CENTERED MOMENTS OF WEIGHTED ONE-LEVEL DENSITIES OF $GL(2)$ L-FUNCTIONS

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ABSTRACT. Katz and Sarnak conjectured that the behavior of zeros near the central point of any family of L-functions is well-modeled by the behavior of eigenvalues near 1 of some classical compact group (either the symplectic, unitary, or even, odd, or full orthogonal group). In 2018, Knightly and Reno proved that the symmetry group can vary depending on how the L-functions in the family are weighted. They observed both orthogonal and symplectic symmetry in the one-level densities of families of cuspidal newform L-functions for different choices of weights. We observe the same dependence of symmetry on weights in the nth centered moments of these one-level densities, for smooth test functions whose Fourier transforms are supported in $\left(-\frac{1}{2n}, \frac{1}{2n}\right)$. To treat the new terms that emerge in our n-level calculations when $n > 1$, i.e., the cross terms that emerge from n-fold products of primes rather than individual primes, we generalize Knightly and Reno's weighted trace formula from primes to arbitrary positive integers. We then perform a delicate analysis of these cross terms to distinguish their contributions to the main and error terms of the nth centered moments. The final novelty here is an elementary combinatorial trick that we use to rewrite the main number theoretic terms arising from our analysis, facilitating comparisons with random matrix theory.

CONTENTS

1. Introduction

Many studies have been undertaken over the last few decades to make precise the connection between number theory and random matrix theory. Before situating our work in this chronology, we recount the most relevant results and how they prompted the refinement of this connection.

In 1972, Montgomery [**Mon73**] initiated the connection between number theory and random matrix theory, proving that, for suitable test functions, the pair correlation of the zeros of the Riemann zeta function $\zeta(s)$ agrees with the pair correlation of the eigenvalues of the Gaussian Unitary Ensemble (GUE). In his seminal work, he furthermore conjectured a correspondence between any local statistic of the zeros of $\zeta(s)$ and the eigenvalues of the GUE. In 1987, Odlyzko [**Odl87**; **Odl01**] provided numerical evidence for this conjecture, verifying the correspondence for particular local statistics: the pair correlation and the nearest neighbor spacing distribution. Subsequent works by Hejhal [**Hej94**] on the triple correlation of $\zeta(s)$ and Rudnick and Sarnak [**RS96**] on the n-level correlation of any automorphic cuspidal L-function, all for suitably restricted test functions, provided strong evidence for Montgomery's conjecture in generality. These studies suggested a remarkable universality in number theory: the statistical profile of any L-function matches that of one and only one of the numerous random matrix ensembles, the GUE.

At the time, however, there were a few reasons to believe that this conjecture did not capture the connection in full. For one, certain local statistics, such as the n-level correlation, are insensitive to the behavior of any finite set of zeros. As there are many important problems in number theory concerning only finite sets of zeros (e.g., the Birch and Swinnerton-Dyer conjecture [**BS63**; **BS65**] is concerned only with the low-lying zeros of elliptic curve L-functions), this marked a considerable shortcoming of the conjecture on the number theory side. Katz and Sarnak [**KS99b**; **KS99a**] confirmed that indeed more attention is needed in this regard. They showed that the n-level correlation of eigenvalues is the same coming from the GUE and all five classical compact groups (unitary, symplectic, and orthogonal, split and unsplit by sign). They addressed this possibility of confounding by defining a new statistic called the one-level density, which is not only distinguishable across the classical compact groups but also sensitive to changes in low-lying zeros (near the central point). For any L-function $L(s, f)$, denoting its non-trivial zeros by $\rho_f = 1/2 + i\gamma_f$[1],

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[1]If the Grand Riemann Hypothesis (GRH) is true, then γ would be real, and the non-trivial zeros ρ could be ordered accordingly. There are interesting interpretations of this ordering in the context of random matrix theory and nuclear physics, actuating connections to the eigenvalues of

they defined its one-level density by

$$D(f, \phi) \; := \; \sum_{\rho_f} \phi\left(\frac{\gamma_f \log Q_f}{2\pi}\right), \tag{1.1}$$

where ϕ is an even Schwartz (test) function, and Q_f is the analytic conductor of f. The low-lying zeros of $L(s, f)$ have imaginary part approximately $1/\log Q_f$ and the average spacing between zeros with imaginary part T is known to be approximately $1/\log T$. Hence, the scaled zeros, $\frac{\gamma_f \log Q_f}{2\pi}$, have average spacing 1, meaning there are only finitely many up to any given constant. As it is not instructive to statistically survey the finitely many low-lying zeros of a single L-function, Katz and Sarnak passed to the study of collections of naturally related (similarly behaved) L-functions, "families" so to speak. In doing so, they were able to execute averages and take limits, as is customary in analytic number theory, to identify the common behavior underlying the L-functions in the family.

To facilitate our discussion, we establish a standard notation for averages, writing the average of a map Q over a finite collection $\mathcal{C}$ with weight w as

$$\mathcal{A}_{\mathcal{C}}^w(Q) \; := \; \frac{\sum_{c \in \mathcal{C}} w(c) Q(c)}{\sum_{c \in \mathcal{C}} w(c)}. \tag{1.2}$$

For unweighted averages, i.e., when w is identically 1 on $\mathcal{C}$, we omit the superscript in the notation (1.2).

In refining Montgomery's conjecture based on their seminal study, Katz and Sarnak formulated the now celebrated density conjecture, stating that the behavior of zeros near $1/2$ in any family of L-functions matches the behavior of eigenvalues near 1 in one of the five classical compact groups. More precisely, consider a family $\mathcal{F} = \bigcup_n \mathcal{F}_n$ of L-functions, where each sub-family $\mathcal{F}_n$ is finite. Let the average one-level density of each sub-family be given by

$$\mathcal{A}_{\mathcal{F}_n}(D(\cdot, \phi)) \; := \; \frac{\sum_{L(s,f) \in \mathcal{F}_n} D(f, \phi)}{|\mathcal{F}_n|}. \tag{1.3}$$

Then, there exists a symmetry group G among the classical compact groups O, SO(even), SO(odd), Sp, and U such that for any test function ϕ with compactly supported Fourier transform $\widehat{\phi}$,

$$\lim_{n \to \infty} \mathcal{A}_{\mathcal{F}_n}(D(\cdot, \phi)) \; = \; \int_{-\infty}^{\infty} \phi(x) W_G(x) dx. \tag{1.4}$$

Here, $W_G(x)$ is the limiting distribution of the one-level density of the eigenvalues coming from the random matrices in G as rank tends to ∞. These distributions are

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classical compact groups and the energy levels of heavy nuclei. With this being said, we do not assume the truth of the GRH in what follows, performing our study in generality.

given by

$$W_{\mathrm{U}}(x) \;=\; 1, \tag{1.5}$$

$$W_{\mathrm{Sp}}(x) \;=\; 1 - \frac{\sin(2\pi x)}{2\pi x}, \tag{1.6}$$

$$W_{\mathrm{SO(even)}}(x) \;=\; 1 + \frac{\sin(2\pi x)}{2\pi x}, \tag{1.7}$$

$$W_{\mathrm{SO(odd)}}(x) \;=\; \delta_0 + 1 - \frac{\sin(2\pi x)}{2\pi x}, \tag{1.8}$$

$$W_{\mathrm{O}}(x) \;=\; 1 + \frac{1}{2}\delta_0(x), \tag{1.9}$$

where δ_0 is the Dirac distribution at 0. When $\widehat{\phi}$ is supported in $(-1,1)$, which will be the case throughout our study, $W_{\mathrm{O}}(x)$ and $W_{\mathrm{Sp}}(x)$, and $W_{\mathrm{SO(even)}}$ and $W_{\mathrm{SO(odd)}}$ coincide as distributions. This can be realized through Plancherel's theorem [**ILS00**].

There is now an enormous body of work showing that the one-level densities of various families of L-functions (coming from Dirichlet characters, elliptic curves, cuspidal newforms, Maass forms, number fields, etc.) agree with the scaled limit of one of the five classical compact groups; for examples, see [**Alp+15**; **Bar+17**; **BCL24**; **DM06**; **FM15**; **GK12**; **ILS00**; **KR18**; **RR07**; **You06**]. With this being said, the one-level density is not the only local statistic that provides information about low-lying zeros. Katz and Sarnak defined a higher-order analog for the one-level density called the n-level density. For any L-function $L(s,f)$, its n-level density entails n-tuples of its zeros $\rho_f^{(1)}, \ldots, \rho_f^{(n)}$ and n test functions $\phi_1, \ldots, \phi_n$:

$$D_n(f; \phi_1, \ldots, \phi_n) \;:=\; \sum_{\substack{\rho_f^{(1)}, \ldots, \rho_f^{(n)} \\ \gamma_f^{(i)} \neq \pm\gamma_f^{(j)}}} \phi_1\!\left(\frac{\gamma_f^{(1)} \log Q_f}{2\pi}\right) \cdots \phi_n\!\left(\frac{\gamma_f^{(n)} \log Q_f}{2\pi}\right). \tag{1.10}$$

The n-level density of a family is usually calculated in terms of its one-level density using the principle of inclusion-exclusion [**Rub01**; **Gao08**]. This approach, however, relies on our knowledge of the distribution of the signs of the functional equations in the family, which is beyond current theory. In this view, Hughes and Rudnick [**HR02**] initiated the study of a more tractable n-level statistic: the nth centered moment of the one-level density. More precisely, for the general sub-family $\mathcal{F}_n$ of L-functions given above, the nth centered moment of its one-level density is defined as

$$\mathcal{A}_{\mathcal{F}_n}\left[(D(\cdot, \phi) - \mathcal{A}_{\mathcal{F}_n}(D(\cdot, \phi)))^n\right] \;:=\; \frac{\sum_{L(s,f) \in \mathcal{F}_n}(D(f, \phi) - \mathcal{A}_{\mathcal{F}_n}(D(\cdot, \phi)))^n}{|\mathcal{F}_n|}. \tag{1.11}$$

The nth centered moment of the one-level density (1.11) is equivalent to the n-level density (1.10) when $\phi_1 = \cdots = \phi_n$, explaining why these statistics encode the same useful arithmetic information, e.g., the order of vanishing at the central point. There are cases [**Che+25**] in which taking different $\phi_1, \ldots, \phi_n$ has proved more productive than taking $\phi_1 = \cdots = \phi_n$ in calculating the n-level density, cases in which the equivalence of the nth centered moment of the one-level density is broken.

