HYPER-BISHOPS, HYPER-ROOKS, AND HYPER-QUEENS: PERCENTAGE OF SAFE SQUARES ON HIGHER DIMENSIONAL CHESS BOARDS

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ABSTRACT. The n queens problem considers the maximum number of safe squares on an $n \times n$ chess board when placing n queens; the answer is only known for small n. Miller, Sheng and Turek considered instead n randomly placed rooks, proving the proportion of safe squares converges to $1/e^2$. We generalize and solve when randomly placing n hyper-rooks and n^{k-1} line-rooks on a k-dimensional board, using combinatorial and probabilistic methods, with the proportion of safe squares converging to $1/e^k$. We prove that the proportion of safe squares on an $n \times n$ board with bishops in 2 dimensions converges to $2/e^2$. This problem is significantly more interesting and difficult; while a rook attacks the same number of squares wherever it's placed, this is not so for bishops. We expand to the k-dimensional chessboard, defining line-bishops to attack along 2-dimensional diagonals and hyper-bishops to attack in the $k-1$ dimensional subspace defined by its diagonals in the $k-2$ dimensional subspace. We then combine the movement of rooks and bishops to consider the movement of queens in 2 dimensions, as well as line-queens and hyper-queens in k dimensions.

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1. INTRODUCTION

Chess is a deep well of ideas for mathematical problems, inspiring entire textbooks such as [Wat04] and [Pet11]. A famous problem is the n queens problem, originally proposed by Bezzel under the pen-name “Schadenfreude" in [Bez48], which considers the maximum number of safe squares on an $n \times n$ chess board when placing n queens. This problem has attracted substantial interest, described in [BS09], but the solution is only known for small n, as shown in [LV11]. A similar problem is the dominating queens problem, which looks at the minimum number of queens required to cover an $n \times n$ chess board.

In 2020, Miller, Sheng, and Turek showed in [MST21] that for an $n \times n$ board with n randomly placed rooks, the percentage of safe squares converges to $1/e^2$ as $n \to \infty$. Inspired by this paper, we adopt their techniques to several generalizations of the problem. We begin by restating some of their key definitions and notation.

Definition 1.1. *A board configuration, denoted $\mathcal{B}$, is a choice of placements of attacking pieces. For a board configuration on a k-dimensional chessboard, we define the binary indicator variable $X_{x_1,\dots,x_k}$ by*

$$X_{x_1,\dots,x_k}(\mathcal{B}) \;:=\; \begin{cases} 1 & (x_1,\dots,x_k) \text{ is safe under } \mathcal{B} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

On a board with side length n, we denote by $S_n(\mathcal{B})$ the number of safe spaces on the board for configuration $\mathcal{B}$, so

$$S_n(\mathcal{B}) \;:=\; \sum_{x_1,\dots,x_n=1}^{n} X_{x_1,\dots,x_n}(\mathcal{B}), \tag{1.2}$$

and

$$\mathbb{E}[S_n] \;=\; \sum_{x_1,\dots,x_n=1}^{n} \mathbb{E}[X_{x_1,\dots,x_n}(\mathcal{B})]. \tag{1.3}$$

Lastly, we define

$$\mu_n \;:=\; \frac{1}{n^k} \sum_{x_1,\dots,x_n=1}^{n} \mathbb{E}[X_{x_1,\dots,x_n}(\mathcal{B})], \tag{1.4}$$

so μ_n is the expected proportion of safe squares on the board.

We first expand to bishops and queens in 2 dimensions, and determine the expected percentage of safe squares. Notably, the problem becomes much more interesting here. While a rook at any place on the board attacks the same number of spaces, bishops and queens attack differently depending on which square they are placed in. As an illustration, note that on an $n \times n$ board, with n odd, a bishop in the center attacks $2(n-1)$ squares while a bishop placed on the edge attacks $n-1$ squares. Additionally, the probability of being placed on a square near the inside differs from the probability of being placed on an outer square.

Theorem 1.2. *As n approaches infinity, the mean number of safe squares on an $n \times n$ chessboard with n randomly placed bishops is asymptotically $2n^2/e^2$, and the expected proportion of safe squares converges to $2/e^2$.*

Theorem 1.3. *As n approaches infinity, the mean number of safe squares on an $n \times n$ chessboard with n randomly placed queens is asymptotically $2n^2/e^4$, and the expected proportion of safe squares converges to $2/e^4$.*

For both of the above problems, we prove that the variance is tends to 0 as n approaches infinity, which means that the percentage of safe squares for a given board $\mathcal{B}$ is almost certainly very close to the expected value as n grows large. We prove this for placing dn^{k-m} pieces in k dimensions, each of which attack some an^m squares, meaning that it generalizes to higher dimensions.

Theorem 1.4. *Let $n, k, m, d, a \in \mathbb{Z}_{>0}$. Define μ_n as in Definition 1.1, with dn^{k-m} attacking pieces placed, each of which attack an^m spaces. Then, the variance of the random variable with mean μ_n approaches 0 as n approaches infinity.*

For our results in higher dimensions, we first introduce some notation and terminology.

1.1. **Notation and Basic Definitions in Higher Dimensions.** As we move into the k^{th} dimension, we first provide some definitions. There are a variety of possible extensions, all with their own strengths and weaknesses, so choosing to define pieces or setups differently then we do may prove an interesting path for future work.

Definition 1.5 (Higher dimensional boards)**.** *A k-dimensional board has k dimensions with equal integer side length n. Boards are created by stacking alternating boards in the $(k-1)$-dimensional subspace so that no two adjacent squares are the same color.*

Note that this is how a standard 2-dimensional chessboard is created. Squares of alternating color are placed in a line; then, the colors are shifted in the line above so that no square is adjacent to a square of the same color. This definition is particularly strong as it guarantees that any subspace of a k-dimensional chessboard is still a chessboard. It also ensures that bishops in k dimensions, as we later define them, maintain their parity. We illustrate what a $5 \times 5 \times 5$ chessboard would look like in Figure 1.1.

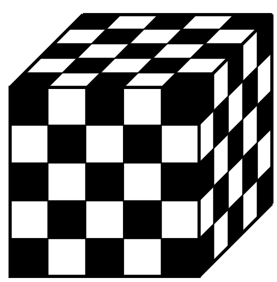

FIGURE 1. Depiction of a $5 \times 5 \times 5$ chessboard.

Definition 1.6 (Higher dimensional line-pieces)**.** *For any dimension $k > 1$, a k-dimensional line-piece moves in perpendicular planes as it does on a 2-dimensional chessboard. It can move in any 2-dimensional plane, with all other coordinates being held constant. When determining the expected number of safe squares, we place n^{k-1} line-pieces.*

Note here that we place n^{k-1} line-pieces because the number of pieces required to dominate the board is of order n^{k-1}. If we were to place some dn^{k-2} line-pieces, then each attacks some an spaces, and since n tends to infinity, then dan^{k-1} never covers a positive proportion of all n^k spaces.

Definition 1.7 (Higher dimensional hyper-pieces)**.** *For any dimension k, a k-dimensional hyper-piece attacks in $(k-1)$-dimensional subspaces. As an example, since rooks on 2-dimensional chessboard attack any piece that share a line with them, hyper-rooks on a 3-dimensional chessboard attack any piece that share a plane with them. When determining the expected number of safe squares, we place n hyper-pieces.*

We then find that for our line-rooks and hyper-rooks, after adjusting for their relative board covering power, they cover an equal amount of the board on average. More explicitly, we have the following theorem.

Theorem 1.8. *The mean number of safe spaces on an k-dimensional chessboard with side length n and either n hyper-rooks or n^{k-1} line-rooks placed on it is asymptotically n^k/e^k. Hence, the probability a space is safe converges to $1/e^k$.*

Finally, we provide definitions for line-bishops and line-queens, as well as hyper-bishops and hyper-queens in k dimensions. This problem quickly becomes fascinating. While a rook at any given space attacks the same number of squares, and so generalizes easily, the number of squares a bishop attacks varies widely, often with little symmetry. Due to this, we are unable to determine the expected proportion of spaces hit in the k dimensions, and instead analyze 3 dimensions.

