# LOW LYING ZEROS OF $L$-FUNCTIONS WITH ORTHOGONAL SYMMETRY

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Abstract. We investigate the moments of a smooth counting function of the zeros near the central point of $L$-functions of weight $k$ cuspidal newforms of prime level $N$. We split by the sign of the functional equations and show that for test functions whose Fourier transform is supported in $(-\frac{1}{n}, \frac{1}{n})$, as $N \to \infty$ the first $n$ centered moments are Gaussian. By extending the support to $(-\frac{1}{n-1}, \frac{1}{n-1})$, we see non-Gaussian behavior; in particular the odd centered moments are non-zero for such test functions. If we do not split by sign, we obtain Gaussian behavior for support in $(-\frac{2}{n}, \frac{2}{n})$ if $2k \geq n$. The $n^{\text{th}}$ centered moments agree with Random Matrix Theory in this extended range, providing additional support for the Katz-Sarnak conjectures. The proof requires calculating multidimensional integrals of the non-diagonal terms in the Bessel-Kloosterman expansion of the Petersson formula. We convert these multidimensional integrals to one-dimensional integrals already considered in the work of Iwaniec-Luo-Sarnak, and derive a new and more tractable expression for the $n^{\text{th}}$ centered moments for such test functions. This new formula facilitates comparisons between number theory and random matrix theory for test functions supported in $(-\frac{1}{n-1}, \frac{1}{n-1})$ by simplifying the combinatorial arguments. As an application we obtain bounds for the percentage of such cusp forms with a given order of vanishing at the central point.

## 1. Introduction

Let $H_k^\star(N)$ be the set of all holomorphic cusp forms of weight $k$ which are newforms of level $N$. Every $f \in H_k^\star(N)$ has a Fourier expansion

$$f(z) \ = \ \sum_{n=1}^{\infty} a_f(n) e(nz). \tag{1.1}$$

Set $\lambda_f(n) = a_f(n) n^{-(k-1)/2}$. The $L$-function associated to $f$ is

$$L(s,f) \ = \ \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}. \tag{1.2}$$

The completed $L$-function is

$$\Lambda(s,f) \ = \ \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(s,f), \tag{1.3}$$

and it satisfies the functional equation $\Lambda(s,f) = \epsilon_f \Lambda(1-s,f)$ with $\epsilon_f = \pm 1$. Therefore $H_k^\star(N)$ splits into two disjoint subsets, $H_k^+(N) = \{f \in H_k^\star(N) : \epsilon_f = +1\}$ and $H_k^-(N) = \{f \in H_k^\star(N) : \epsilon_f = -1\}$. Each $L$-function has a set of non-trivial zeros $\rho_f = \frac{1}{2} + \mathrm{i}\gamma_f$. The Generalized Riemann Hypothesis is the statement that all $\gamma_f \in \mathbb{R}$ for all $f$.

Assuming GRH, the zeros of any such $L$-function lie on the critical line, and therefore it is possible to investigate statistics of the normalized zeros (that is, the zeros which have been stretched out to be one apart on average). The general philosophy, born out in many examples (see for example [CFKRS, KeSn]), is that statistical behavior of families of $L$-functions can be modeled by ensembles of random matrices. The spacing statistics of high zeros of automorphic

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cuspidal $L$-functions (see [Mon, Hej, RS]), for certain test functions, agree with the corresponding statistics of eigenvalues of unitary matrices chosen with Haar measure (or equivalently, complex Hermitian matrices whose independent entries are chosen according to Gaussian distributions). Initially this led to the belief that this was the only matrix ensemble relevant to number theory; however Katz and Sarnak ([KS1, KS2]) prove that these statistics are the same for all classical compact groups. These statistics, the $n$-level correlations, are insensitive to finitely many zeros; thus, differences in behavior at the central point $s = \frac{1}{2}$ are missed by such investigations, and a new statistic, sensitive to behavior near the central point, is needed to distinguish families of $L$-functions. In many cases ([ILS, Ru, Ro, HR2, FI, Mil2, Yo, DM, Gü, Gao]) the behavior of the low lying zeros (zeros near the central point) of families of $L$-functions are shown to behave similarly to eigenvalues near 1 of classical compact groups (unitary, symplectic and orthogonal). The different groups exhibit different behavior near 1.

Let $\phi$ be an even Schwartz function such that its Fourier transform has compact support. We are interested in moments of the smooth counting function (also called the one-level density or linear statistic)

$$D(f;\phi) \;=\; \sum_{\gamma_f} \phi\left(\frac{\log R}{2\pi}\gamma_f\right) \tag{1.4}$$

when averaged over either $H_k^*(N)$ (the unsplit case), or $H_k^+(N)$ or $H_k^-(N)$ (the split cases) as $N \to \infty$ through the primes, with $k$ held fixed. Here $\gamma_f$ runs through the non-trivial zeros of $L(s,f)$, and $R$ is its analytic conductor ($R = k^2N$ for these families). We rescale the zeros by $\log R$ as this is the order of the number of zeros with imaginary part less than a large absolute constant. Because of the rapid decay of $\phi$, most of the contribution in (1.4) is from zeros near the central point. We use the uniform average over $f \in H_k^\sigma(N)$ (for $\sigma$ one of $*$, $+$ or $-$), in the sense that if $Q$ is a function defined on $f \in H_k^\sigma(N)$, then the average of $Q$ over $H_k^\sigma(N)$ is

$$\langle Q(f)\rangle_\sigma \;:=\; \frac{1}{|H_k^\sigma(N)|}\sum_{f\in H_k^\sigma(N)} Q(f). \tag{1.5}$$

We discuss in detail in Remarks 2.11 and 6.1 why we chose to uniformly weigh each $f \in H_k^\sigma(N)$ and not use harmonic averaging as in [Ro], though both approaches yield the same support.

Our first theorem evaluates the centered moments of $D(f,\phi)$ over $f \in H_k^*(N)$.

**Theorem 1.1.** *Assume GRH for $L(s,f)$. For $n \geq 1$ an integer, if* $\operatorname{supp}(\widehat{\phi}) \subset (-\frac{1}{n}\frac{2k-1}{k}, \frac{1}{n}\frac{2k-1}{k})$ *then the $n^{\text{th}}$ centered moment of $D(f;\phi)$, when averaged over the elements of $H_k^*(N)$, converges as $N \to \infty$ through prime values to the $n^{\text{th}}$ centered moment of the Gaussian distribution with mean*

$$\widehat{\phi}(0) + \frac{1}{2}\int_{-1}^{1}\widehat{\phi}(y)\,\mathrm{d}y \tag{1.6}$$

*and variance*

$$\sigma_\phi^2 \;=\; 2\int_{-1/2}^{1/2}|y|\widehat{\phi}(y)^2\,\mathrm{d}y. \tag{1.7}$$

**Remark 1.2.** We assume GRH for $L(s,f)$ to simplify the arguments below; however, we may remove this assumption by arguing as on page 88 of [ILS] (specifically, either use the Petersson formula to handle the $p^2$ terms in the explicit formula, or crude estimates for $L(s, \operatorname{sym}^2 f \otimes \operatorname{sym}^2 f)$).

Thus, with a little more work, Theorem 1.1 can be made unconditional. By assuming GRH for Dirichlet $L$-functions, in Theorem E.1 we increase the support to $(-\frac{2}{n}, \frac{2}{n})$, provided $2k \geq n$. The relation between $n$ and $k$ arises from some technicalities in controlling error terms; these obstructions are usually not apparent in studying just the $n = 1$ case.

If we split by sign, then the same argument still gives Gaussian moments, but with a greater restriction on the support of the test function $\widehat{\phi}$. Later we increase the support by invoking GRH for Dirichlet $L$-functions.

**Theorem 1.3.** *Under GRH for $L(s,f)$, if $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n}, \frac{1}{n})$ then the $n^{\text{th}}$ centered moment of $D(f;\phi)$, when averaged over the elements of either $H_k^+(N)$ or $H_k^-(N)$, converges as $N \to \infty$ through prime values to the $n^{\text{th}}$ centered moment of the Gaussian distribution with mean*

$$\widehat{\phi}(0) + \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(y)\, \mathrm{d}y \tag{1.8}$$

*and variance*

$$\sigma_\phi^2 \;=\; 2\int_{-1/2}^{1/2} |y|\widehat{\phi}(y)^2\, \mathrm{d}y. \tag{1.9}$$

Hughes and Rudnick [HR1] prove a similar result within random matrix theory. For a Schwartz function $\phi$ on the real line, define

$$F_M(\theta) \;:=\; \sum_{j=-\infty}^{\infty} \phi\left(\frac{M}{2\pi}(\theta + 2\pi j)\right), \tag{1.10}$$

which is $2\pi$-periodic and localized on a scale of $\frac{1}{M}$. For $U$ an $M \times M$ unitary matrix with eigenvalues $e^{\mathrm{i}\theta_n}$, set

$$Z_\phi(U) \;:=\; \sum_{n=1}^{M} F_M(\theta_n). \tag{1.11}$$

Note that going from $e^{\mathrm{i}\theta_n}$ to $\theta_n$ is well defined, since $F_M(\theta)$ is $2\pi$-periodic. We often consider $U$ to be a special orthogonal matrix when the eigenvalues occur in complex-conjugate pairs, and thus will be doubly counted. $Z_\phi(U)$ is the random matrix equivalent of $D(f;\phi)$. In [HR1] it was proved that

**Theorem 1.4.** *If $\text{supp}(\widehat{\phi}) \subseteq [-\frac{1}{n}, \frac{1}{n}]$ then the first $n$ moments of $Z_\phi(U)$ averaged with Haar measure over $\mathrm{SO}(M)$ (with $M$ either even or odd) converge to the moments of the Gaussian distribution with mean*

$$\widehat{\phi}(0) + \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(y)\, \mathrm{d}y \tag{1.12}$$

*and variance*

$$\sigma_\phi^2 \;=\; 2\int_{-1/2}^{1/2} |y|\widehat{\phi}(y)^2\, \mathrm{d}y. \tag{1.13}$$

*In particular, the odd moments vanish, and for $2m \le n$ the $2m^{\text{th}}$ moment is $(2m-1)!!\ \sigma_\phi^{2m}$.*

*If $\text{supp}\,\widehat{\phi} \subseteq [-\frac{2}{n}, \frac{2}{n}]$, then the $n^{\text{th}}$ moment of $Z_\phi(U)$ when averaged over the mean[1] of* SO(even) *and* SO(odd) *converges to the $n^{\text{th}}$ moment of a Gaussian random variable with mean and variance given by* (1.12) *and* (1.13) *respectively.*

Thus Theorems 1.1 (and E.1), 1.3 and 1.4 provide evidence for the connection between number theory and random matrix theory, specifically that the behavior of zeros near the central point is well modeled by that of eigenvalues near 1 of a classical compact group. It was remarked in [HR1] that the $n^{\text{th}}$ moment of $Z_\phi$ when averaged over either SO(even) or SO(odd) ceases to be Gaussian once the support of $\widehat{\phi}$ is greater than $[-\frac{1}{n}, \frac{1}{n}]$ (we make this remark precise in Theorem 1.7). Similarly we prove Theorem 1.3 is essentially sharp by showing the odd centered moments of $D(f;\phi)$ are non-zero if the support of $\widehat{\phi}$ is strictly greater than $[-\frac{1}{n}, \frac{1}{n}]$, and thus the distribution of $D(f;\phi)$ is non-Gaussian in that case. Furthermore, the $n^{\text{th}}$ centered moments when averaged over $H_k^+(N)$ or $H_k^-(N)$ are different as soon as the support of $\widehat{\phi}$ exceeds $[-\frac{1}{n}, \frac{1}{n}]$. This was anticipated in the work of Iwaniec, Luo and Sarnak [ILS], who proved the following theorem (Theorem 1.1 of [ILS]):

[1]By the mean of SO(even) and SO(odd) we mean the ensemble where half the matrices are SO(even) and the other half SO(odd).

**Theorem 1.5.** *If* $\mathrm{supp}(\widehat{\phi}) \subset (-2,2)$*, then the first moment agrees with random matrix theory; explicitly, under GRH for* $L(s,f)$ *and for Dirichlet L-functions,*

$$\lim_{N\to\infty} \langle D(f;\phi)\rangle_+ \ = \ \widehat{\phi}(0) + \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(y)\,\mathrm{d}y \tag{1.14}$$

$$\lim_{N\to\infty} \langle D(f;\phi)\rangle_- \ = \ \widehat{\phi}(0) + \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(y)\,\mathrm{d}y + \int_{|y|\geq 1} \widehat{\phi}(y)\,\mathrm{d}y. \tag{1.15}$$

*(Here* $N$ *tends to infinity through the square-free integers).*

While the assumption of GRH for Dirichlet $L$-functions is essential above (it is needed to analyze the Kloosterman sums), on page 88 of [ILS] they give two arguments which remove the assumption of GRH for $L(s,f)$.