The general approach in calculating these density statistics for families of L-functions is to find first an explicit formula converting the sum over zeros in (1.1) to a sum over primes, then an asymptotic (trace) formula for the resulting sum over primes. Finding the explicit formula is usually straightforward, e.g., [**ILS00**], but finding the trace formula usually entails intricate equidistribution laws governing the family. In the specific case of families of automorphic forms, trace formulae naturally involve weights. For example, harmonic weights naturally arise in the Petersson trace formula for families of cuspidal newforms [**ILS00**], and analytic weights naturally arise in the Kuznetsov trace form for families of Maass forms [**KL13**; **GK12**]. In this view, many statistical surveys on families of automorphic L-functions [**Alp+15**; **GK12**; **ILS00**] maintain these weights and calculate weighted, rather than unweighted, densities of their zeros. In all the above surveys, the weights have proved to be innocuous, in that they do not affect the symmetry type of the low-lying zeros, i.e., the uniform and weighted zeros have the same limiting distribution. However, this is not true in general.

In 2012, Kowalski, Saha, and Tsimmerman [**KST12**] demonstrated that more attention is needed in passing from weighted to unweighted densities. They found that the (unweighted) zeros of $GSp(4)$ spinor L-functions have orthogonal symmetry; whereas the same zeros, when assigned harmonic weights, exhibited symplectic symmetry. It is reasonable to expect that the distribution of a collection can change based on how the individual elements are weighted. Knightly and Reno [**KR18**] confirmed this expectation in the commonly studied case of cuspidal newform L-functions. They observed both orthogonal and symplectic symmetry in families of holomorphic cuspidal newforms for different choices of weights.

To be exact, we fix a real, primitive Dirichlet character χ of modulus $D \geq 1$, and a positive integer r relatively prime to D. Consider the Gauss sum $\tau(\chi)$ attached to χ (see (2.16) for the definition), and a basis $\mathcal{F}_k(N)'$ of Hecke cuspidal newforms of weight k and level N. Knightly and Reno considered the following two families:

- $\mathcal{F}_1 = \bigcup_k \mathcal{F}_k(1)'$ with k ranging over even integers satisfying $\tau(\chi)^2 \neq -i^k D$;
- $\mathcal{F}_2 = \bigcup_{k,N} \mathcal{F}_k(N)'$ with k ranging over even integers satisfying $\tau(\chi)^2 = -i^k D$, equivalently $\chi(-1) = -i^k$, and N ranging over primes not dividing rD.

The conditions on k and N serve to simplify our computations, as we demonstrate in the beginning of Section 4. To any Hecke cuspidal newform $f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi i n z}$ in these families, they assigned the weight

$$w_{\chi,r}(f) := \frac{\Lambda\left(\frac{1}{2}, f \times \chi\right)|a_f(r)|^2}{\|f\|^2}, \tag{1.12}$$

where $\Lambda(s, f \times \chi)$ is the completed L-function, defined in (2.14). These weights are non-negative by Guo's theorem [**Guo96**]. It is natural to study weights containing central (twisted) L-values given the original interest in zeros near the central point [**Faz24**], not to mention the Fourier coefficients in the weights also contain information about central L-values due to a famous conjecture of Böcherer [**BS92**]. With this setup, Knightly and Reno considered the average weighted one-level density of each sub-family $\mathcal{F}_k(N)'$:

$$\mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}(D(\cdot,\phi)) := \frac{\sum_{f \in \mathcal{F}_k(N)'} w_{\chi,r}(f)D(f,\phi)}{\sum_{f \in \mathcal{F}_k(N)'} w_{\chi,r}(f)}. \tag{1.13}$$

They proved that for suitably restricted test functions, the limiting distribution of the average weighted one-level densities for both $\mathcal{F}_1$ and $\mathcal{F}_2$ varies depending on the triviality of the twisting character in the weight (1.12):

$$\lim_{k \to \infty} \mathcal{A}_{\mathcal{F}_k(1)'}^{w_{\chi,r}}(D(\cdot,\phi_1)) = \lim_{k+N \to \infty} \mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}(D(\cdot,\phi_1))$$

$$= \begin{cases} \int_{-\infty}^{\infty} \phi_1(x)W_{\mathrm{Sp}}(x)dx & \text{if } \chi \text{ trivial,} \\ \int_{-\infty}^{\infty} \phi_1(x)W_{\mathrm{O}}(x)dx & \text{if } \chi \text{ non-trivial;} \end{cases} \tag{1.14}$$

here, the limits are over the aforementioned values of k and N.

We consider the other arithmetically insightful statistic, the nth centered moment of the one-level density, for the same families and weights in an effort to generalize the dependence between weights and symmetry type. To be exact, we study the weighted n^{th} centered moment of the average one-level density of each sub-family $\mathcal{F}_k(N)'$:

$$\mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}\left[\left(D(\cdot,\phi) - \mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}(D(\cdot,\phi))\right)^n\right]$$

$$:= \frac{\sum_{f \in \mathcal{F}_k(N)'} w_{\chi,r}(f)\left(D(f,\phi) - \mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}(D(\cdot,\phi))\right)^n}{\sum_{f \in \mathcal{F}_k(N)'} w_{\chi,r}(f)}. \tag{1.15}$$

In addition to the notation so far, we adopt that for any test function ϕ,

$$\sigma_\phi^2 := \int_{-\infty}^{\infty} \widehat{\phi}^2(y)|y|\,dy. \tag{1.16}$$

Our main result is that for suitably restricted test functions ϕ, the weighted n^{th} centered moment of the average one-level densities for both $\mathcal{F}_1$ and $\mathcal{F}_2$ converge to the nth centered moment of a Gaussian distribution with variance σ_ϕ^2.

Theorem 1.1. *Let χ be a real, primitive Dirichlet character of modulus $D \geq 1$, and let r be a positive integer relatively prime to D. For any positive integer n and any test functions ϕ with $\mathrm{supp}(\widehat{\phi}) \subset \left(-\frac{1}{2n}, \frac{1}{2n}\right)$,*

$$\lim_{k \to \infty} \mathcal{A}_{\mathcal{F}_k(1)'}^{w_{\chi,r}} \left[\left(D(\cdot, \phi) - \mathcal{A}_{\mathcal{F}_k(1)'}^{w_{\chi,r}}(D(\cdot, \phi)) \right)^n \right] = \begin{cases} (n-1)!! \, \sigma_\phi^n & \text{if } n \text{ even}, \\ 0 & \text{if } n \text{ odd}, \end{cases} \tag{1.17}$$

$$\lim_{k+N \to \infty} \mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}} \left[\left(D(\cdot, \phi) - \mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}(D(\cdot, \phi)) \right)^n \right] = \begin{cases} (n-1)!! \, \sigma_\phi^n & \text{if } n \text{ even}, \\ 0 & \text{if } n \text{ odd}. \end{cases} \tag{1.18}$$

Remark 1.2. It would be interesting to increase the support beyond $(-\frac{1}{n}, \frac{1}{n})$ because we can start to differentiate $W_{\mathrm{O}}(x)$ and $W_{\mathrm{Sp}}(x)$ as distributions only beyond this range, as noted in the opening discussion. In the current scope, Theorem 1.1 does not detect the difference between the orthogonal and symplectic distributions, and hence the dependence between weights (particularly the triviality of the twisting character therein) and symmetry type.

Remark 1.3. The number theoretic statement of Theorem 1.1 aligns with random matrix theory: for similarly restricted test functions ϕ, the nth centered moment of the one-level densities for both the orthogonal and symplectic matrix ensembles [**HM07**; **HR07**] also converge to the nth centered moment of a Gaussian distribution with variance σ_ϕ^2. However, when the support is beyond $[-\frac{1}{n}, \frac{1}{n}]$, these random matrix statistics are known to no longer follow Gaussian behavior. Extending the support and realizing the same non-Gaussian behavior on the number theory side, i.e., in the setting of Theorem 1.1, would provide striking evidence for the conjectured connection between these two sub-fields.

Remark 1.4. When $n = 1$, we recover from Theorem 1.1 one of the two main results from the study of Knightly and Reno: (1.14). In their other main result [**KR18**], Knightly and Reno considered different weights:

$$w_\chi'(f) = \frac{\Lambda\left(\frac{1}{2}, f \times \chi\right) \Lambda\left(\frac{1}{2}, f\right)}{\|f\|^2}. \tag{1.19}$$

It would be instructive to study the nth centered moment of the one-level density with these weights as well, to further demonstrate the dependence between weights and symmetry type.

To prove Theorem 1.1, we follow the general approach for density calculations outlined above. In Section 2, we review this relevant mathematical context in more detail and collect some standard results on modular forms.

In Section 3, we apply the explicit formula ar the one-level density of cuspidal newform L-functions (Lemma 3.1), which leads us from an average of zeros to an average of Hecke eigenvalues over the sub-families $\mathcal{F}_k(N)'$. While Knightly and Reno [**KR18**] also considered averages of Hecke eigenvalues over $\mathcal{F}_k(N)'$, they did so only at prime powers because it sufficed for their one-level calculations. The averages that emerge in our n-level calculations entail Hecke eigenvalues at arbitrary integers. This complexity is generated by cross terms coming from n-fold products of primes rather than individual primes (one-fold products of primes, so to speak). It may be instructive to compare (3.7) and [**KR18**] in this regard.

In Section 4, we appeal to a formula given in [**JK15**] for a related weighted average of Fourier coefficients over $\mathcal{F}_k(N)'$. Leveraging the relation between Fourier coefficients and Hecke eigenvalues, we derive an asymptotic trace formula (Lemma 4.3) for the average over $\mathcal{F}_k(N)'$ of Hecke eigenvalues at arbitrary positive integers, not just at prime powers. Several new and interesting number theoretic terms arise in our derivation (see Lemmas 4.1 and 4.2).

In Section 5, we analyze these number theoretic terms in several cases, mirroring the proof of [**HM07**]. We do this separately for non-trivial and trivial twisting characters, in Subsections 5.1 and 5.2 respectively, because the distributions are provably different (1.14) and the analyses are demonstrably different between these cases.

