.4 shows that, for every positive integer k, there are infinitely many integers whose z-orbits have length k, since the length of the z-orbit of m is one plus the fixed point order of m. Figure 2 illustrates how integers from 1 to 100 are distributed among the Λ_k, $0 \leq k \leq 6$. How are the numbers from 1 to n are distributed approximately among the Λ_k, $k \geq 0$?

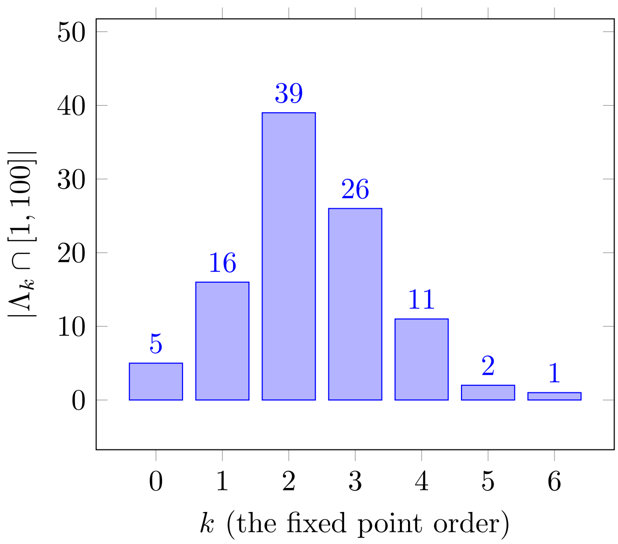

Figure 2. The distribution of integers from 1 to 100 among Λ_k, $0 \leq k \leq 6$.

From Table 2, the first few terms of the sequence $\lambda_k = \min \Lambda_k, k \geq 0$, are

$$1,\ 6,\ 4,\ 3,\ 2,\ 11,\ 89,\ 1069,\ 2137,\ 4273,\ 59833,\ \ldots.$$

It is straightforward to see that $z(\lambda_{k+1}) \geq \lambda_k$ for all $k \geq 1$, and it is probably true that $z(\lambda_{k+1}) = \lambda_k$ for k large enough. As of March 3, 2025 this sequence is not in the OEIS.

3. Infinitely many orbits end in each fixed point

In this section, we prove that Ω_x is an infinite set for all fixed points $x > 5$, where Ω_x denotes the set of integers whose z-orbit reaches x (Theorem 3.3). We need the following lemma in the computation of the values of z on odd prime powers [FM, Th. 2.4].

Lemma 3.1 ([FM]). *Let p be an odd prime number and $\alpha \geq 1$. Then $z(p^\alpha) = p^{\max(\alpha-t,0)} z(p)$ where t is the number of times that p divides $F_{z(p)}$, $t \geq 1$. In particular, $z(p^\alpha) = p^\beta z(p)$ for some $0 \leq \beta < \alpha$.*

By a theorem of Carmichael [C, Y], if $n \neq 1, 2, 6, 12$, then F_n has a primitive divisor; i.e., there exists a prime $p(n)$ such that $p(n) \mid F_n$ but $p(n) \nmid F_k$ for all $0 < k < n$; in other words, F_n is the first Fibonacci number that is divisible by $p(n)$, and so $z(p(n)) = n$. Recall that if $n_1 \mid n_2$, then $F_{n_1} \mid F_{n_2}$, a fact we use in the proof of the next lemma.

Lemma 3.2. *If $b > 1$, then F_{5^b} has at least b distinct odd prime factors. In particular, F_{5^b} has at least one odd prime factor other than 5.*

Proof. Since $5^c \mid 5^b$ for integers $1 \leq c \leq b$, we have $F_{5^c} \mid F_{5^b}$. On the other hand, $z(p(5^c)) = 5^c$, and so $p(5^c)$ is an odd prime factor of F_{5^b}, where $1 \leq c \leq b$. If $c_1 > c_2 \geq 1$, then $p(5^{c_2}) | F_{5^{c_2}}$ and $5^{c_2} < 5^{c_1}$, which means that $p(5^{c_2})$ divides an earlier Fibonacci number than $F_{5^{c_1}}$. It follows that $p(5^{c_1}) \neq p(5^{c_2})$ if $c_1 > c_2 \geq 1$. Therefore, the prime numbers $p(5), p(5^2), \ldots, p(5^b)$ are distinct odd prime factors of F_{5^b}. □

