

UPPER BOUNDS FOR THE LOWEST FIRST ZERO IN FAMILIES OF CUSPIDAL NEWFORMS

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ABSTRACT. Assuming the Generalized Riemann Hypothesis, the non-trivial zeros of L -functions lie on the critical line with the real part $1/2$. We find an upper bound of the lowest first zero in families of even cuspidal newforms of prime level tending to infinity. We obtain explicit bounds using the n -level densities and results towards the Katz-Sarnak density conjecture. We prove that as the level tends to infinity, there is at least one form with a normalized zero within 0.218503 of the average spacing. We also obtain the first-ever bounds on the percentage of forms in these families with a fixed number of zeros within a small distance near the central point.

CONTENTS

1. Introduction	1
1.1. Background	1
1.2. Modular Forms	2
1.3. Random Matrix Theory	3
1.4. Centered Moments	5
1.5. Main Results	6
2. Preliminaries	8
3. Proof of Theorem 1.6	10
3.1. Construction of test function	10
3.2. Proof of Theorem 1.6	11
3.3. Explicit bounds on first zero	12
4. Proof of Theorem 1.8	15
4.1. Explicit bounds for percentages	16
5. Future work	17
Appendix A. Expanding the Space of Test Functions	18
References	19

1. INTRODUCTION

1.1. Background. The zeros and poles of L -functions carry substantial information about various quantities in number theory, from the distribution of primes to class numbers to the Mordell-Weil group (at least conjecturally; see [BSD1, BSD2]). The starting point of many of these investigations is the Generalized Riemann Hypothesis (GRH): all non-trivial zeros in the critical strip have real part $1/2$. As the distribution

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of these zeros is of great importance, a natural question is: *assuming GRH, how far up must we go on the critical line before we are assured that we see the first zero?*

The first critical zero of the Riemann zeta function is surprisingly large compared to other L -functions, around $1/2 + 14.135i$. Stephen D. Miller [Mi] proved an upper bound on the first zero of the form $1/2 + i\gamma$ for all L -functions of real archimedean type of $-\gamma_0 < \gamma < \gamma_0$, where $\gamma_0 \approx 14.13$. This is generalized in [B–] to hold for general L -functions, for $-\gamma_1 < \gamma < \gamma_1$ with $\gamma_1 \approx 22.661$; see also [B, HR]. For elliptic curves, Mestre [Mes] proved the first zero occurs by $O(1/\log \log N_E)$, where N_E is the conductor of E , though we expect the correct scale to study the zeros near the central point is the significantly smaller $1/\log N_E$.

Complementing the previous studies which gave bounds for the lowest first zero for individual L -functions, we explore what improvements arise when we examine all forms in a family and investigate how high up we must go to ensure we see the lowest of the lowest zeros. We prove upper bounds on the lowest first zero above the central point among all the forms for the family of even cuspidal newforms of prime level tending to infinity. Additionally, similar calculations work for the family of odd cuspidal newforms with a trivial modification since odd forms have a zero at the central point by symmetry so the known contribution may be removed from the computations. Moreover, we obtain results on the number of zeros near the central point (low-lying zeros) by obtaining an upper bound on the percentage of forms in the family that have a certain number of zeros in a small interval near the central point. We bound the percentages of forms in these families that can have “many” zeros “close” to the central point. The smaller the window and the greater the number of zeros the less likely this is to happen; our goal is to quantify how unlikely.

Below we describe the families of L -functions we study, and then state our results. We assume the reader is familiar with the literature; see [IK] for a review of needed properties of L -functions, and [BFMT-B, KS2] for a history and results on connections between the behavior of zeros near the central point and eigenvalues of random matrix ensembles. The methods of this paper are an extension of the arguments in [GM, Mes].

1.2. Modular Forms. We quickly review some background material and notation. Given a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a complex number z , we set $\gamma z = (az + b)/(cz + d)$. The matrices T and S , with respective actions

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z = z + 1, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z = -1/z, \quad (1.1)$$

generate the group $\mathrm{SL}_2(\mathbb{Z})$, the set of 2×2 integer matrices with determinant 1 (under multiplication); below we look at functions that transform nicely under this group, as well as its subgroups.

Definition 1.1. *Let f be a function on the upper-half of the complex plane $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. A modular form of weight k is a meromorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$ such that*

$$f(\gamma z) = (cz + d)^k f(z) \quad (1.2)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Additionally, f is bounded as $\Im(z)$ goes to infinity.

Notice that if we take $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have $f(z + 1) = f(z)$, which means that f is periodic with period 1. The cusps forms are modular forms where $f(z)$ approaches 0 when $\Im(z) \rightarrow \infty$.

We can generalize the above definition through additional restrictions on matrices under which our functions should transform nicely. A modular form of level N has the additional property that we only care about the transformation rule for $f(\gamma z)$ when the lower left entry of γ (what we called c above) equals zero modulo N .

Definition 1.2 (Cuspidal Newforms). Let $H_k^*(N)$ be the set of holomorphic cusp forms of weight k that are newforms¹ of level N . For every $f \in H_k^*(N)$, we have a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz) \quad (1.3)$$

with $e(z) = e^{2\pi iz}$. We set $\lambda_f(n) = a_f(n)n^{-(k-1)/2}$. The L -function associated to f is

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n)n^{-s}. \quad (1.4)$$

The completed L -function is

$$\Lambda(s, f) := \left(\frac{\sqrt{N}}{2\pi} \right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(s, f). \quad (1.5)$$

Each $\Lambda(s, f)$ satisfies a functional equation $\Lambda(s, f) = \epsilon_f \Lambda(1-s, f)$, where ϵ_f is either $+1$ or -1 . We can thus separate $H_k^*(N)$ into two subsets:

$$H_k^+(N) = \{f \in H_k^*(N) : \epsilon_f = +1\} \quad \text{and} \quad H_k^-(N) = \{f \in H_k^*(N) : \epsilon_f = -1\}.$$

The first set is the even forms (which we study below), while the second is the odd ones. We adjusted the coefficients of our forms by removing the factor $n^{-(k-1)/2}$ so that the functional equation relates s to $1-s$.

Since the work of Montgomery and Dyson [Mon], there is extensive literature that many properties of L -functions, from their values to the distribution of spacings between their zeros, are well-modeled by ensembles of matrices (for details see §1.3 as well as the references mentioned earlier). For suitable statistics and restricted test functions, there is a unique matrix ensemble (the scaling limit of a classical compact group, or a slight modification of one) which we can associate to a given family of L -functions; we slightly abuse notation and say that ensemble is the associated symmetry group (even though it is possible the agreement fails for another statistic or less restricted test functions). The symmetry group corresponding to $H_k^*(N)$ is the Orthogonal group, denoted by O . For the subset $H_k^+(N)$, its associated symmetry group is the scaling limit of the Special Orthogonal group of even-sized matrices, represented as $SO(\text{even})$. For $H_k^-(N)$, its associated symmetry group is the scaling limit of the Special Orthogonal group of odd-sized matrices, represented as $SO(\text{odd})$.

1.3. Random Matrix Theory. Montgomery's work in 1972 unveiled a fascinating link between the distribution of zeros of the Riemann zeta function and the distribution of eigenvalues of random Hermitian matrices. He conjectured that the n -level correlation function of the Riemann zeta function's zeros and that of the Gaussian Unitary Ensemble (GUE) coincide. For suitably restricted test functions, this conjecture was later proven by Hejhal [Hej] for the 3-level of the Riemann zeta function and Rudnick and Sarnak [RS1] for all n for all automorphic L -functions). As our goal is to study other statistics than the n -level correlations, we content ourselves to just remark that this was the first statistic studied, and knowing these correlations for all n yields the spacing distribution between adjacent zeros. Odlyzko calculated the zeros of the Riemann zeta function at the height of the $10^{20\text{th}}$ zero and observed that the main term of the spacing of nearby zeros agrees with that of the GUE (see [H, BFMT-B, Con] and their references for more details on these connections).

Although the n -level correlations provide amazing results of the distribution of zeros of one form, these are defined through limits of sums over the zeros on the critical line and are thus insensitive to finitely many zeros. In particular, they cannot say anything about the behavior near the central point, which is often of great arithmetic interest. Katz and Sarnak [KS1, KS2] then observed that the statistics of zeros near the central point of many L -functions also align with the eigenvalues near 1 of classical compact groups, and thus the

¹This means they are not lifted trivially from a modular form of level $M|N$.