In the case of trivial twisting characters, it is not obvious a priori that the number theoretic terms resulting from our analysis match up exactly with the expected random matrix theoretic terms (1.18). This apparent mismatch is observed in most, if not all, n-level calculations [**Rub01**; **Gao08**; **HM07**; **HR02**; **Sos00**]. This is because the random matrix theoretic terms are derived for arbitrary support, cf. [**KS99a**; **KS99b**]; whereas the number theoretic terms are derived for restricted support, out of technical necessity. Besides, since these calculations usually use some combinatorial argument (such as the principle of inclusion-exclusion) to express the n-level statistic in terms of the corresponding one-level statistic, it is instructive to reverse this combinatorial argument in the end, i.e., unravel the combinatorial expression that emerges after the one-level statistic is calculated. In this vein, Soshnikov [**Sos00**] developed a combinatorial trick relating to generating series, which has been referenced and adapted in several subsequent n-level calculations [**HM07**; **HR02**]; the essence of this trick is a deep combinatorial fact called the Hunt-Dyson formula. Novelly, our combinatorial strategy for unraveling the number theoretic terms in Subsection 5.2 does not rely on this trick. We use only elementary methods to rewrite the resulting combinatorial factors and motivate our otherwise unassuming

applications of the binomial theorem. This represents the main challenge in going from Knightly and Reno's one-level calculation, which did not require any combinatorial argument, to our n-level calculation. It would be interesting to assess the applicability of our elementary combinatorial argument in other n-level calculations, like the ones referenced above.

Remark 1.5. In view of the many incentives to extending support, e.g., Remarks 1.2 and 1.3, it is worth mentioning that the current support restrictions are in place only to bound the contribution of the error term in the asymptotic trace formula (see Lemma 5.1). All other contributions are analyzed unconditionally, for arbitrary support. Extending the support, therefore, entails finding more lower-order terms in the respective trace formula and thereby analyzing a smaller error term.

2. Notation and Preliminaries

We follow the definitions and notations in [**IK04**] throughout. We define the Nth congruence subgroup of $SL_2(\mathbb{Z})$ by

$$\Gamma_0(N) \ := \ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ \middle| \ c \equiv 0 \pmod{N} \right\}. \tag{2.1}$$

A holomorphic cusp form of weight k and level N is a function f on the complex upper half-plane $\mathbb{H}$ that transforms "nicely" under the action of $\Gamma_0(N)$:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z). \tag{2.2}$$

It admits a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}, \tag{2.3}$$

where the coefficients $a_f(n)$ are complex numbers normalized so that $a_f(1) = 1$.

We denote the space of all holomorphic cusp forms of weight k and level N by $S_k(N)$. This space is equipped with the Petersson inner product, making it Hilbert. The inner product is given by

$$\langle f, g \rangle \ := \ \frac{1}{\nu(N)} \int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)}\, y^k \, \frac{dx\,dy}{y^2}, \tag{2.4}$$

where $\nu(N) := [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$. Consider the Hecke operators which, for each $n \in \mathbb{N}$, act on the linear space $S_k(N)$ and are given by

$$T_n f(z) \ := \ n^{k-1} \sum_{\substack{ad=n \\ (a,N)=1}} \sum_{b=0}^{d-1} d^{-k} f\left(\frac{az+b}{d}\right). \tag{2.5}$$

The theory of functional analysis provides that the Hecke operators are simultane-ously diagonalizable, allowing us to speak of an orthogonal basis $\mathcal{F}_k(N)$ of forms which are simultaneous eigenforms of all these Hecke operators (Hecke eigenforms for short). We emphasize that $\mathcal{F}_k(N)$ is finite; in fact, from [**ILS00**],

$$|\mathcal{F}_k(N)| \sim \frac{k-1}{12}\phi(N) + O\left((kN)^{5/6}\right), \tag{2.6}$$

where ϕ is Euler's totient function. For any Hecke eigenform $f \in \mathcal{F}_k(N)$, we refer to the eigenvalue of f under T_n as the nth Hecke eigenvalue of f. It is straightforward from Definitions 2.3 and 2.5 that the Hecke eigenvalues of f are closely related to the Fourier coefficients of f:

$$a_f(n) = \lambda_f(n)n^{(k-1)/2}. \tag{2.7}$$

Furthermore, the Hecke eigenvalues of f possess useful multiplicative properties:

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right); \tag{2.8}$$

in particular, if $(m,n) = 1$, then

$$\lambda_f(m)\lambda_f(n) = \lambda_f(mn), \tag{2.9}$$

and if p is a prime not dividing N, then

$$\lambda_f(p)^{2k} = \sum_{r=0}^{k}\left(\binom{2k}{k-r} - \binom{2k}{k-r-1}\right)\lambda_f(p^{2k}), \tag{2.10}$$

$$\lambda_f(p)^{2k+1} = \sum_{r=0}^{k}\left(\binom{2k+1}{k-r} - \binom{2k+1}{k-r-1}\right)\lambda_f(p^{2k+1}). \tag{2.11}$$

Proofs of (2.10) and (2.11) can be found in [**Guy00**]. These properties motivate the definition of the L-function associated to f:

$$L(s,f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}, \quad \Re(s) > 1. \tag{2.12}$$

Interesting variants of this L-function can be obtained by twisting it with Dirichlet characters. Formally speaking, for a fixed integer $D \geq 1$ with $(D,N) = 1$, and a primitive Dirichlet character χ of modulus D, the L-function of f twisted by χ is given by

$$L(s, f \times \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)a_f(n)}{n^{s+\frac{k-1}{2}}}, \quad \Re(s) > 1. \tag{2.13}$$

We complete this twisted L-function with an appropriate Gamma factor, analytically continuing it to the entire complex plane:

$$\Lambda(s, f \times \chi) \ := \ (2\pi)^{-s-\frac{k-1}{2}} \Gamma \left(s + \frac{k-1}{2} \right) L(s, f \times \chi). \tag{2.14}$$

The completed L-function satisfies a functional equation relating s to $1 - s$. For example, when $N = 1$, the functional equation is

$$\Lambda(s, f \times \chi) \ = \ \frac{i^k}{D^{2s-1}} \frac{\tau(\chi)^2}{D} \, \Lambda(1 - s, f \times \overline{\chi}), \tag{2.15}$$

where $\tau(\chi)$ is the Gauss sum attached to χ:

$$\tau(\chi) \ := \ \sum_{m=1}^{D} \chi(m) e^{2\pi i m / D}. \tag{2.16}$$

With all this said, however, given any form in $S_k(N)$, it is possible to induce a form in $S_k(M)$ for all $M > N$ with $N \mid M$; the induced form is aptly called an "oldform." The forms orthogonal to the space spanned by oldforms are called "newforms". This theory, developed by Atkin and Lehner in 1970 [**AL70**], is relevant to our study because oldforms of level M are related more naturally to the newforms (of level $N < M$) which induce them than the newforms of level M, by way of their analytic conductors. (As mentioned in Section 1, the analytic conductor of an L-function encodes the approximate imaginary part of its low-lying zeros.) In particular, the analytic conductor of any oldform of level M induced from a newform of level N is equal to the analytic conductor of any newform of level N. In view of the Katz-Sarnak philosophy of studying families of (naturally related) L-functions, we filter out the oldforms in $\mathcal{F}_k(N)$ and direct our interest to the remaining set $\mathcal{F}_k(N)'$ of exclusively newforms, all of which have analytic conductor:

$$Q_{k,N} \ := \ k^2 N. \tag{2.17}$$

We now describe the setup for the rest of our paper. We fix a real, primitive Dirichlet character χ of modulus $D \geq 1$, and a positive integer r relatively prime to D. Reiterating Section 1, we follow Knightly and Reno [**KR18**] in considering two families, each composed of the finite sub-families $\mathcal{F}_k(N)'$ discussed above:

- $\mathcal{F}_1 = \bigcup_k \mathcal{F}_k(1)'$ with k ranging over even integers satisfying $\tau(\chi)^2 \neq -i^k D$;
- $\mathcal{F}_2 = \bigcup_{k,N} \mathcal{F}_k(N)'$ with k ranging over even integers satisfying $\tau(\chi)^2 = -i^k D$, equivalently $\chi(-1) = -i^k$, and N ranging over primes not divisible by rD.

We assign weights to the Hecke newforms f in these families,

$$w_{\chi,r}(f) \ := \ \frac{\Lambda\left(\frac{1}{2}, f \times \chi\right) |a_f(r)|^2}{\|f\|^2}, \tag{1.12}$$

and examine their influence on the nth centered moment of their one-level densities (1.15).

3. Explicit Formula

In this section, we restrict our attention to a single sub-family $\mathcal{F}_k(N)'$ of either $\mathcal{F}_1$ or $\mathcal{F}_2$. Given any Hecke eigenform $f \in \mathcal{F}_k(N)'$, an explicit formula for the one-level density of $L(s, f)$ is given in [**ILS00**]:

$$
\begin{aligned}
D(f; \phi) \ = \ \widehat{\phi}(0) + \frac{1}{2}\phi(0) + O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right) &- 2\sum_{p \nmid N} \lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log Q_{k,N}}\right) \frac{\log p}{\sqrt{p}\log Q_{k,N}} \\
&- 2\sum_{p\nmid N} \lambda_f(p^2)\widehat{\phi}\left(\frac{2\log p}{\log Q_{k,N}}\right) \frac{\log p}{p\log Q_{k,N}}.
\end{aligned}
$$
(3.1)

Using crude estimates for the higher symmetric power L-function $L(s, \mathrm{sym}^2(f) \otimes \mathrm{sym}^2(f))$, we can absorb the second moment of the Hecke eigenvalues (3.1) into the error term $O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right)$, as explained in [**ILS00**].