Note that if $\mathrm{supp}(\widehat{\phi}) \subset (-1,1)$ then $\lim \langle D(f;\phi)\rangle_+ = \lim \langle D(f;\phi)\rangle_-$, but they are different if the support of $\widehat{\phi}$ is outside this interval. Thus in order to test the expected belief that averages over $H_k^+(N)$ correspond to averages over SO(even) and averages over $H_k^-(N)$ correspond to averages over SO(odd), it is essential that the calculations in [ILS] have support greater than 1, as for smaller support the 1-level densities of the orthogonal groups are indistinguishable. In this paper we further test this correspondence. Our main result is

**Theorem 1.6.** *Let* $n \geq 2$*,* $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$*,* $D(f;\phi)$ *be as in* (1.4)*, and define*

$$R_n(\phi) \ = \ (-1)^{n-1}2^{n-1}\left[\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x}\,\mathrm{d}x - \frac{1}{2}\phi(0)^n\right] \tag{1.16}$$

$$\sigma_\phi^2 \ = \ 2\int_{-1}^{1} |y|\widehat{\phi}(y)^2\,\mathrm{d}y. \tag{1.17}$$

*Assume GRH for* $L(s,f)$ *and for all Dirichlet L-functions. As* $N\to\infty$ *through the primes,*

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \left\langle \left(D(f;\phi) - \langle D(f;\phi)\rangle_\pm\right)^n \right\rangle_\pm = \begin{cases} (2m-1)!!\ \sigma_\phi^{2m} \pm R_{2m}(\phi) & \text{if } n=2m \text{ is even} \\ \pm R_{2m+1}(\phi) & \text{if } n = 2m+1 \text{ is odd.}\end{cases} \tag{1.18}$$

Note if $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n}, \frac{1}{n})$ then $R_n(\phi) = 0$ and we recover the Gaussian behavior of Theorem 1.3. Also $R_n(\phi)$ is not identically zero for test functions $\phi$ such that $\mathrm{supp}(\widehat{\phi}) \not\subset [-\frac{1}{n}, \frac{1}{n}]$. The assumption of GRH for $L(s,f)$ is for ease of exposition, and can be removed; see Remark 1.2.

Finally we show that the random matrix moments of $Z_\phi$ correctly model the moments of $D(f;\phi)$ (at least for the support of $\widehat{\phi}$ restricted as in Theorem 1.6), in the sense that the $n^{\mathrm{th}}$ centered moment of $D(f;\phi)$ averaged over $H_k^+(N)$ equals the $n^{\mathrm{th}}$ centered moment of $Z_\phi$ averaged over SO(even), and $H_k^-(N)$ similarly corresponds to SO(odd).

**Theorem 1.7.** *The means of* $Z_\phi(U)$ *when averaged with respect to Haar measure over* SO(even) *or* SO(odd) *are*

$$\mu_+ := \lim_{\substack{M\to\infty \\ M\text{even}}} \mathbb{E}_{\mathrm{SO}(M)}\left[Z_\phi(u)\right] \ = \ \widehat{\phi}(0) + \frac{1}{2}\int_{-\infty}^{\infty} \widehat{\phi}(y)\,\mathrm{d}y \tag{1.19}$$

$$\mu_- := \lim_{\substack{M\to\infty \\ M\text{odd}}} \mathbb{E}_{\mathrm{SO}(M)}\left[Z_\phi(u)\right] \ = \ \widehat{\phi}(0) + \frac{1}{2}\int_{-\infty}^{\infty} \widehat{\phi}(y)\,\mathrm{d}y + \int_{|y|\geq 1} \widehat{\phi}(y)\,\mathrm{d}y. \tag{1.20}$$

*Let* $R_n(\phi)$ *and* $\sigma_\phi^2$ *be as in* (1.16) *and* (1.17)*. For* $n \geq 2$ *if* $\mathrm{supp}(\widehat{\phi}) \subseteq [-\frac{1}{n-1}, \frac{1}{n-1}]$ *then the* $n^{\mathrm{th}}$ *centered moment of* $Z_\phi(U)$ *converges to*

$$\lim_{\substack{M\to\infty \\ M\text{even}}} \mathbb{E}_{\mathrm{SO}(M)}\left[(Z_\phi(U) - \mu_+)^n\right] = \begin{cases} (2m-1)!!\ \sigma_\phi^{2m} + R_{2m}(\phi) & \text{if } n = 2m \text{ is even} \\ R_{2m+1}(\phi) & \text{if } n = 2m+1 \text{ is odd}\end{cases} \tag{1.21}$$

*and*

$$\lim_{\substack{M\to\infty \\ M \text{odd}}} \mathbb{E}_{\mathrm{SO}(M)}\left[(Z_\phi(U)-\mu_-)^n\right] = \begin{cases} (2m-1)!!\ \sigma_\phi^{2m} - R_{2m}(\phi) & \text{if } n=2m \text{ is even} \\ -R_{2m+1}(\phi) & \text{if } n = 2m+1 \text{ is odd.} \end{cases} \tag{1.22}$$

It is conjectured that the $n^{\text{th}}$ centered moments from number theory agree with random matrix theory for any Schwartz test function; our results above may be interpreted as providing additional evidence.

Our goal is to reduce as many calculations as possible to ones already done in the seminal work of [ILS], where their delicate analysis of the Kloosterman and Bessel terms in the Petersson formula allowed them to go well beyond the diagonal. We quickly review notation and state some needed estimates. We then calculate the relevant number theory quantities, concentrating on the new terms that did not arise in [ILS]. Using properties of the Fourier and Mellin transforms and convolutions, we reduce our multidimensional integrals of Kloosterman-Bessel terms to one-dimensional integrals considered in [ILS].

**Remark 1.8.** Random matrix theory provides exact formulas for the moments for test functions of any support, derivable from the $n$-level densities (in particular, the determinant expansions of these); see [KS1, KS2] for details. However, *a priori* it is not obvious that these results agree with those obtained in number theory for test functions as restricted in our theorems. For example, much of the analysis in Rubinstein [Ru] and Gao [Gao] of the $n$-level densities of quadratic Dirichlet $L$-functions is devoted to analyzing the resulting combinatorial expressions to show agreement with random matrix theory; the centered moments are combinatorially much easier to analyze.

To simplify showing agreement between number theory and random matrix theory we further develop the combinatorics used in the work of Hughes-Rudnick [HR1] and Soshnikov [Sosh], and, by desymmetrizing certain integrals which arise, derive some needed Fourier transform identities. Doing so allows us to handle support in $[-\frac{1}{n-1}, \frac{1}{n-1}]$ on the random matrix side. While this makes our results more restrictive than the exact determinant expansions of Katz-Sarnak, these new formulas are significantly more convenient for comparisons with number theory, involving simple one-dimensional integrals of convolutions of our test function rather than sums of determinants. This allows us to avoid the combinatorial analysis of the number theory terms in [Ru, Gao]. In the course of proving the agreement between number theory and random matrix theory, we derive (1.18), a new and, for test functions with $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$, significantly more tractable expansion for the $n^{\text{th}}$ centered moments than the determinant expansions.

In §5 we see that the first natural boundary in analyzing the $n^{\text{th}}$ centered moment for SO(even) and SO(odd) in random matrix theory is for test functions supported in $[-\frac{1}{n}, \frac{1}{n}]$; the next natural boundary (where new terms arise) occurs for test functions supported in $[-\frac{1}{n-1}, \frac{1}{n-1}]$. It is essential that we are able to perform the number theory analysis for test functions whose support exceeds $[-\frac{1}{n}, \frac{1}{n}]$. While it is desirable to obtain as large support as possible, by breaking $[-\frac{1}{n}, \frac{1}{n}]$ in Theorem 1.6 we see the new terms arise in the number theory expansions as well, and agree perfectly with random matrix theory.

Instead of investigating centered moments we could study the $n$-level densities. Assuming GRH, the imaginary parts of the zeros of an $L$-function associated to a modular form $f \in H_k^+(N)$ can be written as $\gamma_f^{(j)}$ where $0 \le \gamma_f^{(1)} \le \gamma_f^{(2)} \le \dots$, and $\gamma_f^{(-j)} = -\gamma_f^{(j)}$. If $f \in H_k^-(N)$ there is an additional zero $\gamma_f^{(0)} = 0$ (note there is no forced zero at the central point for $f \in H_k^+(N)$). The symmetrized $n$-level density is

$$\frac{1}{|H_k^\pm(N)|} \sum_{f\in H_k^\pm(N)} \sum_{\substack{j_1,\dots,j_n \\ j_i \ne \pm j_k}} \phi_1\left(\frac{\log R}{2\pi}\gamma_f^{(j_1)}\right)\cdots\phi_n\left(\frac{\log R}{2\pi}\gamma_f^{(j_n)}\right), \tag{1.23}$$

where the $\phi_i$ are even Schwartz functions whose Fourier transforms have compact support. Since our families are of constant sign, we understand the number of zeros at the central point (unlike, say, for generic one-parameter families of $L$-functions of elliptic curves). While our arguments immediately generalize to the case when the $\phi_i$ are not all equal, we chose to study the $n^{\text{th}}$ centered moments to facilitate comparison with random matrix theory in the range where the Bessel-Kloosterman terms contribute.

Another application of centered moments is in estimating the order of vanishing of $L$-functions at the central point. Miller [Mil1] noticed that as $n$ increases, the $n$-level densities provide better and better estimates for bounding the order of vanishing at the central point; unfortunately, as $n$ increases the bounds are better only for *high* (growing with $n$) vanishing at the central point. We obtain similar bounds from the $n^{\text{th}}$ centered moments. Explicitly, from Theorem 1.6 we immediately obtain

**Corollary 1.9.** *Consider the families of weight $k$ cuspidal newforms split by sign, $H_k^{\pm}(N)$. Assume GRH for all Dirichlet $L$-functions and all $L(s,f)$. For each $n$ there are constants $r_n$ and $c_n$ such that as $N \to \infty$ through the primes, for $r \geq r_n$ the probability of at least $r$ zeros at the central point is at most $c_n r^{-n}$; equivalently, the probability of fewer than $r$ zeros at the central point is at least $1 - c_n r^{-n}$.*

The paper is organized as follows. In §2 we review the needed number theory results, write down the expansions for the centered moment sums, and collect many of the estimates that we need later. We then prove our number theory results, Theorems 1.1 (the unsplit case) and 1.3 (the split case with restricted support) in §3, and Theorem 1.6 (the unsplit case where we go beyond the diagonal by analyzing the Bessel-Kloosterman terms in the Petersson expansion) in §4; we show these agree with random matrix theory (Theorem 1.7) in §5. In §6 we prove Corollary 1.9, and show that it provides better bounds than [ILS] for the percentage of odd cuspidal newforms with at least 5 zeros at the central point.

## 2. Number Theory Preliminaries

### 2.1. Notation.

**Definition 2.1** (Gauss Sums)**.** *For $\chi$ a character modulo $q$ and $e(x) = e^{2\pi i x}$,*

$$G_\chi(n) \;=\; \sum_{a \bmod q} \chi(a) e(an/q), \tag{2.1}$$

*and $|G_\chi(n)| \leq \sqrt{q}$.*

**Definition 2.2** (Ramanujan Sums)**.** *If $\chi = \chi_0$ (the principal character modulo $q$) in* (2.1)*, then $G_{\chi_0}(n)$ becomes the Ramanujan sum*

$$R(n,q) \;=\; {\sum_{a \bmod q}}^{*} e(an/q) \;=\; \sum_{d|(n,q)} \mu(q/d) d, \tag{2.2}$$

*where $*$ restricts the summation to be over all $a$ relatively prime to $q$.*

**Definition 2.3** (Kloosterman Sums)**.** *For integers $m$ and $n$,*

$$S(m,n;q) \;=\; {\sum_{d \bmod q}}^{*} e\left(\frac{md}{q} + \frac{n\overline{d}}{q}\right), \tag{2.3}$$

*where $d\overline{d} \equiv 1 \bmod q$. We have*

$$|S(m,n;q)| \;\leq\; (m,n,q)\sqrt{\min\left\{\frac{q}{(m,q)}, \frac{q}{(n,q)}\right\}}\;\tau(q), \tag{2.4}$$

*where $\tau(q)$ is the number of divisors of $q$; see Equation 2.13 of* [ILS].