GUE is not the full story. They introduced the n -level densities statistic, where, unlike the correlations, most of the contribution is from the zeros very close to the central point and is thus very sensitive to finitely many zeros.

Definition 1.3 (n -Level Density). *The n -level density of an L -function $L(s, f)$ for a test function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by*

$$D_n(f; \Phi) := \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} \Phi \left(\frac{\log c_f}{2\pi} \gamma_{f, j_1}, \dots, \frac{\log c_f}{2\pi} \gamma_{f, j_n} \right), \quad (1.6)$$

where c_f is the analytic conductor of f and the non-trivial zeros² of $L(s, f)$ are denoted by $\frac{1}{2} + \gamma_f^{(j)}$. In many applications, we assume Φ factors as a product of n copies of a function ϕ : $\Phi(x_1, \dots, x_n) = \prod_{j=1}^n \phi_j(x_j)$ where often³ the ϕ_j all equal the same ϕ . We denote a zero scaled by a fixed multiple⁴ of the logarithm of the analytic conductor by a tilde; that is, $\tilde{\gamma}_f^{(j)} = \gamma_f^{(j)} \frac{\log c_f}{2\pi}$.

The analytic conductor arises from the functional equation of the L -function; near the central point, the spacing between adjacent zeros is on the order of the reciprocal of its logarithm. This provides the natural scale to study the distance of zeros from the central point; on average, it is around $1/\log c_f$ from one such zero to another. We are interested in how often the first or first few zeros are less than a fixed percentage of this quantity. In other words: *How often is the first zero less than half the average spacing?* For families of even N cuspidal newforms, the analytic conductor doesn't depend on the form, and this greatly simplifies the analysis (see [M] for the challenges that arise when the conductor varies, even by small amounts on a log-scale).

Note: *Whenever we talk about bounds on the location of zeros near the central point, we always mean relative to the average spacing; thus a bound of .25 means .25 times the reciprocal of the logarithm of the analytic conductor.*

We balance between obtaining optimal results and having calculations that can be done in closed form. These compromises require a test function Φ with certain characteristics that yield excellent bounds while still having feasible calculations that we expect get us close to the optimal result. As the zeros are symmetric about the real axis, we need only study even Φ functions.⁵ We always choose Φ to be a Schwartz function, so all of its derivatives decay faster than $|x|^{-A}$ for any $A > 0$.

The test function necessarily has compact support for its Fourier transform, defined in (2.1), so we may execute the number theoretic sums that arise. If the test function is concentrated near the central point and rapidly decaying, we glean a significant amount of information on the zeros close to the real axis. For instance, a delta spike would retrieve contributions solely from zeros at the central point and remove any noise from zeros with large imaginary components. Similar to the trade-off in Heisenberg's Uncertainty Principle (the more one knows a particle's position the less one knows about its momentum), there is a Fourier transform analogue: the more precisely one localizes a function, the more spread out its Fourier transform is. For us, this means that the closer we get to a delta spike, we have to handle more terms in the resulting number theory expansions, and unfortunately our averaging formulas are not able to do so beyond a certain point. We will see that there are additional criteria; some of these are not necessary in general but are required for our method; we can only optimize over a subset of the test functions.

²We are assuming GRH; therefore, all zeros of L -functions shall lie on the critical strip.

³See [CGJMT] for an interesting example where it is essential that the ϕ_j have different support!

⁴The presence of the 2π factor is to simplify expressions based on the definition of the Fourier transform.

⁵Any test function is the sum of an even and an odd function and the sum of the odd part over the zeros vanish, so there is no need to include it.

For Theorem 1.8, we use the naive test function, defined as follows. In all applications, it is clear what the largest support σ_n is such that we know the n -level density or n^{th} centered moments. To indicate this dependence on σ_n , we include a subscript n below.

Definition 1.4 (Naive Test Function). *The naive test function for the n -level is defined by*

$$\phi_{\text{naive};n}(x) := \left(\frac{\sin(\pi\sigma_n x)}{\pi\sigma_n x} \right)^2, \quad (1.7)$$

and its Fourier transform is

$$\widehat{\phi}_{\text{naive};n}(y) = \frac{1}{\sigma_n} \left(y - \frac{|y|}{\sigma_n} \right) \quad (1.8)$$

for $|y| < \sigma_n$ and zero otherwise.

Since the naive test function is non-negative, it is useful in determining upper bounds as we can drop non-negative contributions of the density. Some previous studies such as [BCDMZ, FM, ILS] attempt to find the optimal non-negative test functions for certain level densities. However, the improvement in results is quite small, so we are satisfied with just using the naive test function as it greatly simplifies the exposition. If necessary, additional work through manipulating test functions may yield slight improvements.

For Theorem 1.6, we have a different set of characteristics for the test function ϕ . Instead of being completely non-negative, we now only want ϕ to be non-negative within a certain distance from the central point and non-positive otherwise (and positive at the origin). Such a setup allows us to drop non-positive contributions to the density, thus obtaining a lower bound. To find such ϕ , it is easier to first construct its finitely supported Fourier transform; we provide details in §3.1.

1.4. Centered Moments. The n^{th} centered moments are alternative statistics (but essentially equivalent after some combinatorics) to the n -level densities. Hughes-Miller [HM] show their expansions are better suited to obtain bounds than the determinant expansions of Katz-Sarnak. For the following theorem from [C–], we may only take σ up to 2, though the random matrix theoretic correspondence suggests that arbitrarily large support can be taken. We state the results below for arbitrary σ , so improvements in those calculations immediately transfer to improvements here.

Theorem 1.5. *Assume GRH for Dirichlet L -functions and for cuspidal newforms and their symmetric squares⁶. Let $n \geq 2$ and $\text{supp}(\phi) \subset (-\frac{\sigma}{n}, \frac{\sigma}{n})$, where by [C–] we may take $\sigma = 2$. Define*

$$\sigma_\phi^2 := 2 \int_{-\infty}^{\infty} |y| \widehat{\phi}(y)^2 dy \quad (1.9)$$

and

$$\begin{aligned} R(m, i; \phi) := & 2^{m-1} (-1)^{m+1} \sum_{l=0}^{i-1} (-1)^l \binom{m}{l} \\ & \left(-\frac{1}{2} \phi^m(0) + \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_2) \cdots \widehat{\phi}(x_{l+1}) \right. \\ & \left. \int_{-\infty}^{\infty} \phi^{m-l}(x_1) \frac{\sin(2\pi x_1(1 + |x_2| + \cdots + |x_{l+1}|))}{2\pi x_1} dx_1 \cdots dx_{l+1} \right) \end{aligned} \quad (1.10)$$

and

$$S(n, a, \phi) := \sum_{l=0}^{\lfloor \frac{a-1}{2} \rfloor} \frac{n!}{(n-2l)!l!} R(n-2l, a-2l, \phi) \left(\frac{\sigma_\phi^2}{2} \right)^l. \quad (1.11)$$

⁶With more work, along the lines of the remark after equations (4.23) and (4.23) of [ILS], one can remove the appeal to GRH for the symmetric square L -functions by careful averaging.

By $\langle Q(f) \rangle_{N;\pm}$ we mean the average of $Q(f)$ over all f in the family of even (odd) cuspidal newforms of level N for the positive (negative) sign (the number of such forms is proportional to N). Then

$$\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \langle (D(f; \phi) - \langle D(f; \phi) \rangle_{N;\pm})^n \rangle_{N;\pm} = 1_{n \text{ even}} (n-1)!! \sigma_\phi^n \pm S(n, a; \phi), \quad (1.12)$$

where

$$1_{n \text{ even}} := \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

1.5. Main Results. We obtain results on the number and location of low-lying zeros of the family of cuspidal newforms of level N through n^{th} centered moment. In Theorem 1.6, we give an upper bound $(-\omega, \omega)$ on the lowest first zero of the family through odd-centered moments. We provide additional results in Theorem 1.8 for zeros near the central point and provide an upper bound on the percentage of forms in the family with at least r zeros within a certain interval $(-\rho, \rho)$ near the central point.

