Geometric proof for the irrationality of $\sqrt{7}$ (and other odds)

Double Blind No Name 1, Double Blind No Name 2, Double Blind No Name 3, and Double Blind No Name 4



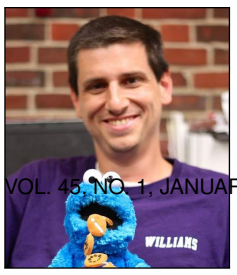
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Summary. The proof of the irrationality of $\sqrt{2}$ is one of the first proofs most mathematicians encounter. Though usually done with algebra, involving a descent argument leading to a contradiction, there are other approaches. An interesting one is Stanley Tennenbaum's geometric proof of overlapping squares. Using regular polygons (triangles, pentagons and hexagons), similar constructions have been found to prove the irrationality of $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$ and $\sqrt{10}$. We modify his construction using squares to obtain the first geometric proof of the irrationality of $\sqrt{7}$, show it extends to all non-square odd integers strictly between 1 and 9, and discuss the obstruction that arises for larger odd integers.

In the 1950s Stanley Tennenbaum (see [**5**]) came up with a beautiful geometric proof of the irrationality of $\sqrt{2}$, namely we cannot write it as a ratio of two positive integers. This approach was extended by Miller and Montague [**8, 9**] to handle other numbers, such as $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$ and $\sqrt{10}$; note $\sqrt{8} = 2\sqrt{2}$ is clearly irrational from the proof of $\sqrt{2}$, while $\sqrt{4} = 2$ and $\sqrt{9} = 3$ are clearly rational. Miller and Montague handled $\sqrt{5}$ by pentagons and $\sqrt{3}$, $\sqrt{6}$ and $\sqrt{10}$ by noting these are Triangular numbers[1] and using equilateral triangles.

There are two interesting items to note. First, their triangular argument cannot work for all Triangular numbers, as $T_8 = 36$ has a rational square-root! Thus the construction has to break down at some point.

Second, the method is silent on $\sqrt{7}$. Can we give a geometric proof here as well? Chen, Miller and Wu [**4**] explored other shapes, and while they were able to prove the irrationality of $\sqrt{6}$ with hexagons, they could not find a heptagon calculation for $\sqrt{7}$.

While it may be possible to make heptagons work, fortunately there is no need. We show how to modify the geometric proof of the irrationality of the $\sqrt{2}$ to obtain that for $\sqrt{7}$, still using only squares. These arguments generalize to all non-square odd integers less than 9; similar to the issues Miller and Montague encountered, our proof breaks for larger values.

Below we first quickly review the standard algebraic irrationality proofs, commenting on some of the subtleties. We then state the main result of this note, and then provide proofs in the following section.

Remark 1. The problem of a geometric proof of the irrationality of $\sqrt{7}$ has been open for a long time. The fourth named author often gives it as a challenge problem in all his courses; the construction below was obtained with minimal feedback from him by the other three authors when they took his Math 310: Introduction to Modern Algebra at Siena University in Fall 2025.

[1]The n^{th} Triangular number T_n is $1 + 2 + \cdots + n = n(n+1)/2$; see sequence A000217 of [**11**].

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Remark 2 (Challenge!). We may view this work as adding another term to a special sequence of numbers, namely those whose square-roots can be shown to be irrational using geometric arguments in the spirit of Tennenbaum. We thus know that sequence begins $2, 3, 5, 6, 7, 8, 10$, with our addition determining that 7 is in this list. We challenge the reader to extend the list further; while earlier work proved 15 belongs, what of 11 and 13? A great resource to study special sequences of integers is the famous OEIS [**11**], the On-line Encyclopedia of Integer Sequences. If you enter the first few terms of a sequence it tells you all the ones it knows that start that way, along with numerous references and properties.

For example, entering 1, 2, 3, 5, 8 yields over 700 candidates, including the Fibonacci numbers, A000045, but also the number of ways to partition $2n + 1$ into distinct positive integers, A078408. Many sequences start alike and frequently one has to compute several terms to narrow it down (another fun one is A000124, which begins 1, 2, 4 but whose next term is 7; this is the solution to a nice geometric question: what is the maximum number of pieces a circle can be divided into with n lines?)