Theorem 3.3. *Let $b \geq 0$ and m be a positive integer such that $5 \nmid m$ and every prime factor of m divides F_{5^b}. Let $\eta(m)$ be the largest exponent appearing in the prime factorization of m.*

i) If $k = \eta(m)$, then $z^k(m \cdot 5^b) = 5^b$.
ii) If $a \geq 0$ and $k = \max\{\eta(m), \lceil a/2 \rceil\}$, then $z^k(m \cdot 2^a \cdot 12 \cdot 5^b) = 12 \cdot 5^b$.

Proof. Let $\mathcal{F}$ be the set of all positive integers n such that $5 \nmid n$ and every prime factor of n is a prime factor of F_{5^b}. Therefore, $m \in \mathcal{F}$ and $\eta(m) = \max\{\alpha_1, \ldots, \alpha_r\}$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $\alpha_1, \ldots, \alpha_r \geq 1$, and $p_1, \ldots, p_r$ are distinct prime factors of F_{5^b} (also $1 \in \mathcal{F}$ and $\eta(1) = 0$). It follows from Lemma 3.2 that if $b > 1$, then $\mathcal{F}$ is an infinite set.

For part (i), if $m = 1$, the claim follows from the fact that 5^b is a fixed point of z for all $b \geq 0$. Thus, suppose that $m > 1$. By Lemma 3.1, there exist $0 \leq \beta_i < \alpha_i$ such that $z(p_i^{\alpha_i}) = p_i^{\beta_i} z(p_i)$, $1 \leq i \leq r$. Since $p_i | F_{5^b}$, we have $z(p_i) | 5^b$. It follows that $\operatorname{lcm}\left(z(p_i^{\alpha_i}), 5^b\right) = \operatorname{lcm}\left(p_i^{\beta_i} z(p_i), 5^b\right) = p_i^{\beta_i} \cdot 5^b$ for all $1 \leq i \leq r$. Then

$$\begin{aligned} z(m \cdot 5^b) = z(p_1^{\alpha_1} \cdots p_r^{\alpha_r} \cdot 5^b) &= \operatorname{lcm}\left(z(p_1^{\alpha_1}), \ldots, z(p_r^{\alpha_r}), z(5^b)\right) \\ &= \operatorname{lcm}\left(p_1^{\beta_1} z(p_1), \ldots, p_r^{\beta_r} z(p_r), 5^b\right) \\ &= p_1^{\beta_1} \cdots p_r^{\beta_r} \cdot 5^b \\ &= m_1 \cdot 5^b, \end{aligned}$$

where $m_1 \in \mathcal{F}$ and $\eta(m_1) \leq \eta(m) - 1$. By a finite induction, for each $1 \leq i \leq k$ there exists $m_i \in \mathcal{F}$ such that $z^i(m \cdot 5^b) = m_i \cdot 5^b$ and $\eta(m_i) \le \eta(m) - i$. Therefore, with $i = k = \eta(m)$, we obtain $z^k(m \cdot 5^b) = 5^b$. This completes the proof of part (i).

For part (ii), suppose that $a > 0$ (the proof when $a = 0$ is similar and is omitted). If $m = 1$, then

$$\begin{aligned} z\left(2^a \cdot 12 \cdot 5^b\right) &= z\left(2^{a+2} \cdot 3 \cdot 5^b\right) \\ &= \operatorname{lcm}\left(z(2^{a+2}), z(3), z(5^b)\right) \\ &= \operatorname{lcm}\left(2^a \cdot 3, 4, 5^b\right) \\ &= 2^{\max(a-2,0)} \cdot 12 \cdot 5^b, \end{aligned}$$

and so repeating this argument $\lceil a/2 \rceil$ times proves the claim. If $m > 1$, one has