**Definition 2.4** (Fourier Transform)**.** *We use the following normalization:*

$$\widehat{\phi}(y) \;=\; \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i xy}\, dx, \qquad \phi(x) \;=\; \int_{-\infty}^{\infty} \widehat{\phi}(y) e^{2\pi i xy}\, dy. \tag{2.5}$$

**Definition 2.5** (Characteristic Function)**.** *For $A \subset \mathbb{R}$, let*

$$\mathbb{1}_{\{x\in A\}} \;=\; \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

The Bessel function of the first kind occurs frequently in this paper, and so we collect here some standard bounds for it (see, for example, [GR, Wat]).

**Lemma 2.6.** *Let $k \geq 2$ be an integer. The Bessel function satisfies*

(1) $J_{k-1}(x) \ll 1$.
(2) $J_{k-1}(x) \ll x$.
(3) $J_{k-1}(x) \ll x^{k-1}$.
(4) $J_{k-1}(x) \ll x^{-\frac{1}{2}}$.
(5) $2J'_\nu(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$.

2.2. **Fourier coefficients.** Let $k$ and $N$ be positive integers with $k$ even and $N$ prime. We denote by $S_k(N)$ the space of all cusp forms of weight $k$ for the Hecke congruence subgroup $\Gamma_0(N)$ of level $N$. That is, $f$ belongs to $S_k(N)$ if and only if $f$ is holomorphic in the upper half-plane, satisfies

$$f\left(\frac{az+b}{cz+d}\right) \;=\; (cz+d)^k f(z) \tag{2.7}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) := \left\{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \;:\; \gamma \equiv 0 \bmod N\right\}$, and vanishes at each cusp of $\Gamma_0(N)$. See [I2] for more details about cusp forms.

Let $f \in S_k(N)$ be a cuspidal newform of weight $k$ and level $N$; in our case this means $f$ is a cusp form of level $N$ but not of level 1. It has a Fourier expansion

$$f(z) \;=\; \sum_{n=1}^{\infty} a_f(n) e(nz), \tag{2.8}$$

with $f$ normalized so that $a_f(1) = 1$. We normalize the coefficients by defining

$$\lambda_f(n) \;=\; a_f(n) n^{-(k-1)/2}. \tag{2.9}$$

$H_k^*(N)$ is the set of all $f \in S_k(N)$ which are newforms. We split this set into two subsets, $H_k^+(N)$ and $H_k^-(N)$, depending on whether the sign of the functional equation of the associated $L$-function (see §1 for details) is plus or minus. From Equation 2.73 of [ILS] we have for $N > 1$ that

$$|H_k^\pm(N)| \;=\; \frac{k-1}{24}N + O\left((kN)^{\frac{5}{6}}\right). \tag{2.10}$$

For simplicity we shall deal only with the case $N$ prime, a fact which we will occasionally remind the reader of (though, as in [ILS], similar arguments work for $N$ square-free). For a newform of level $N$, $\lambda_f(N)$ is related to the sign of the form ([ILS], Equation 3.5):

**Lemma 2.7.** *If $f \in H_k^*(N)$ and $N$ is prime, then*

$$\epsilon_f \;=\; -\mathrm{i}^k \lambda_f(N)\sqrt{N}. \tag{2.11}$$

As $\epsilon_f = \pm 1$, (2.11) implies $|\lambda_f(N)| = \frac{1}{\sqrt{N}}$. Essential in our investigations will be the multiplicative properties of the Fourier coefficients.

**Lemma 2.8.** *Let $f \in H_k^*(N)$. Then*

$$\lambda_f(m)\lambda_f(n) \;=\; \sum_{\substack{d|(m,n)\\(d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right). \tag{2.12}$$

*In particular, if $(m,n) = 1$ then*

$$\lambda_f(m)\lambda_f(n) \;=\; \lambda_f(mn), \tag{2.13}$$

*and if $p$ is a prime not dividing the level $N$, then*

$$\begin{aligned}\lambda_f(p)^{2m} &= \sum_{r=0}^{m}\left(\binom{2m}{m-r}-\binom{2m}{m-r-1}\right)\lambda_f(p^{2r})\\ \lambda_f(p)^{2m+1} &= \sum_{r=0}^{m}\left(\binom{2m+1}{m-r}-\binom{2m+1}{m-r-1}\right)\lambda_f(p^{2r+1}). \qquad (2.14)\end{aligned}$$

We discovered the coefficients for the expansion of $\lambda_f(p)^n$ from [Guy]. Note for a prime $p \nmid N$,

$$\lambda_f(p)^2 \;=\; \lambda_f(p^2)+1, \qquad (2.15)$$

and $\lambda_f(p^{2m})$ is a sum of $\lambda_f(p^{2r})$ (i.e., only even powers) while $\lambda_f(p^{2m+1})$ is a sum of $\lambda_f(p^{2r+1})$ (i.e., only odd powers). Consider

$$\Delta^{\sigma}_{k,N}(n) \;=\; \sum_{f\in H_k^{\sigma}(N)} \lambda_f(n), \quad \sigma\in\{+,-,*\}. \qquad (2.16)$$

Note we are *not* dividing by the cardinality of the family, which is of order $N$. Splitting by sign and using Lemma 2.7 we have that if $N$ is prime and $(N,n)=1$,

$$\begin{aligned}\Delta^{\pm}_{k,N}(n) &= \sum_{f\in H_k^*(N)} \frac{1}{2}(1\pm\epsilon_f)\lambda_f(n)\\ &= \frac{1}{2}\Delta^*_{k,N}(n) \;\mp\; \frac{\mathrm{i}^k\sqrt{N}}{2}\Delta^*_{k,N}(nN). \qquad (2.17)\end{aligned}$$

Thus, to execute sums over $f\in H_k^{\pm}(N)$, it suffices to understand sums over all $f\in H_k^*(N)$. Propositions 2.1, 2.11 and 2.15 of [ILS] yield a useful form of the Petersson formula:

**Lemma 2.9** ([ILS]). *Let $X,Y$ be parameters to be determined later subject to $X<N$. If $N$ is prime and $(n,N^2)|N$ then*

$$\Delta^*_{k,N}(n) \;=\; \Delta'_{k,N}(n)+\Delta^{\infty}_{k,N}(n), \qquad (2.18)$$

*where*

$$\begin{aligned}\Delta'_{k,N}(n) &= \frac{(k-1)N}{12\sqrt{n}}\,\delta_{n,\square_Y}\\ &\quad+\frac{(k-1)N}{12}\sum_{\substack{(m,N)=1\\ m\le Y}}\frac{2\pi\mathrm{i}^k}{m}\sum_{\substack{c\equiv 0 \bmod N\\ c\ge N}}\frac{S(m^2,n;c)}{c}J_{k-1}\left(4\pi\frac{\sqrt{m^2n}}{c}\right), \qquad (2.19)\end{aligned}$$

*where $\delta_{n,\square_Y}=1$ only if $n=m^2$ with $m\le Y$ and 0 otherwise. The remaining piece, $\Delta^{\infty}_{k,N}(n)$, is called the* complementary sum.

*If $(a_q)$ is a sequence satisfying*

$$\sum_{\substack{(q,nN)=1\\ q<Q}}\lambda_f(q)a_q \;\ll\; (nkN)^{\epsilon'}, \quad \log Q \;\ll\; \log kN \qquad (2.20)$$

*for every[2] $f\in H_k^*(1)\cup H_k^*(N)$, the implied constant depending on $\epsilon'$ only, if $(n,N^2)|N$, then by Lemma 2.12 of* [ILS]

$$\sum_{\substack{(q,nN)=1\\ q<Q}}\Delta^{\infty}_{k,N}(nq)a_q \;\ll\; \frac{kN}{\sqrt{(n,N)}}\left(\frac{1}{X}+\frac{1}{\sqrt{Y}}\right)(nkNXY)^{\epsilon'}. \qquad (2.21)$$

In the applications we will take $X$ to be either $N-1$ or $N^{\epsilon}$ and $Y=N^{\epsilon}$, where $\epsilon,\epsilon'$ are chosen so that the right hand side of (2.21) is $O(N^{1-\epsilon''})$ for some $\epsilon''>0$ if $n\nmid N$, and is $O(N^{-\epsilon''})$ if $n|N$. In Lemma A.1 we show that the complementary sum does not contribute for all cases that arise in this paper. We write $c=bN$ for $c\equiv 0 \bmod N$.

---

[2]We need $f\in H_k^*(1)$ (as well as $f\in H_k^*(N)$, as these $f$ arise in the combinatorics in expanding the $\Delta^*_{k,N}$.

Using the estimate on Kloosterman sums, (2.4), the bounds on the Bessel function $J_{k-1}(x) \ll x$ and $J_{k-1}(x) \ll x^{k-1}$ from Lemma 2.6, and (2.10), we can trivially estimate $\frac{1}{|H_k^*(N)|}\Delta'_{k,N}(n)$. We obtain the following lemma:

**Lemma 2.10.** *Assume* $(n, N) = 1$. *Then*

$$\frac{1}{|H_k^*(N)|}\ \Delta'_{k,N}(n) \ =\ \frac{1}{\sqrt{n}}\ \delta_{n,\square_Y} + O\left(n^{(k-1)/2}N^{-k+1/2+\epsilon}\right), \tag{2.22}$$

*and*

$$\frac{1}{|H_k^*(N)|}\ \Delta'_{k,N}(Nn)\ \ll\ \sqrt{n}N^{-\frac{3}{2}+\epsilon}. \tag{2.23}$$

**Remark 2.11.** We chose to uniformly average over $f \in H_k^*(N)$ in (2.16). We obtain similar results if instead we use harmonic averaging as in [Ro] or Theorems 1.9 and 1.10 of [ILS], specifically

$$\langle Q(f)\rangle_{\pm,\text{harmonic}}\ =\ \sum_{f\in H_k^{\pm}(N)} \frac{\Gamma(k-1)}{(4\pi)^{k-1}(f,f)_N}\ Q(f), \tag{2.24}$$

where $(f, f)_N$ is the Petersson inner product on cusp forms of weight $k$ and level $N$. The advantage of harmonic averaging is that it facilitates the analysis of the $p^2$ terms in the explicit formula; specifically, we would not need to assume GRH for $L(s, f)$. We have chosen to use uniform averages for several reasons. The first is that, as in Theorem 1.1 of [ILS], the assumption of GRH for $L(s, f)$ can be removed relatively easily by appealing to either the Petersson formula or properties of $L(s, \mathrm{sym}^2 f \otimes \mathrm{sym}^2 f)$. The second is that much effort was spent in [ILS] in removing these arithmetic weights (see their comment on page 66), and removing the weights is essential to bound the order of vanishing at the central point (see Corollary 1.9 and Remark 6.1). Finally, when we uniformly average, our transformation of the multidimensional integrals lead to one-dimensional integrals that are directly comparable to the uniformly averaged cases in [ILS].

The one-dimensional integral referred to above is:

**Lemma 2.12.** *Let* $\Psi$ *be an even Schwartz function with* $\mathrm{supp}(\widehat{\Psi}) \subset (-2, 2)$. *Then*

$$\sum_{m\le N^\epsilon}\frac{1}{m^2}\sum_{(b,N)=1}\frac{R(m^2,b)R(1,b)}{\varphi(b)}\int_{y=0}^{\infty}J_{k-1}(y)\widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}y}{\log R}$$
$$=\ -\frac{1}{2}\left[\int_{-\infty}^{\infty}\Psi(x)\frac{\sin 2\pi x}{2\pi x}\mathrm{d}x - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right), \tag{2.25}$$

*where* $R(n, c)$ *is given by* (2.2), $R = k^2N$ *and* $\varphi$ *is Euler's totient function.*

This follows from Equations 7.5 and 7.6 of [ILS]. The explicit formula converts sums over zeros to sums over primes. Later we convert these prime sums to integrals, and then the above lemma allows us to evaluate the final expressions.