Algebraic approaches and issues One of the joys of mathematics is that often there are multiple ways to prove the same result, with different approaches highlighting different features of the original problem, or being better suited to certain generalizations. A terrific example involves one of the first proofs students see, the irrationality of the square root of 2 (namely, that $\sqrt{2}$ cannot be written as the ratio of two integers).

The most common approach is a proof by contradiction; for this and other standard proofs see [**1, 2, 3, 7, 12**]. Assuming $\sqrt{2} = a/b$ for two relatively prime integers a and b, one quickly sees that $2b^2 = a^2$ and therefore $2|a^2$ and hence $2|a$. This allows us to write $a = 2\widetilde{a}$, and then a similar argument shows $b = 2\widetilde{b}$, contradicting a and b are relatively prime.

After finishing the above proof, one should immediately ask for what other integers n is $\sqrt{n}$ irrational? Unfortunately, our argument subtly assumed if $2|a^2$ then $2|a$. This is a special case of a more general truth: if a prime p divides a product xy then $p|x$ or $p|y$ (this is an inclusive or; it could divide both). Such a result clearly need not be true if p is not prime; for example $6|2 \cdot 9$ but 6 divides neither 2 nor 9.[2]

Unfortunately some work is needed to prove if a prime divides a product then it divides at least one factor; see for example [**10**]. For any *fixed* prime we can prove this by brute force by looking at congruence classes[3], but we need some work to handle the general case; this is probably why the ancient Greek Theodorus of Cyrene only gave proofs of the irrationality of $\sqrt{n}$ up to $n = 17$. Interestingly, in our proof for the irrationality of $\sqrt{n}$ for $n \in \{3, 5, 7\}$ we do use $2|xy$ implies $2|x$ or $2|y$, but that result for one prime suffices for all these n.

Geometric proofs We quickly review Tennenbaum's proof, for fun paraphrasing from [**4**] who paraphrase from [**8**]. Assume $\sqrt{2} = a/b$. Without loss of generality we may assume this fraction has the smallest numerator (or equivalently the smallest denominator: if $a/b = a'/b'$ then if $b < b'$ we must have $a < a'$ as $a = a'(b/b') < a$) that works. Squaring gives $a^2 = 2b^2$.

[2] For a more interesting example (the fourth named author first saw this in [**6**], page 151, Problem 6c, which attributes this observation to a note by Hale Trotter in Math. Monthly in April 1988): $\sin^2 x = (1 - \cos x)(1 + \cos x)$ yet $\sin x$ divides neither factor on the right. The more standard example works in the ring $\mathbb{Z}[\sqrt{-5}]$, where $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$, yet no factor of the left divides any on the right.

[3] For example, if $p = 2$ assume $p|xy$. If either x or y is even the claim is true; if both are odd then we may write $x = 2a + 1$, $y = 2b + 1$ and thus $xy = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$, which is odd and not divisible by 2.

Geometrically, this means the area of two squares of side length b equals the area of one square of side length a. We illustrate this in Figure 1, note the total area covered

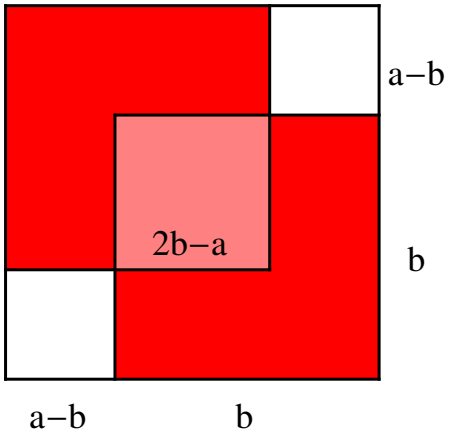


Figure 1. Geometric proof of the irrationality of $\sqrt{2}$. As we are assuming $\sqrt{2} = a/b$, this forces $b < a$, leading to the picture above.

by the squares with side length b (which double counts the overlapping region) equals the area of the larger square with side length a. Thus the doubly counted part, which is a square of side length $2b - a$, has area equal to that of the two white, uncovered squares (each of side length $a - b$).