$$\begin{aligned} z\left(m \cdot 2^a \cdot 12 \cdot 5^b\right) &= z\left(p_1^{\alpha_1} \cdots p_r^{\alpha_r} \cdot 2^{a+2} \cdot 3 \cdot 5^b\right) \\ &= \operatorname{lcm}\left(z(p_1^{\alpha_1}), \ldots, z(p_r^{\alpha_r}), z(2^{a+2}), z(3), z(5^b)\right) \\ &= \operatorname{lcm}\left(p_1^{\beta_1} z(p_1), \ldots, p_r^{\beta_r} z(p_r), 2^a \cdot 3, 4, 5^b\right) \\ &= p_1^{\beta_1} \cdots p_r^{\beta_r} \cdot 2^{\max(a-2,0)} \cdot 12 \cdot 5^b \\ &= m_1 \cdot 2^{\max(a-2,0)} \cdot 12 \cdot 5^b, \end{aligned}$$

where $m_1 \in \mathcal{F}$ and $\eta(m_1) \le \eta(m) - 1$. By a finite induction, for each $1 \le i \le k$, there exists $m_i \in \mathcal{F}$ such that $z^i(m) = m_i \cdot 2^{\max(a-2i,0)} \cdot 12 \cdot 5^b$ and $\eta(m_i) \le \eta(m) - i$. Therefore, with $i = k = \max\{\eta(m), \lceil a/2 \rceil\}$, we obtain $z^k(m) = 12 \cdot 5^b$. □

In Theorem 3.3, the explicit elements that we found in Ω_x to prove its infinitude are all multiples of x itself. Of course it is not necessary for an element of Ω_x to be a multiple of x. For example, the first few elements of Ω_{12} are

$$\Omega_{12}: \ 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 12,\ 13,\ 14,\ 16,\ 17,\ 18,\ 19,\ 21,\ \ldots.$$

One conjectures that Ω_x contains infinitely many relatively prime elements for every fixed point $x > 5$.

4. All integers have finite fixed point order

We now give a new proof of Theorem 4.3: the finiteness of the fixed point order map. Recall that if p is an odd prime other than 5, then $z(p)|(p-1)$ or $z(p)|(p+1)$ [Luc]. In particular, either $z(p) = p \pm 1$ or $z(p) \le (p+1)/2$. We use this result in the proof of the following lemma.

Lemma 4.1. *Let $n = 2^a \cdot 5^b \cdot p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $r \ge 2$, where $p_1, \ldots, p_r$ are distinct odd prime numbers not equal to 5. Suppose that $b \ge 0$ and $\alpha_i \ge 1$ for all $1 \le i \le r$. Moreover, suppose that either $a = 0$, or $a \ge 3$, or $a = 2$ and $3|n$. Then $z(n) < n$.*

Proof. The proof is divided into three cases.

Case 1. Suppose that $a = b = 0$. By Lemma 3.1, there exist $0 \le \beta_i < \alpha_i$ such that $z(p_i^{\alpha_i}) = p_i^{\beta_i} z(p_i)$, $1 \le i \le r$. For each $1 \le i \le r$, we have $z(p_i) = p_i \pm 1$ or

$z(p_i) \leq (p_i+1)/2$. If $z(p_i) \leq (p_i+1)/2$ for all $1 \leq i \leq r$, then $z(p_i^{\alpha_i}) = p_i^{\beta_i} z(p_i) \leq p_i^{\alpha_i - 1}(p_i+1)/2 < p_i^{\alpha_i}$, and so, by Lemma 2.1, we have

$$z(n) \;=\; \operatorname{lcm}\left(z(p_1^{\alpha_1}), \ldots, z(p_r^{\alpha_r})\right) \;\leq\; \prod_{i=1}^{r} z(p_i^{\alpha_i}) \;<\; \prod_{i=1}^{r} p_i^{\alpha_i} \;\leq\; n.$$

Thus, without loss of generality, suppose that there exists $1 \leq k \leq r$ such that $z(p_i) = p_i \pm 1$ for $1 \leq i \leq k$ and $z(p_i) \leq (p_i+1)/2$ for $k+1 \leq i \leq r$. Then

$$z(n) \;=\; \operatorname{lcm}\left(z(p_1^{\alpha_1}), \ldots, z(p_r^{\alpha_r})\right) \;\leq\; \operatorname{lcm}\left(z(p_1^{\alpha_1}), \ldots, z(p_k^{\alpha_k})\right) \prod_{i=k+1}^{r} z\left(p_i^{\alpha_i}\right). \qquad (5)$$