Previous work focused on using the 1-level density to obtain such bounds; the n -level densities and n^{th} centered moments are far more difficult to work with, leading to significantly harder integrals as well as combinatorial challenges in removing the contributions of certain forms and isolating the desired results. However, as the support increases, these results justify the efforts and are significantly better than those from the 1-level. Other work has shown the value of this approach with better results for bounding the probability of vanishing to order r or greater at the central point; see [DM, LM]. Instead of trying to bound the lowest zero of one form, our question is different as we are searching for the lowest of the low and thus we obtain significantly better results.

Theorem 1.6. *Assume GRH. Let $h(y)$ be an even function that is at least twice continuously differentiable with h supported in $[-1, 1]$, and monotonically decreasing from 0 to 1. Let $f(y) := h(2yn/\sigma)$ and $g(y) := (f * f)(y)$, the convolution of f with f . Letting $\phi_\omega(x) = (1 - (x/\omega)^2)\widehat{g}(x)$, note that ω is the first zero of ϕ_ω . Therefore its Fourier transform $\widehat{\phi}_\omega$ satisfies $\widehat{\phi}_\omega(y) = g(y) + (2\pi\omega)^{-2}g''(y)$. Then we have $\text{supp}(\widehat{\phi}_\omega) \subset (-\sigma/n, \sigma/n)$, $\phi_\omega(x)$ is non-negative when $|x| \leq \omega$, and $\phi_\omega(x)$ is non-positive when $|x| > \omega$. Then for an odd $n = 2m + 1$, whenever ω satisfies the inequality*

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2} \int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y) dy \right)^n < 1_{n \text{ even}} (n-1)!! \sigma_{\phi_\omega}^n + S(n, a; \phi_\omega), \quad (1.13)$$

then there exists at least one form with at least one normalized zero in the interval $(-\omega, \omega)$. In particular, if we denote $(-\omega_{\min}, \omega_{\min})$ to be the interval near the central point that contains at least one normalized zero, then the one-level density gives us an explicit expression that there is at least one zero in this interval whenever ω_{\min} satisfies

$$\omega > \left(-\frac{\sigma \int_0^1 h(u)^2 du + \frac{\sigma^2}{4} \int_0^{2/\sigma} \int_{v-1}^1 h(u)h(v-u) du dv}{\frac{1}{\sigma} \int_0^1 h(u)h''(u) du + \frac{1}{4} \int_0^{2/\sigma} \int_{v-1}^1 h(u)h''(v-u) du dv} \right)^{-\frac{1}{2}} \pi^{-1}. \quad (1.14)$$

For the support $\sigma = 2$ (which is the largest known value we can take for this family if we only assume GRH, though it is conjectured to hold for all σ) and $h = \cos(\pi x/2)$ for $|x| \leq 1$ and 0 otherwise (the optimum function to use for one-level density), the smallest first nontrivial normalized zero of the family of even cuspidal newforms has imaginary part γ in the interval $[-\gamma_{\min}, \gamma_{\min}]$ with $\gamma_{\min} \approx 0.21864$.

Increasing the space of test functions yields improvements on omega. Letting $h(y)$ be three times differentiable and ϕ_ω be a degree 6 polynomial yields a minimum value of ω of approximately 0.218503.

While increasing the level n gives better results, we can only use $\sigma = 2$ (currently the best-known result under the assumption of only GRH) and compare the bounds arising from using results from levels 1, 3, and

5. However, as we increase the level, there are worse results for the bounds on the lowest zero. We prove that ω and σ are inversely proportional, which means that if we take a larger σ , ϕ_ω can be more focused around the central point, better approximating the delta spike, and then the benefit of raising to the power n comes into play and yields smaller intervals. The random matrix theory conjectures imply that we can take σ arbitrarily large (as aforementioned, results are proven in the number theory side for sums over zeros under GRH only for σ up to 2, though assuming additional conjectures allows a slight increase). Thus, if we use a σ greater than 2, we see that eventually as the support becomes sufficiently large, the higher levels yield better results.

Remark 1.7. *We examined additional results from levels 3 and 5. Unfortunately, higher levels involve difficult integrals to evaluate that do not possess closed-form evaluations, especially when we are using a complicated test function involving Fourier transforms and convolution. While we cannot use theoretical arguments or software packages such as Mathematica to evaluate in closed form the multi-dimensional integrals on the right-hand side, by using Riemann sums we can calculate the integrals accurately enough by having the error term sufficiently small and evaluate the integrals up to a small enough error for our purposes. In the computations below we use Simpson’s Rule. As the error is proportional to the fifth power of the size of the subdivisions, our values should be very close to the truth; however, to use these values with confidence would require a small amount of additional work to determine the bounding multiplicative constants (we chose a fine enough partition so that subdividing each interval in half did not produce a noticeable change). We found that the result from the 3-level is better than that from the 1-level when $\sigma = 7$. Furthermore, when $\sigma = 6$, the 5-level statistics beat the 1 and 3 levels. These results highlight the importance of deriving the theory on bounds arising from higher moments.*

Although previous investigations have mostly been concerned with bounds on the first zero above the central point, Goes-Miller [GM] obtained upper and lower bounds for the number of zeros in one-parameter families of elliptic curves in windows near the central point. We build on their work and bound the percentage of forms with a large number of zeros in a small window about the central point – there is tremendous freedom in assigning values to “large” and “small” above. In brief, we obtain excellent bounds stating that the probability is very low of many more zeros in a small window than expected. As the averaging formulas for cuspidal newforms are better than those for elliptic curve L -functions, we perform number theoretic calculations for larger σ and thus obtain better results than in the elliptic curve case. Specifically, we bound the percentage of forms in the family of cuspidal newforms \mathcal{F}_N with r normalized zeros on the interval $(-\rho, \rho)$. If ρ is sufficiently small, we expect there to be few such zeros.

Theorem 1.8. *Assume GRH. Let $P_{r,\rho}(\mathcal{F})$ denote the limit when N tends infinity of the percent of forms in \mathcal{F}_N that have at least r normalized zeros on the interval $(-\rho, \rho)$. For $r \geq \mu(\phi, \mathcal{F})/\phi(\rho)$ and $n = 2m$ even,*

$$P_{r,\rho}(\mathcal{F}) \leq \frac{1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n, a; \phi)}{(r\phi(\rho) - \mu(\phi, \mathcal{F}))^n}, \quad (1.15)$$

where ϕ is a non-negative even Schwartz test function.

Previous studies [DM, HM, LM] bounded the order of vanishing through n -level densities and n^{th} centered moments and observed an interesting phenomenon when one increased the level⁷. While it is predicted that higher levels would yield better bounds, one only sees the improvement for higher-order vanishing (note that this is similar to the effect we observed above where there must be at least one form with at least one zero sufficiently close). In fact, better bounds for small analytic ranks are produced through lower levels. We see a similar pattern in our bounds resulting from Theorem 1.8.

If we plot the percentage versus the number of normalized zeros for a fixed interval $(-\rho, \rho)$ (see Figure 1), we see an interesting correlation. Although the bounds from the higher levels begin higher than those from lower levels when r is small, they decrease faster and eventually yield better results as r increases

⁷In this discussion, by *level*, we mean the integer n as in n -level density.

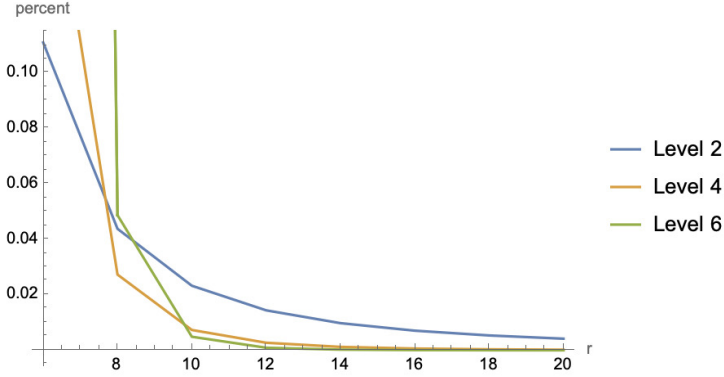


FIGURE 1. Percentage vs. number of zeros for a fixed $\rho = .4$. For ease of viewing, we have connected the discrete data to piece-wise linear curves.

(recall that the larger r is, the less likely it is to have at least that many normalized zeros close to the central point).