We conclude that $(2b - a)^2 = 2(a - b)^2$ or $\sqrt{2} = (2b - a)/(a - b)$. From this we find $\sqrt{2}$ equals the ratio of two smaller integers. Thus we have found a fraction whose numerator, $2b - a$, is smaller than a, a contradiction completing the proof. Clearly $2b - a > 0$; if not then $2b \leq a$ and thus $4b^2 \leq a^2 = 2b^2$. For the upper bound, note $2b - a < a$, as otherwise $2b - a \geq a$ which gives $b \geq a$, and thus $\sqrt{2} \leq 1$, an absurdity. Thus $\sqrt{2}$ is irrational.

As remarked above, several generalizations of Tennenbaum's arguments handled a variety of numbers, but none were able to do $\sqrt{7}$. We remedy this omission by giving a geometric proof for $\sqrt{n}$ when $n \in \{3, 5, 7\}$, illustrating it for $n = 7$ and discussing why our proof does not work for larger odd integers. While we do not need the general statement that if a prime divides a product it divides at least one factor, we do need the special case when the prime is 2 for a parity argument. Fortunately the claim is straightforward to prove for any fixed prime, see Footnote 3.

Geometric Irrationality Proofs of $\sqrt{n}$ for $n \in \{3, 5, 7\}$

Theorem 1. *Tennenbaum's argument can be modified to give a geometric proof of the irrationality of $\sqrt{n}$ for $n \in \{3, 5, 7\}$, and this generalization fails for larger odd n.*

Assume, for the sake of contradiction, that $\sqrt{n}$ is rational. By the definition of rational numbers, this means there are positive integers p and q such that

$$\sqrt{n} = \frac{p}{q}. \tag{1}$$

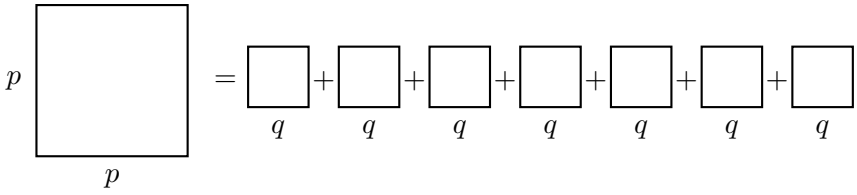
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We chose our fraction p/q from among all fractions which equal $\sqrt{n}$ to be the one with the smallest numerator (or equivalently the smallest denominator).[4] In particular, this immediately implies p and q are relatively prime, else we could obtain smaller numbers by removing the common factor. We prove $\sqrt{n}$ is irrational by contradiction. We show that assuming $\sqrt{n}$ is rational leads to it equaling another fraction with smaller denominator, contradicting the minimality of our fraction p/q.

Squaring (1) and clearing the denominator yields

$$nq^2 \;=\; p^2, \tag{2}$$

which we interpret geometrically: the area of a square of side length p is equal to the combined area of n squares with side length q. We illustrate this below for the case $n = 7$.



We obtain a contradiction by finding a smaller pair of integers p' and q' than p and q whose ratio p'/q' is also $\sqrt{n}$.

Geometric interpretation Consider a $p \times p$ square representing the right-hand side of (2). Place n (for definiteness in the picture we show the interesting case of $n = 7$) $q \times q$ squares evenly spaced along the diagonal from the bottom left to the top right so that they fit inside the $p \times p$ square as shown in Figure 2. Thus we keep moving the bottom left corner of each $q \times q$ square up a constant amount along the diagonal of the initial square. As the first square is $q \times q$ and we have $n - 1$ movements of size s say, we must have

$$q + (n-1)s \;=\; p,$$

as we have to end at the opposite corner. Thus the step size $s = (p - q)/(n - 1)$.

[4] We do need to justify this statement, but fortunately this isn't too bad. By assumption there exists at least one representation of $\sqrt{n}$ as a ratio of two positive integers, p/q. We now check all integers less than p, a finite set, to see if any work; for any candidate numerator $\widetilde{p}$ note the denominator is at most $\widetilde{p}/\sqrt{n} < \widetilde{p}$, giving us a finite set and thus we may take the smallest representative in this set.