Since $z(p_i^{\alpha_i}) = p_i^{\beta_i} z(p_i) = p_i^{\beta_i}(p_i \pm 1)$, each $z(p_i^{\alpha_i})$ is even, $1 \leq i \leq k$, and so

$$\operatorname{lcm}\left(z(p_1^{\alpha_1}), \ldots, z(p_k^{\alpha_k})\right) \;\leq\; \frac{1}{2^{k-1}} \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i+1). \qquad (6)$$

It follows from (5) and (6) that

$$z(n) \;\leq\; \frac{1}{2^{k-1}} \prod_{i=1}^{k} p_i^{\alpha_i - 1}\left(p_i+1\right) \prod_{i=k+1}^{r} p_i^{\alpha_i - 1}\left(\frac{p_i+1}{2}\right) \;\leq\; 2n \prod_{i=1}^{r}\left(\frac{p_i+1}{2p_i}\right). \qquad (7)$$

For $1 \leq i \leq r$, if $p_i \geq 7$, then $(p_i+1)/(2p_i) \leq 4/7$, and if $p_i = 3$, then $(p_i+1)/(2p_i) = 2/3$. It follows that

$$\prod_{i=1}^{r}\left(\frac{p_i+1}{2p_i}\right) \;\leq\; \frac{2}{3}\left(\frac{4}{7}\right)^{r-1} \;<\; \frac{1}{2}.$$

It then follows from (7) that $z(n) < n$.

Case 2. Suppose that $b \geq 0$ is arbitrary and $a = 0$ or $a \geq 3$. If $a = 0$, then $z(2^a) = 2^a$. If $a \geq 3$, then $z(2^a) = 2^{a-2} \cdot 3 \leq 2^a$. It follows that

$$\begin{aligned} z(n) \;&=\; \operatorname{lcm}\left(z(2^a), z(5^b), z(p_1^{\alpha_1}), \ldots, z(p_r^{\alpha_r})\right) \qquad (8)\\ &\leq\; z(2^a) \cdot z(5^b) \cdot \operatorname{lcm}\left(z(p_1^{\alpha_1}) \ldots z(p_r^{\alpha_r})\right)\\ &\leq\; 2^a \cdot 5^b \cdot z\left(p_1^{\alpha_1} \cdots p_r^{\alpha_r}\right)\\ &<\; 2^a \cdot 5^b \cdot p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \end{aligned}$$

by *Case 1*, hence $z(n) < n$.

Case 3. Suppose that $a = 2$ and $3|n$. Without loss of generality, we can assume that

$p_1 = 3$. In this case, since $z(4) = 6$ and $z(3^{\alpha_1}) = 3^{\alpha_1 - 1} \cdot 4$ [M3, Th. 1.1], we have

$$\begin{aligned} z(n) &= \operatorname{lcm}\left(z(4), z(5^b), z(3^{\alpha_1}), z(p_2^{\alpha_2}), \ldots, z(p_r^{\alpha_r})\right) \\ &\le z(5^b) \cdot \operatorname{lcm}\left(6, 3^{\alpha_1 - 1} \cdot 4, z(p_2^{\alpha_2}), \ldots, z(p_r^{\alpha_r})\right) \\ &\le z(5^b) \cdot 3 \cdot \operatorname{lcm}\left(z(3^{\alpha_1}), z(p_2^{\alpha_1}), \ldots, z(p_r^{\alpha_r})\right) \\ &\le 5^b \cdot 3 \cdot z\left(3^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}\right) \\ &< 5^b \cdot 4 \cdot 3^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}, \end{aligned}$$

again by *Case 1*, hence $z(n) < n$ in this case as well. □

Since $z(p) | p \pm 1$ for any odd prime number $p \neq 5$, either $z(p) = p + 1$ or $z(p) < p$ [Luc]. For composite numbers, z is generally non-increasing, at least on the set of multiples of 12.