For example, compare the bounds from level 2 to level 6 for a fixed $\rho = .4$. We see that when $r = 6$, the 2-level gives a bound of 0.111085 (i.e., at most approximately 11% have a zero within 40% of the average spacing near the central point) while the 6-level yields 1.58572 (which is a useless bound as it exceeds a hundred percent!). However, when we consider $r = 20$, we see a significantly better bound from level 6 which gives 0.0000121805 compared to level 2 which yields 0.0042038. Thus, the higher levels yield greater decay in the probability of the unlikely case of a large number of zeros very close to the central point, far better than the results from the 2-level density.

We also see that for a fixed number of zeros, the higher level densities give better bounds for a larger interval $(-\rho, \rho)$ near the central point (see the tables in §4.1). For instance, for a fixed $r = 8$, the 4-level density gives a bound of 0.0331395 at $\rho = .5$, smaller than the 6-level which yields 0.0534808. However, when we increase ρ to be .9, we have the 6-level giving a better bound of 0.847282 rather than 2.14456 from level four. See Tables 1 and 2.

2. PRELIMINARIES

First, we establish the necessary notation and present the required preliminary results, starting with discussing certain properties of the Fourier transform. As the Fourier transform of a convolution is the product of the Fourier transforms, the difficult process of integration is converted to the simpler task of multiplication. However, this conversion later necessitates computing an inverse Fourier transform. The following properties are standard and are collected for convenience and to identify normalizations; see [SS] for details.

Definition 2.1 (Fourier Transform). *The Fourier transform of a function $\phi(x)$ is defined by*

$$\mathfrak{F}(\phi(x))(y) := \widehat{\phi}(y) = \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i xy} dx. \quad (2.1)$$

A simple calculation yields the following.

Lemma 2.2 (Scaling, Derivative). *Scaling the Fourier transform by a constant c satisfies*

$$\mathfrak{F}(\phi(cx))(y) = \frac{1}{|c|} \cdot \widehat{\phi}\left(\frac{y}{c}\right), \quad (2.2)$$

while for derivatives we have

$$\mathfrak{F}(\phi'(x))(y) = -2\pi i y \widehat{\phi}(y), \quad \text{and} \quad \mathfrak{F}(\phi''(x))(y) = -(2\pi y)^2 \widehat{\phi}(y). \quad (2.3)$$

Definition 2.3 (Convolution). *The convolution of $a(x)$ and $b(x)$ is*

$$(a * b)(x) := \int_{-\infty}^{\infty} a(t)b(x-t)dt. \quad (2.4)$$

*Note if a and b are functions supported in $(-s, s)$ then $a * b$ is supported in $(-2s, 2s)$.*

The following theorem converts convolutions to easier quantities to study.

Theorem 2.4 (Convolution Theorem). *For functions $a(x)$ and $b(x)$,*

$$\mathfrak{F}(a * b) = \mathfrak{F}(a)\mathfrak{F}(b). \quad (2.5)$$

We will use Leibniz's theorem for differentiating under the integral sign to represent derivatives of convolutions as solutions to integrals.

Lemma 2.5. (*Leibniz Integral Rule*) *Consider the integral with respect to t of a bivariate function $f(x, t)$. Letting the bounds of the integrands depend on functions, $a(x)$ and $b(x)$, then:*

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, t) dt = f(x, b(x)) \cdot \frac{d}{dx} b(x) - f(x, a(x)) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt. \quad (2.6)$$

We introduce the Fourier transform pair of the sinc function and the rectangular function.

Definition 2.6 (Sinc function). *The normalized sinc function is*

$$\text{sinc } x := \begin{cases} \frac{\sin \pi x}{\pi x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases} \quad (2.7)$$

Definition 2.7 (Rectangular function). *The rectangular function, also known as the normalized boxcar function, is*

$$\text{rect} \left(\frac{x}{a} \right) := \begin{cases} 0 & \text{if } |x| > \frac{a}{2} \\ \frac{1}{2} & \text{if } |x| = \frac{a}{2} \\ 1 & \text{if } |x| < \frac{a}{2}. \end{cases} \quad (2.8)$$

Lemma 2.8. *The sinc function and the rectangular function are the Fourier transform of each other:*

$$\mathfrak{F}(\text{sinc}(x)) = \int_{-\infty}^{\infty} \text{sinc}(x)e^{-2\pi ixy} dx = \text{rect}(x) \quad (2.9)$$

and

$$\mathfrak{F} \left(\text{rect} \left(\frac{x}{a} \right) \right) = \int_{-\infty}^{\infty} \text{rect} \left(\frac{x}{a} \right) e^{-2\pi ixy} dx = a \text{sinc}(ax). \quad (2.10)$$

Plancherel's Theorem is sometimes useful when integrating a product of two functions, converting the integral of a product to the integral of the product of their Fourier transforms. We use this on $L^2(\mathbb{R})$, which are functions whose integral of the square of their absolute value is finite; as our test functions are bounded and decay rapidly, they are all in this space.

Theorem 2.9 (Plancherel's Theorem). *For two $L^2(\mathbb{R})$ functions $a(x)$ and $b(x)$,*

$$\int_{-\infty}^{\infty} a(x)\overline{b(x)}dx = \int_{-\infty}^{\infty} \mathfrak{F}((a(x))\overline{\mathfrak{F}(b(x))})dx. \quad (2.11)$$

In our analysis, we have equalities where the left-hand side involves sums over zeros, while the right-hand side involves sums of integrals whose integrands are comprised of products of quantities related to our test functions and the n -level density kernels. As these integrals are impossible to evaluate in closed form for our test functions, we use Riemann sums to approximate the integrals by choosing the step size sufficiently small we can ensure that the required number of digits of our answer is correct.

Theorem 2.10 (Riemann Sum). *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function defined on a closed interval $[a, b]$ and $a, b \in \mathbb{R}$. Let $P = (x_0, x_1, \dots, x_n)$ be the partition of $[a, b]$ such that*

$$a := x_0 < x_1 < \dots < x_n =: b. \quad (2.12)$$

The Riemann sum S of f over $[a, b]$ with partition P is

$$S(f; P) := \sum_{k=1}^n f(x_k^*) \Delta x_k, \quad (2.13)$$

where $\Delta x_k := x_k - x_{k-1}$ and $x_k^ \in [x_{k-1}, x_k]$. The sum converges to the integral of f over $[a, b]$ as $\max_k \Delta x_k$ approaches zero.*

Many of our terms involve combinatorial factors, including the double factorial.

Definition 2.11 (Double Factorial). *The double factorial of n is the product of all integers up to n that shares the same parity (odd or even):*

$$n!! := \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n - 2k) = n(n-2)(n-4)(n-6) \dots. \quad (2.14)$$

Definition 2.12. *Let $\mathcal{F}_{N,r}^{(\rho)}$ denote the family of forms in \mathcal{F}_N that have at least r zeros within a distance ρ . For a fixed r , as N tends to infinity through the primes, define $P_{r,\rho}(\mathcal{F})$ to be the limit of the percentage of forms in \mathcal{F}_N that have at least r zeros within a distance ρ :*

$$P_{r,\rho}(\mathcal{F}) := \lim_{N \rightarrow \infty} \frac{|\mathcal{F}_{N,r}^{(\rho)}|}{|\mathcal{F}_N|}. \quad (2.15)$$

3. PROOF OF THEOREM 1.6

3.1. Construction of test function. We first outline the construction of the test function ϕ_ω that meets the required conditions. We initially adopt a similar process from [GM] and [HR], and then later discuss how to modify to increase the space of test functions and obtain better results.

We start on the Fourier transform side and construct a $\widehat{\phi}(y)$ that has finite support $(-\frac{\sigma}{n}, \frac{\sigma}{n})$, since it is not easy to find a function that satisfies all the conditions for ϕ directly but it is easy to start on the Fourier transform side with a function with restricted support. Let h be an even function that is at least twice continuously differentiable, supported in $[-1, 1]$, and monotonically decreasing from 0 to 1. For fixed ω and σ , let $f(y) := h(2yn/\sigma)$ and $g(y) := (f * f)(y)$ (the convolution of f with itself (2.3)). Note $\widehat{g}(x) = \widehat{f}(x)^2 \geq 0$. Let $\widehat{\phi}_\omega(y)$ equal $g(y) + (2\pi\omega)^{-2}g''(y)$ (where $\widehat{\phi}_\omega(y)$ is the Fourier transform of $\phi_\omega(x)$).