2. Combinatorial Work

Many of the results in [MST21] follow from

$$\lim_{n\to\infty} \binom{n^2 - an - b}{n} \Big/ \binom{n^2}{n} \;=\; \frac{1}{e^a}.$$

We generalize their proof to show that a similar statement holds in the k^{th} dimension.

Lemma 2.1. *For positive integers a, k, m, c, d and any integer b, with $k > m > k - c$, we have*

$$\lim_{n\to\infty} \binom{n^k - an^m + bn^{k-c}}{dn^{k-m}} \Big/ \binom{n^k}{dn^{k-m}} \;=\; \frac{1}{e^{da}}. \tag{2.1}$$

Proof. Note that

$$\begin{aligned}
&\binom{n^k - an^m + bn^{k-c}}{dn^{k-m}} \Big/ \binom{n^k}{dn^{k-m}} \\
=\;& \frac{(n^k - an^m + bn^{k-c})!}{(dn^{k-m})!(n^k - an^m + bn^{k-c} - dn^{k-m})!} \cdot \frac{(dn^{k-m})!(n^k - dn^{k-m})!}{(n^k)!} \\
=\;& \frac{(n^k - an^m + bn^{k-c})!}{(n^k - an^m + bn^{k-c} - dn^{k-m})!} \cdot \frac{(n^k - dn^{k-m})!}{(n^k)!} \\
=\;& \frac{(n^k - dn^{k-m})(n^k - dn^{k-m} - 1)\cdots(n^k - dn^{k-m} - an^m + bn^{k-c} + 1)}{(n^k)(n^k - 1)\cdots(n^k - an^m + bn^{k-c} + 1)} \\
=\;& \prod_{i=0}^{an^m - bn^{k-c} - 1} \frac{n^k - dn^{k-m} - i}{n^k - i} \\
=\;& \prod_{i=0}^{an^m - bn^{k-c} - 1} \left(1 - \frac{dn^{k-m}}{n^k - i}\right).
\end{aligned} \tag{2.2}$$

Knowing that $\lim_{n\to\infty}(1 - d/n^m)^{an^m} = 1/e^{da}$, we look to express $(1 - (dn^{k-m})/(n^k - i))$ as $(1 - d/n^m - \delta)$, where δ is some small correction. To this end, we note

$$\frac{dn^{k-m}}{n^k - i} \;=\; \frac{d}{n^m} + \frac{di}{n^m(n^k - i)}. \tag{2.3}$$

Since the product is defined for all $i \leq an^m - bn^{k-c} - 1$, we see that

$$\left(1 - \frac{d}{n^m} - \frac{d(an^m - bn^{k-c} - 1)}{n^m(n^k - an^m + bn^{k-c} + 1)}\right)^{an^m - bn^{k-c}} \leq \prod_{i=0}^{an^m - bn^{k-c} - 1} \left(1 - \frac{dn^{k-m}}{n^k - i}\right) \leq \left(1 - \frac{d}{n^m}\right)^{an^m - bn^{k-c}}. \quad (2.4)$$

Thus

$$\lim_{n\to\infty} \left(1 - \frac{d}{n^m}\right)^{an^m - bn^{k-c}} = \lim_{n\to\infty} \left(\left(1 - \frac{d}{n^m}\right)^{n^m}\right)^a \cdot \left(\left(1 - \frac{d}{n^m}\right)^{n^{k-c}}\right)^b = \frac{1}{e^{da}}. \quad (2.5)$$

Therefore, for any n^{k-c} such that $m > k - c$, $\lim_{n\to\infty}((1 - \frac{1}{n^m})^{n^{k-c}})^b = 1$. Hence, any n with degree less than m does not impact the limit.

We now consider the lower bound. Factoring out $(1 - \frac{d}{n^m})$, whose behavior we understand, we see that what is left approaches 1, and thus does not change the limit. We find

$$\begin{aligned} & 1 - \frac{d}{n^m} - \frac{d(an^m - bn^{k-c} - 1)}{n^m(n^k - an^m - bn^{k-c} - 1)} \\ &= \left(1 - \frac{d}{n^m}\right)\left(1 - \frac{d(an^m - bn^{k-c} - 1)}{n^m(n^k - an^m - bn^{k-c} - 1)} \cdot \frac{n^m}{n^m - d}\right) \\ &= \left(1 - \frac{d}{n^m}\right)\left(1 - \frac{d(an^m - bn^{k-c} - 1)}{n^k - an^m - bn^{k-c} - 1} \cdot \frac{1}{n^m - d}\right). \end{aligned} \quad (2.6)$$

So from (2.4) and substituting with the above equation, we have

$$\begin{aligned} & \left(1 - \frac{d}{n^m}\right)^{an^m - bn^{k-c}} \left(1 - \frac{d(an^m - bn^{k-c} - 1)}{n^k - an^m - bn^{k-c} - 1} \cdot \frac{1}{n^m - d}\right)^{an^m - bn^{k-c}} \\ & \leq \prod_{i=0}^{an^m - bn^{k-c} - 1} \left(1 - \frac{dn^{k-m}}{n^k - i}\right) \\ & \leq \left(1 - \frac{d}{n^m}\right)^{an^m - bn^{k-c}}. \end{aligned} \quad (2.7)$$

We now show via Squeeze Theorem that

$$\lim_{n\to\infty} \left(1 - \frac{d(an^m - bn^{k-c} - 1)}{n^k - an^m - bn^{k-c} - 1} \cdot \frac{1}{n^m - d}\right)^{an^m - bn^{k-c}} = 1. \quad (2.8)$$

Trivially,

$$\lim_{n\to\infty} \left(1 - \frac{d(an^m - bn^{k-c} - 1)}{n^k - an^m - bn^{k-c} - 1} \cdot \frac{1}{n^m - d}\right)^{an^m - bn^{k-c}} \leq (1)^{an^m - bn^{k-c}} = 1. \quad (2.9)$$

For the other direction, let $\varepsilon > 0$. Then, as n approaches infinity, $\varepsilon > \frac{d(an^m - bn^{k-c}-1)}{n^k - an^m - bn^{k-c}-1}$ since $k > m$. Then, $\frac{\varepsilon}{n^m - d} > \frac{d(an^m - bn^{k-c}-1)}{(n^k - an^m - bn^{k-c}-1)(n^m-d)}$, so consider

$$\lim_{n\to\infty}\left(1 - \frac{d(an^m - bn^{k-c}-1)}{n^k - an^m - bn^{k-c}-1}\cdot\frac{1}{n^m-d}\right)^{an^m - bn^{k-c}} > \lim_{n\to\infty}\left(1-\frac{\varepsilon}{n^m-d}\right)^{an^m-bn^{k-c}}$$

$$= \lim_{n\to\infty}\left(\left(1-\frac{\varepsilon}{n^m}\right)^{n^m}\right)^a\left(\left(1-\frac{\varepsilon}{n^m}\right)^{n^{k-c}}\right)^b. \quad (2.10)$$

Then, (2.5) gives us

$$\begin{aligned}
\lim_{n\to\infty}\left(\left(1-\frac{\varepsilon}{n^m}\right)^{n^m}\right)^a\left(\left(1-\frac{\varepsilon}{n^m}\right)^{n^{k-c}}\right)^b &= \lim_{n\to\infty}\left(\left(1-\frac{\varepsilon}{n^m}\right)^{n^m}\right)^a\lim_{n\to\infty}\left(\left(1-\frac{\varepsilon}{n^m}\right)^{n^{k-c}}\right)^b \\
&= \lim_{n\to\infty}\left(\left(1-\frac{\varepsilon}{n^m}\right)^{n^m}\right)^a\cdot 1 \\
&= e^{-\varepsilon a}. \quad (2.11)
\end{aligned}$$

Then, as ε approaches 0, $e^{-\varepsilon a}\to e^0 = 1$, proving that

$$\lim_{n\to\infty}\left(1 - \frac{d(an^m - bn^{k-c}-1)}{n^k - an^m - bn^{k-c}-1}\cdot\frac{1}{n^m-d}\right)^{an^m - bn^{k-c}} = 1. \quad (2.12)$$

Therefore,

$$\lim_{n\to\infty}\frac{\binom{n^k - an^m + bn^{k-c}}{dn^{k-m}}}{\binom{n^k}{dn^{k-m}}} = \frac{1}{e^{da}}. \quad (2.13)$$

□

Remark 2.2. *This lemma is extremely powerful because of the behavior it describes. Each term can be related to the chess problems.*

n^k	*Total spaces on chess board.*
$an^m - bn^{k-c}$	*Spaces attacked by piece.*
dn^{k-m}	*Pieces placed on board.*

We count the number of board setups in which a space is safe and divide by the total number of possible board configurations, ending up with the probability that the space is safe.