Lemma 4.2. *If n is a multiple of 12, then $z(n) \le n$.*

Proof. Let $n = 12 \cdot 5^b \cdot m$, where $b \ge 0$ and m is not divisible by 5. First, suppose that m is odd. If $m = 1$, then n is a fixed point of z, and so $z(n) = n$ in this case. If m has any prime factors other than 3, then the claim follows from Lemma 4.1. Thus, suppose that $m = 3^c$, where $c \ge 1$, and so $n = 4 \cdot 5^b \cdot 3^{c+1}$. Since $z(3^{c+1}) = 3^c \cdot 4$ [M3, Th. 1.1], by Lemma 2.1, we have

$$\begin{aligned} z(n) = z\left(4 \cdot 5^b \cdot 3^{c+1}\right) &= \operatorname{lcm}\left(z(4) \cdot z\left(5^b\right) \cdot z\left(3^{c+1}\right)\right) \\ &= \operatorname{lcm}\left(6, 5^b, 3^c \cdot 4\right) \\ &= 4 \cdot 5^b \cdot 3^c < n. \end{aligned}$$

Next, suppose that m is even and let $m = 2^a \cdot m'$, where $a \ge 1$ and m' is odd. Therefore, we have $n = 2^{a+2} \cdot 5^b \cdot m'$. If m' has any prime factors other than 3, the claim follows from Lemma 4.1. Thus, suppose that $m' = 3^c$, $c \ge 1$. Then

$$\begin{aligned} z(n) = z(2^{a+2} \cdot 5^b \cdot 3^c) &= \operatorname{lcm}\left(z\left(2^{a+2}\right), z\left(5^b\right), z\left(3^c\right)\right) \\ &= \operatorname{lcm}\left(2^a \cdot 3, 5^b, 3^{c-1} \cdot 4\right) \\ &\le 2^{a+1} \cdot 5^b \cdot 3^c < n, \end{aligned}$$

and the lemma follows. □

We are now ready to prove the finiteness of the fixed point order map.

Theorem 4.3. *All positive integers n have finite fixed point order.*

Proof. We first prove the claim for even integers n. If n is even, then $z(n)$ is divisible by $z(2) = 3$, and so $z^2(n)$ is divisible by $z(3) = 4$. Then $z^3(n)$ is divisible by $z(4) = 6$ and $z^4(n)$ is divisible by $z(6) = 12$. Consequently, $z^k(n)$ is divisible by 12 for all $k \ge 4$. It then follows from Lemma 4.2 that $z^{k+1}(n) \le z^k(n)$ for all $k \ge 4$. In other words, the sequence $z^4(n), z^5(n), \ldots$ is a non-increasing sequence of positive integers, and so there must exist $k \ge 0$ such that $z^{k+1}(n) = z^k(n)$, hence $z^k(n)$ is a fixed point of z.

To prove the claim for odd numbers, let n be the smallest odd counterexample to the statement of the theorem. In particular, $z^k(n)$ is odd for all $k \ge 1$; otherwise, if $z^k(n)$ is even for some $k \geq 1$, a contradiction arises from the even case we just discussed (as we showed every even number has an iterate which is a fixed point).

It follows from Lemma 4.1 that n has at most one prime factor other than 5. If not then $z(n) < n$ as n has at least two odd factors not equal to 5. By assumption $z(n)$ is odd, and if it iterates to a fixed point then so too does n, but as we are assuming n is the smallest odd integer not iterating to a fixed point we obtain a contradiction as $z(n)$ is a smaller such odd number. Therefore there is an odd prime $p \neq 5$ such that $n = 5^b \cdot p^\alpha$, where $b \geq 0$, $\alpha > 0$ (if $\alpha = 0$ then n iterates to a fixed point as it is a power of 5). By Lemma 3.1, there exists $0 \leq \beta < \alpha$ such that $z(p^\alpha) = p^\beta z(p) \leq p^{\alpha-1} z(p)$. If $z(p) \neq p \pm 1$, then $z(p) \leq (p+1)/2$, hence

$$\begin{aligned} z(n) \ = \ \operatorname{lcm}\left(z(5^b), z(p^\alpha)\right) \ &\leq \ z\left(5^b\right) z\left(p^\alpha\right) \\ &\leq \ 5^b \cdot p^{\alpha-1}(p+1)/2 \ < \ n. \end{aligned}$$

Since $z(n)$ is odd and n is the least counterexample, we arrive at a contradiction. On the other hand, if $z(p) = p \pm 1$, then $z(n) = \operatorname{lcm}\left(z(5^b), p^\alpha(p \pm 1)\right)$ is even. This is also a contradiction (from our argument for the even case), and the theorem follows. □

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