We show that $\text{supp } \widehat{\phi}_\omega(y) \subset (-\frac{\sigma}{n}, \frac{\sigma}{n})$. Notice, since $\text{supp}(f) \subset (-\frac{\sigma}{2n}, \frac{\sigma}{2n})$ and $g = f * f$, $\text{supp}(g) \subset (-\frac{\sigma}{n}, \frac{\sigma}{n})$. Since $\widehat{\phi}_\omega(y) = g(y) + (2\pi\omega)^{-2}g''(y)$ and the support of g'' is contained in that of g , the support of $\widehat{\phi}_\omega(y)$ is contained in $(-\frac{\sigma}{n}, \frac{\sigma}{n})$, as desired.

We find $\phi_\omega(x)$ by taking the Fourier transform⁸ of $\widehat{\phi}_\omega(y)$. The Fourier transform of $g''(y)$ is $-(2\pi y)^2 \widehat{g}(y)$. Combining the above, we have the Fourier transform of $\widehat{\phi}_\omega(y) = g(y) + (2\pi\omega)^{-2}g''(y)$ is $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$, shown in Figure 2.

Remark 3.1. *Following the framework of constructing the test function above, one can adopt more elaborate constructions, such as adding a quartic term in ϕ , to obtain potentially better test functions that yield better results, but at the cost of more involved computations. We explore this idea later.*

⁸A useful property of the Fourier transform is $\mathfrak{F}(\mathfrak{F}(\phi(x))) = \phi(-x)$. Since ϕ is even, we can find the original function using this property directly.

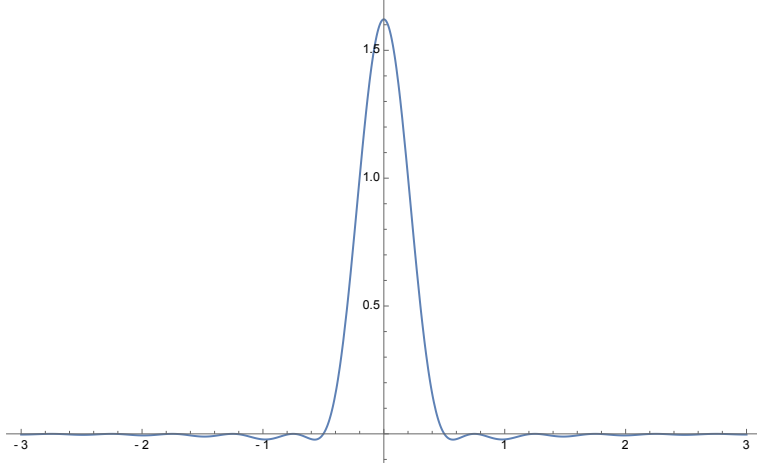


FIGURE 2. Plot of $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$, for $h = \cos(\frac{\pi y}{2})$, $\sigma = 2$, $\omega = .5$, and $n = 1$.

3.2. Proof of Theorem 1.6. We use odd centered moments to obtain a lower bound on the number of normalized zeros within a small distance ω from the central point. Then we prove the theorem by showing a contradiction if there are no zeros in the interval $(-\omega, \omega)$.

As remarked in the introduction, there is a long history of results on the n -level densities and the combinatorially easier centered moments. The first results are due to Iwaniec-Luo-Sarnak [ILS], who computed the 1-level density with support up to 2. Their result was later extended by Hughes-Miller [HM] to the n -level densities and centered moments with support up to $1/(n-1)$. Recently, there was a breakthrough by Cohen et. al. [C-], who were finally able to handle the combinatorics and prove that the n -level densities and centered moments agree with random matrix theory for support up to $2/n$; see Theorem 1.5. We extract the following useful consequence of their work.

Corollary 3.2. *We can write (1.12) as*

$$\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \left(\sum_j \phi(\tilde{\gamma}_{f,j} c_n) - \mu(\phi, \mathcal{F}) \right)^n = 1_{n \text{ even}} (n-1)!! \sigma_\phi^n \pm S(n, a; \phi), \quad (3.1)$$

where $\mu(\phi, \mathcal{F})$ is the mean of the family of cuspidal newforms.

Proof. The difference between the mean of the family in the limit and the mean when the level is N is negligible; this follows from an application of the Binomial Theorem and the Cauchy-Schwartz inequality. We can therefore simplify the calculations and replace $\mu(\phi, \mathcal{F}_N)$ with $\mu(\phi, \mathcal{F})$. For a complete proof see [DM]. \square

Theorem 3.3 (See [ILS, HM]). *The mean over $SO(\text{even})$ is*

$$\mu(\phi, \mathcal{F}) := \widehat{\phi}(0) + \frac{1}{2} \int_{-\infty}^{\infty} \widehat{\phi}(y) dy. \quad (3.2)$$

We restate our first main result, Theorem 1.6, for convenience, and then give the proof.

Theorem 1.6. *Assume GRH. For n odd, whenever ω satisfies the inequality*

$$-\left(\widehat{\phi}_\omega(0) + \frac{1}{2} \int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y) dy \right)^n < S(n, a; \phi_\omega), \quad (3.3)$$

there exists at least one form with one normalized zero on the interval $(-\omega, \omega)$.

Proof. As remarked earlier, we focus only on the family of even forms; there are trivial modifications if one studies the odd forms (since all forms with an odd functional equation are guaranteed a zero at the central point, we would need to remove that contribution).

We start with equation (3.1). As ϕ_ω is non-negative when $|x| \leq \omega$ and non-positive when $|x| > \omega$, the contribution of the scaled zeroes to the centered moment when $|x| \geq \omega$ is non-positive. Since the power is odd, if we drop those non-positive contributions, it does not increase the left-hand side; *this is why we restrict to odd levels*. Thus, considering a sum restricted to the positive contributions $|\gamma_{f,j}| \leq \omega$ will have a lower bound equal to the original sum as removing the negative contributions $|\gamma_{f,j}| > \omega$ shall only increase the inner sum in (3.1), giving that

$$\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \left(\sum_{|\gamma_{f,j}| \leq \omega} \phi_\omega(\tilde{\gamma}_{f,j} c_n) - \mu(\phi_\omega, \mathcal{F}) \right)^n \geq S(n, a; \phi_\omega). \quad (3.4)$$

Assume that no forms have a normalized zero in the interval $(-\omega, \omega)$. Then the scaled zeroes with imaginary part at most ω in absolute value contribute 0 to the centered moment, and thus

$$\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} (-\mu(\phi_\omega, \mathcal{F}))^n \geq S(n, a; \phi_\omega). \quad (3.5)$$

Since the mean is independent of f , we can extract it from the summation:

$$(-\mu(\phi_\omega, \mathcal{F}))^n \lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} 1 \geq S(n, a; \phi_\omega). \quad (3.6)$$

As $\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} 1 = \lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \frac{|\mathcal{F}_N|}{|\mathcal{F}_N|} = 1$, we find

$$(-\mu(\phi_\omega, \mathcal{F}))^n \geq S(n, a; \phi_\omega). \quad (3.7)$$

Because of the compact support of $\hat{\phi}_\omega$, by Theorem 3.3, this is

$$-\left(\hat{\phi}_\omega(0) + \frac{1}{2} \int_{-\sigma/n}^{\sigma/n} \hat{\phi}_\omega(y) dy \right)^n \geq S(n, a; \phi_\omega). \quad (3.8)$$

Thus, if we choose ω that satisfies the following inequality

$$-\left(\hat{\phi}_\omega(0) + \frac{1}{2} \int_{-\sigma/n}^{\sigma/n} \hat{\phi}_\omega(y) dy \right)^n < S(n, a; \phi_\omega), \quad (3.9)$$

we get a contradiction, indicating there has to be at least one form with one normalized zero on the interval $(-\omega, \omega)$. \square

Remark 3.4. *The theorem above improves itself as the zeros are distributed symmetrically along the critical line, and thus if $1/2 + i\gamma$ is a zero so too is $1/2 - i\gamma$. In fact, there are at least two zeros in such a window.*

3.3. Explicit bounds on first zero. We examine the inequality in (3.9) and give an explicit formula for the one-level calculation of the bound ω .