3. Bishops and Queens in the 2nd Dimension

3.1. **Probability of a Safe Square for Bishops and Queens.** While in the rook case a rook hits the same number of squares for any (i, j) it gets placed on, the same is not true for bishops or queens. Recall the expected percentage of safe squares is

$$\mu_n := \frac{1}{n^2}\sum_{i,j=1}^{n}\mathbb{E}[X_{i,j}]. \quad (3.1)$$

Note that the expected number of safe squares is different for each value of (i, j).

We first consider the odd board case, although we show later that parity becomes irrelevant as n approaches infinity. We define the number of squares a bishop hits in terms of r rings, defining the 0^{th} ring to contain the square $((n-1)/2, (n-1)/2)$ and recursively define $(r+1)^{\text{st}}$ ring to contain the squares along the border of the r^{th} ring, as demonstrated by the colored rings in Figure 2. Then the outermost ring is at $r = (n-1)/2$.

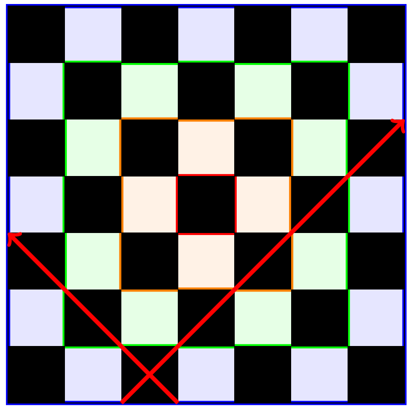

FIGURE 2. Depiction of the rings on a 7 by 7 chessboard, as well as the attacking path of a bishop at (3,1).

For all $r \neq 0$, the r^{th} ring has width $2r+1$ and therefore perimeter $4(2r+1)-4 = 4(2r)$, since squares overlap at the corners. Note that the center square in the 0^{th} ring attacks $2(n-1)$ squares and occupies one. Additionally, we see that for each ring movement outward, a bishop attacks 2 less squares on that side and attacks no more on the other side; however, any movement within a ring does not change the number of squares a bishop attacks, as it gains an equal amount of squares it attacks as it loses. Therefore, any bishop in the r^{th} ring attacks $2n-2r-1$ squares.

Thus, the expected percentage of safe squares on an odd $n \times n$ chessboard with n randomly placed bishops is:

$$\mu_n \;=\; \frac{1}{n^2} \cdot \frac{\binom{n^2-2n+1}{n}}{\binom{n^2}{n}} + \sum_{r=1}^{(n-1)/2} \frac{4(2r)}{n^2} \frac{\binom{n^2-2n+2r+1}{n}}{\binom{n^2}{n}}. \tag{3.2}$$

Before we evaluate the limit as n approaches infinity, we provide naive bounds. When a bishop is placed in the center ring, with $r=0$, it attacks $2n-1$ squares. If all bishops were this powerful, then by Lemma 2.1, the expected percentage of safe squares would be $1/e^2$. Any bishop placed on the outermost ring, such that $r = (n-1)/2$, attacks $2n-2((n-1)/2)-1 = n$ squares. If all bishops were this powerful, the expected percentage of safe squares would be $1/e$. Since randomly placed bishops are at most as powerful as a center bishop, and at least as powerful as an outer bishop, we bound the expected percentage of safe squares by

$$\frac{1}{e^2} \;\leq\; \mu_n \;\leq\; \frac{1}{e}. \tag{3.3}$$

With these bounds in mind, we now evaluate the limit.

We begin by simplifying. We first note that the probability of being placed on the center square is $1/n^2$ and it attacks $2n-1$ spaces, meaning that as n approaches infinity, it does not contribute to the main term. Additionally, as shown in Lemma 2.1, any term of degree less than $k-1$ becomes negligible as n approaches infinity, so we can ignore such terms as in the limit they contribute a factor of 1. We take the limit

$$\lim_{n\to\infty} \sum_{r=1}^{(n-1)/2} \left(\frac{8r}{n^2} \frac{\binom{n^2-2n+2r}{n}}{\binom{n^2}{n}} \right) \;=\; \lim_{n\to\infty} \sum_{r=1}^{(n-1)/2} \left(\frac{8r}{n^2} \prod_{\alpha=0}^{2n-2r} \frac{n^2-n-\alpha}{n^2-\alpha} \right). \tag{3.4}$$

By substituting in $\alpha = 2n$ and $\alpha = 0$ we find that

$$\left(1 - \frac{n}{n^2-2n}\right)^{2n-2r} \;\leq\; \prod_{\alpha=0}^{2n-2r} \frac{n^2-n-\alpha}{n^2-\alpha} \;\leq\; \left(1-\frac{1}{n}\right)^{2n-2r}. \tag{3.5}$$

Substituting these in we have

$$\frac{8}{n^2}\left(1-\frac{n}{n^2-2n}\right)^{2n}\sum_{r=1}^{(n-1)/2} r\left(1-\frac{n}{n^2-2n}\right)^{-2r} \;\le\; \sum_{r=1}^{(n-1)/2}\left(\frac{8r}{n^2}\prod_{\alpha=0}^{2n-2r}\frac{n^2-n-\alpha}{n^2-\alpha}\right)$$
$$\le\; \frac{8}{n^2}\left(1-\frac{1}{n}\right)^{2n}\sum_{r=1}^{(n-1)/2} r\left(1-\frac{1}{n}\right)^{-2r}. \quad (3.6)$$

Then, by expanding the sum, we see that this implies

$$\frac{8}{n^2}\left(1-\frac{n}{n^2-2n}\right)^{2n}\frac{(1-\frac{n}{n^2-2n})^{-2}(\frac{n-1}{2}(1-\frac{n}{n^2-2n})^{-n-1}-\frac{n+1}{2}(1-\frac{n}{n^2-2n})^{-n+1}+1)}{(1-(1-\frac{n}{n^2-2n})^{-2})^2}$$
$$\le\; \sum_{r=1}^{(n-1)/2}\left(\frac{8r}{n^2}\prod_{\alpha=0}^{2n-2r}\frac{n^2-n-\alpha}{n^2-\alpha}\right)$$
$$\le\; \frac{8}{n^2}\left(1-\frac{1}{n}\right)^{2n}\frac{(1-\frac{1}{n})^{-2}(\frac{n-1}{2}(1-\frac{1}{n})^{-n-1}-\frac{n+1}{2}(1-\frac{1}{n})^{-n+1}+1)}{(1-(1-\frac{1}{n})^{-2})^2}. \quad (3.7)$$

Then, we take the limit as n approaches infinity (see Lemmas A.1 and A.2 for more detail) and find:

$$8\cdot\frac{1}{e^2}\cdot\frac{1}{4} \;\le\; \lim_{n\to\infty}\sum_{r=1}^{(n-1)/2}\left(\frac{8r}{n^2}\prod_{\alpha=0}^{2n-2r}\frac{n^2-n-\alpha}{n^2-\alpha}\right) \;\le\; 8\cdot\frac{1}{e^2}\cdot\frac{1}{4}. \quad (3.8)$$

Since, we showed that constant terms do not impact the final limit, by the squeeze theorem:

$$\lim_{n\to\infty}\sum_{r=1}^{(n-1)/2}\left(\frac{8r}{n^2}\prod_{\alpha=0}^{2n-2r-1}\frac{n^2-n-\alpha}{n^2-\alpha}\right) \;=\; \frac{2}{e^2} \;\approx\; 27.067\%. \quad (3.9)$$

Evaluating the limit for the even case yields the same result. We can define the even case similarly. The 0^{th} ring has 4 squares, and the r^{th} ring has $4(2r+1)$ spaces. Because we index from 0, the outermost ring is the $(n/2-1)^{\text{st}}$ ring. A bishop in the 0^{th} ring attacks $n-1$ squares along one diagonal and $n-2$ along the other, and occupies one; for every ring movement outward, it attacks 2 less squares.