As stated before, Knightly and Reno studied the average weighted one-level density of $\mathcal{F}_k(N)'$, proving that it depends on the triviality of the twisting character in the weight [**KR18**]:

$$
\mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}(D(\cdot, \phi)) \ = \
\begin{cases}
\widehat{\phi}(0) - \frac{1}{2}\phi(0) + O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right) & \text{if } \chi \text{ trivial,} \\
\widehat{\phi}(0) + \frac{1}{2}\phi(0) + O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right) & \text{if } \chi \text{ non-trivial.}
\end{cases}
$$
(3.2)

We are interested in the weighted n^{th} centered moment of this average (weighted) one-level density:

$$
\mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}\left[\left(D(\cdot, \phi) - \mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}(D(\cdot, \phi))\right)^n\right].
$$
(3.3)

Applying (3.1) and (3.2), we determine that (3.3) equals

$$
O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right) +
\begin{cases}
\sum_{t=0}^n \binom{n}{t}(-2)^t \mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}\left[\left(\sum_{p\nmid N}\lambda.(p)\widehat{\phi}\left(\frac{\log p}{\log(k^2 N)}\right) \frac{\log p}{\sqrt{p}\log Q_{k,N}}\right)^t\right] \\
\text{if } \chi \text{ trivial,} \\
(-2)^n \mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}\left[\left(\sum_{p\nmid N}\lambda.(p)\widehat{\phi}\left(\frac{\log p}{\log Q_{k,N}}\right) \frac{\log p}{\sqrt{p}\log Q_{k,N}}\right)^n\right] \\
\text{if } \chi \text{ non-trivial.}
\end{cases}
$$
(3.4)

In this view, we attend to averages of the form

$$
\mathcal{A}^{w_{\chi,r}}_{\mathcal{F}_k(N)'}\left[\left(\sum_{p \nmid N} \lambda.(p)\,\widehat{\phi}\left(\frac{\log p}{\log Q_{k,N}}\right)\frac{\log p}{\sqrt{p}\log Q_{k,N}}\right)^t\right],
\tag{3.5}
$$

for $0 \le t \le n$.

Lemma 3.1. *For any $0 \le t \le n$, the average in (3.5) is equal to*

$$
\sum_{\substack{1\le \ell \le t,\\ n_1+\cdots+n_\ell=t}}\frac{t!}{\ell!\,n_1!\cdots n_\ell!}\sum_{\substack{(q_1,\ldots,q_\ell)\\ q_j \nmid N}}\left[\left(\prod_{j=1}^{\ell}\widehat{\phi}\left(\frac{\log q_j}{\log Q_{k,N}}\right)^{n_j}\frac{\log^{n_j} q_j}{q_j^{n_j/2}\log^{n_j} Q_{k,N}}\right)\right.
$$

$$
\left.\times \sum_{\substack{0\le m_j\le n_j,\\ m_j\equiv n_j(2)}}\left(\prod_{j=1}^{\ell}c_{n_j,m_j}\right)\mathcal{A}^{w_{\chi,r}}_{\mathcal{F}_k(N)'}\left(\lambda.\left(\prod_{j=1}^{\ell}q_j^{m_j}\right)\right)\right],
\tag{3.6}
$$

where each c_{n_j,m_j} is bounded by an absolute constant in t.

Proof. We expand the product of t sums in (3.5) to get a sum over t-tuples:

$$
\sum_{\substack{(p_1,\ldots,p_t)\\ p_i\nmid N}}\left[\left(\prod_{i=1}^{t}\widehat{\phi}\left(\frac{\log p_i}{\log Q_{k,N}}\right)\frac{\log p_i}{\sqrt{p_i}\log Q_{k,N}}\right)\mathcal{A}^{w_{\chi,r}}_{\mathcal{F}_k(N)'}\left(\prod_{i=1}^{t}\lambda.(p_i)\right)\right].
\tag{3.7}
$$

To appeal to the multiplicative properties of Hecke eigenvalues (2.9), we consider the prime factorization: $\prod_{i=1}^{t}p_i=\prod_{j=1}^{\ell}q_j^{n_j}$. However, to change the index of summation from p_i to q_j, we require the combinatorial factor indicating the number of t-tuples $(p_1,\ldots,p_t)$ with a given prime factorization $\prod_{j=1}^{\ell}q_j^{n_j}$:

$$
\frac{1}{\ell!}\binom{t}{n_1}\binom{t-n_1}{n_2}\cdots\binom{n_\ell}{n_\ell}=\frac{t!}{\ell!\,n_1!\cdots n_\ell!}.
\tag{3.8}
$$

With this, we make the desired change of index in (3.7) to get

$$
\sum_{\substack{1\le \ell \le t,\\ n_1+\cdots+n_\ell=t}}\frac{t!}{\ell!\,n_1!\cdots n_\ell!}\sum_{\substack{(q_1,\ldots,q_\ell)\\ q_j\nmid N}}\left[\left(\prod_{j=1}^{\ell}\widehat{\phi}\left(\frac{\log q_j}{\log Q_{k,N}}\right)^{n_j}\frac{\log^{n_j}q_j}{q_j^{n_j/2}\log^{n_j}Q_{k,N}}\right)\right.
$$

$$
\left.\times\mathcal{A}^{w_{\chi,r}}_{\mathcal{F}_k(N)'}\left(\prod_{j=1}^{\ell}\lambda.(q_j)^{n_j}\right)\right].
\tag{3.9}
$$

As in (2.10) and (2.11), we write $\lambda.(q_j)^{n_j}$ as a linear combination of Hecke eigenvalues at the powers of q_j:

$$\lambda.(q_j)^{n_j} \;=\; \sum_{\substack{0 \le m_j \le n_j \\ m_j \equiv n_j(2)}} c_{n_j,m_j} \; \lambda.(q_j^{m_j}), \tag{3.10}$$

where each c_{n_j,m_j} is bounded by an absolute constant in $n_j \le t$. Making this substitution in (3.9) gives (3.6). $\qquad\square$

Lemma 3.1 leads our study to weighted averages of Hecke eigenvalues over $\mathcal{F}_k(N)'$.

4. Weighted Trace Formula

In this section, we derive an asymptotic trace formula for the weighted average of the mth Hecke eigenvalue (m arbitrary) over $\mathcal{F}_k(N)'$:

$$\mathcal{A}^{w_{\chi,r}}_{\mathcal{F}_k(N)'}(\lambda.(m)) \;:=\; \frac{\sum_{f \in \mathcal{F}_k(N)'} w_{\chi,r}(f)\lambda_f(m)}{\sum_{f \in \mathcal{F}_k(N)'} w_{\chi,r}(f)}. \tag{4.1}$$

Trace formulae are more readily attainable for $\mathcal{F}_k(N)$ than $\mathcal{F}_k(N)'$. This is because the equidistribution laws that govern Hecke eigenvalues—and thereby influence trace formulae—are better understood for the space of all forms $S_k(N)$ rather than the space of newforms alone. Studying families of newforms usually entails extensive bookkeeping and adapting trace formulae from the ambient space to the space of newforms, e.g., [**ILS00**]. Fortunately, we do not have to do any bookkeeping in our study, owing to the strategic conditioning on k and N in our definitions of $\mathcal{F}_1$ and $\mathcal{F}_2$.

For level 1, it is straightforward that $\mathcal{F}_k(1) = \mathcal{F}_k(1)'$ because there are no levels lower than 1, and hence no forms to induce from.

For prime level $N > 1$, as Knightly and Reno explained in [**KR18**], the condition involving $\tau(\chi)$ in the definition of $\mathcal{F}_2$ licenses us to bypass bookkeeping. Every form $f \in \mathcal{F}_k(N)$ is either a newform of level N or an oldform of level N induced from a newform g of level 1. In the latter case, since $\tau(\chi)^2 = -i^k D$, the functional equation (2.15) for $\Lambda(s, g \times \chi)$ forces $w_{\chi,r}(g)$ to vanish. For the induced form $g_N(z) := g(Nz)$ too, the weight vanishes:

$$\Lambda\left(\frac{1}{2}, g_N \times \chi\right) \;=\; \frac{\chi(N)}{N^{k/2}}\Lambda\left(\frac{1}{2}, g \times \chi\right) \;=\; 0 \;\implies\; w_{\chi,r}(g_N) = 0. \tag{4.2}$$

It follows that the weight vanishes for all forms in the span of $\{g, g_N\}$, f in particular. This is to say that $w_{\chi,r}(f)$ vanishes for all oldforms $f \in \mathcal{F}_k(N)$; hence any $w_{\chi,r}$-weighed average over $\mathcal{F}_k(N)$ reduces to a $w_{\chi,r}$-weighted average over $\mathcal{F}_k(N)'$, and we can refer to these interchangeably.

In the interest of finding a weighted trace formula for (4.1), we call attention to a related formula in [**JK15**]. By applying certain vertical equidistribution laws for $S_k(N)$, Jackson and Knightly derived a formula for the sum of Fourier coefficients over $\mathcal{F}_k(N)$ weighted by

$$w'_{\chi,r}(f) \; := \; \frac{\Lambda\left(\frac{1}{2}, f \times \chi\right) \overline{a_f(r)}}{\|f\|^2}, \quad f \in \mathcal{F}_k(N). \tag{4.3}$$

Knightly and Reno analyzed this to obtain an asymptotic formula [**KR18**]. To be explicit, if $N = 1$, then for all $m \geq 1$,

$$\sum_{f \in \mathcal{F}_k(1)} w'_{\chi,r}(f) a_f(m) \; = \; \frac{2^{k-1}(2\pi)^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}{(k-2)!}\left(1 + i^k \frac{\tau(\chi)^2}{D}\right)(rm)^{\frac{k}{2}-1}\chi(rm)\,\sigma_1((r,m))$$

$$+ \; O\left(\frac{(4\pi r m)^{k-1}D^{\frac{k}{2}-\frac{1}{2}}}{(k-2)!}\right); \tag{4.4}$$

and if $k > 2$ and $N > 1$, then for all $m \geq 1$,

$$\frac{1}{\nu(N)}\sum_{f \in \mathcal{F}_k(N)} w'_{\chi,r}(f) a_f(m) \; = \; \frac{2^{k-1}(2\pi)^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}{(k-2)!}\,(rm)^{\frac{k}{2}-1}\chi(rm)\,\sigma_1((r,m))$$

$$+ \; O\left(\frac{(4\pi r m)^{k-1}D^{\frac{k}{2}-\frac{1}{2}}}{N^{\frac{k}{2}}(k-2)!}\right), \tag{4.5}$$

where the implied constants depend only on r and D, and σ_1 is the divisor sum function. (For the interested reader, we have specialized [**KR18**] for trivial nebentypus, real χ, and $s = 1/2$, and absorbed constants in r and D into the implied constant.)

In the following three lemmas, we adapt these asymptotic trace formulae for the weighted average of our interest (4.1).