Theorem 3.5. *Assume GRH. The first normalized zero of the family of cuspidal newforms is in the interval $(-\omega_{\min}, \omega_{\min})$, where*

$$\omega > \left(-\frac{\sigma \int_0^1 h(u)^2 du + \frac{\sigma^2}{4} \int_0^{2/\sigma} \int_{v-1}^1 h(u)h(v-u) du dv}{\frac{1}{\sigma} \int_0^1 h(u)h''(u) du + \frac{1}{4} \int_0^{2/\sigma} \int_{v-1}^1 h(u)h''(v-u) du dv} \right)^{-\frac{1}{2}} \pi^{-1}. \quad (3.10)$$

Proof. Note that $\int_{-\sigma/n}^{\sigma/n} \widehat{\phi}_\omega(y) dy = \phi_\omega(0)$. Thus for $n = 1$, we can rewrite the inequality in (3.9) as

$$-\widehat{\phi}_\omega(0) - \frac{1}{2}\phi_\omega(0) < S(1, 1; \phi_\omega). \quad (3.11)$$

By Theorem 1.5, the right hand side of (3.11) is

$$S(1, 1; \phi_\omega) = R(1, 1; \phi_\omega) = -\frac{1}{2}\phi_\omega(0) + \int_{-\infty}^{\infty} \phi_\omega(x) \frac{\sin(2\pi x)}{2\pi x} dx. \quad (3.12)$$

The $-\phi_\omega(0)/2$ terms cancel, so we focus on simplifying the second term. Notice that directly integrating against the sine function is hard, so instead we use Plancherel's Theorem (Theorem 2.9) and rewrite the integral as an integral of the product of their Fourier transforms. The Fourier transform of $\phi_\omega(x)$ is $\widehat{\phi}_\omega(y)$. To perform the integration of $\phi_\omega(x)$ against $\sin(2\pi x)/2\pi x$, note the latter is the sinc function from Definition 2.6. Since the Fourier transform of the sinc function is the rectangular function, with some re-scaling by Theorem 2.2 we get

$$\int_{-\infty}^{\infty} \phi_\omega(x) \frac{\sin(2\pi x)}{2\pi x} dx = 2 \int_0^{\infty} \widehat{\phi}_\omega(y) \frac{1}{2} \text{rect}\left(\frac{y}{2}\right) dy. \quad (3.13)$$

Notice the rectangular function $\text{rect}(y/2)$ equals 1 from -1 to 1 and 0 everywhere else, so we may rewrite the above integral as $\int_0^1 \widehat{\phi}_\omega(y) dy$. Thus the RHS simplifies to $-\frac{1}{2}\phi_\omega(0) + \int_0^1 \widehat{\phi}_\omega(y) dy$. Combining with what we have for the LHS from (3.11), we get that the bound for the smallest first normalized zero is when

$$-\widehat{\phi}_\omega(0) < \int_0^1 \widehat{\phi}_\omega(y) dy. \quad (3.14)$$

We now simplify $\widehat{\phi}_\omega(0)$ in terms of an arbitrary h that fulfills our conditions. From the definition of $\widehat{\phi}_\omega$ we have

$$\widehat{\phi}_\omega(0) = g(0) + (2\pi\omega)^{-2}g''(0). \quad (3.15)$$

Through the construction of g and by taking advantage of the evenness of h and f , we get

$$g(0) = \int_{-\infty}^{\infty} f(t)f(0-t)dt = 2 \int_0^{\sigma/2} h\left(\frac{2t}{\sigma}\right)^2 dt = \sigma \int_0^1 h(u)^2 du. \quad (3.16)$$

Utilizing the Leibniz Rule of Integration, we obtain a formula for $g''(y)$ for $0 \leq y \leq \sigma$:

$$\begin{aligned} g''(y) &= \frac{d^2}{dy^2} \int_{y-\sigma/2}^{\sigma/2} f(t)f(y-t) dt \\ &= \frac{d}{dy} \left(-f\left(y - \frac{\sigma}{2}\right) f\left(\frac{\sigma}{2}\right) + \int_{y-\sigma/2}^{\sigma/2} f(t)f'(y-t) dt \right) \\ &= -f'\left(y - \frac{\sigma}{2}\right) f\left(\frac{\sigma}{2}\right) - f\left(y - \frac{\sigma}{2}\right) f'\left(\frac{\sigma}{2}\right) + \int_{y-\sigma/2}^{\sigma/2} f(t)f''(y-t) dt. \end{aligned}$$

Therefore, since $f(\sigma/2) = 0$ and $f(-\sigma/2) = 0$ by definition, we obtain the value for $y = 0$:

$$\begin{aligned}
g''(0) &= \int_{-\sigma/2}^{\sigma/2} f(t)f''(0-t) dt \\
&= 2 \int_0^{\sigma/2} f(t)f''(t) dt \\
&= 2 \int_0^{\sigma/2} h\left(\frac{2t}{\sigma}\right) \frac{4}{\sigma^2} h''\left(\frac{2t}{\sigma}\right) dt \\
&= \frac{4}{\sigma} \int_0^1 h(u)h''(u) du.
\end{aligned} \tag{3.17}$$

Combining all of the above, we find that

$$-\widehat{\phi}_\omega(0) = -\sigma \int_0^1 h(u)^2 du - (2\pi\omega)^{-2} \frac{4}{\sigma} \int_0^1 h(u)h''(u) du. \tag{3.18}$$

We now simplify the RHS of (3.14). By the expansion of $\widehat{\phi}_\omega$,

$$\int_0^1 \widehat{\phi}_\omega(y) dy = \int_0^1 g(y) dy + (2\pi\omega)^{-2} \int_0^1 g''(y) dy. \tag{3.19}$$

To rewrite the integrals in terms of h , we detail the procedure to simplify the first term from above; the second term follows analogously:

$$\begin{aligned}
\int_0^1 g(y) dy &= \int_0^1 \int_{-\infty}^{\infty} f(t)f(y-t) dt dy \\
&= \int_0^1 \int_{y-\sigma/2}^{\sigma/2} h\left(\frac{2t}{\sigma}\right) h\left(\frac{2(y-t)}{\sigma}\right) dt dy \\
&= \frac{\sigma}{2} \int_0^1 \int_{2y/\sigma-1}^1 h(u)h\left(\frac{2y}{\sigma}-u\right) du dy \\
&= \frac{\sigma^2}{4} \int_0^{2/\sigma} \int_{v-1}^1 h(u)h(v-u) du dv.
\end{aligned} \tag{3.20}$$

Utilizing the previous techniques to compute derivatives and analyze integrals yields similar results. Eventually, the right-hand side becomes

$$\int_0^1 \widehat{\phi}_\omega(y) dy = \frac{\sigma^2}{4} \int_0^{2/\sigma} \int_{v-1}^1 h(u)h(v-u) du dv + (2\pi\omega)^{-2} \int_0^{2/\sigma} \int_{v-1}^1 h(u)h''(v-u) du dv. \tag{3.21}$$

Putting the simplified versions of two sides of (3.14) together and isolating ω , we get the expression for the minimum ω :

$$\omega > \left(-\frac{\sigma \int_0^1 h(u)^2 du + \frac{\sigma^2}{4} \int_0^{2/\sigma} \int_{v-1}^1 h(u)h(v-u) du dv}{\frac{1}{\sigma} \int_0^1 h(u)h''(u) du + \frac{1}{4} \int_0^{2/\sigma} \int_{v-1}^1 h(u)h''(v-u) du dv} \right)^{-\frac{1}{2}} \pi^{-1}. \tag{3.22}$$

□

Remark 3.6. Since number theory has only been able to compute the needed quantities for σ up to 2 (under the assumption of GRH), we get $\omega_{\min}(2, h) \approx 0.21864$ for $h = \cos(\pi y/2)$. We also looked at results for levels 3 and 5 by directly integrating both sides of the inequality, hoping to see better bounds than the 1-level. Since the right-hand side involves a multi-dimensional integral of a very complicated function (ϕ_ω , involving Fourier transforms and convolutions), we estimate it through Riemann sums; see 2.13 and also Remark 1.7 for comments on the accuracy of the resulting computations. Although it cannot give exact

values, the approximation is sufficiently close as we have a small error term. For our evaluation, we use the simple function $h(y) = 1 - y^2$ to construct our test function ϕ_ω . This is because the complex nature of the right-hand side makes evaluation extremely slow. Since optimizing the function has little improvement in the final results, we are satisfied with using the chosen h for now (though ongoing studies focus on finding the optimal function to use). However, although we expect higher levels to yield better bounds for sufficiently large support, that is not the case here. As we increase the level, the bound gets worse for support $\sigma = 2$: for $n = 3$, $\omega \approx .34$, and for $n = 5$, $\omega \approx .85$.