Thus, the expected percentage of safe squares on an even $n\times n$ chessboard with n randomly placed bishops is

$$\mu_n \;=\; \sum_{r=0}^{(n/2-1)}\frac{4(2r+1)}{n^2}\frac{\binom{n^2-2n+2r+2}{n}}{\binom{n^2}{n}}$$
$$=\; \frac{4}{n^2}\frac{\binom{n^2-2n+2}{n}}{\binom{n^2}{n}}+\sum_{r=1}^{(n/2-1)}\frac{4(2r)}{n^2}\frac{\binom{n^2-2n+2r+2}{n}}{\binom{n^2}{n}}+\sum_{r=1}^{(n/2-1)}\frac{4}{n^2}\frac{\binom{n^2-2n+2r+2}{n}}{\binom{n^2}{n}}. \quad (3.10)$$

Again we simplify, noting that the first term is negligible as n approaches infinity, and we can disregard constants by Lemma 2.1. The last term also tends to 0 as n approaches infinity, as we show below:

$$
\begin{aligned}
\lim_{n\to\infty} \sum_{r=1}^{(n/2-1)} \frac{4}{n^2} \frac{\binom{n^2-2n+2r+2}{n}}{\binom{n^2}{n}} &= \lim_{n\to\infty} \sum_{r=1}^{(n/2-1)} \frac{4}{n^2} \prod_{\alpha=0}^{2n-2r} \frac{n^2-n-\alpha}{n^2-\alpha} \\
&\leq \lim_{n\to\infty} \sum_{r=1}^{(n/2-1)} \frac{4}{n^2} \left(1-\frac{1}{n}\right)^{2n-2r} \\
&= \lim_{n\to\infty} \frac{4}{n^2} \left(1-\frac{1}{n}\right)^{n} \frac{n^2(1-1/n)^n+2n-1}{2n-1} \\
&= 0. \qquad (3.11)
\end{aligned}
$$

Thus, for the even case, we take the limit

$$
\lim_{n\to\infty} \sum_{r=1}^{(n/2-1)} \left(\frac{8r}{n^2} \frac{\binom{n^2-2n+2r}{n}}{\binom{n^2}{n}} \right). \qquad (3.12)
$$

As n approaches infinity, the difference between $n/2-1$ and $(n-1)/2$ is negligible, meaning that in both even and odd cases the expected proportion of squares attacked is $2/e^2$.

Note that the squares in which the rook movement of a queen and the bishop movement of a queen attack are disjoint sets. A queen in the r^{th} ring attacks $2(n-1)$ squares with rook movement and $2(n-1)-2r$ with bishop movement and occupies one space. From this, it follows that the percentage of safe squares on an $n \times n$ board with n randomly placed queens is

$$
\sum_{r=0}^{(n-1)/2} \frac{4(2r)}{n^2} \frac{\binom{n^2-4n+2r+1}{n}}{\binom{n^2}{n}}. \qquad (3.13)
$$

Following the proof above, we evaluate this limit of this expression as n approaches infinity to be $2/e^4$.

3.2. **Variance for 2D Bishops and Queens.** As we determine variance throughout the paper, we use a similar method for all pieces in any k dimensions. To reduce repetition, we prove a general version here, noting that it applies to bishops and queens in the two dimensions.

Theorem 3.1. *Let $n, k, m, d, a \in \mathbb{Z}^{>0}$. Define μ_n as the average percentage of safe squares on a k-dimensional chessboard with side length n, with dn^{k-m} attacking pieces placed, each of which attack an^m spaces. Then, the variance of the random variable with mean μ_n approaches 0 as n approaches infinity.*

Proof. By the definition of variance and standard properties:

$$
\begin{aligned}
\operatorname{Var}\left(\frac{S_n}{n^k}\right) &= \frac{\operatorname{Var}(S_n)}{n^{2k}} \\
&= \frac{1}{n^{2k}} \left(\sum_{i_1,i_2,\ldots,i_k=1}^{n} \operatorname{Var}(X_{i_1,i_2,\ldots,i_k}) + \sum_{i_1,\ldots,i_k,j_1,\ldots,j_k=1}^{n} \operatorname{Cov}(X_{i_1,i_2,\ldots,i_k}, X_{j_1,j_2,\ldots,j_k}) \right). \qquad (3.14)
\end{aligned}
$$

We then look at the variance and covariance terms separately.

Variance Term From the definition of variance, and because $X_{i_1,...,i_k}$ is a binary indicator function,

$$\begin{aligned}\text{Var}(X_{i_1,...,i_k}) &= \mathbb{E}(X^2_{i_1,...,i_k}) - \mathbb{E}(X_{i_1,...,i_k})^2 \\ &= \mathbb{E}(X_{i_1,...,i_k}) - \mathbb{E}(X_{i_1,...,i_k})^2 \\ &= \mu_n - \mu_n^2.\end{aligned}$$

Thus, the contribution of the variance term to S_n is

$$\sum_{i_1,i_2,...,i_k=1}^{n} \text{Var}(X_{i_1,i_2,...,i_k}) = n^k(\mu_n - \mu_n^2). \tag{3.15}$$

Covariance Term We now consider the covariance term. The binary indicator variables $X_{i_1,...,i_k}$ and $X_{j_1,...,j_k}$ both consider all possible board configurations, so have the same expected value. Then, the following holds:

$$\begin{aligned}&\text{Cov}(X_{i_1,...,i_k}, X_{j_1,...,j_k}) \\ &= \mathbb{E}[(X_{i_1,...,i_k} - \mu_{i_1,...,i_k})(X_{j_1,...,j_k} - \mu_{j_1,...,j_k})] \\ &= \mathbb{E}[X_{i_1,...,i_k}X_{j_1,...,j_k}] - \mathbb{E}[\mu_{j_1,...,j_k}X_{i_1,...,i_k}] - \mathbb{E}[\mu_{i_1,...,i_k}X_{j_1,...,j_k}] - \mu_{i_1,...,i_k}\mu_{j_1,...,j_k} \\ &= \mathbb{E}[X_{i_1,...,i_k}X_{j_1,...,j_k}] - \mu_n^2.\end{aligned} \tag{3.16}$$

As there are n^k choices for $(i_1,...,i_k)$ and $n^k - 1$ choices for $(j_1,...,j_k)$, there are $n^{2k} - n^k$ distinct pairs. We consider two cases.

Case 1: Pieces do not attack each other. First, we calculate how many such pairs there are. There are n^k choices for $(i_1,...,i_k)$, which attack some $a_i n^m - b_i n^{k-c}$ squares. So then, there are $n^k - a_i n^m + b_i n^{k-c}$ choices for $(j_1,...,j_k)$, meaning there are $n^k(n^k - a_i n^m + b_i n^{k-c})$ pairs that do not attack each other. Since every piece attacks in some dimension less than k, the number of non-attacking pairs is always of degree $2k$, regardless of the dimension or type of piece.