Lemma 4.1. *Suppose $N = 1$. Then, for any positive integer $m \nmid N$, the sum of the mth Hecke eigenvalue over $\mathcal{F}_k(1)'$ is equal to*

$$\sum_{f \in \mathcal{F}_k(1)'} w_{\chi,r}(f)\lambda_f(m) \; = \; \frac{2^{k-1}(2\pi r^2)^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)\sigma_1(r)}{(k-2)!}\left(1 + i^k \frac{\tau(\chi)^2}{D}\right)m^{-\frac{1}{2}}\chi(m)\,\sigma_1((r,m))$$

$$+ \; O\left(\frac{m^{\frac{k-1}{2}}V^k}{(k-2)!}\right), \tag{4.6}$$

where the implied constant and V depend only on r and D.

Proof. For all f, we have that $w_{\chi,r}(f) = w'_{\chi,r}(f)a_f(r)$. We use this relation between weights and the relation (2.7) between Fourier coefficients and Hecke eigenvalues to pass between the sum of interest (4.6) and the sum already analyzed (4.4):

$$
\begin{aligned}
\sum_{f\in\mathcal{F}_k(1)'} w_{\chi,r}(f)\lambda_f(m) &= \sum_{f\in\mathcal{F}_k(1)'} w_{\chi,r}(f)\lambda_f(m) \\
&= \sum_{f\in\mathcal{F}_k(1)} w'_{\chi,r}a_f(r)(f)\lambda_f(m) \\
&= \sum_{f\in\mathcal{F}_k(1)} w'_{\chi,r}(f)r^{\frac{k-1}{2}}\lambda_f(r)\lambda_f(m). \quad (4.7)
\end{aligned}
$$

Appealing to the multiplicative properties (2.9) of Hecke eigenvalues,

$$
\begin{aligned}
\sum_{f\in\mathcal{F}_k(1)'} w_{\chi,r}(f)\lambda_f(m) &= \sum_{f\in\mathcal{F}_k(1)} w'_{\chi,r}(f)r^{\frac{k-1}{2}}\lambda_f(r)\lambda_f(m) \\
&= \sum_{f\in\mathcal{F}_k(1)} w'_{\chi,r}(f)r^{\frac{k-1}{2}} \sum_{d|(r,m)} \lambda_f\left(\frac{rm}{d^2}\right) \\
&= \sum_{f\in\mathcal{F}_k(1)} w'_{\chi,r}(f)r^{\frac{k-1}{2}} \sum_{d|(r,m)} \left(\frac{rm}{d^2}\right)^{-\frac{k-1}{2}} a_f\left(\frac{rm}{d^2}\right). \quad (4.8)
\end{aligned}
$$

Changing the order of these finite summations,

$$
\begin{aligned}
\sum_{f\in\mathcal{F}_k(1)'} w_{\chi,r}(f)\lambda_f(m) &= \sum_{f\in\mathcal{F}_k(1)} w'_{\chi,r}(f)r^{\frac{k-1}{2}} \sum_{d|(r,m)} \left(\frac{rm}{d^2}\right)^{-\frac{k-1}{2}} a_f\left(\frac{rm}{d^2}\right) \\
&= \sum_{d|(r,m)} \left(\frac{m}{d^2}\right)^{-\frac{k-1}{2}} \sum_{f\in\mathcal{F}_k(1)} w'_{\chi,r}(f)a_f\left(\frac{rm}{d^2}\right). \quad (4.9)
\end{aligned}
$$

Appealing to (4.4) and recalling that χ is real, i.e., $\chi^2 = 1$, we find that (4.9) equals

$$
\begin{aligned}
&\frac{2^{k-1}(2\pi)^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}{(k-2)!}\left(1+i^k\frac{\tau(\chi)^2}{D}\right) \sum_{d|(r,m)} \left(\frac{m}{d^2}\right)^{-\frac{k-1}{2}} \left(\frac{r^2m}{d^2}\right)^{\frac{k}{2}-1} \chi\left(\frac{r^2m}{d^2}\right) \sigma\left(\left(r,\frac{rm}{d^2}\right)\right) \\
&+ O\left(\frac{(4\pi r)^{k-1}D^{\frac{k}{2}-\frac{1}{2}}}{(k-2)!} \sum_{d|(r,m)} \left(\frac{m}{d^2}\right)^{-\frac{k-1}{2}} \left(\frac{rm}{d^2}\right)^{k-1}\right) \\
&= \frac{2^{k-1}(2\pi r^2)^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}{(k-2)!}\left(1+i^k\frac{\tau(\chi)^2}{D}\right) m^{-\frac{1}{2}}\chi(m) \sum_{d|(r,m)} d\,\sigma_1\left(\left(r,\frac{rm}{d^2}\right)\right)
\end{aligned}
$$

$$+ O\left(m^{\frac{k-1}{2}}\frac{(4\pi r^2)^{k-1}D^{\frac{k}{2}-\frac{1}{2}}}{(k-2)!}\sum_{d|(r,m)}d^{-k+1}\right)$$

$$= \frac{2^{k-1}(2\pi r^2)^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}{(k-2)!}\left(1+i^k\frac{\tau(\chi)^2}{D}\right)m^{-\frac{1}{2}}\chi(m)\sum_{d|(r,m)}d\,\sigma_1\left(\left(r,\frac{rm}{d^2}\right)\right)$$

$$+ O\left(\frac{m^{\frac{k-1}{2}}V^k}{(k-2)!}\right), \tag{4.10}$$

where V is a constant depending on r and D. We now recall the following divisor sum identity: for an arithmetic function f and a completely multiplicative function g,

$$f(m)f(n) = \sum_{d|(m,n)} f\left(\frac{mn}{d^2}\right)g(d). \tag{4.11}$$

Taking $g(n) = n$ and $f(n) = \sigma_1((r,n))$ in the above identity allows us to simplify the main term in (4.10), giving that it equals

$$\frac{2^{k-1}(2\pi r^2)^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)\sigma_1(r)}{(k-2)!}\left(1+i^k\frac{\tau(\chi)^2}{D}\right)m^{-\frac{1}{2}}\chi(m)\sigma_1((r,m)) + O\left(\frac{m^{\frac{k-1}{2}}V^k}{(k-2)!}\right). \tag{4.12}$$

This is precisely the formula in the statement of the lemma. $\qquad\square$

Lemma 4.2. *Suppose $k > 2$ and $N > 1$. Then, for any positive integer $m \nmid N$, the sum of the mth Hecke eigenvalue over $\mathcal{F}_k(N)'$ is equal to*

$$\frac{1}{\nu(N)}\sum_{f\in\mathcal{F}_k(N)'}w_{\chi,r}(f)\lambda_f(m) = \frac{2^{k-1}(2\pi r^2)^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)\sigma_1(r)}{(k-2)!}m^{-\frac{1}{2}}\chi(m)\,\sigma_1((r,m))$$

$$+ O\left(\frac{m^{\frac{k-1}{2}}V^k}{N^{\frac{k}{2}}(k-2)!}\right), \tag{4.13}$$

where the implied constant and V depend only on r and D, and σ_1 is the divisor sum function.

Proof. This proof proceeds similarly to the proof of Lemma 4.1, leveraging the relation between the weights in question, the relation (2.7) between Fourier coefficients and Hecke eigenvalues, and the multiplicative properties (2.9) of Hecke eigenvalues to adapt (4.5) to get (4.13). $\qquad\square$

Lemma 4.3. *Let $m = \prod_{j=1}^{\ell} q_j^{m_j} \nmid N$, where $q_1, \ldots, q_j$ are the distinct primes dividing m. Then, the average (4.1) of the mth Hecke eigenvalue over $\mathcal{F}_k(N)'$ is equal to*

$$\mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}\left(\lambda.\left(\prod_{j=1}^{\ell} q_j^{m_j}\right)\right) = \left(\prod_{j=1}^{\ell} q_j^{m_j}\right)^{-\frac{1}{2}} \left(\prod_{j=1}^{\ell} \chi(q_j)^{m_j}\right) \sigma_1\left(\left(r, \prod_{j=1}^{\ell} q_j^{m_j}\right)\right)$$

(4.14)

$$+ \ O\left(\frac{\left(\prod_{j=1}^{\ell} q_j^{m_j}\right)^{\frac{k-1}{2}} W^k}{N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}}\right),$$

(4.15)

where the implied constant and W depend only on r and D.

Proof. We prove this statement when $k > 2$ and $N > 1$. We would like to compute

$$\mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}(\lambda.(m)) = \frac{\sum_{f \in \mathcal{F}_k(N)'} w_{\chi,r}(f) \lambda_f(m)}{\sum_{f \in \mathcal{F}_k(N)'} w_{\chi,r}(f)}.$$

(4.16)

By Lemma 4.2, the denominator (setting $m = 1$) equals

$$\frac{2^{k-1}(2\pi r^2)^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)\sigma_1(r)}{(k-2)!} + O\left(\frac{V^k}{N^{\frac{k}{2}}(k-2)!}\right) =: M_D + E_D,$$

(4.17)

and the numerator equals

$$\frac{2^{k-1}(2\pi r^2)^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)\sigma_1(r)}{(k-2)!} m^{-\frac{1}{2}}\chi(m)\,\sigma_1((r,m)) + \ O\left(\frac{m^{\frac{k-1}{2}}V^k}{N^{\frac{k}{2}}(k-2)!}\right) =: M_N + E_N.$$

(4.18)

Rewriting our target expression (4.16) in terms of M_N, E_N, M_D, and E_D,

$$\mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}(\lambda.(m)) = \frac{M_N + E_N}{M_D + E_D}$$

$$= \frac{M_N}{M_D} + \frac{E_N - \frac{M_N}{M_D}E_D}{M_D + E_D}.$$

(4.19)

It is straightforward that

$$\frac{M_N}{M_D} = m^{-\frac{1}{2}}\chi(m)\,\sigma_1((r,m))$$

$$= \left(\prod_{j=1}^{\ell} q_j^{m_j}\right)^{-\frac{1}{2}} \left(\prod_{j=1}^{\ell} \chi(q_j)^{m_j}\right) \sigma_1\left(\left(r, \prod_{j=1}^{\ell} q_j^{m_j}\right)\right),$$

(4.20)

matching the main term (4.14) in the lemma. We finish by demonstrating that the second term in (4.19) has the desired rate of decay (4.15). For clarity of the analysis, we let

$$C := \frac{V^k}{N^{\frac{k}{2}}(k-2)!},$$ (4.21)

under which,

$$
\begin{aligned}
\frac{E_N - \frac{M_N}{M_D}E_D}{M_D + E_D} &\ll \frac{\left(m^{\frac{k-1}{2}} - m^{-\frac{1}{2}}\chi(m)\sigma_1((r,m))\right)C}{M_D + E_D} \\
&\ll_r \frac{m^{\frac{k-1}{2}}C}{M_D + E_D} \\
&= m^{\frac{k-1}{2}}\frac{C/M_D}{1 + E_D/M_D}.
\end{aligned}
$$ (4.22)

The argument in [**JK15**] gives that

$$\frac{E_D}{M_D} \ll \frac{C}{M_D} \ll_{r,D} \frac{W^k}{N^{\frac{k-1}{2}}k^{\frac{k}{2}-1}},$$ (4.23)

where W depends only on r and D. It directly follows that (4.22) has the claimed decay in the statement of the lemma.