2.3. **Density and Moment Sums.** Let $f \in H_k^*(N)$, and let $\Lambda(s, f)$ be its associated completed $L$-function, (1.3). The Generalized Riemann Hypothesis states that all the zeros of $\Lambda(s, f)$ (i.e., the non-trivial zeros of $L(s, f)$) are of the form $\rho_f = \frac{1}{2} + \mathrm{i}\gamma_f$ with $\gamma_f \in \mathbb{R}$. The analytic conductor of $\Lambda(s, f)$ is $R = k^2N$, and its smooth counting function (also called the 1-level density) is

$$D(f;\phi)\ \ =\ \ \sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right), \tag{2.26}$$

where $\phi$ an even Schwartz function whose Fourier transform has compact support and the sum is over all the zeros of $\Lambda(s, f)$. Because $\phi$ decays rapidly, the main contribution to (2.26) is from zeros near the central point. The explicit formula applied to $D(f; \phi)$ gives (see Equation 4.25 of [ILS])

$$D(f;\phi)\ =\ \widehat{\phi}(0) + \frac{1}{2}\phi(0) - P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right), \tag{2.27}$$

where

$$P(f;\phi) \ = \ \sum_{p\nmid N} \lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{2.28}$$

While the derivation of (2.27) in [ILS] uses GRH for $L(s,\mathrm{sym}^2 f)$, as they remark this formula can be established on average over $f$ by an analysis of the Petersson formula or from properties of $L(s,\mathrm{sym}^2 f\otimes\mathrm{sym}^2 f)$ (see page 88 of [ILS]). For ease of exposition we shall assume GRH for $L(s,f)$ below. We trivially absorbed the $p=N$ term into the error. If $\mathrm{supp}(\widehat{\phi})\subset(-1,1)$, [ILS] show the $P(f,\phi)$ term does not contribute, and hence $\lim_{N\to\infty}\langle D(f;\phi)\rangle_\sigma = \widehat{\phi}(0)+\frac12\phi(0)$ for any $\sigma\in\{+,-,*\}$. Thus, to study the centered moments, we must evaluate

$$\begin{aligned}\langle (D(f;\phi)-\langle D(f;\phi)\rangle_\sigma)^n\rangle_\sigma \ &= \ \left\langle\left(-P(f;\phi)+\ O\left(\frac{\log\log R}{\log R}\right)\right)^n\right\rangle_\sigma \\ &= \ (-1)^n\langle P(f;\phi)^n\rangle_\sigma \ + \ O\left(\frac{\log\log R}{\log R}\right).\end{aligned} \tag{2.29}$$

The last line follows from Hölder's inequality and the fact that $\langle P(f;\phi)^n\rangle_\sigma \ll 1$ (which follows from (2.27) and that $\langle |D(f;\phi)|\rangle_\sigma \ll 1$). By using Hölder's inequality, we can prove (2.29) without having to construct a positive majorizing test function with suitable support, as is often done (see, for example, [RS, Ru]). See Appendix B for details. We split by sign and use Lemma 2.7 to obtain

$$\begin{aligned}\sum_{f\in H_k^\pm(N)} P(f;\phi)^n \ &= \ \sum_{f\in H_k^*(N)}\frac{1\pm\epsilon_f}{2}P(f;\phi)^n \\ &= \ \frac12\sum_{f\in H_k^*(N)}P(f;\phi)^n \ \mp\ \frac12\sum_{f\in H_k^*(N)}\mathrm{i}^k\sqrt{N}\lambda_f(N)P(f;\phi)^n.\end{aligned} \tag{2.30}$$

Since $|H_k^+(N)|\sim|H_k^-(N)|\sim\frac12|H_k^*(N)|$ as $N\to\infty$ by (2.10), we have

$$\langle P(f;\phi)^n\rangle_\pm \ \sim \ \langle P(f;\phi)^n\rangle_* \ \mp\ \mathrm{i}^k\sqrt{N}\langle\lambda_f(N)P(f;\phi)^n\rangle_*. \tag{2.31}$$

In conclusion, if $\mathrm{supp}(\widehat{\phi})\subset(-1,1)$, we have

$$\lim_{N\to\infty}\langle (D(f;\phi)-\langle D(f;\phi)\rangle_*)^n\rangle_* \ = \ (-1)^n\lim_{N\to\infty}S_1^{(n)} \tag{2.32}$$

and

$$\lim_{N\to\infty}\left\langle\left(D(f;\phi)-\langle D(f;\phi)\rangle_\pm\right)^n\right\rangle_\pm \ = \ (-1)^n\lim_{N\to\infty}S_1^{(n)} \ \pm\ (-1)^{n+1}\lim_{N\to\infty}S_2^{(n)} \tag{2.33}$$

(assuming all limits exist), where

$$S_1^{(n)} \ := \ \sum_{p_1\nmid N,\dots,p_n\nmid N}\prod_{j=1}^n\left(\widehat{\phi}\left(\frac{\log p_j}{\log R}\right)\frac{2\log p_j}{\sqrt{p_j}\log R}\right)\left\langle\prod_{j=1}^n\lambda_f(p_i)\right\rangle_* \tag{2.34}$$

and

$$S_2^{(n)} \ := \ \mathrm{i}^k\sqrt{N}\sum_{p_1\nmid N,\dots,p_n\nmid N}\prod_{j=1}^n\left(\widehat{\phi}\left(\frac{\log p_j}{\log R}\right)\frac{2\log p_j}{\sqrt{p_j}\log R}\right)\left\langle\lambda_f(N)\prod_{j=1}^n\lambda_f(p_i)\right\rangle_*. \tag{2.35}$$

## 3. Mock-Gaussian behavior: Proof of Theorems 1.1 and 1.3

In this section we prove Theorems 1.1 and 1.3, which states that for test functions with suitable support, the centered moments of $D(f;\phi)$ are Gaussian. By (2.33) we must therefore study the limits of $S_1^{(n)}$ and $S_2^{(n)}$ as $N\to\infty$ through the primes, with $\mathrm{supp}(\widehat{\phi})\subseteq(-\frac1n,\frac1n)$.

Because there is no $S_2^{(n)}$ term when we do not split by sign, Theorem 1.1 is equivalent to the following lemma, which we now prove.

**Lemma 3.1.** *Let $S_1^{(n)}$ be defined as in (2.34), and assume GRH for $L(s,f)$. Then if* $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n}\frac{2k-1}{k}, \frac{1}{n}\frac{2k-1}{k})$,

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} S_1^{(n)} \;=\; \begin{cases} (2m-1)!!\ \sigma_\phi^{2m} & \text{if } n = 2m \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{3.1}$$

*where*

$$\sigma_\phi^2 \;=\; 2\int_{-\infty}^{\infty} |y|\widehat{\phi}(y)^2\, \mathrm{d}y. \tag{3.2}$$

*Proof.* We split the sum over primes into sums over powers of distinct primes. Let $p_1\cdots p_n = q_1^{n_1}\cdots q_\ell^{n_\ell}$ with the $q_j$ distinct, so

$$\prod_{j=1}^{n} \lambda_f(p_i) \;=\; \prod_{j=1}^{\ell} \lambda_f(q_j)^{n_j}. \tag{3.3}$$

By the multiplicativity of $\lambda_f$ (Lemma 2.8), $\lambda_f(q_j)^{n_j}$ can be written as a sum of $\lambda_f(q_j^{m_j})$ where the $m_j$ are non-negative integers less than or equals to $n_j$ with $m_j \equiv n_j \bmod 2$.

The only way for $\prod_{i=1}^n \lambda_f(p_i)$ to have a constant term (i.e., $\lambda_f(1)$) is for $p_1\cdots p_n$ to equal a perfect square; this will be the main term. This can only happen when $n = 2m$ is an even integer. In this case each prime occurs an even number of times, and the primes can be paired. Assume first that each $n_j = 2$ so that each prime occurs exactly twice. The number of ways to pair $2m$ elements in pairs is $\frac{1}{m!}\binom{2m}{2}\binom{2m-2}{2}\cdots\binom{2}{2} = \frac{(2m)!}{2^m m!} = (2m-1)!!$; note these are the even moments of the standard Gaussian. Using the Prime Number Theorem to evaluate the prime sums, and the fact that $\widehat{\phi}$ is even, we see that the contribution from these terms is

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} (2m-1)!! \left( \sum_{p\nmid N} \widehat{\phi}\left(\frac{\log p}{\log R}\right)^2 \left(\frac{2\log p}{\sqrt{p}\log R}\right)^2 \right)^m \;=\; (2m-1)!! \left(2\int_{-\infty}^{\infty} \widehat{\phi}(y)^2 |y|\, \mathrm{d}y\right)^m; \tag{3.4}$$

note the integral is the variance $\sigma_\phi^2$ because of the support condition on $\widehat{\phi}$. The other possibility is that some $n_j \geq 4$. In this case we obtain a formula similar to (3.4), the only changes being a different combinatorial factor than $(2m-1)!!$ outside, and we have sums such as

$$\sum_{p\nmid N} \widehat{\phi}\left(\frac{\log p}{\log R}\right)^{n_j} \left(\frac{2\log p}{\sqrt{p}\log R}\right)^{n_j}. \tag{3.5}$$

If $n_j = 2$ then (3.5) is $O(1)$ by the Prime Number Theorem; however, (3.5) is $O\left(\log^{-4} R\right)$ if $n_j \geq 4$. Thus the contribution from the terms where at least one $n_j \geq 4$ is negligible.

The other contributions from expanding $\prod_{i=1}^n \lambda_f(p_i)$ are of the form

$$\sum_{\substack{q_1\nmid N,\dots,q_\ell\nmid N \\ q_j \text{distinct}}} \prod_{j=1}^{\ell} \widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j} \left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j} \langle \lambda_f(q_1^{m_1}\cdots q_\ell^{m_\ell})\rangle_* \tag{3.6}$$

with $\ell \geq 1$ (i.e., there is at least one prime) and $m_j \geq 1$ for at least one $j$ (i.e., this is not a constant term). We show in the limit as $N\to\infty$ that these terms do not contribute. By (1.5) and (2.18),

$$\langle \lambda_f(q_1^{m_1}\cdots q_\ell^{m_\ell})\rangle_* \;=\; \frac{1}{|H_k^*(N)|}\left(\Delta'_{k,N}(q_1^{m_1}\cdots q_\ell^{m_\ell}) + \Delta^\infty_{k,N}(q_1^{m_1}\cdots q_\ell^{m_\ell})\right). \tag{3.7}$$

Let $X = N-1$ and $Y = N^\epsilon$. By Lemma A.1, which assumes GRH for $L(s,f)$ (and in fact is why we assume GRH), for $\epsilon$ sufficiently small the complementary sum piece contributes

$$\frac{1}{|H_k^*(N)|}\ \Delta^\infty_{k,N}(q_1^{m_1}\cdots q_\ell^{m_\ell}) \;\ll\; O(N^{-\epsilon''}). \tag{3.8}$$

For $\Delta'_{k,N}$, by (2.22)

$$\frac{1}{|H_k^*(N)|}\ \Delta'_{k,N}(q_1^{m_1}\cdots q_\ell^{m_\ell}) \ = \ \frac{1}{q_1^{m_1/2}\cdots q_\ell^{m_\ell/2}}\ \delta_{q_1^{m_1}\cdots q_\ell^{m_\ell},\ \square_Y} + \ O\left((q_1^{m_1}\cdots q_\ell^{m_\ell})^{(k-1)/2}N^{-k+1/2+\epsilon}\right), \quad (3.9)$$

where the first term is present only if all $m_j$ are even (implying all $n_j$ are even as $m_j \equiv n_j \bmod 2$).

First we show the sum over squares is $\ll \log^{-2} R$. The squares contribute to $S_1^{(n)}$

$$\prod_{j=1}^{\ell} \sum_{\substack{q_j \\ q_j\neq q_k}} \widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j} \frac{2^{n_j}\log^{n_j} q_j}{q_j^{(n_j+m_j)/2}\log^{n_j} R}\ . \quad (3.10)$$

Note each $m_j$ is even. The contribution from terms with either $n_j \geq 2$ and $m_j = 0$ or $m_j \geq 2$ is $O(1)$, exactly as above. However, we have assumed that at least one $m_j \geq 1$ (and since $m_j$ must be even here, $m_j \geq 2$). The prime sum of such a term converges, and so its contribution will be $O\left(\log^{-n_j} R\right)$. The product of all these contributions is at most $O\left(\log^{-2} R\right)$, as required.

Now we bound the contribution to $S_1^{(n)}$ from the $O$-term in (3.9). Recall that $\sum_{j=1}^{\ell} n_j = n$. If $\mathrm{supp}(\widehat{\phi}) \subset (-\alpha,\alpha)$, the contribution is largest when $m_j = n_j$, which is bounded by

$$\begin{aligned}
&\ll\ N^{-k+\frac12+\epsilon} \sum_{q_1\ll R^\alpha,\dots,q_\ell\ll R^\alpha}\ \prod_{j=1}^{\ell}\left[\left(\frac{\log q_j}{\sqrt{q_j}\log R}\right)^{n_j} q_j^{m_j(k-1)/2}\right] \\
&\ll\ N^{-k+\frac12+\epsilon}\prod_{j=1}^{\ell}\left(\sum_{q\ll R^\alpha} q^{-\frac{n_j k}{2}-n_j}\right) \\
&\ll\ N^{-k+\frac12+\epsilon} R^{\frac{nk\alpha}{2}-n+\ell}. \qquad (3.11)
\end{aligned}$$

The worst case is when $\ell = n$. Since $R = k^2N$, if $\alpha < \frac{2k-1}{nk}$ this vanishes as $N\to\infty$. Hence if $\alpha < \frac{1}{n}\frac{2k-1}{k}$

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} S_1^{(n)} \ = \ \begin{cases} (2m-1)!!\left(2\int_{-\infty}^{\infty}\widehat{\phi}(y)^2|y|\ dy\right)^m & \text{if } n = 2m \text{ is even} \\ 0 & \text{if } n \text{ is odd.}\end{cases} \quad (3.12)$$

This completes the proof of Lemma 3.1 and Theorem 1.1. □

In Theorem E.1, by assuming GRH for Dirichlet $L$-functions, we extend the support in Theorem 1.1 to $(-\frac{2}{n},\frac{2}{n})$ for $2k \geq n$.