A piece at $(i_1,...,i_k)$ attacks $a_i n^m - b_i n^{k-c}$ squares, while a piece at $(j_1,...,j_k)$ attacks $a_j n^m - b_j n^{k-c}$ squares. Then, the probability of a square being safe for a given $(i_1,...,i_k)$ and $(j_1,...,j_k)$ is $\binom{n^k - a_i n^m - a_j n^m + b_i n^{k-c} + b_j n^{k-c}}{n^{k-c}} / \binom{n^k}{n^{k-c}}$, as it is the board configurations where pieces are placed at spaces that attack neither $(i_1,...,i_k)$ nor $(j_1,...,j_k)$, over all possible board configurations. Additionally, the probability of a bishop or queen being placed in the r^{th} ring is $8r/n^k$. We evaluate the limit:

$$\begin{aligned}&\lim_{n\to\infty} \sum_{r_k,r_k=0}^{(n-1)/2,(n-1)/2} \cdots \sum_{r_2,r_2=0}^{r_3,r_3} \left(\frac{64 r_2 r_2}{n^{2k}} \prod_{\alpha=0}^{a_i n^m + a_j n^m - b_i n^{k-c} - b_j n^{k-c}} \frac{n^k - n - \alpha^{k-c}}{n^k - \alpha n^{k-c}} \right) \\ &= \lim_{n\to\infty} \sum_{r_k=0}^{(n-1)/2} \cdots \sum_{r_2=0}^{r_3} \left(\frac{8r_2}{n^k} \prod_{\alpha=0}^{a_i n^m - b_i n^{k-c}} \frac{n^k - n - \alpha}{n^k - \alpha} \right) \cdot \\ &\sum_{r_k=0}^{(n-1)/2} \cdots \sum_{r_2=0}^{r_3} \left(\frac{8r_2}{n^k} \prod_{\alpha=0}^{a_j n^m - b_j n^{k-c}} \frac{n^k - n - \alpha^{k-c}}{n^k - \alpha n^{k-c}} \right) \\ &= \mu_n \cdot \mu_n \\ &= \mu_n^2.\end{aligned} \tag{3.17}$$

Case 2: Pieces attack each other. There are $n^{2k} - n^k$ distinct pairs of spaces, and for pieces that attack an^m spaces, we have $n^{2k} - an^{k+m}$ pairs that do not attack each other, meaning that there are $an^{k+m} - n^k$ pairs that do attack each other. However, as we take variance, we divide by n^{2k}, meaning that any terms of degree less than n^{2k} tend to 0, so it is not necessary to calculate the percentage of safe squares when two pieces attack each other. Therefore, we do not consider it as a term when we calculate variance.

Variance Conclusion We calculate $\text{Var}(S_n/n^k) = \text{Var}(S_n)/n^{2k}$. Then, from the previous sections:

$$\begin{aligned}\lim_{n\to\infty}\frac{\text{Var}(S_n)}{n^{2k}} &= \lim_{n\to\infty}\frac{1}{n^{2k}}\left(\sum_{i_1,\ldots,i_k=1}^{n}\text{Var}(X_{i_1,\ldots,i_k}) + \sum_{i_1,\ldots,i_k,j_1,\ldots,j_k=1}^{n}\text{Cov}(X_{i_1,\ldots,i_k}, X_{j_1,\ldots,j_k})\right)\\ &= \lim_{n\to\infty}\frac{1}{n^{2k}}\left(n^k(\mu_n-\mu_n^2) + (n^{2k}-k_in^{k-c}n^k)\mu_n^2 - n^k(n^k-1)\mu_n^2)\right)\\ &= \lim_{n\to\infty}(\mu_n-\mu_n^2)\left(\frac{1}{n^k}\right) + \mu_n^2\left(1-\frac{k_jn^{2k-c}}{n^{2k}}\right) - \mu_n^2\left(1-\frac{1}{n^k}\right). \end{aligned} \tag{3.18}$$

We note the constant terms cancel out, since $\mu_n^2 - \mu_n^2 = 0$. Then, all the other terms approach 0 as n approaches infinity, so the variance can be made as close to 0 as desired, meaning the average percentage of safe squares converges for n^{k-1} line-pieces and n hyper-pieces in any k dimensions. □

It follows that the percentage of safe squares for bishops and queens also converges in two dimensions.

4. Line-Pieces in Higher Dimensions

As we consider higher dimensional chessboards, we first analyze pieces that attack linearly, placing n^{k-1} pieces on a k-dimensional, side length n chessboard, as the number of pieces required to dominate the board is of order n^{k-1}. Any piece that attacks linearly moves through all k dimensions, but only moves in a line.

4.1. Line-Rooks.

Here, we both refine our definition of line-rooks, and aim to show Theorem 1.8, though the proof is almost entirely handled by Lemma 2.1. The remainder is simply counting the number of spaces a line-rook may attack.

Definition 4.1 (Line-Rooks). *A line-rook attacks any square that shares n^{k-1} planes with it, which is equivalent to having all but one coordinate be equal.*

As an example, we show the movement of a line-rook placed at $(3, 3, 3)$ on a $5 \times 5 \times 5$ board in Figure 3. The line-rook moves only along the bolded lines.

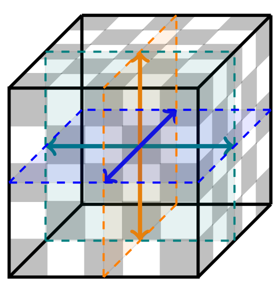

FIGURE 3. Movement of a line-rook placed at $(3, 3, 3)$ on a $5 \times 5 \times 5$ chessboard.

For any $(i_1, i_2, \ldots, i_k)$, there are $kn - c$ possible rook placements that attack it, noting that the constant c term from overlap becomes negligible as n approaches infinity. We place n^{k-1} pieces and then divide by the total possible number of board combinations. We then use Lemma 2.1 to evaluate

$$\lim_{n\to\infty}\binom{n^k-kn+c}{n^{k-1}}\bigg/\binom{n^k}{n^{k-1}} = \frac{1}{e^k}. \tag{4.1}$$

Note that because of overlap, it does not take all n^{k-1} line-rooks to dominate a n^k board. As shown in [Eng97], for $k = 3$, only $n^2/2$ rooks are needed to dominate an $n \times n \times n$ chessboard, disregarding parity as n tends to infinity. The number of rooks needed to dominate a k-dimensional board in general is not yet known.

We can consider an alternate limit in 3 dimensions, where instead of placing n^2 line-rooks, we place $n^2/2$ rooks. By Lemma 2.1, this becomes

$$\lim_{n\to\infty} \binom{n^3 - 3n + c}{n^2/2} \Big/ \binom{n^3}{n^2/2} = \frac{1}{e^{3/2}}. \tag{4.2}$$

4.2. **Line-Bishops.** For line-bishops, we note that a two dimensional line-bishop at (i, j) attacks any $(i \pm c, j \pm c)$. To expand on this in k dimensions, we use the following definition.

Definition 4.2 (Line-Bishops in k dimensions)**.** *In k dimensions, a k-dimensional line-bishop attacks as a normal bishop inside any plane it resides in, and does not attack any other spaces.*

In 3 dimensions, this is equivalent to attacking along 6 lines, 2 for each of the xy, xz, and xy planes the bishop lies within. Note that this definition maintains parity, as every square on a k-dimensional chessboard is an alternating color, so then every square on any 2-dimensional board within the larger board must also alternate color.

If we consider all planes that the line-bishop attacks on a k-dimensional chessboard, we find that we must take all permutations of 2 dimensions out of the total k dimensions. Hence, there are $k!/2!$ two-dimensional planes for a line-bishop to attack along. As we have defined it, a line-bishop placed in the center attacks as a regular bishop hitting $2n - 1$ spaces in each of these $k!/2$ planes, for a total of $k!n - (k! - 1)$ spaces attacked. Since $k! - 1$ is dwarfed as n approaches infinity, we consider the central bishop as attacking $k!n$ spaces.