4. PROOF OF THEOREM 1.8

We obtain an upper bound through even centered moments and using the naive test function ϕ defined in (1.7).

Theorem 4.1. *Assume GRH. Let $P_{r,\rho}(\mathcal{F})$ denote the percent of forms in the family that has at least r normalized zeros on the interval $(-\rho, \rho)$ near the central point. For an even n and $r \geq \mu(\phi, \mathcal{F})/\phi(\rho)$,*

$$P_{r,\rho}(\mathcal{F}) \leq \frac{1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi)}{(r\phi(\rho) - \mu(\phi, \mathcal{F}))^n}. \quad (4.1)$$

Proof. We first rewrite (3.1) as

$$\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \left(\sum_{|\tilde{\gamma}_{f,j}| \leq \rho} \phi(\tilde{\gamma}_{f,j}c_n) + T_f(\phi) - \mu(\phi, \mathcal{F}) \right)^n = 1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi), \quad (4.2)$$

where $T_f(\phi)$ is the contribution from the scaled $\phi(\gamma_{f,j}c_n)$ when $|\gamma_{f,j}| > \rho$.

We look at forms with at least r normalized zeros within a distance ρ from the central point, and denote the set of these forms by $\mathcal{F}_{N,r}^{(\rho)}$. Since n is even, by dropping all the form with less than r normalized zeros in the interval $(-\rho, \rho)$, we are dropping a non-negative sum; *this is why we work with an even moment*. Since doing so cannot increase the size of the left-hand side, we obtain the following upper bound:

$$\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_{N,r}^{(\rho)}} \left(\sum_{|\tilde{\gamma}_{f,j}| \leq \rho} \phi(\tilde{\gamma}_{f,j}c_n) + T_f(\phi) - \mu(\phi, \mathcal{F}) \right)^n \leq 1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi). \quad (4.3)$$

We then replace the summation of $\phi(\tilde{\gamma}_{f,j}c_n)$ with $r\phi(\rho)$, only counting the contribution of r zeros. This involves several steps, and importantly none of these lead to an increase in the LHS. The lack of increase is because ϕ is non-negative and n is even, so ϕ is decreasing from $(0, \rho)$. Therefore, when we replace the contribution of the test function at these zeros by $\phi(\rho)$, we do not increase the sum. It is important to note that r is an even number, since by symmetry, if $1/2 + i\gamma$ is a zero, so too is $1/2 - i\gamma$.

We also want to be able to drop the term $T_f(\phi)$ and not increase the left-hand side. However, although ϕ is non-negative, if the sum of the first two terms in the parentheses is smaller than the absolute value of the mean, we cannot drop the $T_f(\phi)$ term and be assured that the LHS did not increase⁹. Thus we restrict r to be greater than or equal to $\mu(\phi, \mathcal{F})/\phi(\rho)$, and further decrease the LHS by dropping $T_f(\phi)$:

$$\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_{N,r}^{(\rho)}} (r\phi(\rho) - \mu(\phi, \mathcal{F}))^n \leq 1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi). \quad (4.4)$$

As the summand is independent of f , we move it outside the sum and find

$$(r\phi(\rho) - \mu(\phi, \mathcal{F}))^n \lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_{N,r}^{(\rho)}} 1 \leq 1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi). \quad (4.5)$$

⁹For example, if we have $(2 + 3 - 4)^2$ that is 1, which is smaller than $(2 - 4)^2 = 4$.

Notice, by (2.15), $\lim_{N \rightarrow \infty} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_{N,r}^{(\rho)}} 1 = \lim_{N \rightarrow \infty} \frac{|\mathcal{F}_{N,r}^{(\rho)}|}{|\mathcal{F}_N|} = P_{r,\rho}(\mathcal{F})$. Isolating the percentage yields

$$P_{r,\rho}(\mathcal{F}) \leq \frac{1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi)}{(r\phi(\rho) - \mu(\phi, \mathcal{F}))^n}. \quad (4.6)$$

□

4.1. Explicit bounds for percentages. We provide explicit bounds calculated using Theorem 1.8. Notice that all number of zeros are even by symmetry.

TABLE 1. Probability of forms having a given number of normalized zeros 0.2 away from the central point, using the naive test function with a support of $2/n$. “N/A” means that the level cannot give the bound because of the restriction in Theorem 1.8

Number of zeros	2-level	4-level	6-level
2	6.651738	N/A	N/A
4	0.104108	2.370419	$2.147231 \cdot 10^4$
6	0.029617	0.069998	0.809927
8	0.013769	0.011079	0.022517
10	0.007924	0.003152	0.002427
12	0.005142	0.001213	$4.79813 \cdot 10^{-4}$
14	0.003605	$5.612212 \cdot 10^{-4}$	$1.340970 \cdot 10^{-4}$
16	0.002666	$2.942389 \cdot 10^{-4}$	$4.688515 \cdot 10^{-5}$
18	0.002052	$1.687747 \cdot 10^{-4}$	$1.917773 \cdot 10^{-5}$
20	0.001627	$1.036100 \cdot 10^{-4}$	$8.809943 \cdot 10^{-6}$

TABLE 2. Probability of forms having a given number of normalized zeros 0.4 away from the central point, using the naive test function with support of $2/n$.

Number of zeros	2-level	4-level	6-level
4	0.665694	8.334733	$1.744392 \cdot 10^3$
6	0.111085	0.145883	1.585718
8	0.043857	0.020351	0.036592
10	0.023310	0.005469	0.003680
12	0.014430	0.002038	0.000702
14	0.009804	0.000924	0.000192
16	0.007093	0.000475	$6.606784 \cdot 10^{-5}$
18	0.005369	0.000271	$2.673289 \cdot 10^{-5}$
20	0.004204	0.000165	$1.218053 \cdot 10^{-5}$

As shown in both Table 1 and 2, we see the pattern that for bounding a small number of zeros, the 2-level does a significantly better job than the 4 or 6-level. In fact, both bounds for the 4 and 6-level when $r = 4$ are useless as they already exceed a hundred percent! Moreover, levels 4 and 6 cannot even produce bounds for $r = 1$ because of the restriction of $r \geq \mu(\phi, \mathcal{F})/\phi(\rho)$ in our theorem. However, as we look at a larger number of zeros in the interval, the 4-level does eventually yield a better bound than the 2-level, and similarly eventually the 6-level beats both the 2-level and the 4-level in giving the smallest bound.

TABLE 3. Probability of forms having a given number of normalized zeros 0.8 away from the central point, using the naive test function with support of $2/n$. “N/A” means that the level cannot give the bound because of the restriction in Theorem 1.8.

Number of zeros	2-level	4-level	6-level
6	N/A	10.849910	48.154279
16	N/A	0.004235	$2.83230 \cdot 10^{-4}$
26	N/A	$3.541901 \cdot 10^{-4}$	$6.716802 \cdot 10^{-6}$
28	420.045063	$2.486819 \cdot 10^{-4}$	$3.943864 \cdot 10^{-6}$
30	20.991406	$1.796948 \cdot 10^{-4}$	$2.418466 \cdot 10^{-6}$
32	6.651738	$1.330555 \cdot 10^{-4}$	$1.538761 \cdot 10^{-6}$
34	3.220871	$1.006126 \cdot 10^{-4}$	$1.010576 \cdot 10^{-6}$

We also see the pattern that for the same number of zeros, the chance of them existing in a smaller range $(-\rho, \rho)$ is much lower than in a larger interval. For instance, using $n = 2$, the probability of having 8 zeros on the interval $(-0.2, 0.2)$ is approximately 0.013769, while for the interval $(-0.4, 0.3)$ it is much larger, about 0.043857.