Theorem 1.3 is equivalent to showing that $S_2^{(n)}$ is negligible for $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n},\frac{1}{n})$, which we now prove.

**Lemma 3.2.** *Assume GRH for $L(s,f)$, and let $S_2^{(n)}$ be defined as in* (2.35)*. If* $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n},\frac{1}{n})$*, then*

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} S_2^{(n)} = 0. \quad (3.13)$$

*Proof.* The same argument for $S_1^{(n)}$ works for $S_2^{(n)}$, but now there can be no squares because we have $\lambda_f(N)$ and none of the primes equal $N$. $S_2^{(n)}$ is made up of terms like

$$\sqrt{N}\sum_{\substack{q_1\nmid N,\dots,q_\ell\nmid N \\ q_j \text{distinct}}}\ \prod_{j=1}^{\ell}\widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j}\left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j}\langle\lambda_f(Nq_1^{m_1}\dots q_\ell^{m_\ell})\rangle_*\,. \quad (3.14)$$

We again use Lemma 2.9 to evaluate the average over $\lambda_f$. By Lemma A.1 (which requires GRH for $L(s,f)$) the complementary sum is $O(N^{-1-\epsilon''})$, which is negligible when multiplied by $N^{1/2}$.

By (2.23) the remaining piece is bounded by

$$\ll N^{-1+\epsilon} \sum_{q_1 \ll R^\alpha, \dots, q_\ell \ll R^\alpha} \prod_{j=1}^{\ell} \left[ \left( \frac{2 \log q_j}{\sqrt{q_j} \log R} \right)^{n_j} q_j^{m_j/2} \right] \ll N^{-1+\epsilon} \left( \sum_{q \ll R^\alpha} \frac{\log q}{\log R} \right)^n$$
$$\ll N^{-1+\epsilon} R^{n\alpha}, \tag{3.15}$$

as the worst term occurs when $n_j = m_j = 1$. This contribution is vanishingly small if $\alpha < \frac{1}{n}$ (recall $R = k^2 N$). □

Therefore, by (2.32) and Lemma 3.1, if $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n}\frac{2k-1}{k}, \frac{1}{n}\frac{2k-1}{k})$ then

$$\lim_{\substack{N \to \infty \\ N \text{ prime}}} \langle (D(f;\phi) - \langle D(f;\phi) \rangle_*)^n \rangle_* = \begin{cases} (2m-1)!! \left( 2 \int_{-\infty}^{\infty} \widehat{\phi}(y)^2 |y| \, \mathrm{d}y \right)^m & \text{if } n = 2m \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{3.16}$$

and by (2.33) and Lemmas 3.1 and 3.2, if $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n}, \frac{1}{n})$ then

$$\lim_{\substack{N \to \infty \\ N \text{ prime}}} \left\langle \left( D(f;\phi) - \langle D(f;\phi) \rangle_\pm \right)^n \right\rangle_\pm = \begin{cases} (2m-1)!! \left( 2 \int_{-\infty}^{\infty} \widehat{\phi}(y)^2 |y| \, \mathrm{d}y \right)^m & \text{if } n = 2m \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{3.17}$$

Because of the support condition on $\widehat{\phi}$, the integral in (3.17) is the same as the variance in Theorem 1.3, which completes the proof of that theorem.

**Remark 3.3.** By choosing $k$ sufficiently large, we can take the support of $\widehat{\phi}$ as close to $(-\frac{2}{n}, \frac{2}{n})$ as desired in Theorem 1.1; by using GRH for Dirichlet $L$-functions in Theorem E.1 we show that if $k$ is sufficiently large relative to $n$ ($2k \geq n$) then we may take any $\widehat{\phi}$ with $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$. This is a natural boundary to expect, as [ILS] obtained $(-2, 2)$ when $n = 1$. For mock-Gaussian behavior (Theorem 1.3), we do not need to be able to handle support as large as that; however, support exceeding $(-\frac{1}{n}, \frac{1}{n})$ will be essential in calculating the centered moments in the extended regime of Theorem 1.6.

## 4. Going Beyond the Diagonal: Proof of Theorem 1.6

We calculate the $n^{\text{th}}$ centered moment of $D(f;\phi)$ when $n \geq 2$ and $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$, and we will not worry about terms which do not contribute in this region. We outline the arguments below. We assume GRH for $L(s,f)$ for ease of exposition, though as stated in Remark 1.2, following [ILS] we may remove this assumption with additional effort. In §4.1 we reduce the proof of Theorem 1.6 to the limit of $S_2^{(n)}$, which we analyze in the following subsections. In §4.2 we apply the Petersson formula, and in §4.3 we analyze the Kloosterman terms by using Dirichlet characters. In §4.4 we see that the contributions from the non-principal characters are negligible. By using the Mellin transform and shifting contours, we convert the prime sums to integrals in Lemma 4.9 in §4.5. The proof of Theorem 1.6 is completed by evaluating these integrals in §4.6, where by changing variables Lemma 2.12 is applicable.

### 4.1. Preliminaries.

As [ILS] has already handled the case when $n = 1$, we assume $n \geq 2$ below. Let $\mathrm{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$ with $\sigma \leq 1$. By (2.33),

$$\lim_{N \to \infty} \left\langle \left( D(f;\phi) - \langle D(f;\phi) \rangle_\pm \right)^n \right\rangle_\pm = (-1)^n \lim_{N \to \infty} S_1^{(n)} \pm (-1)^{n+1} \lim_{N \to \infty} S_2^{(n)}. \tag{4.1}$$

To prove Theorem 1.6 we need to handle support up to $\frac{1}{n-1}$. If $n \geq 3$ and $k \geq 2$, then $\frac{1}{n-1} \leq \frac{1}{n}\frac{2k-1}{k}$, and thus Lemma 3.1 evaluates $S_1^{(n)}$ for $\sigma < \frac{1}{n-1}$. If $n = 2$, however, then $\frac{1}{n-1} > \frac{1}{n}\frac{2k-1}{k}$, and thus there is a decrease in support. This is easily surmounted by using Theorem E.1 instead of Lemma 3.1. Theorem E.1 assumes GRH for Dirichlet $L$-functions; however, we shall be assuming GRH for Dirichlet $L$-functions when we study $S_2^{(n)}$.

Thus all that remains to prove Theorem 1.6 is to show that if $\sigma < \frac{1}{n-1}$ then

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} S_2^{(n)} \;=\; 2^{n-1}\left[\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x}\,\mathrm{d}x - \frac{1}{2}\phi(0)^n\right], \tag{4.2}$$

and this we shall proceed to do in a series of lemmas, culminating in Lemma 4.11. This will complete the proof of the $n^{\text{th}}$ centered moment in Theorem 1.6.

**Remark 4.1.** When we do not split by sign (as in Theorems 1.1 and E.1), we can prove results up to $\frac{2}{n}$; because of the more complicated terms in the Bessel-Kloosterman expansion, we can only handle the split cases up to $\frac{1}{n-1}$. As the two supports are equal when $n = 2$, investigating small $n$ can be quite misleading as to what support one should expect for general $n$.

### 4.2. Applying the Petersson Formula.

**Lemma 4.2.** *Let $S_2^{(n)}$ be defined as in (2.35), and assume GRH for $L(s,f)$. If* $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$*, then*

$$S_2^{(n)} \;=\; \frac{2^{n+1}\pi}{\sqrt{N}} \sum_{p_1,\dots,p_n} \sum_{m\le N^\epsilon} \frac{1}{m} \sum_{b=1}^{\infty} \frac{S(m^2, p_1\cdots p_n N; Nb)}{b} J_{k-1}\left(\frac{4\pi m\sqrt{p_1\cdots p_n}}{b\sqrt{N}}\right)$$
$$\times \prod_{j=1}^{n}\left(\widehat{\phi}\left(\frac{\log p_j}{\log R}\right)\frac{\log p_j}{\sqrt{p_j}\log R}\right) + O(N^{-\epsilon}). \tag{4.3}$$

*Proof.* The multiplicativity of $\lambda_f$ (Lemma 2.8) shows that $S_2^{(n)}$ is made up of terms of the form

$$\mathrm{i}^k\sqrt{N} \sideset{}{'}\sum_{\substack{q_1\nmid N,\dots,q_\ell\nmid N \\ q_j \text{distinct}}} \prod_{j=1}^{\ell} \widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j}\left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j} \langle \lambda_f(Nq_1^{m_1}\cdots q_\ell^{m_\ell})\rangle_*\,, \tag{4.4}$$

where $m_j \le n_j$, $m_j \equiv n_j \mod 2$ and $\sum n_j = n$; here and below $\sum'$ means the sum is taken over distinct primes only. We will show that the contribution from terms with at least one $n_j \ge 2$ is vanishingly small as $N\to\infty$ when $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$.

We expand $\langle \lambda_f(Nq_1^{m_1}\cdots q_\ell^{m_\ell})\rangle_*$ via the Petersson formula (Lemma 2.9). By Lemma A.1, which relies on GRH for $L(s,f)$, the complementary sums are of size $O(N^{-1-\epsilon''})$ for $X = Y = N^\epsilon$, which is negligible when multiplied by $N^{1/2}$. We are left with the $\Delta'_{k,N}(Nq_1^{m_1}\cdots q_\ell^{m_\ell})$ terms. That is, (4.4) can be replaced by

$$E := \mathrm{i}^k\sqrt{N} \sideset{}{'}\sum_{\substack{q_1\nmid N,\dots,q_\ell\nmid N \\ q_j \text{distinct}}} \left[\prod_{j=1}^{\ell} \widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j}\left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j}\right] \frac{1}{|H_k^*(N)|}\Delta'_{k,N}(Nq_1^{m_1}\cdots q_\ell^{m_\ell}). \tag{4.5}$$

Assume that $\text{supp}(\widehat{\phi}) \subseteq [-\sigma,\sigma]$. Note that $Nq_1^{m_1}\cdots q_\ell^{m_\ell}$ can never equal a square, since none of the $q_j$ divide $N$. Applying (2.23) we obtain

$$\frac{1}{|H_k^*(N)|}\,\Delta'_{k,N}(Nq_1^{m_1}\cdots q_\ell^{m_\ell}) \;\ll\; N^{-3/2+\epsilon} q_1^{m_1/2}\cdots q_\ell^{m_\ell/2}, \tag{4.6}$$

and so

$$E \;\ll\; \sum_{q_1\ll R^\sigma,\dots,q_\ell\ll R^\sigma} \left[\prod_{j=1}^{\ell}\left(\frac{\log q_j}{\sqrt{q_j}\log R}\right)^{n_j}\right] \frac{1}{N^{1-\epsilon}} q_1^{m_1/2}\cdots q_\ell^{m_\ell/2}. \tag{4.7}$$

The sum in (4.7) is maximized if $m_j = n_j$ and as many as possible of the $n_j = 1$, because this maximizes $\ell$ and hence the number of sums. For the cases where at least one $n_j \ge 2$, the worst case is when $\ell = n-1$, whence the sum in (4.7) contributes

$$\frac{1}{(\log R)^n N^{1-\epsilon}} \sum_{q_1\ll R^\sigma,\dots,q_{n-1}\ll R^\sigma} (\log q_1)\cdots(\log q_{n-2})\,(\log q_{n-1})^2 \ll N^{-1+\epsilon}R^{(n-1)\sigma}. \tag{4.8}$$

If $\sigma < \frac{1}{n-1}$ this has a negligible contribution in the large $N$ limit. Therefore if $\sigma < \frac{1}{n-1}$ the only way for (4.5) not to vanish as $N \to \infty$ is if all the $n_j = 1$. In other words we have shown that

$$S_2^{(n)} = \frac{\mathrm{i}^k\sqrt{N}}{|H_k^*(N)|} \sideset{}{'}\sum_{\substack{p_1\nmid N,\dots,p_n\nmid N\\ p_j \text{ distinct}}} \prod_{j=1}^n \left(\widehat{\phi}\left(\frac{\log p_j}{\log R}\right)\frac{2\log p_j}{\sqrt{p_j}\log R}\right)\Delta'_{k,N}(Np_1\cdots p_n) + O(N^{-\epsilon}). \quad (4.9)$$