The proof for $N = 1$ is similar; the extra factor of $1 + \frac{i^k \tau(\chi)^2}{D}$ in the main term (4.4) will cancel coming from both the numerator and the denominator. $\square$

5. Combinatorial Analysis

In this section, we present our proof of Theorem 1.1, consolidating Lemmas 3.1 and 4.3.

To review, we are interested in

$$\mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}}\left[\left(\sum_{p\nmid N}\lambda.(p)\,\widehat{\phi}\left(\frac{\log p}{\log Q_{k,N}}\right)\frac{\log p}{\sqrt{p}\log Q_{k,N}}\right)^t\right],$$ (5.1)

which, by Lemma 3.1, is known to equal

$$\sum_{\substack{1 \le \ell \le t, \\ n_1 + \cdots + n_\ell = t}} \frac{t!}{\ell!\,n_1!\cdots n_\ell!} \sum_{\substack{(q_1,\ldots,q_\ell) \\ q_j\nmid N}}\left[\left(\prod_{j=1}^\ell \widehat{\phi}\left(\frac{\log q_j}{\log Q_{k,N}}\right)^{n_j}\frac{\log^{n_j}q_j}{q_j^{n_j/2}\log^{n_j}Q_{k,N}}\right)\right.
$$

$$\times \sum_{\substack{0 \leq m_j \leq n_j, \\ m_j \equiv n_j (2)}} \left(\prod_{j=1}^{\ell} c_{n_j, m_j} \right) \mathcal{A}_{\mathcal{F}_k(N)'}^{w_{\chi,r}} \left(\lambda . \left(\prod_{j=1}^{\ell} q_j^{m_j} \right) \right) \Bigg].$$
$$(5.2)$$

Applying the weighted trace formula from Lemma 4.3 to expand (5.2), we find that the contribution of the main term (4.14) to (5.2) is

$$\sum_{\ell=1}^{t} \sum_{n_1 + \cdots + n_\ell = t} \sum_{\substack{0 \leq m_j \leq n_j \\ m_j \equiv n_j (2) \\ 1 \leq j \leq \ell}} \frac{t!}{\ell! \, n_1! \cdots n_\ell!} \left(\prod_{j=1}^{\ell} c_{n_j, m_j} \right)$$

$$\times \sum_{\substack{(q_1, \ldots, q_\ell) \\ q_j \nmid N}} \left(\prod_{j=1}^{\ell} \widehat{\phi} \left(\frac{\log q_j}{\log Q_{k,N}} \right)^{n_j} \frac{\chi(q_j)^{m_j} \log^{n_j} q_j}{q_j^{(n_j + m_j)/2} \log^{n_j} Q_{k,N}} \sigma_1((r, q_j^{m_j})) \right);$$
$$(5.3)$$

and the contribution of the error term (4.15) to (5.2) is

$$\ll_{r,n,D} \frac{W^k}{N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}} \sum_{\substack{(q_1, \ldots, q_\ell) \\ q_j \nmid N}} \left(\prod_{j=1}^{\ell} \widehat{\phi} \left(\frac{\log q_j}{\log Q_{k,N}} \right)^{n_j} \frac{\log^{n_j} q_j}{q_j^{n_j/2} \log^{n_j} Q_{k,N}} q_j^{\frac{m_j(k-1)}{2}} \right). \quad (5.4)$$

Note that we have absorbed the combinatorial sum in (5.2) into the implied constant in (5.4) as it is bounded by a polynomial in $t \leq n$. It is also worth reiterating that W is a constant depending only on r and D.

We start by bounding the error term contribution (5.4). As explained in Remark 1.5, this step poses the only obstruction to extending support.

Lemma 5.1. *For any test function ϕ with $supp(\widehat{\phi}) \subset \left(-\frac{1}{2n}, \frac{1}{2n} \right)$, the error term contribution (5.4) vanishes when either $k \to \infty$ or $k + N \to \infty$.*

Proof. By the principle of inclusion-exclusion, we know that

$$\frac{W^k}{N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}} \sum_{\substack{(q_1, \ldots, q_\ell) \\ q_j \nmid N}} \left(\prod_{j=1}^{\ell} \widehat{\phi} \left(\frac{\log q_j}{\log Q_{k,N}} \right)^{n_j} \frac{\log^{n_j} q_j}{q_j^{n_j/2} \log^{n_j} Q_{k,N}} q_j^{\frac{m_j(k-1)}{2}} \right)$$

$$\ll \frac{W^k}{N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}} \prod_{j=1}^{\ell} \left(\sum_{q \nmid N} \widehat{\phi} \left(\frac{\log q}{\log Q_{k,N}} \right)^{n_j} \frac{\log^{n_j} q}{q^{n_j/2} \log^{n_j} Q_{k,N}} q_j^{\frac{m_j(k-1)}{2}} \right). \quad (5.5)$$

Besides, if $\text{supp}(\widehat{\phi}) \subset (-\alpha, \alpha)$, then (5.5) is bounded by

$$
\ll \frac{W^k}{N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}} \prod_{j=1}^{\ell} \left(\sum_{q \ll Q_{k,n}^\alpha} q^{\frac{m_j(k-1)}{2} - \frac{n_j}{2}} \right)
$$

$$
\ll \frac{W^k}{N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}} \prod_{j=1}^{\ell} \left(\sum_{q \ll Q_{k,n}^\alpha} q^{\frac{n_j k}{2} - n_j} \right)
$$

$$
\ll \frac{W^k}{N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}} \prod_{j=1}^{\ell} Q_{k,N}^{\frac{n_j k \alpha}{2} - n_j \alpha + 1}
$$

$$
\ll \frac{W^k}{N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}} Q_{k,N}^{\frac{n k \alpha}{2} - n \alpha + n}, \tag{5.6}
$$

recalling that $m_j \leq n_j$, $\ell \leq n$, and $\sum n_j = n$. Since $Q_{k,N} = k^2 N$, if $\alpha < \frac{1}{2n}$, then (5.6) vanishes as $k \to \infty$ or $k + N \to \infty$. $\qquad\square$

Lemma 5.1 licenses us to focus exclusively on the main term contribution (5.3) in our analysis of (5.2). Applying the principle of inclusion-exclusion to (5.3), we swap the sum and the product, reversing the idea in (3.7) and picking up the error term $O\left(\frac{\log\log(3N)}{\log Q_{k,N}} \right)$:

$$
\sum_{\ell=1}^{t} \sum_{n_1 + \cdots + n_\ell = t} \sum_{\substack{0 \leq m_j \leq n_j \\ m_j \equiv n_j (2) \\ 1 \leq j \leq \ell}} \frac{t!}{\ell!\, n_1! \cdots n_\ell!} \left(\prod_{j=1}^{\ell} c_{n_j, m_j} \right)
$$

$$
\times \prod_{j=1}^{\ell} \left(\sum_{q \nmid N} \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}} \right)^{n_j} \frac{\chi(q)^{m_j} \log^{n_j} q}{q^{(n_j + m_j)/2} \log^{n_j} Q_{k,N}} \, \sigma_1\left((r, q^{m_j}) \right) \right).
\tag{5.7}
$$

We justify this step in tedious detail in Appendix A. We thereby reduce the problem to analyzing

$$
\sum_{q \nmid N} \widehat{\phi}\left(\frac{\log q}{\log R} \right)^{n_j} \frac{\chi(q)^{m_j} \log^n q}{q^{(n+m)/2} \log^n R} \, \sigma_1\left((r, q^m) \right)
\tag{5.8}
$$

for specific cases of n_j and m_j. We exhibit this analysis separately for non-trivial and trivial χ, in Lemmas 5.2 and 5.3 respectively.

Lemma 5.2. *Let χ be a nontrivial character of modulus $D \geq 1$. Let n be a positive integer and m be a nonnegative integer such that $m \leq n$ and $m \equiv n$ (mod 2). Then*

$$\sum_{\substack{q\ prime;\\ q\nmid N}} \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right)^n \frac{\chi(q)^m \log^n q}{q^{(m+n)/2} \log^n Q_{k,N}} \sigma_1\left((r, q^m)\right)$$

$$= \begin{cases} \frac{\sigma_\phi^2}{4} + O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right) & (m, n) = (0, 2), \\ O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right) & (m, n) = (1, 1), \\ O\left(\frac{1}{\log^2 Q_{k,N}}\right) & otherwise. \end{cases} \tag{5.9}$$

.

Proof. First, we consider $(m, n) = (0, 2)$. In this case, we apply Abel's summation formula in tandem with the prime number theorem, recalling that ϕ is even and $\widehat{\phi}$ has compact support. Therefore, the sum under consideration equals

$$\sum_{\substack{q\ prime;\\ q\nmid N}} \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right)^2 \frac{\log^2 q}{q \log^2 Q_{k,N}} = \frac{\sigma_\phi^2}{4} + O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right). \tag{5.10}$$

This equation is also given in [**HM07**].