We now expand our previous concept of rings to k dimensions, to help analyze the spaces seen by any line-bishop. We begin with our concept of rings in the the second dimension which we call r_2. As established earlier, for every increase of one to r_2, a bishop attacks two fewer squares. We then define $k - 2$ dimensions of rings from r_3 to r_k, with any r_i ring existing in an i-dimensional subspace of the chessboard. Just as with r_2, within each i-dimensional subspace there are $\lfloor n/2 \rfloor$ values for r_i, and in k dimensions, a piece on a board has some value for each $r_2, r_3, ..., r_k$. The value of any r_i is determined by the piece's distance from the center point of some i-dimensional subspace of the board; if there are multiple subspaces with i dimensions, we define r_i to be the innermost ring, so it takes the minimal value over all i-dimensional subspaces. This ensures that bishops placed centrally are more powerful than bishops placed in the outer rings.

As an example, for a three-dimensional chessboard, a line-bishop placed at the center square along one of the faces of an $n \times n \times n$ chessboard, as depicted in Figure 4, would have $r_2 = 0$ and $r_3 = \lfloor n/2 \rfloor$. Although the bishop is placed in the outermost ring for the xz and yz planes, defining $r_2 = 0$ ensures that the bishop attacks the correct amount of spaces in the xy plane.

For each movement out in the r_i ring, the line-bishop attacks $(i - 1)^2 \cdot (i - 2)!$ fewer pieces. To see this, consider a bishop moved from the center coordinate $(c_1, c_2, \ldots, c_i)$ to the coordinate $(c_1 - 1, c_2, \ldots, c_i)$. To begin counting the number of attacking spaces it loses, consider all $i!/2$ planes, and note that by our definition of rings in 2 dimensions, a bishop attacks 2 less spaces in each plane, therefore attacking $2(i)!/2$ less spaces for each movement outward. However, this counting method overestimates, as it also considers planes that do not include the c_1 axis. To correct for this, we subtract $2(c - 1)!/2$, which is all of the 2 dimensional planes that remain unaffected by the movement outwards along the c_1 axis. Therefore, for every movement outward in the i^{th} ring, a bishop attacks $2(i!/2 - (i - 1)!/2) = i! - (i - 1)!$ less spaces.

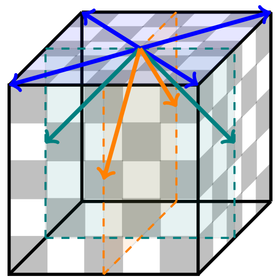

FIGURE 4. Movement of a line-bishop placed at $(3,3,5)$ on a $5\times5\times5$ chessboard, meaning that $r_2=0$ and $r_3=2$.

Then, for any bishop in rings $(r_2,\ldots,r_k)$, we must subtract the following sum:

$$s \;:=\; 2r_2+\sum_{i=3}^{k}(i!-(i-1)!)r_i. \tag{4.3}$$

For the sake of concise notation, we refer to this sum as s for the remainder of the line-bishop and line-queen proofs.

As a check, we note that a bishop in the corner has value $n/2$ for all r, meaning it attacks, $k!n-n/2(2+\sum_{i=3}^{k}(i!-(i-1)!)$ spaces. This simplifies to $k!n/2$, since by telescoping we have

$$\sum_{i=3}^{k}(i!-(i-1)!) \;=\; k!-2!. \tag{4.4}$$

For a k-dimensional chess board, there are $2^{k-3}(k-1)k$ two-dimensional faces, see for example [Ban96], and as shown in the 2-dimensional bishop case, there are $8r_2$ spaces for the $r_2{}^{\text{th}}$ ring. We define the expected percentage of safe squares on a chessboard in k dimensions as

$$\frac{1}{n^k}\sum_{r_k=0}^{n/2}\sum_{r_{k-1}=0}^{r_k}\cdots\sum_{r_2=0}^{r_3}2^{k-3}(k-1)k8r_2\frac{\binom{n^k-(k!-\frac{s}{n})n}{n^{k-1}}}{\binom{n^k}{n^{k-1}}}. \tag{4.5}$$

As an example, we define and computationally determine the expected percentage of safe squares as n approaches infinity for $k=3$ below:

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{r_3=0}^{n/2}\sum_{r_2=0}^{r_3}48r_2\prod_{\alpha=0}^{6n-4r_3-2r_2}\left(1-\frac{n^2}{n^3-\alpha}\right) \;=\; \frac{-1+9e^2-2e^3}{3e^6} \;\approx\; 2.0929\%. \tag{4.6}$$

We notice an interesting phenomenon here. Observe that in three dimensions, a centrally placed bishop attacks $6n-5$ spaces, as opposed to a rook, which only attacks $3n-2$ spaces. A bishop placed in the corner also attacks $3n-2$ spaces, meaning that in three dimensions, any given line-bishop is at least as powerful as a line-rook. We verify this by evaluating the line-rooks limit in three dimensions, noticing that they leave $\approx 4.9787\%$ of the board safe, compared to the $\approx 2.0929\%$ safe for line-bishops. Moreover, for all $k>2$, line-bishops become stronger than line-rooks. Since line-bishops change in two coordinates as they move, while line-rooks change only in one, the number of diagonals for a bishops to attack along increases as a factorial, while the number of straight paths for a rook to attack along only increases linearly.

4.3. **Line-Queens.** Again, the rook movement and the bishop movement of a queen are completely disjoint. Therefore, we keep the definition of rings from the line-bishop definition, and add the kn pieces hit by the rook movement, resulting in a limit of

$$L := \lim_{n\to\infty} \frac{1}{n^k} \sum_{r_k=0}^{n/2} \sum_{r_{k-1}=0}^{r_k} ... \sum_{r_2=0}^{r_3} (2^{k-3}(k-1)k) 8r_2 \frac{\binom{n^k-(k+k!-\frac{s}{n})n}{n^{k-1}}}{\binom{n^k}{n^{k-1}}}$$

$$= \lim_{n\to\infty} \frac{1}{n^k} \sum_{r_k=0}^{n/2} \sum_{r_{k-1}=0}^{r_k} ... \sum_{r_2=0}^{r_3} (2^{k-3}(k-1)k) 8r_2 \prod_{\alpha=0}^{kn+k!n-s} \left(1 - \frac{n^{k-1}}{n^k-\alpha}\right). \tag{4.7}$$

Following the proof of the expected value in two dimensions, we bound the product inside the summand by substituting in values for a:

$$\left(1 - \frac{n^{k-1}}{n^k - kn - k!n + s}\right)^{kn+k!n-s} \leq \prod_{\alpha=0}^{kn+k!n-s} \left(1 - \frac{n^{k-1}}{n^k-\alpha}\right) \leq \left(1-\frac{1}{n}\right)^{kn+k!n-s}. \tag{4.8}$$

As the kn degree of the factor is independent of any r_k, it can be factored out of the summand. From that, we bound the limit:

$$\lim_{n\to\infty} \left(1 - \frac{n^{k-1}}{n^k - kn - k!n + s}\right)^{kn} \frac{1}{n^k} \sum_{r_k=0}^{n/2} \cdots \sum_{r_2=0}^{r_3} (2^{k-3}(k-1)k) 8r_2 \left(1 - \frac{n^{k-1}}{n^k - kn - k!n + s}\right)^{k!n-s}$$
$$\leq L$$
$$\leq \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{kn} \frac{1}{n^k} \sum_{r_k=0}^{n/2} \cdots \sum_{r_2=0}^{r_3} (2^{k-3}(k-1)k) 8r_2 \left(1-\frac{1}{n}\right)^{k!n-s}. \tag{4.9}$$

Following the two dimensional bishop limit, we find:

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{kn} = \lim_{n\to\infty} \left(1 - \frac{n^{k-1}}{n^k - kn - k!n + s}\right)^{kn} = \frac{1}{e^k}. \tag{4.10}$$

Since $1/e^k$ is the percentage of squares hit by a line-rook in k dimensions, then, the expected percentage of safe squares for line-queens in the k dimensions is simply the expected percentage of safe squares for line-rooks multiplied by the expected percentage of safe squares for line-bishops.