We remove the distinctness condition by trivially summing the contribution when two or more primes coincide. If $p_{n-1} = p_n$, say, then by (4.6) this contributes

$$\ll \sum_{p_1,\dots,p_{n-1}\ll R^\sigma} \left[\prod_{j=1}^{n-2}\left(\frac{\log p_j}{\sqrt{p_j}\log R}\right)\right]\left(\frac{\log p_{n-1}}{\sqrt{p_{n-1}}\log R}\right)^2 \frac{1}{N^{1-\epsilon}}\, p_1^{1/2}\cdots p_{n-2}^{1/2}\, p_{n-1}, \quad (4.10)$$

which is of size $N^{-1+\epsilon}R^{(n-1)\sigma}$ and is vanishingly small if $\sigma < 1/(n-1)$. Since $R = k^2N$ and $N$ is a prime, the compact support condition on $\widehat{\phi}$ means the condition $p_j \nmid N$ is automatically satisfied for sufficiently large $N$. Finally, since (2.10) shows that $|H_k^*(N)| \sim N(k-1)/12$, applying (2.19) with $X = Y = N^\epsilon$ yields the lemma. □

**Remark 4.3.** If $\sigma > \frac{1}{n-1}$, the contribution to the $n^{\text{th}}$ centered moment arising from powers of primes needs to be considered; however, other calculations below (Lemma 4.9) can only be analyzed for $\sigma < \frac{1}{n-1}$. In §5 we see this is a natural boundary, and that new terms are expected to arise once the support exceeds $[-\frac{1}{n-1}, \frac{1}{n-1}]$.

**Lemma 4.4.** *For* $\mathrm{supp}(\widehat{\phi}) \subseteq (-\frac{5}{2n}, \frac{5}{2n})$*, the contribution in* (4.3) *from the terms when* $(b,N) \neq 1$ *is* $O(N^{-\epsilon})$*.*

*Proof.* Since $N$ is prime, if $(b,N) \neq 1$ then $(b,N) = jN$ for $j = 1,2,\dots$. If $\mathrm{supp}\,\widehat{\phi} \subset (-\sigma,\sigma)$ then these terms contribute to $S_2^{(n)}$ an amount bounded by

$$\ll \frac{1}{\sqrt{N}} \sum_{p_1,\dots,p_n\le N^\sigma} \sum_{m\le N^\epsilon} \frac{1}{m}\sum_{j=1}^\infty \frac{|S(m^2,p_1\cdots p_nN; jN^2)|}{jN}\left|J_{k-1}\left(\frac{4\pi m\sqrt{p_1\cdots p_n}}{jN^{3/2}}\right)\right| \frac{1}{\sqrt{p_1\cdots p_n}}. \quad (4.11)$$

By the bound for Kloosterman sums (2.4), $|S(m^2,p_1\cdots p_nN;jN^2)| \ll j^{\frac12+\epsilon}N^{\frac12+\epsilon}$. This is because $(m^2,p_1\cdots p_nN, jN^2) \ll m^2 \ll N^\epsilon$ and $\tau(c) \ll c^\epsilon$. Lemma 2.6(2) gives $J_{k-1}(x) \ll x$, and thus the contribution is bounded by

$$\ll N^{-5/2+\epsilon'}N^{n\sigma}\left(\sum_{m\le N^\epsilon} 1\right)\left(\sum_{j=1}^\infty \frac{1}{j^{3/2-\epsilon}}\right) \ll N^{-5/2+n\sigma+\epsilon''}, \quad (4.12)$$

which is vanishingly small if $\sigma < 5/2n$. □

Combining Lemmas 4.2 and 4.4, we have under GRH for $L(s,f)$, if $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$ then

$$S_2^{(n)} = \frac{2^{n+1}\pi}{\sqrt{N}} \sum_{p_1,\dots,p_n}\sum_{m\le N^\epsilon}\frac{1}{m}\sum_{(b,N)=1}\frac{S(m^2,p_1\cdots p_nN;Nb)}{b}J_{k-1}\left(\frac{4\pi m\sqrt{p_1\cdots p_n}}{b\sqrt{N}}\right)$$
$$\times \prod_{j=1}^n\left(\widehat{\phi}\left(\frac{\log p_j}{\log R}\right)\frac{\log p_j}{\sqrt{p_j}\log R}\right) + O(N^{-\epsilon}). \quad (4.13)$$

We now show that the terms with $\log b \gg \log N$ are negligible. We need to restrict $b$ because later (equation (4.22)) we have sums of $1/b$, and this ensures that sum is not too large.

**Lemma 4.5.** *For* $\mathrm{supp}(\widehat{\phi}) \subseteq (-\frac{1000}{n}, \frac{1000}{n})$*, the contribution in* (4.13) *from the* $b \ge N^{2006}$ *terms is* $O(N^{-3})$*.*

*Proof.* By the bound for Kloosterman sums (2.4), $S(m^2, p_1 \cdots p_n N, bN) \ll b^{\frac{1}{2}+\epsilon} N^\epsilon$. This is because $(m^2, p_1 \cdots p_n N, bN) \ll m^2 \ll N^\epsilon$ and $\tau(c) \ll c^\epsilon$. Lemma 2.6(3) gives $J_{k-1}(x) \ll x^{k-1}$, which bounds the summand in (4.3) by

$$\frac{1}{\sqrt{N}} \frac{1}{m} \frac{b^{\frac{1}{2}+\epsilon} N^\epsilon}{b} m^{k-1} (p_1 \cdots p_n)^{\frac{k-1}{2}} N^{-\frac{k-1}{2}} b^{-(k-1)} \frac{1}{\sqrt{p_1 \cdots p_n}} = m^{k-2} b^{-k+\frac{1}{2}} (p_1 \cdots p_n)^{\frac{k}{2}-1} N^{-\frac{k}{2}} N^\epsilon. \quad (4.14)$$

If $\mathrm{supp}(\widehat{\phi}) \subset [-\sigma, \sigma]$ then the $\widehat{\phi}\left(\frac{\log p_j}{\log R}\right)$ term in (4.3) restricts the $p_j$-sum to be over $p_j \ll N^\sigma$. Executing the summations over the primes and summing over $b \geq N^{2006}$ yields

$$N^{-\frac{k}{2}+\epsilon} \sum_{m \leq N^\epsilon} m^{k-2} \sum_{b \geq N^{2006}} b^{-k+\frac{1}{2}} \prod_{j=1}^{n} \sum_{p_j \ll N^\sigma} p_j^{\frac{k}{2}-1} \ll N^{-\frac{k}{2}+\epsilon'} N^{(-k+\frac{3}{2})2006} N^{\frac{k}{2} n\sigma} \ll N^{\frac{k}{2}(n\sigma - 1004+\epsilon')}. \quad (4.15)$$

Therefore the contribution as $N \to \infty$ in (4.13) from the terms when $b \geq N^{2006}$ is negligibly small when $\sigma < \frac{1000}{n}$. □

4.3. **Expanding the Kloosterman Sums.** The next lemma converts the Kloosterman sums into Gauss sums, and in §4.5 in Lemma 4.9 we convert the resulting prime sums into an integral, which we evaluate in §4.6, completing the proof of Theorem 1.6.

**Lemma 4.6.** *Under GRH for $L(s, f)$, if $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$ then*

$$S_2^{(n)} = -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{p_1,\dots,p_n} \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{\substack{(b,N)=1 \\ b < N^{2006}}} \frac{1}{b\varphi(b)} \sum_{\chi (\mathrm{mod}\, b)} \chi(N) G_\chi(m^2) G_\chi(1) \overline{\chi}(p_1 \cdots p_n) \times J_{k-1}\left(\frac{4\pi m \sqrt{p_1 \cdots p_n}}{b\sqrt{N}}\right) \prod_{j=1}^{n} \left(\widehat{\phi}\left(\frac{\log p_j}{\log R}\right) \frac{\log p_j}{\sqrt{p_j} \log R}\right) + O(N^{-\epsilon}). \quad (4.16)$$

*Proof.* By Lemma C.1 we have for $(p_1 \cdots p_n, b) = 1$ and $(b, N) = 1$ that

$$S(m^2, p_1 \cdots p_n N; Nb) = \frac{-1}{\varphi(b)} \sum_{\chi (\mathrm{mod}\, b)} \chi(N) G_\chi(m^2) G_\chi(1) \overline{\chi}(p_1 \cdots p_n). \quad (4.17)$$

If $(p_1 \cdots p_n, b) > 1$ then the left hand side of (4.17) is non-zero but the right hand side vanishes; however, the contribution to (4.13) when $(p_1 \cdots p_n, b) > 1$ is negligible if $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$. To see this, note that the worst case is when just one prime divides $b$ and the other $n-1$ primes range freely. We may assume $p_1 | b$, and write $b = rp_1$ (since $p_1$ is a prime). As $J_{k-1}(x) \ll x$, such terms contribute to (4.13) an amount bounded by

$$\frac{1}{\sqrt{N}} \sum_{p_1,\dots,p_n \leq N^\sigma} \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{r=1}^{\infty} \frac{m^2 \sqrt{r}}{rp_1} \frac{\sqrt{p_1 \dots p_n}}{rp_1 \sqrt{N}} \frac{1}{\sqrt{p_1 \dots p_n}} \ll N^{-1+\epsilon'} \left(\sum_{p \leq N^\sigma} 1\right)^{n-1} \ll N^{-1+(n-1)\sigma+\epsilon'}, \quad (4.18)$$

which is vanishingly small if $\sigma < \frac{1}{n-1}$.

Thus we may use (4.17) in (4.13) for all $(p_1, \dots, p_n, b, m)$, which yields the lemma. Note that the minus sign comes from the $-1$ in (4.17) from Lemma C.1. □

4.4. **Handling the Non-Principal Characters.**

**Lemma 4.7.** *Under GRH for Dirichlet L-functions, if $\mathrm{supp}(\widehat{\phi}) \subseteq \left(-\frac{2}{n}, \frac{2}{n}\right)$ then the contribution from the non-principal characters to $S_2^{(n)}$ in (4.16) is negligible.*

*Proof.* We use $J_{k-1}(x) \ll x$ to bound the contribution from the non-principal characters in (4.16) by

$$\ll \frac{1}{\sqrt{N}} \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{\substack{(b,N)=1 \\ b<N^{2006}}} \frac{1}{b} \frac{1}{\varphi(b)} \sum_{\substack{\chi (\bmod b) \\ \chi \neq \chi_0}} \left|G_\chi(m^2) G_\chi(1)\right|$$
$$\times \frac{m}{b\sqrt{N}} \prod_{j=1}^{n} \left| \sum_{p_j \neq N} \overline{\chi}(p_j) \log p_j \cdot \frac{1}{\log R} \widehat{\phi}\left(\frac{\log p_i}{\log R}\right) \right|. \quad (4.19)$$

As $\chi \neq \chi_0$ (the principal character with modulus $b$), by GRH for Dirichlet $L$-functions we have for $\log xNb \ll R$ that $\sum_{p \leq x} \overline{\chi}(p) \log p = O(x^{\frac12} \log^2(bNx)) = O(x^{\frac12} R^\epsilon)$. We now use partial summation and the compact support of $\widehat{\phi}$. The boundary term vanishes, and we are left with

$$\begin{aligned} \sum_{p_j \neq N} \overline{\chi}(p_j) \log p_j \cdot \frac{1}{\log R} \widehat{\phi}\left(\frac{\log p_j}{\log R}\right) &\ll R^\epsilon \int_2^{R^\sigma} u^{\frac12} \left| \frac{\mathrm{d}}{\mathrm{d}u} \widehat{\phi}\left(\frac{\log u}{\log R}\right) \right| \mathrm{d}u \\ &\ll R^\epsilon \int_2^{R^\sigma} u^{-\frac12} \left| \frac{1}{\log R} \widehat{\phi}'\left(\frac{\log u}{\log R}\right) \right| \mathrm{d}u \\ &\ll R^{(\frac12+\epsilon)\sigma}. \end{aligned} \quad (4.20)$$

As $R = k^2 N$, the contribution from the $n$ prime sums in (4.16) is $\ll N^{\frac{\sigma n}{2}+\epsilon'}$.