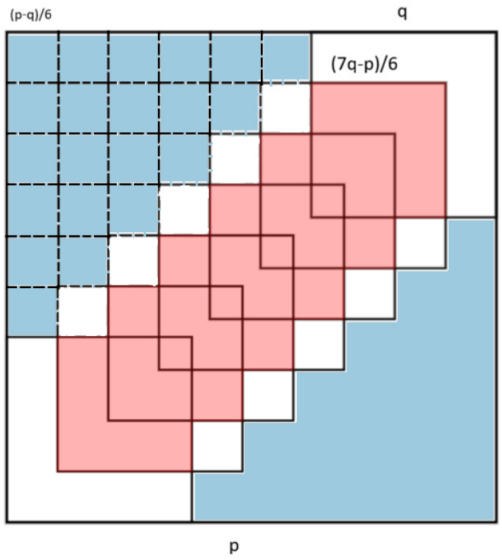


Figure 2. Geometric interpretation: a $p \times p$ square with seven equally spaced $q \times q$ squares starting at the bottom left and ending at the top right. We will show the overlaps and uncovered regions generate smaller squares, which correspond to a strictly smaller solution in the same form as (p, q). Here the blue area denotes the uncovered area, the white the singly counted area, and the red the area covered multiple times.

Remark 3. The regions not covered by the $q \times q$ squares form smaller squares of side length $(p - q)/(n - 1)$, as this is our shift up and to the right.

Remark 4. The overlapping regions along the diagonal produce $n - 1$ equal squares of side length

$$q - \frac{p-q}{n-1} = \frac{nq-p}{n-1},$$

arising from taking one shift $\frac{p-q}{n-1}$ to the right, and an equal shift, $\frac{p-q}{n-1}$ up.

Our key insight is the following: *Instead of tracking every piece of area that is covered two, three or four times, finding those dimensions and multiplicities and thus those areas, it suffices to compare the* ***total*** *uncovered and over-counted areas, as they must be equal.*

The sum of the areas of all the overlaps, regardless of whether its pieces are double, triple, or quadruple counted, must equal the area of the uncovered region as we are assuming $nq^2 = p^2$:

$$\text{(total uncovered area)} = \text{(total over-counted area)}.$$

What are the dimensions of the small uncovered squares? We start with a square that is $q \times q$ and shift it up and to the right by $(nq - p)/(n - 1)$. Thus these uncovered squares have side length

$$q - \frac{nq-p}{n-1} = \frac{p-q}{n-1}.$$

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There are $n-1$ columns of these uncovered smaller squares; the first column has $n-1$ of these squares, the second column has $n-2$ and so on. Thus the total is

$$2\left((n-1)+(n-2)+\cdots+1\right) \;=\; 2\frac{(n-1)n}{2} \;=\; (n-1)n$$

(the factor of 2 is because we have two sets of uncovered squares, in the bottom right and the top left).

Thus the $(n-1)n$ smaller squares of side length $(p-q)/(n-1)$ have the same area as the $n-1$ of our double counted squares of side length $(nq-p)/(n-1)$:

$$(n-1)n\left(\frac{p-q}{n-1}\right)^2 \;=\; (n-1)\left(\frac{nq-p}{n-1}\right)^2.$$

or after removing the common factor of $n-1$

$$n\left(\frac{p-q}{n-1}\right)^2 \;=\; \left(\frac{nq-p}{n-1}\right)^2. \tag{3}$$

In the next subsection we use a parity argument to obtain a contradiction, completing the proof of Theorem 1.

Parity arguments and integer reduction Recall from (2) we have $nq^2 = p^2$. Inspired by (3), we define $\widetilde{p}$ and $\widetilde{q}$ by

$$\widetilde{q} \;:=\; \left(\frac{p-q}{n-1}\right), \quad \text{and} \quad \widetilde{p} \;:=\; \left(\frac{nq-p}{n-1}\right).$$

There is a complication: we do not know that $\widetilde{p}$ and $\widetilde{q}$ are integers. Thus even if we can show they are smaller than p and q, that doesn't suffice to find a contradiction. Fortunately we can show related quantities are integers and smaller. We study

$$q' \;:=\; \frac{p-q}{2} \quad \text{and} \quad p' \;:=\; \frac{nq-p}{2}$$

(in the argument below we see why these are the natural objects to use). We show that p' and q' are smaller *integers* than p and q respectively, that is $p' < p$ and $q' < q$. This yields the desired contradiction, as we assumed p and q are relatively prime and the smallest pair whose ratio is $\sqrt{n}$.