Second, we consider $(m, n) = (1, 1)$. We follow the same method from the previous case and appeal to Dirichlet's theorem on primes in arithmetic progressions (the value of χ is ± 1 on exactly half the primes, upto lower order terms). This gives

$$\sum_{\substack{q\ prime;\\ q\nmid N}} \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right) \frac{\chi(q) \log q}{q \log Q_{k,N}} = \sum_{q\ prime} \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right) \frac{\chi(q) \log q}{q \log Q_{k,N}} + O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right)$$

$$= \sum_{q:\chi(p)=1} \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right) \frac{\log q}{q \log Q_{k,N}} - \sum_{q:\chi(p)=-1} \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right) \frac{\log q}{q \log Q_{k,N}} + O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right)$$

$$= O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right); \tag{5.11}$$

this equation is also given in [**KR18**].

Finally, we consider $(m, n) \neq (0, 2), (1, 1)$. Since m and n are integers with the same parity, we know that $m + n \geq 4$ in this case, and the sum on the right-hand side of (5.12) converges:

$$\sum_{\substack{q\ prime;\\ q\nmid N}} \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right)^n \frac{\chi(q)^m \log^n q}{q^{(m+n)/2} \log^n Q_{k,N}} \sigma_1\left(\gcd\left(r, q^m\right)\right) \ll_r \frac{1}{\log^n Q_{k,N}} \sum_{q\ prime} \frac{\log^n q}{q^{(m+n)/2}}$$

$$= O\left(\frac{1}{\log^2 Q_{k,N}}\right).$$
$$(5.12)$$

This completes the analysis for χ nontrivial. $\qquad\square$

Lemma 5.3. *Let χ_0 be the trivial character of modulus $D \geq 1$. Let n be a positive integer and m be a nonnegative integer such that $m \leq n$ and $m \equiv n \pmod 2$. Then*

$$\sum_{\substack{q\ prime; \\ q\nmid N}} \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right)^n \frac{\chi_0(q)^m \log^n q}{q^{(m+n)/2} \log^n Q_{k,N}} \sigma_1\left(\gcd\left(r, q^m\right)\right)$$

$$= \begin{cases} \frac{\sigma_\phi^2}{4} + O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right) & (m,n) = (0,2), \\ \frac{\phi(0)}{2} + O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right) & (m,n) = (1,1), \\ O\left(\frac{1}{\log^2 Q_{k,N}}\right) & otherwise. \end{cases}$$
$$(5.13)$$

Proof. The analysis for $(m,n) \neq (1,1)$ is identical to Lemma 5.2, so we need only tend to $(m,n) = (1,1)$. We compute the corresponding sum using the methods from the previous lemma:

$$\sum_{\substack{q\ prime; \\ q\nmid N}} \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right) \frac{\chi_0(q)\log q}{q \log Q_{k,N}} = \sum_{q\ prime} \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right) \frac{\log q}{q \log Q_{k,N}} + O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right)$$

$$= \frac{\phi(0)}{2} + O\left(\frac{\log\log(3N)}{\log Q_{k,N}}\right).$$
$$(5.14)$$

This equation is also given in [**KR18**]. $\qquad\square$

Considering these lemmas, we calculate the nth centered moment separately for non-trivial and trivial χ in Subsections 5.1 and 5.2, respectively. For clarity of the exposition, we let

$$T(q, n_j, m_j, \chi) := \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right)^{n_j} \frac{\chi(q)^{m_j} \log^n q}{q^{(n+m)/2} \log^n Q_{k,N}} \sigma_1\left(\gcd\left(r, q^m\right)\right)$$
$$(5.15)$$

going forward.

5.1. **Non-Trivial Twisting Character.** In view of Lemma 5.2, we split our consideration across two cases: $(m_j, n_j) \neq (0,2)$ for some j, and $(m_j, n_j) = (0,2)$ for all j.

In the former case, consider the j for which $(m_j, n_j) \neq (0, 2)$. For this j, Lemma 5.2 yields

$$\sum_{q \nmid N} T(q, n_j, m_j, \chi) = O\left(\frac{\log \log(3N)}{\log Q_{k,N}}\right). \tag{5.16}$$

For all the other j, Lemma 5.2 gives

$$\sum_{q \nmid N} T(q, n_j, m_j, \chi) = O(1). \tag{5.17}$$

Multiplying the order of magnitude over all j, we find that the main term does not contribute in this case, asymptotically speaking:

$$\sum_{\substack{1 \leq \ell \leq t; \\ n_1 + \cdots + n_\ell = t; \\ m_j \equiv n_j (2)}} \frac{t!}{\ell! n_1! \cdots n_\ell!} \left(\prod_{j=1}^{\ell} c_{n_j, m_j}\right) \prod_{j=1}^{\ell} \left(\sum_{q \nmid N} T(q, n_j, m_j, \chi)\right) = O\left(\frac{\log \log(3N)}{\log Q_{k,N}}\right); \tag{5.18}$$

we absorb the combinatorial sum into the constant as it is bounded by a polynomial in $t \leq n$.

The latter case, $(m_j, n_j) = (0, 2)$ for all j, only arises when t is even and $\ell = t/2$ since $\sum_j n_j = t$. In this case, for all j, we specialize (2.10) to determine that $c_{n_j, m_j} = 1$; and Lemma 5.2 implies

$$\sum_{q \nmid N} T(q, n_j, m_j, \chi) = \frac{\sigma_\phi^2}{4} + O\left(\frac{\log \log(3N)}{\log Q_{k,N}}\right). \tag{5.19}$$

Overall, our analysis supports that the main term of (5.2) is

$$\frac{t!}{(t/2)! \, 2^{\ell/2}} \cdot \left(\frac{\sigma_\phi^2}{4}\right)^{t/2} + O\left(\frac{\log \log(3N)}{\log Q_{k,N}}\right) = (t-1)!! \left(\frac{\sigma_\phi^2}{4}\right)^{t/2} + O\left(\frac{\log \log(3N)}{\log Q_{k,N}}\right). \tag{5.20}$$

Plugging (5.20) into (3.4), we conclude that the main term of the weighted n^{th} centered moment of the average one-level density in this case is

$$(-2)^n (n-1)!! \left(\frac{\sigma_\phi^2}{4}\right)^{n/2} = (n-1)!! \, \sigma_\phi^n, \tag{5.21}$$

proving Theorem 1.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

5.2. Trivial Twisting Character.

Although the analysis in this case is simpler than the previous case, the combinatorial arguments made to synthesize the various main term contributions in this case are more unassuming and interesting.

Again, in view of Lemma 5.2, we split our consideration across two cases: $m_j + n_j \geq 3$ for some j, and $m_j + n_j \leq 2$ for all j.

In the former case, consider the j for which $n_j + m_j \geq 3$. For this j, Lemma 5.3 yields

$$\sum_{q \nmid N} \widehat{\phi} \left(\frac{\log q}{\log R} \right)^{n_j} \frac{\log^{n_j} q}{q^{(n_j + m_j)/2} \log^{n_j} R} = O\left(\frac{1}{\log^2 R} \right) \tag{5.22}$$

For all the other j, Lemma 5.2 gives

$$\sum_{q \nmid N} T(q, n_j, m_j, \chi) = O(1). \tag{5.23}$$

Multiplying the order of magnitude over all j, we find that the main term does not contribute in this case, as in (5.18).

Next, we consider the case $m_j + n_j \leq 2$ for all j. Since m_j and n_j have the same parity, the set of possibilities for (m_j, n_j) is $\{(0, 2), (1, 1)\}$; specializing (2.10) for these possibilities gives $c_{n_j, m_j} = 1$. For each j, based on Lemma 5.3, if $n_j = 1$ it contributes $\phi(0)/2$ to the main term; if $n_j = 2$ it contributes $\sigma_\phi^2/4$ to the main term. If the number of j's with $n_j = 2$ is $0 \leq s \leq \lfloor t/2 \rfloor$, then the number of j's with $n_j = 1$ is $t - 2s$ because $\sum_j n_j = t$; this case occurs $\binom{t-s}{s}$ times. Tallying the main term contribution in all these cases, we get

$$\sum_{s=0}^{\lfloor t/2 \rfloor} \frac{t!}{2^s (t-s)!} \binom{t-s}{s} \left(\frac{\phi(0)}{2} \right)^{t-2s} \left(\frac{\sigma_\phi^2}{4} \right)^s. \tag{5.24}$$

Recalling (3.4), we conclude the main term of the weighted n^{th} centered moment of the average one-level density is given by the following combinatorial sum:

$$\mathfrak{C} := \sum_{t=0}^{n} \binom{n}{t} (-2)^t \sum_{s=0}^{\lfloor t/2 \rfloor} \frac{t!}{2^s (t-s)!} \binom{t-s}{s} \left(\frac{\phi(0)}{2} \right)^{t-2s} \left(\frac{\sigma_\phi^2}{4} \right)^s. \tag{5.25}$$

We unravel $\mathfrak{C}$ not with the generating series trick of Soshnikov [**Sos00**], which is characteristic in n-level calculations, but with an elementary, unrelated combinatorial trick of our own.

Regrouping the factors in each term of $\mathfrak{C}$ so that the respective index is most prominent in its summation,

$$\mathfrak{C} = \phi(0)^n \sum_{t=0}^{n} \binom{n}{t} (-1)^t \sum_{s=0}^{\lfloor t/2 \rfloor} \frac{t!}{(t-s)!} \binom{t-s}{s} \left(\frac{\sigma_\phi^2}{2\phi(0)^2} \right)^s. \tag{5.26}$$

Seeing the various parts of the typical term in the inner summation, particularly $t!$, $(t-2s)!$, and the cancelable $(t-s)!$, we expect to rewrite it in the form $\binom{t}{2s} (\cdot)^{2s}$ to

subsequently use the binomial theorem, the preferred tool for this type of problem:

$$
\begin{aligned}
\frac{t!}{(t-s)!}\binom{t-s}{s}\left(\frac{\sigma_\phi^2}{2\phi(0)^2}\right)^s &= \frac{t!}{(t-s)!}\frac{(t-s)!}{s!(t-2s)!}\left(\frac{\sigma_\phi^2}{2\phi(0)^2}\right)^s \\
&= \frac{t!}{s!(t-2s)!}\left(\frac{\sigma_\phi^2}{2\phi(0)^2}\right)^s \\
&= \binom{t}{2s}\frac{(2s)!}{s!}\left(\frac{\sigma_\phi^2}{2\phi(0)^2}\right)^s.
\end{aligned} \tag{5.27}
$$