By Lemma C.2,

$$\frac{1}{\varphi(b)} \sum_{\substack{\chi (\bmod b) \\ \chi \neq \chi_0}} \left|G_\chi(m^2) G_\chi(1)\right| \ll b. \quad (4.21)$$

Substituting the character and prime sum bounds into (4.19) and executing the sum on $m$ yields

$$N^{-\frac12} N^{\epsilon''} \sum_{\substack{(b,N)=1 \\ b<N^{2006}}} \frac{b}{b^2 \sqrt{N}} N^{\frac{\sigma n}{2}} \ll N^{-1+\frac{n}{2}\sigma+\epsilon'''}. \quad (4.22)$$

As we have $\sum b^{-1}$ above, it is essential that $b$ is at most a fixed power of $N$; this is accomplished by Lemma 4.5. Therefore the non-principal characters do not contribute to (4.16) for $\text{supp}(\widehat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right)$. □

The next lemma shows if $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$, then we may add in the contribution of powers of primes with negligible error. This aids the passage to $L$-functions in the next section.

**Lemma 4.8.** *Under GRH for $L(s,f)$ and all Dirichlet $L$-functions, if* $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$, *then*

$$S_2^{(n)} = -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{\substack{(b,N)=1 \\ b<N^{2006}}} \frac{R(m^2,b) R(1,b)}{b\varphi(b)}$$
$$\times \sum_{n_1,\dots,n_n} \left( \prod_{j=1}^{n} \widehat{\phi}\left(\frac{\log n_j}{\log R}\right) \frac{\chi_0(n_j)\Lambda(n_j)}{\sqrt{n_j}\log R} \right) J_{k-1}\left(\frac{4\pi m\sqrt{n_1 \dots n_n}}{b\sqrt{N}}\right) + O(N^{-\epsilon}). \quad (4.23)$$

*Proof.* The support condition follows from taking the minimum of the supports of Lemmas 4.2, 4.4, 4.5, 4.6 and 4.7, and (4.24) below.

Let $\chi_0$ be the principal character modulo $b$. Since it is real, $\overline{\chi_0} = \chi_0$. From (2.2), the definition of $R(\alpha,b)$, we have $R(\alpha,b) = G_{\chi_0}(\alpha)$. Thus $G_{\chi_0}(m^2)G_{\chi_0}(1) = R(m^2,b)R(1,b)$. Since $(b,N)=1$, $\chi_0(N)=1$. Lemmas 4.6 and 4.7 imply (4.23), with the restriction that the sums are taken over primes.

We must show that the squares and higher powers of the primes add a negligible contribution to (4.23). Fix a tuple $(\ell_1,\dots,\ell_n)$ of positive integers and consider $\prod_{j=1}^n n_j^{\ell_j}$. We may assume

$\ell_1 \le \cdots \le \ell_n$ and at least one $\ell_j \ge 2$, as otherwise all $n_j^{\ell_j}$ are prime; note there are $2^n - 1$ such tuples. Using $J_\nu(x) \ll 1$ (Lemma 2.6(1)), the contribution from this tuple is at most

$$N^{-\frac{1}{2}+\epsilon'} \sum_{\substack{n_1^{\ell_1},\dots,n_n^{\ell_n} \le N^\sigma \\ \ell_1,\dots,\ell_r=1;\ell_{r+1},\dots,\ell_n\ge 2}} \frac{1}{\sqrt{n_1^{\ell_1}\cdots n_n^{\ell_n}}} \ll N^{-\frac{1}{2}+\epsilon'} N^{\frac{r}{2}\sigma}, \tag{4.24}$$

which is negligible for $\sigma < \frac{1}{n-1}$ as $r \le n-1$. This completes the proof of the lemma. □

4.5. **Converting from Sums to Integrals.** In this subsection we prove the following lemma, which will be used to finish the evaluation of $S_2^{(n)}$ in §4.6.

**Lemma 4.9.** *Under the Riemann Hypothesis for $\zeta(s)$, if $\mathrm{supp}(\widehat{\phi}) \subseteq (-\frac{1}{n-1}, \frac{1}{n-1})$, then*

$$\sum_{n_1,\dots,n_n} \left(\prod_{i=1}^n \widehat{\phi}\left(\frac{\log n_i}{\log R}\right) \frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right) J_{k-1}\left(\frac{4\pi m\sqrt{n_1\dots n_n}}{b\sqrt{N}}\right) = I_n(\widehat{\phi}) + O\left(N^{\frac{(n-1)\sigma}{2}+\epsilon}\right) \tag{4.25}$$

*uniformly for $m \le N^\epsilon$ and $b \ge 1$, where*

$$I_n(\widehat{\phi}) = \frac{b\sqrt{N}}{2\pi m}\int_0^\infty J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}x}{\log R} \tag{4.26}$$

*and $\Phi_n(x) = \phi(x)^n$. Note for $\sigma < \frac{1}{n-1}$ that the main term is larger than the error term.*

*Proof.* Let $G_{k-1}(s)$ be the Mellin transform of the Bessel function. By (6.561.14) of [GR] it is

$$\begin{aligned} G_{k-1}(s) &= \int_0^\infty J_{k-1}(x)x^{s-1}\mathrm{d}x \\ &= 2^{s-1}\Gamma\left(\frac{k-1+s}{2}\right)\Big/\Gamma\left(\frac{k+1-s}{2}\right), \quad \mathfrak{Re}(s+k-1)>0,\ \mathfrak{Re}(s)<\frac{3}{2}. \end{aligned} \tag{4.27}$$

Since we have $k \ge 2$, we may take $\mathfrak{Re}(s) \in [0,1]$. The inverse transform is

$$J_{k-1}(x) = \frac{1}{2\pi \mathrm{i}}\int_{\mathfrak{Re}(s)=1} G_{k-1}(s)x^{-s}\mathrm{d}s, \tag{4.28}$$

and so our task is to evaluate

$$\begin{aligned} &\sum_{n_1,\dots,n_n}\left[\prod_{i=1}^n \widehat{\phi}\left(\frac{\log n_i}{\log R}\right)\frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right] J_{k-1}\left(\frac{4\pi m\sqrt{n_1\cdots n_n}}{b\sqrt{N}}\right) \\ =\ &\sum_{n_1,\dots,n_n}\left[\prod_{i=1}^n \widehat{\phi}\left(\frac{\log n_i}{\log R}\right)\frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right]\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{Re}(s)=1} G_{k-1}(s)\left(\frac{4\pi m\sqrt{n_1\cdots n_2}}{b\sqrt{N}}\right)^{-s}\mathrm{d}s \\ =\ &\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{Re}(s)=1}\left[\prod_{i=1}^n\sum_{n_i}\widehat{\phi}\left(\frac{\log n_i}{\log R}\right)\frac{\chi_0(n_i)\Lambda(n_i)}{n_i^{(1+s)/2}\log R}\right]\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)\ \mathrm{d}s\ . \end{aligned} \tag{4.29}$$

Using the Mellin transform allows us to move the summations inside the product. In (4.33) we derive a useful integral version of the $n_i$-sums.

Note that for $\mathfrak{Re}(z) > 1$,

$$L(z,\chi_0) = \prod_p\left(1-\frac{\chi_0(p)}{p^z}\right)^{-1} = \zeta(z)\prod_{p|b}\left(1-\frac{1}{p^z}\right), \tag{4.30}$$

and so $L(z,\chi_0)$ has a simple pole at $z=1$, and zeros at $z=0$ and all the zeros of the Riemann zeta function. Consider the integral

$$I = -\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{Re}(z)=2}\phi\left(\frac{(2z-s-1)\log R}{4\pi\mathrm{i}}\right)\frac{L'}{L}(z,\chi_0)\,\mathrm{d}z, \tag{4.31}$$

where we have extended $\phi$ by setting

$$\phi(x+\mathrm{i}y) \;=\; \int_{-\infty}^{\infty} \widehat{\phi}(u)e^{2\pi\mathrm{i}(x+\mathrm{i}y)u}\,\mathrm{d}u. \tag{4.32}$$

Since $\widehat{\phi}$ is a Schwartz function of compact support, $\phi(x+\mathrm{i}y)$ decays rapidly as $x \to \pm\infty$ for any fixed $y$. Thus the integral in (4.31) is absolutely convergent, and all contour shifts are well defined. On the line of integration the $L$-function can be written as a Dirichlet series, and we have

$$\begin{aligned} I \;&=\; \frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{Re}(z)=2} \phi\left(\frac{(2z-s-1)\log R}{4\pi\mathrm{i}}\right)\sum_{r=1}^{\infty}\frac{\Lambda(r)\chi_0(r)}{r^z}\,\mathrm{d}z \\ &=\; \sum_{r=1}^{\infty}\Lambda(r)\chi_0(r)\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{Re}(z)=2}\phi\left(\frac{(2z-s-1)\log R}{4\pi\mathrm{i}}\right)r^{-z}\,\mathrm{d}z \\ &=\; \sum_{r=1}^{\infty}\Lambda(r)\chi_0(r)\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{Re}(z)=(1+\mathfrak{Re}(s))/2}\phi\left(\frac{(2z-s-1)\log R}{4\pi\mathrm{i}}\right)r^{-z}\,\mathrm{d}z \\ &=\; \sum_{r=1}^{\infty}\frac{\Lambda(r)\chi_0(r)}{r^{(1+s)/2}\log R}\,\widehat{\phi}\left(\frac{\log r}{\log R}\right). \end{aligned} \tag{4.33}$$

Interchanging of the order of summation and integration in (4.33) is justified by the absolute convergence. The $\widehat{\phi}(\log r/\log R)$ factor arises from expanding the integral in (4.33); because $\mathfrak{Re}(z) = \frac{1+\mathfrak{Re}(s)}{2}$, the argument of $\phi$ is real and the resulting integral is just the Fourier transform.

An alternative evaluation of the integral in (4.31) is to shift the contour to the line $\mathfrak{Re}(z) = c$ with $1/2 < c < 1$. This contour shift picks up the pole at $z = 1$ and nothing else (under the Riemann Hypothesis).

We may therefore conclude that

$$\begin{aligned} \sum_{r=1}^{\infty}\widehat{\phi}\left(\frac{\log r}{\log R}\right)\frac{\chi_0(r)\Lambda(r)}{r^{(1+s)/2}\log R} \;&=\; \phi\left(\frac{1-s}{4\pi\mathrm{i}}\log R\right) \\ &\quad - \frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{Re}(z)=c}\phi\left(\frac{(2z-1-s)\log R}{4\pi\mathrm{i}}\right)\frac{L'}{L}(z,\chi_0)\,\mathrm{d}z. \end{aligned} \tag{4.34}$$

Denoting the integral in (4.34) by $\mathcal{E}(s)$, we see that (4.29) equals

$$\begin{aligned} &\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{Re}(s)=1}\left(\phi\left(\frac{1-s}{4\pi\mathrm{i}}\log R\right)+\mathcal{E}(s)\right)^n\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s}G_{k-1}(s)\,\mathrm{d}s \\ &\quad = \sum_{j=0}^{n}\binom{n}{j}\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{Re}(s)=1}\phi\left(\frac{1-s}{4\pi\mathrm{i}}\log R\right)^{n-j}\mathcal{E}(s)^j\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s}G_{k-1}(s)\,\mathrm{d}s. \end{aligned} \tag{4.35}$$

The main term is when $j = 0$. Letting $\Phi_n(x) := \phi(x)^n$ and using (4.27) to write $G_{k-1}(s)$ in terms of the Bessel function, we see that it equals

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi\left(\frac{-t\log R}{4\pi}\right)^n\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-1-\mathrm{i}t}G_{k-1}(1+\mathrm{i}t)\,\mathrm{d}t \\ &= \frac{b\sqrt{N}}{8\pi^2 m}\int_{-\infty}^{\infty}\phi\left(\frac{-t\log R}{4\pi}\right)^n\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-\mathrm{i}t}\int_0^{\infty}J_{k-1}(x)x^{\mathrm{i}t}\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{b\sqrt{N}}{8\pi^2 m}\int_0^{\infty}J_{k-1}(x)\int_{-\infty}^{\infty}\Phi_n\left(\frac{-t\log R}{4\pi}\right)\exp\left(\mathrm{i}t\log(bx\sqrt{N}/4\pi m)\right)\,\mathrm{d}t\,\mathrm{d}x \\ &= \frac{b\sqrt{N}}{2\pi m\log R}\int_0^{\infty}J_{k-1}(x)\int_{-\infty}^{\infty}\Phi_n(u)\exp\left(-2\pi\mathrm{i}u\,\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\,\mathrm{d}u\,\mathrm{d}x; \end{aligned} \tag{4.36}$$

interchanging the order of integration is justified by Fubini's Theorem. As the inner integral is simply the Fourier transform of $\Phi_n$, the main term equals

$$\frac{b\sqrt{N}}{2\pi m \log R} \int_0^\infty J_{k-1}(x) \widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right) \mathrm{d}x, \tag{4.37}$$

which we have denoted $I_n(\widehat{\phi})$.