It's enough to prove $p' < p$, as that immediately implies $q' < q$. To see this, we rewrite (2) as $q = p/\sqrt{n}$, and then if $p' < p$ we find

$$q' \;=\; \frac{p'}{\sqrt{n}} \;<\; \frac{p}{\sqrt{n}} \;=\; q.$$

We finally use the fact that n is an odd integer less than 9. Since n is odd, $n-1$ is even and thus we can find an n' such that $n-1 = 2n'$.

Using simple algebra we rewrite (3) as

$$n\left(\frac{p-q}{n-1}\right)^2 \;=\; \left(\frac{nq-p}{n-1}\right)^2$$

$$n\frac{(p-q)^2}{(n-1)^2} = \frac{(nq-p)^2}{(n-1)^2}$$

$$n\frac{(p-q)^2}{(2n')^2} = \frac{(nq-p)^2}{(2n')^2}$$

$$n\frac{(p-q)^2}{2^2} = \frac{(nq-p)^2}{2^2},$$

which yields

$$n\left(\frac{p-q}{2}\right)^2 = \left(\frac{nq-p}{2}\right)^2 \quad \text{or} \quad \frac{(nq-p)/2}{(p-q)/2} = \sqrt{n}. \tag{4}$$

Proof. We first show p' and q' are integers. Clearly p and q cannot both be even, as they are relatively prime. If exactly one were even consider $nq^2 = p^2$. Since n is odd (we are again finally using that assumption!) one side is even and one side is odd, which is impossible (we are using our parity argument, namely if 2 divides a product it must divide at least one factor; see Footnote 3.). Thus p and q are both odd. Hence $p-q$ and $nq-p$ are even and thus divisible by 2, implying $q' = (p-q)/2$ and $p' = (nq-p)/2$ are integers.

We now obtain a contradiction. We show $p' = (nq-p)/2 < p$, thus giving us a smaller solution to (2) as $\sqrt{n} = (nq-p)/(p-q)$. We compare the two quantities, noting in each step the direction of the inequality does not change:

$$\begin{aligned}
&\frac{nq-p}{2} \text{ versus } p \\
&nq-p \text{ versus } 2p \\
&nq \text{ versus } 3p \\
&nq \text{ versus } 3q\sqrt{n} \quad \text{as } \sqrt{n} = p/q \\
&\sqrt{n} \text{ versus } 3 \\
&n \text{ versus } 9.
\end{aligned}$$

By assumption $n < 9$ so the left hand sides above are always less than the right hand sides, and we have thus found a smaller solution to (2), a contradiction to our choices of p and q, completing the proof of Theorem 1. ∎

Failure to generalize to even n We quickly discuss why our argument fails for even n; we have already seen that we need $n < 9$ to ensure the new candidates are smaller than the assumed minimal solution to (2). We give two examples, illustrating different issues.

What if we try to modify our approach for $\sqrt{4}$? The clearest geometric reason it fails is that, since $\sqrt{4} = 2$, the large $p \times p$ square can be tiled perfectly by four $q \times q$ squares with no overlaps and no gaps. Thus the diagonal placement that produced

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uncovered and over-counted regions in the $\sqrt{7}$ case produces no leftover regions at all when $n = 4$.

Algebraically, we can see exactly where the argument fails inside the structure of our proof if we try and place the squares across the diagonal. If we attempt to mimic the same construction used for $n = 7$, the corresponding overlap–gap equation, (3), becomes

$$4\left(\frac{p-q}{3}\right)^2 \;=\; \left(\frac{4q-p}{3}\right)^2.$$

This suggests defining

$$q' \;:=\; \frac{p-q}{3} \qquad \text{and} \qquad p' \;:=\; \frac{4q-p}{3}.$$

Now assume, as in the main proof, that

$$\sqrt{4} \;=\; \frac{p}{q} \qquad \text{so that} \qquad p \;=\; 2q.$$

Substituting $p = 2q$ into the expressions for p' and q' gives

$$q' \;=\; \frac{2q-q}{3} \;=\; \frac{q}{3}, \qquad p' \;=\; \frac{4q-2q}{3} \;=\; \frac{2q}{3}.$$

At this point the method breaks down for two different reasons.