Then, for line-queens, the expected percentage of safe squares is simply the expected proportion of safe squares for a line-bishop divided by e^k. We show $k = 3$ as an example below:

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{r_3=1}^{n/2} \sum_{r_2=1}^{r_3} 48r_2 \prod_{\alpha=0}^{9n-4r_3-2r} \left(1 - \frac{n^2}{n^3-\alpha}\right) = \frac{-1+9e^2-2e^3}{3e^9} \approx 0.1042\%. \tag{4.11}$$

4.4. **Variance of Line-Pieces.** We proved earlier that variance always converges to 0 as it becomes $\mu_n^2 - \mu_n^2 = 0$ as n approaches infinity. Then the expected proportions of safe squares converge to the values stated for all pieces above.

5. Hyper-pieces in Higher Dimensions

We consider an alternative way of expanding to k dimensions by considering what we call "hyper-pieces". Any hyper-piece in k dimensions attacks in $(k-1)$-dimensional subspaces, and we place n hyper-pieces in order to obtain dominance of the board. We are able to find the average of hyper-rooks, but the other pieces evade our efforts beyond simple bounding arguments.

5.1. **Hyper-rooks.** We define hyper-rooks and give the proof of Theorem 1.8.

Definition 5.1 (Hyper-rooks)**.** *A hyper-rook attacks any piece that shares at least one coordinate with it.*

This means that it attacks in a $(k-1)$-dimensional plane, which has n^{k-1} squares, and since there are k dimensions in the board, there are k of these planes, subtracting some an^{k-2} for overlap points of all dimensions. This means that for k dimensions, a hyper-rook leaves $n^k - kn^{k-1} - an^{k-2}$ safe spaces. Therefore, the expected percentage of safe squares on a k-dimension board chessboard with side length n and n hyper-rooks as n approaches infinity is

$$\lim_{n\to\infty} \binom{n^k - kn^{k-1} - an^{k-2}}{n} \Big/ \binom{n^k}{n} \;=\; \frac{1}{e^k} \tag{5.1}$$

by Lemma 2.1.

5.2. **Hyper-bishops.** One way to analyze a bishop at (i, j) is as a piece that attacks along the lines of

$$\begin{aligned} (x-i)+(y-j) &= 0, \\ (x-i)-(y-j) &= 0. \end{aligned} \tag{5.2}$$

One option to extend this into higher dimensions is to add on extra coordinates, in all possible diagonal subspaces. For example, in 3 dimensions, the planes to describe hyper-bishop movement at (i, j, k) would be

$$\begin{aligned} (x-i)+(y-j)+(z-k) &= 0, \\ (x-i)+(y-j)-(z-k) &= 0, \\ (x-i)-(y-j)+(z-k) &= 0, \\ (x-i)-(y-j)-(z-k) &= 0. \end{aligned} \tag{5.3}$$

Definition 5.2 (Hyper-Bishops)**.** *In general, for a k-dimensional chessboard, a hyper-bishop at $(a_1, a_2, \ldots, a_k)$ can attack the areas defined by any possible version of*

$$(x_1 - a_1) \pm (x_2 - a_2) \pm \cdots \pm (x_k - a_k) \;=\; 0. \tag{5.4}$$

This definition meets a number of key analogies to 2-dimensional bishops. It only attacks squares of the same color, and it projects downward, so that looking at any 2-dimensional subspace of the board, it moves as a 2-dimensional bishop would. Unfortunately, the one major disadvantage of this definition is that counting the number of squares seen in k dimensions becomes challenging as we lose the symmetrical pattern of rings that we saw for line-bishops.

Instead, we naively bound the number of squares attacked for 3 dimensions, by considering the number of squares seen by a center piece, which is the most powerful, and a corner piece, which is the least powerful.

To do this, we consider a vertical slicing method, where we analyze the spaces a hyper-bishop attacks on each 2-dimensional sub-board. As an example, we show center hyper-bishop movement on a $5 \times 5 \times 5$ cube in Figure 5 and corner hyper-bishop movement on a $5 \times 5 \times 5$ cube in Figure 6.

We first consider the expected percentage of safe squares if every space is as effective as the center space. We consider n vertical slices of the chessboard, and analyze the hyper-bishop movement in each one. In the center slice, $(n-1)/2$, the hyper-bishop attacks $2n-1$ spaces. For every slice of distance i from $(n-1)/2$, the hyper-bishop attacks in a diagonal on both sides in the ring i. This is analogous to moving as if it were 2 regular bishops on the slice, both placed in ring i, which then gives us $2(2n-2i-1)-2$ spaces seen.

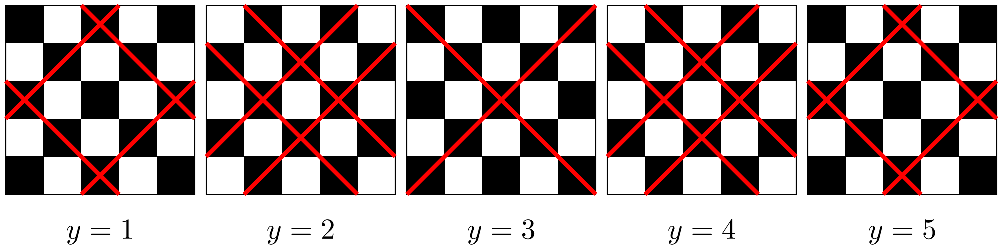

FIGURE 5. Spaces attacked by a hyper-bishop placed at $(3, 3, 3)$ on a $5 \times 5 \times 5$ board.

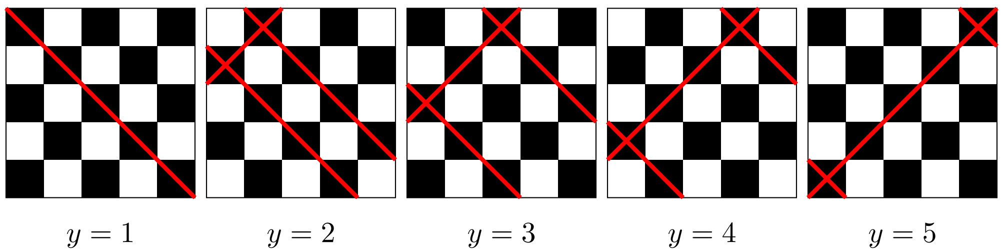

FIGURE 6. Spaces attacked by a hyper-bishop placed at $(1, 1, 1)$ on a $5 \times 5 \times 5$ board.

Therefore, the center hyper-bishop sees the following number of squares on an $n \times n \times n$ board:

$$\begin{aligned} 2n - 1 + 2\sum_{i=1}^{\frac{n-1}{2}} 2(2n - 2i - 2) &= 2n - 1 + 8\left(n \cdot \frac{n-1}{2} - \frac{n-1}{2} - \sum_{i=1}^{\frac{n-1}{2}} i\right) \\ &= 2n - 1 + 4n^2 - 4n - 4n + 4 - 8\frac{\frac{n-1}{2} \cdot \frac{n+1}{2}}{2} \\ &= 3n^2 - 6n + 4. \end{aligned} \tag{5.5}$$

Therefore, if we assume that all pieces see this maximal number of squares, this gives us an expected percentage of safe squares of

$$\lim_{n\to\infty} \frac{\binom{n^3-3n^2+6n-4}{n}}{\binom{n^3}{n}} = \frac{1}{e^3} \tag{5.6}$$

by Lemma 2.1.