Having obtained the desired binomial coefficient, we attend to the other factors in the term:

$$
\begin{aligned}
\frac{t!}{(t-s)!}\binom{t-s}{s}\left(\frac{\sigma_\phi^2}{2\phi(0)^2}\right)^s &= \binom{t}{2s}\frac{(2s)!}{s!}\left(\frac{\sigma_\phi^2}{2\phi(0)^2}\right)^s \\
&= \binom{t}{2s}\frac{(2s)!}{2^s s!}\left(\frac{\sigma_\phi^2}{\phi(0)^2}\right)^s \\
&= \binom{t}{2s}(2s-1)!!\left(\frac{\sigma_\phi^2}{\phi(0)^2}\right)^s.
\end{aligned} \tag{5.28}
$$

To write the term in the desired form, we let X be a Gaussian random variable with mean 0 and variance $\sigma_\phi^2/\phi(0)^2$. The mth centered moment of X is 0 if m is odd, and $(m-1)!!(\sigma_\phi^2/\phi(0)^2)^{m/2}$ if m is even. Therefore,

$$
\begin{aligned}
\frac{t!}{(t-s)!}\binom{t-s}{s}\left(\frac{\sigma_\phi^2}{2\phi(0)^2}\right)^s &= \binom{t}{2s}(2s-1)!!\left(\frac{\sigma_\phi^2}{\phi(0)^2}\right)^s \\
&= \binom{t}{2s}\mathbb{E}\left[X^{2s}\right].
\end{aligned} \tag{5.29}
$$

Having tailored the problem to the binomial theorem, we use it to full avail, twice:

$$
\begin{aligned}
\mathfrak{C} &= \phi(0)^n \sum_{t=0}^n \binom{n}{t}(-1)^t \sum_{s=0}^{\lfloor t/2\rfloor} \frac{t!}{(t-s)!}\binom{t-s}{s}\left(\frac{\sigma_\phi^2}{2\phi(0)^2}\right)^s \\
&= \phi(0)^n \sum_{t=0}^n \binom{n}{t}(-1)^t \sum_{s=0}^{\lfloor t/2\rfloor}\binom{t}{2s}\mathbb{E}\left[X^{2s}\right] \\
&= \phi(0)^n \sum_{t=0}^n \binom{n}{t}(-1)^t \sum_{r=0}^{t}\binom{t}{r}\mathbb{E}\left[X^r\right] \\
&= \phi(0)^n \sum_{t=0}^n \binom{n}{t}(-1)^t \mathbb{E}\left[\sum_{r=0}^{t}\binom{t}{r}1^{t-r}X^r\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \phi(0)^n \sum_{t=0}^n \binom{n}{t} (-1)^t \, \mathbb{E}\left[(1+X)^t\right] \\
&= \phi(0)^n \, \mathbb{E}\left[\sum_{t=0}^n \binom{n}{t} 1^{n-t}(-1-X)^t\right] \\
&= \phi(0)^n \, \mathbb{E}\left[(1-1-X)^n\right] \\
&= \phi(0)^n \, \mathbb{E}\left[(-X)^n\right] \\
&= \begin{cases} 0 & n \text{ odd,} \\ \phi(0)^n (n-1)!! \left(\frac{\sigma_\phi^2}{\phi(0)^2}\right)^{n/2} = (n-1)!! \, \sigma_\phi^n & n \text{ even,} \end{cases}
\end{aligned}
\tag{5.30}
$$

which is precisely the nth centered moment of a Gaussian distribution with variance σ_ϕ^2, completing the proof of Theorem 1.1 in this case too. $\qquad\square$

Appendix A. Switching a Sum and a Product

Switching the following sum and product is what allows us to carry out the combinatorial analysis in Section 5. We present this crucial, albeit cumbersome, step.

Claim A.1. *For all $1 \le j \le \ell$, let m_j and n_j be non-negative integers such that $m_j \le n_j$ and $m_j \equiv n_j \pmod{2}$. Then*

$$
\sum_{\substack{(q_1,\ldots,q_\ell) \\ distinct; \\ q_j \nmid N}} \prod_{j=1}^\ell T(q_j, n_j, m_j, \chi) \;=\; \prod_{j=1}^\ell \left(\sum_{q_j \nmid N} T(q_j, n_j, m_j, \chi)\right) + O\left(\frac{1}{\log^2 Q_{k,N}}\right).
$$

Proof. For clarity of the mainly combinatorial aspects of this proof, we adopt the notation in Section 4:

$$
T(q, n, m, \chi) \;:=\; \widehat{\phi}\left(\frac{\log q}{\log Q_{k,N}}\right)^n \frac{\chi(q)^m \log^n q}{q^{(n+m)/2} \log^n Q_{k,N}} \, \sigma_1\left(\gcd\left(r, q^m\right)\right).
\tag{A.1}
$$

Furthermore, we abbreviate "not necessarily distinct" as "n.n.d.". We would like to prove that

$$
\prod_{j=1}^\ell \left(\sum_{q_j \nmid N} T(q_j, n_j, m_j, \chi)\right) - \sum_{\substack{(q_1,\ldots,q_\ell) \\ distinct; \\ q_j \nmid N}} \prod_{j=1}^\ell T(q_j, n_j, m_j, \chi)
$$

$$
= \sum_{\substack{(q_1,\ldots,q_\ell) \\ \text{n.n.d.;} \\ q_j \nmid N}} \prod_{j=1}^{\ell} T(q_j, n_j, m_j, \chi) - \sum_{\substack{(q_1,\ldots,q_\ell) \\ \text{distinct;} \\ q_j \nmid N}} \prod_{j=1}^{\ell} T(q_j, n_j, m_j, \chi)
$$

$$
= \sum_{\substack{(q_1,\ldots,q_\ell) \\ \text{n.n.d.;} \\ q_j \nmid N; \\ \exists j_1 < j_2: \ q_{j_1} = q_{j_2}}} \left(T(q_{j_1}, n_{j_1}, m_{j_1}, \chi) \cdot T(q_{j_2}, n_{j_2}, m_{j_2}, \chi) \cdot \prod_{\substack{1 \le j \le \ell; \\ j \ne j_1, j_2}} T(q_j, n_j, m_j, \chi) \right)
\tag{A.2}
$$

has order at most $\log^{-2} Q_{k,N}$. By the principle of inclusion-exclusion, we know that

$$
\sum_{\substack{(q_1,\ldots,q_\ell) \\ \text{n.n.d} \\ q_j \nmid N; \\ \exists j_1 > j_2: \ q_{j_1} = q_{j_2}}} \left(T(q_{j_1}, n_{j_1}, m_{j_1}, \chi) \cdot T(q_{j_2}, n_{j_2}, m_{j_2}, \chi) \cdot \prod_{\substack{1 \le j \le \ell; \\ j \ne j_1, j_2}} T(q_j, n_j, m_j, \chi) \right)
$$

$$
\le \sum_{\substack{1 \le j_2 \le \ell; \\ 1 \le j_1 < j_2}} \sum_{\substack{(q_1,\ldots,q_{j_2-1}, \\ q_{j_2+1},\ldots,q_\ell) \\ \text{n.n.d.;} \\ q_j \nmid N}} \left(T(q_{j_1}, n_{j_1}, m_{j_1}, \chi) \cdot T(q_{j_1}, n_{j_2}, m_{j_2}, \chi) \cdot \prod_{\substack{1 \le j \le \ell, \\ j \ne j_1, j_2}} T(q_j, n_j, m_j, \chi) \right).
\tag{A.3}
$$

In fact, for $\ell \ge 3$, the above inequality is strict because we double-count several terms. For one, consider $q_1 = q_2 = q_3$ and $q_3, \ldots, q_\ell$ distinct; the corresponding term $\prod_j T(q_j, n_j, m_j, \chi)$ is counted three times in (A.3) when $(j_1, j_2) = (1, 2), (1, 3)$, and $(2, 3)$. There are many other examples of double-counted terms.

Since $q_1, \ldots, q_{j_2-1}, q_{j_2+1}, \ldots, q_\ell$ are n.n.d. in (A.3), we can freely swap the sum and product:

$$
\sum_{\substack{1 \le j_2 \le \ell; \\ 1 \le j_1 < j_2}} \sum_{\substack{(q_1,\ldots,q_{j_2-1}, \\ q_{j_2+1},\ldots,q_\ell) \\ \text{n.n.d.;} \\ q_j \nmid N}} \left(T(q_{j_1}, n_{j_1}, m_{j_1}, \chi) \cdot T(q_{j_1}, n_{j_2}, m_{j_2}) \cdot \prod_{\substack{1 \le j \le \ell, \\ j \ne j_1, j_2}} T(q_j, n_j, m_j, \chi) \right)
\tag{A.4}
$$

$$
= \sum_{\substack{1 \le j_2 \le \ell; \\ 1 \le j_1 < j_2}} \left[\left(\sum_{q_{j_1} \nmid N} T(q_{j_1}, n_{j_1}, m_{j_1}, \chi) \cdot T(q_{j_1}, n_{j_2}, m_{j_2}, \chi) \right) \cdot \prod_{\substack{1 \le j \le \ell \\ j \ne j_1, j_2}} \left(\sum_{q_j \nmid N} T(q_j, n_j, m_j, \chi) \right) \right].
$$

$$(A.5)$$

From Lemmas 5.2 and 5.3, we know that $\sum T(q_j, n_j, m_j, \chi) = O(1)$ for all $j \ne j_1, j_2$. Furthermore, noting that $m_{j_i} + n_{j_i} \ge 2$ for both $i = 1, 2$, we get that

$$
\sum_{q_{j_1} \nmid N} T(q_{j_1}, n_{j_1}, m_{j_1}, \chi) \cdot T(q_{j_1}, n_{j_2}, m_{j_2}, \chi) \ll_r \sum_{q_{j_1} \nmid N} \frac{\log^{n_{j_1} + n_{j_2}} q}{q^{(m_{j_1} + n_{j_1} + m_{j_2} + n_{j_2})/2} \log^{n_{j_1} + n_{j_2}} Q_{k,N}}
$$

$$
= O\left(\frac{1}{\log^2 Q_{k,N}} \right). \qquad (A.6)
$$

Absorbing the outer sum into the constant (as $\ell \le t \le n$) and multiplying the orders of magnitude of all these factors, we find that the total order of magnitude of (A.5) and hence is at most $\log^{-2} Q_{k,N}$, as claimed. $\qquad \square$

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