Higher levels or moments are also better at bounding the number of normalized zeros within a larger distance from the central point. Because of the restriction of $r \geq \mu(\phi, \mathcal{F})/\phi(\rho)$, lower levels are simply incapable of bounding a small number of zeros! For example, in Table 3 when we fixed a bigger $\rho = .8$, the 2-level can only non-trivially bound the number of zeros when $r > 27$, while the 4-level and 6-level are perfectly fine with r as small as 6. We also see the 6-level producing much better bounds than smaller levels in Table 3. Thus, even though the higher levels are much more difficult to calculate, the compromise is worthwhile as we see its significance when we increase the interval $(-\rho, \rho)$ as well as the number of zeros r .

5. FUTURE WORK

In Theorem 1.6 we focused only on the even forms of the family, as the odd forms are guaranteed to have one zero on the central point. Our result can be extended to odd forms by simply adding that contribution.

There are several directions one can take to improve these results. One is to optimize the test function. Though the optimum test function for the one-level density was determined in previous studies, optimizing the test function for higher-level densities is open. Additionally, a framework of the construction of the test function in Theorem 1.6, it is possible to alter it by adding a quartic term, for instance, to see improvement on the bounds for higher levels as seen in the appendix. Therefore, further work may be done to generalize these techniques and find efficient techniques for calculations regarding the optimization problem; however, previous work highlights that the greatest gains come from increasing the support and using the higher n -level as focused on in this paper. At the cost of some additional standard computations, one can easily add additional terms and obtain similar integrals to approximate; initial explorations indicated that the gain was minimal, and thus we focused on the significant improvements that arise with higher-level densities and moments, justifying the additional work required. In the appendix we show that the additional work leads to only a minimal improvement, from 0.21864 to 0.218503 (about .06%).

For higher levels, one can also try to simplify the limit calculation through complex analysis, making evaluation easier for explicit bounds.

APPENDIX A. EXPANDING THE SPACE OF TEST FUNCTIONS

In Theorem 1.6, the construction of the test function ϕ necessitates $\widehat{g}(x) = O(x^{-4})$ so that $\phi(x) = O(x^{-2})$ and ϕ is even. However, the construction of h limits $\widehat{g}(x)$ to functions satisfying $\widehat{g}(x) = \Theta(x^{-4})$, because any larger decay would not be optimal for ω_{\min} . Therefore, multiplying ϕ by a polynomial term of an even degree allows $\widehat{g}(x)$ to decay at the rate $|x|^{-A}$ for some $A > 4$ while preserving the relation $\phi(x) = O(x^{-2})$. This polynomial term must be positive and even to preserve the conditions of ϕ . We then yield the formula for the test function

$$\phi(x) = \widehat{g}(x)(1 - (x/\omega)^2)(1 + c_1x^2 + c_2x^4 + \dots + c_w x^{2w}), \quad (\text{A.1})$$

where w is the degree of differentiability of $h(x)$ at $x = 1$. We highlight an approach that generalizes $h(x)$ to a polynomial that is w -differentiable at $x = 1$ and varies the coefficients of $h(x)$ and the polynomial term with $\phi(x)$.

Since the argument for Theorem 3.5 uses that $\widehat{g}(x)$ is multiplying $(1 - (x/\omega)^2)$, we define a new function such that $\widehat{g}_w(x) = \widehat{g}(x)(1 + c_1x^2 + c_2x^4 + \dots + c_w x^{2w})$ and $\phi(x) = g_w(x)(1 - (x/\omega)^2)$. Because $g_w(x)$ multiplies $(1 - (x/\omega)^2)$, following the argument of Theorem 3.5 we utilize (3.15) and (3.19) to obtain

$$\omega_{\min} > \frac{1}{2\pi} \left(-\frac{g_w''(0) + \int_0^1 g_w''(x) dx}{\int_0^1 g_w(x) dx + g_w(0)} \right)^{1/2}. \quad (\text{A.2})$$

Since $\widehat{g}_w(x) = \widehat{g}(x)(1 + c_1x^2 + c_2x^4 + \dots + c_w x^{2w})$, we have $\widehat{g}_w(x) = \widehat{g}(x) + c_1\widehat{g}(x)x^2 + c_2\widehat{g}(x)x^4 + \dots + c_w\widehat{g}(x)x^{2w}$. We then use Lemma 2.2 to deduce that

$$\begin{aligned} g_w(x) &= g(x) - c_1(2\pi)^{-2}g''(x) + \dots + c_w(2\pi i)^{-2w} \frac{d^{2w}}{dy^{2w}}g(x) \\ &= g(x) + \sum_{k=1}^w c_k(-4\pi^2)^{-k} \frac{d^{2k}}{dy^{2k}}g(x). \end{aligned} \quad (\text{A.3})$$

When considering the parameters of c_k , remember we need our polynomial to be even and of a given sign in various ranges; that imposes additional restrictions on the c_k 's. Rather than studying all possible polynomials that satisfy the constraints, we restrict ourselves to a tractable subspace. Consider $p_a(x) := \prod_{i=1}^a (\lambda_i x^2 - 1)^2$, a positive even polynomial of degree $4a$ with all real roots where c_k are the coefficients of this polynomial with $1 \leq k \leq 2a = w$. It follows that $\{c_k\}$ depend on the a constraints $\{\lambda_i\}$. In particular, the coefficient c_k may be written as

$$c_k = (-1)^k \sum_{1 \leq r_1 < r_2 < \dots < r_k \leq a} \lambda_{r_1} \lambda_{r_2} \dots \lambda_{r_k}. \quad (\text{A.4})$$

Because all the zeros are real, the coefficients c_k of p_a ensure that the polynomial is positive and even. Additionally, this polynomial reduces the degrees of freedom for k constants to the number of zeros of the polynomial. Conjecturally, the exponential decay of the derivatives requires minimal c_k . Consequently, optimizing these seemingly arbitrary constants reduces computational complexity.

Since the parameters c_k are specified to minimize ω_{\min} , a program may be utilized to minimize with respect to $\{\lambda_i\}$ given w, h . Additionally, we search for a good h of the form

$$h(x) := (1 - x^2)^{2w+1} \left(\prod_{j=1}^s (1 - \alpha_j x^2) + \beta \right), \quad (\text{A.5})$$

where s denotes the number of zeros this polynomial may have and $0 \leq \alpha_j \leq 1$ and $\beta \geq 0$. Thus, a minimization program may be able to take in the constants of σ, s , and w , while optimizing constraints for α_j and λ_i to minimize ω with respect to these parameters.

When letting $w = 1$, $\sigma = 2$, and $s = 4$, a Mathematica program suited for minimization estimates $\omega_{\min} = 0.218503$. Further results show that regardless of a given $h(x)$, there was a noticeable convergence of c_k , implying that the zeros of an optimal g_ω may be approximated accurately through the program. Figure 3 illustrates the h resulting from the program optimizing h via (A.5) and $w, \sigma, s = 1, 2, 4$ respectively. Unfortunately, as remarked earlier, the gain is negligible (on the order of .06%).

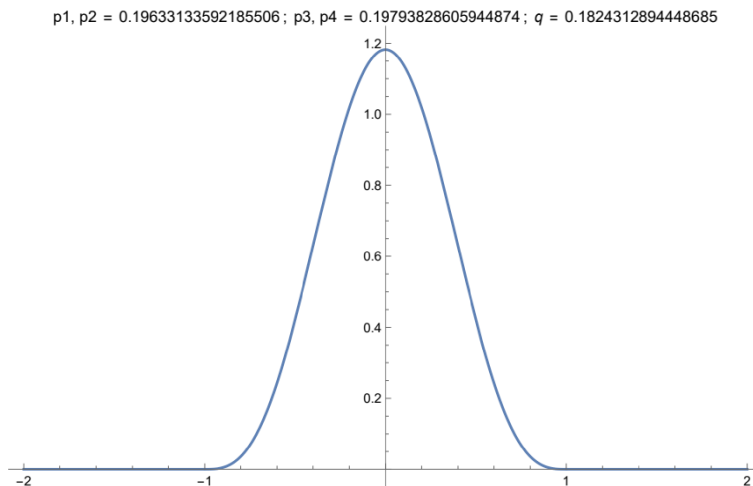


FIGURE 3. Plot of $h(x)$, for $\alpha_1, \alpha_2 = 0.1963$, $\alpha_3, \alpha_4 = 0.1979$, and $\beta = 0.1824$.

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