We obtain a lower bound by considering the weakest possible placements, with all bishops placed in corners. A hyper-bishop placed in one of the lower corners sees n squares on the lowest slice. For the i^{th} slice above, two of the attacking diagonals shift i spaces outward, while one of the attacking diagonals shifts i spaces inward, meaning that it sees $2(n - i + 1) + (i - 2)$ squares, as demonstrated in Figure 6. Just as above, there are n vertical slices. Hence, the total number of squares seen is

$$n + \sum_{i=1}^{n-1}(2(n - i) + (i - 1)) = \frac{3}{2}(n^2 - n) + 1 \tag{5.7}$$

Therefore, our upper bound is now

$$\lim_{n\to\infty} \frac{\binom{n^3-3(n^2-n)/2-1}{n}}{\binom{n^3}{n}} = \frac{1}{e^{3/2}} \tag{5.8}$$

by Lemma 2.1.

Note that both bounds are dependent on the definition of rings from the second dimension. Since the lack of symmetry has prevented us from defining rings for hyper-bishops in k dimensions, finding general bounds for all dimensions proves challenging. We do note a striking similarity between these bounds and our naive bounds in the 2-dimensional case: both follow the pattern of $1/e^k$ and $1/e^{k/2}$ for the bounds. We are optimistic about future work using this definition.

5.3. **Hyper-queens.** Our definition for hyper-queens is once again that of a bishop and a rook placed in the same space, though in this case using the hyper-piece version of both. Unfortunately, our current understanding of hyper-bishops is not sufficient to obtain very strong results. We are able to apply our current bounds on hyper-bishops, however, giving us an upper bound on spaces seen of $6n^2 - 9n$, ignoring the overlap of the rook and bishop since their seen squares are mostly disjoint, and a lower bound of $\frac{9}{2}(n^2 - n)$. This gives bounds on the proportion of safe squares in the limit, namely

$$\lim_{n\to\infty} \frac{\binom{n^3-9(n^2-n)/2}{n}}{\binom{n^3}{n}} = \frac{1}{e^{9/2}} \tag{5.9}$$

and

$$\lim_{n\to\infty} \frac{\binom{n^3-6n^2+9n}{n}}{\binom{n^3}{n}} = \frac{1}{e^6}, \tag{5.10}$$

both of which follow from applications of Lemma 2.1.

6. Future work

The generalization of rook movement to k dimensions for both hyper-rooks and line-rooks is quite fascinating. Not only were both easily generalizable, but in fact they generalized to the same limit, $1/e^k$. Generalizing bishop and queen movement did not lead to as concise of a result, leading us to consider that different definitions of k-dimensional bishop movement may very well be possible.

It is interesting that our naive bounds for hyper-bishops in 3 dimensions, which bound the expected percentage of safe squares between $1/e^3$ and $1/e^{3/2}$, are reminiscent of our naive bounding for 2 dimensional bishops, which were between $1/e^2$ and $1/e$, based on best and worst case assumptions. While precisely counting the number of squares hit is not immediately feasible, providing stronger bounds or bounds for k dimensions would be an important step in further understanding the movement of these pieces.

Additionally, we noted earlier that not all n^{k-1} line-rooks are needed to dominate in k dimensions. The number of line-rooks needed to dominate k dimensions for $k > 3$ remains a fascinating open question in combinatorics. Determining bounds for the number of line-rooks needed to dominate k dimensions would be intrinsically interesting; it would be even more interesting to then apply this new knowledge to the expected percentage of safe squares for randomly placed line-rooks.

Appendix A. Computations

Lemma A.1. *We have*

$$\lim_{n\to\infty} \frac{(1-\frac{1}{n})^{-2}(\frac{n-1}{2}(1-\frac{1}{n})^{-n-1} - \frac{n+1}{2}(1-\frac{1}{n})^{-n+1} + 1)}{n^2(1-(1-\frac{1}{n})^{-2})^2} = \frac{1}{4}.$$

Proof. Note:

$$\begin{aligned}&\frac{(1-\frac{1}{n})^{-2}(\frac{n-1}{2}(1-\frac{1}{n})^{-n-1}-\frac{n+1}{2}(1-\frac{1}{n})^{-n+1}+1)}{n^2(1-(1-\frac{1}{n})^{-2})^2}\\ &=\frac{(1-\frac{1}{n})^{-2}(\frac{n-1}{2}(1-\frac{1}{n})^{-n-1}-\frac{n+1}{2}(1-\frac{1}{n})^{-n+1}+1)}{n^2(1-\frac{1}{n})^{-4}(\frac{-2}{n}+\frac{1}{n^2})^2}.\end{aligned} \tag{A.1}$$

As n approaches infinity, $\frac{n-1}{2}(1-\frac{1}{n})^{-n-1}-\frac{n+1}{2}(1-\frac{1}{n})^{-n+1}$ approaches 0 more rapidly than other terms do. This gives the following:

$$\begin{aligned}&\lim_{n\to\infty}\frac{(1-\frac{1}{n})^{2}(\frac{n-1}{2}(1-\frac{1}{n})^{-n-1}-\frac{n+1}{2}(1-\frac{1}{n})^{-n+1}+1)}{n^2(\frac{-2}{n}+\frac{1}{n^2})^2}\\ &=\lim_{n\to\infty}\frac{(1-\frac{1}{n})^2}{n^2(\frac{-2}{n}+\frac{1}{n^2})^2}\\ &=\lim_{n\to\infty}\frac{\frac{1}{n^2}}{\frac{4}{n^2}}\\ &=\frac{1}{4}.\end{aligned} \tag{A.2}$$

□

Lemma A.2. *We have*

$$\lim_{n\to\infty}\frac{(1-\frac{n}{n^2-2n})^{-2}(\frac{n-1}{2}(1-\frac{n}{n^2-2n})^{-n-1}-\frac{n+1}{2}(1-\frac{n}{n^2-2n})^{-n+1}+1)}{n^2(1-(1-\frac{n}{n^2-2n})^{-2})^2} = \frac{1}{4}.$$

Proof. The proof is nearly identically to the above in Lemma A.1:

$$\begin{aligned}&\frac{(1-\frac{n}{n^2-2n})^{-2}(\frac{n-1}{2}(1-\frac{n}{n^2-2n})^{-n-1}-\frac{n+1}{2}(1-\frac{n}{n^2-2n})^{-n+1}+1)}{n^2(1-(1-\frac{n}{n^2-2n})^{-2})^2}\\ &=\frac{(1-\frac{n}{n^2-2n})^{-2}(\frac{n-1}{2}(1-\frac{n}{n^2-2n})^{-n-1}-\frac{n+1}{2}(1-\frac{n}{n^2-2n})^{-n+1}+1)}{n^2(1-\frac{n}{n^2-2n})^{-4}(\frac{-2n}{n^2-2n}+\frac{n^2}{(n^2-2n)^2})^2}.\end{aligned} \tag{A.3}$$

As n approaches infinity, $\frac{n-1}{2}(1-\frac{n}{n^2-2n})^{-n-1}-\frac{n+1}{2}(1-\frac{n}{n^2-2n})^{-n+1}=0$. Thus:

$$\begin{aligned}&\lim_{n\to\infty}\frac{(1-\frac{n}{n^2-2n})^{2}(\frac{n-1}{2}(1-\frac{n}{n^2-2n})^{-n-1}-\frac{n+1}{2}(1-\frac{n}{n^2-2n})^{-n+1}+1)}{n^2(\frac{-2n}{n^2-2n}+\frac{n^2}{(n^2-2n)^2})^2}\\ &=\lim_{n\to\infty}\frac{(1-\frac{n}{n^2-2n})^2}{(\frac{-2n}{n^2-2n}+\frac{n^2}{(n^2-2n)^2})^2}\\ &=\lim_{n\to\infty}\frac{1}{n^2\frac{4}{n^2}}\\ &=\frac{1}{4}.\end{aligned} \tag{A.4}$$

□

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