1. If q *is not* divisible by 3, then p' and q' are *not integers*, so the descent argument cannot even begin.
2. If q *is* divisible by 3, then $p' = p/3$ and $q' = q/3$, and therefore

$$\frac{p'}{q'} \;=\; \frac{2q/3}{q/3} \;=\; 2 \;=\; \frac{p}{q}.$$

   In this case we have not produced a new pair of integers that contradicts the minimality of p and q; we have merely canceled a common factor of 3. The process does not yield a strictly smaller solution to (2).

What if we try $n = \sqrt{6}$? Arguing as before we obtain

$$6\left(\frac{p-q}{5}\right)^2 \;=\; \left(\frac{6q-p}{5}\right)^2 \quad \text{with} \quad \sqrt{6} \;=\; \frac{p}{q}; \tag{5}$$

thus $p = \sqrt{6}q$ and since $2 < \sqrt{6} < 3$ we easily see $p < 3q$. Multiplying both sides above by 5^2 leads to a new expression for $\sqrt{6}$:

$$\sqrt{6} \;=\; \frac{6q-p}{p-q}. \tag{6}$$

In order to obtain a contradiction we need this to be a rational approximation with smaller numerator, so we require $6q - p < p$ or $6q < 2p$ which of course is the same as $3q < p$; unfortunately $3q > p$, thus we have a new rational approximation with *larger* numerator, and there is thus no contradiction.

Why are we unable to obtain a contradiction here? When n was an odd integer at most 9 we could write $n-1=2n'$ for an integer n'. We could then remove n' from the expressions for $\widetilde{q}$ and $\widetilde{p}$, and still have a factor of 2 in the denominator to make the quantities smaller. We are unable to do that here; if $n=4$ then $n-1$ is the prime 3, while if $n=6$ then the $n-1$ is the prime 5.

Future work

We end with some further questions for the reader to consider; if interested please contact the corresponding author at sjm1@williams.edu.

First, as discussed in Remark 2, what other integers n have geometric arguments proving the irrationality of its square-root? Our construction here has an obstruction that prevents us from tackling $n>8$; perhaps some of the polygonal arguments can be modified, or perhaps a new construction?

Second, to date no one has done a three-dimensional construction; can this argument be modified to prove the irrationality of $\sqrt[3]{2}$ (or $\sqrt[k]{n}$ for some $k>2$)?

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References

1. T. M. Apostol, Irrationality of the square root of two—A geometric proof, Amer. Math. Monthly 107 (2000), no. 9, 841-842.
2. A. Bogomolny, Square root of 2 is Irrational, https://www.cut-the-knot.org/proofs/sq_root.shtml.
3. G. Cairns, Proof Without Words: $\sqrt{2}$ Is Irrational, Math. Mag. 85 (2012), no. 2, 123.
4. Z. Chen, S. J. Miller and C. Wu, Geometric Proofs of the Irrationality of Square-Roots for Select Integers, to appear in the Fibonacci Quarterly. https://arxiv.org/abs/2410.14434.
5. J. H. Conway, The power of mathematics, Power (A. Blackwell and D. MacKay, eds.), Darwin College Lectures 16, Cambridge University Press.
6. S. Lang, Undergraduate Algebra, Springer-Verlag, second edition, 1990.
7. G. H. Hardy and E. Wright, An Introduction to the Theory of Numbers, 5th edition, Oxford Science Publications, Clarendon Press, Oxford, 1995.
8. S. J. Miller and D. Montague, Picturing Irrationality, Math. Mag. 85 (2012), no. 2, 110-114. Expanded version: https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/irrationality50.pdf.
9. S. J. Miller and D. Montague, Irrationality from the Book, Journal of Mathematical Insight 15 (2023), no. 3, 45–67.
10. S. J. Miller and C. Silva, If a prime divides a product..., College Mathematics Journal 48 (2017), no. 2, 123–128. https://arxiv.org/abs/1012.5866v1.
11. N. J. A. Sloane, The On-line Encyclopedia of Integer Sequences, https://oeis.org.
12. R. A. Podestá, A Geometric Proof That $\sqrt{2}$, $\sqrt{5}$ and $\sqrt{7}$ Are Irrational, https://www.researchgate.net/publication/339972098.
13. P. Tennenbaum, private communication to S. J. Miller, 2009.

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