The remaining terms in (4.35) are error terms, arising from $j \in \{1,\dots,n\}$. We shift the integrals over $s$ to the line $\mathfrak{Re}(s) = -\epsilon$ and estimate $\phi$, $\mathcal{E}$ and $G_{k-1}$ in order to bound these terms.

If $\operatorname{supp}\widehat{\phi} \subset [-\sigma,\sigma]$ and $|x| > x_0 > 0$, then integrating by parts $A$ times yields

$$\begin{aligned}\phi\left(\frac{x+\mathrm{i}y}{4\pi\mathrm{i}}\log R\right) &= \int_{-\infty}^{\infty} \widehat{\phi}(u) e^{u(x+\mathrm{i}y)\log R/2}\,\mathrm{d}u \\ &= \int_{-\infty}^{\infty} \widehat{\phi}^{(A)}(u) \left(\frac{-2}{(x+\mathrm{i}y)\log R}\right)^A e^{u(x+\mathrm{i}y)\log R/2}\,\mathrm{d}u \\ &\ll_{x_0} \frac{1}{(1+|y|)^A} \int_{-\infty}^{\infty} \left|\widehat{\phi}^{(A)}(u)\right| R^{xu/2}\,\mathrm{d}u \\ &\ll_{x_0} \frac{N^{|x|\sigma/2}}{(1+|y|)^A}\end{aligned} \tag{4.38}$$

since $\operatorname{supp}(\widehat{\phi}^{(A)}) \subset [-\sigma,\sigma]$, and $R^{|x|\sigma/2} \ll N^{|x|\sigma/2}$.

For $\mathfrak{Re}(z) = c$ with $c \in (\frac{1}{2}, 1]$, we have

$$\frac{L'}{L}(z,\chi_0) \ll_c \log((2+|\mathfrak{Im}(z)|)b) \tag{4.39}$$

which follows from (4.30) and the well-known bound $\frac{\zeta'}{\zeta}(z) \ll_c \log(2+|\mathfrak{Im}(z)|)$ (see, for example, Theorem 14.5 of [T]). Therefore, if $s = -\epsilon + \mathrm{i}t$ and $c = 1/2 + \epsilon'$, we have

$$\begin{aligned}\mathcal{E}(-\epsilon+\mathrm{i}t) &= -\frac{1}{2\pi\mathrm{i}} \int_{\mathfrak{Re}(z)=1/2+\epsilon'} \phi\left(\frac{(2z-1+\epsilon-\mathrm{i}t)\log R}{4\pi\mathrm{i}}\right) \frac{L'}{L}(z,\chi_0)\,\mathrm{d}z \\ &\ll \int_{-\infty}^{\infty} \left|\phi\left(\frac{(2\epsilon'+2\mathrm{i}y+\epsilon-\mathrm{i}t)\log R}{4\pi\mathrm{i}}\right)\right| \log((2+|y|)b)\,\mathrm{d}y \\ &\ll \int_{-\infty}^{\infty} \frac{N^{(2\epsilon'+\epsilon)\sigma/2}}{(1+|2y-t|)^A} \log((2+|y|)b)\,\mathrm{d}y \\ &\ll N^{\epsilon''}\log((2+|t|)b);\end{aligned} \tag{4.40}$$

above we used $2\epsilon'+\epsilon > 0$ so as to be able to apply the bounds from (4.38).

We also need an estimate for $G_{k-1}$. From (4.27) and

$$\Gamma(\sigma+\mathrm{i}t) = \sqrt{2\pi}\,(\mathrm{i}t)^{\sigma-\frac{1}{2}}\,e^{-\frac{\pi}{2}|t|}\,|t/e|^{\mathrm{i}t}\,\left(1+O\left(|t|^{-1}\right)\right), \tag{4.41}$$

we have

$$|G_{k-1}(-\epsilon+\mathrm{i}t)| = 2^{-\epsilon-1}\left|\frac{\Gamma((k-1-\epsilon+\mathrm{i}t)/2)}{\Gamma((k-1-\epsilon-\mathrm{i}t)/2+1+\epsilon)}\right| \ll \frac{1}{(1+|t|)^{1+\epsilon}}. \tag{4.42}$$

Using (4.38), (4.40) and (4.42) in (4.35), we have

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}}\int_{\mathfrak{Re}(s)=-\epsilon} \phi\left(\frac{1-s}{4\pi\mathrm{i}}\log R\right)^{n-j} \mathcal{E}(s)^j \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)\,\mathrm{d}s \\
&\ll \left(\frac{m}{b\sqrt{N}}\right)^{\epsilon}\int_{-\infty}^{\infty}\left|\phi\left(\frac{1+\epsilon-\mathrm{i}t}{4\pi\mathrm{i}}\log R\right)\right|^{n-j} |\mathcal{E}(-\epsilon+\mathrm{i}t)|^j\,|G_{k-1}(-\epsilon+\mathrm{i}t)|\ \mathrm{d}t \\
&\ll \left(\frac{m}{b\sqrt{N}}\right)^{\epsilon}\int_{-\infty}^{\infty}\left(\frac{N^{(1+\epsilon)\sigma/2}}{(1+|t|)^A}\right)^{n-j}\left(N^{\epsilon''}\log((2+|t|)b)\right)^j\frac{1}{(1+|t|)^{1+\epsilon}}\ \mathrm{d}t \\
&\ll N^{(n-j)\sigma/2+\epsilon'''}. \qquad (4.43)
\end{aligned}$$

Note the $t$-integral converges (it is only when $j = n$ that we need to use $\epsilon > 0$ to ensure convergence). The worst term is clearly when $j = 1$, and this yields the desired error term.

This completes the proof of Lemma 4.9. □

**Remark 4.10.** In §4.6 we finish the evaluation of $S_2^{(n)}$. We multiply our terms by $N^{-1/2}$ and execute the summations over $b$ and $m$. Thus in order for the error term in Lemma 4.9 to be negligible we need

$$N^{-\frac{1}{2}}N^{\frac{(n-1)\sigma}{2}+\epsilon} \ \ll\ N^{-\epsilon'}, \qquad (4.44)$$

which forces $\sigma < \frac{1}{n-1}$. We thus see that, in the number theory calculations, $\frac{1}{n-1}$ is a real boundary for this method when we split by sign. This is very different than related problems in [Ru, Gao] and the non-split case of Theorem 1.1; the reason is due to support problems when $n \geq 2$ from handling the Bessel-Kloosterman terms from $\lambda_f(N)$. Thus when we split by sign, we expect our results for support up to $\frac{1}{n-1}$ and not $\frac{2}{n}$.

4.6. **Evaluating $S_2^{(n)}$.** We finish the proof of Theorem 1.6 by completing the evaluation of $S_2^{(n)}$.

**Lemma 4.11.** *Under GRH for $L(s,f)$ and for all Dirichlet $L$-functions, if $n \geq 2$ and $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$, then*

$$S_2^{(n)} \ =\ 2^{n-1}\left[\int_{-\infty}^{\infty}\phi(x)^n\frac{\sin 2\pi x}{2\pi x}\,\mathrm{d}x - \frac{1}{2}\phi(0)^n\right] + O\left(\frac{k\log\log kN}{\log kN}\right). \qquad (4.45)$$

*Proof.* Combining Lemmas 4.8 and 4.9, we have shown that under GRH for $L(s,f)$ and for all Dirichlet $L$-functions, if $n \geq 2$ and $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$, then

$$S_2^{(n)} \ =\ -\frac{2^{n+1}\pi}{\sqrt{N}}\sum_{m\leq N^\epsilon}\frac{1}{m}\sum_{\substack{(b,N)\\ b<N^{2006}}}\frac{R(m^2,b)R(1,b)}{b\varphi(b)}\left(I_n(\widehat{\phi}) + O\left(N^{\frac{(n-1)\sigma}{2}+\epsilon}\right)\right) \qquad (4.46)$$

where

$$I_n(\widehat{\phi}) \ =\ \frac{b\sqrt{N}}{2\pi m}\int_{x=0}^{\infty} J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}x}{\log R} \qquad (4.47)$$

and $\Phi_n(x) = \phi(x)^n$. Since by (2.2) we have $R(m^2,b)R(1,b) \ll m^4$, the contribution from the $O$-term in (4.46) is bounded by

$$N^{-\frac{1}{2}}\sum_{m\leq N^\epsilon}\frac{m^4}{m}\sum_{\substack{(b,N)\\ b<N^{2006}}}\frac{1}{b^2}\left(N^{\frac{(n-1)\sigma}{2}+\epsilon}\right) \ \ll\ N^{\frac{n-1}{2}\cdot(\sigma-\frac{1}{n-1}+\epsilon'')}, \qquad (4.48)$$

which is $O(N^{-\epsilon'''})$ for $\sigma < \frac{1}{n-1}$.

We are left with evaluating the main term. The rapid decay of $I_n(\widehat{\phi})$ with respect to $b$ allows us to extend the $b$-sum of the main term of (4.46) to all $b$ relatively prime to $N$. From (4.47) we have

$$I_n(\widehat{\phi}) \ \ll\ \frac{b\sqrt{N}}{2\pi m}\int_0^\infty\left|\widehat{\Phi_n}\left(\frac{2\log(u\sqrt{N}/4\pi m)}{\log R}\right)\right|\frac{\mathrm{d}u}{b} \ \ll\ \frac{\sqrt{N}}{m}. \qquad (4.49)$$

From (2.2) we have $R(m^2,b)R(1,b) \ll m^4$. The $m$-sum is $O(N^{4\epsilon})$, the factor of $N^{-\frac{1}{2}}$ cancels the factor of $N^{\frac{1}{2}}$, and we have a $b$-sum of $b^{-2}$ (which is negligible for the terms with $b \geq N^{2006}$).

As $\Phi_j(x) = \phi(x)^j$, $\widehat{\Phi_n}$ is the convolution of $\phi$ with itself $n$ times. In particular we have

$$\widehat{\Phi_n}(u) \ = \ \int_{-\infty}^{\infty} \widehat{\Phi_{n-1}}(w)\widehat{\phi}(u-w)\,\mathrm{d}w. \tag{4.50}$$

Note that the support of $\widehat{\Phi_n}$ is at most $n$ times that of $\widehat{\phi}$, which means for $n \geq 2$ it is less than $\frac{n}{n-1} \leq 2$. Therefore we may apply Lemma 2.12. We find that the main term of $S_2^{(n)}$ is

$$\begin{aligned}
&-\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{m\leq N^\epsilon} \frac{1}{m} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{b\varphi(b)}\ I_n(\widehat{\phi}) \\
&= \ -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{m\leq N^\epsilon} \frac{1}{m} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{b\varphi(b)} \frac{b\sqrt{N}}{2\pi m} \int_0^\infty J_{k-1}(y)\widehat{\Phi_n}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}y}{\log R} \\
&= \ -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{m\leq N^\epsilon} \frac{1}{m} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{b\varphi(b)} \frac{b\sqrt{N}}{2\pi m} \int_0^\infty J_{k-1}(y)\widehat{\Phi_n}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}y}{\log R} \\
&= \ -2^n\cdot\left(-\frac{1}{2}\right)\cdot\left[\int_{-\infty}^{\infty}\Phi_n(x)\frac{\sin 2\pi x}{2\pi x}\,\mathrm{d}x - \frac{1}{2}\Phi_n(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right) \\
&= \ 2^{n-1}\left[\int_{-\infty}^{\infty}\phi(x)^n\frac{\sin 2\pi x}{2\pi x}\,\mathrm{d}x - \frac{1}{2}\phi(0)^n\right] + O\left(\frac{k\log\log kN}{\log kN}\right).
\end{aligned} \tag{4.51}$$

This completes the proof of the lemma, and Theorem 1.6. □

**Remark 4.12.** Note that if $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n},\frac{1}{n})$ then $\mathrm{supp}(\widehat{\Phi_n}) \subset (-1,1)$. In this region the Kloosterman-Bessel terms are negligible, and the contribution to the centered moment in (2.33) from $S_2^{(n)}$ vanishes. As $\frac{1}{n-1} > \frac{1}{n}$ for $n \geq 2$, we have entered the non-trivial region where these contributions do not vanish. Thus the mock Gaussian result of Theorem 1.3 is sharp.

## 5. Random Matrix Theory: Proof of Theorem 1.7

In this section we prove Theorem 1.7 by calculating the centered moments of $Z_\phi(U)$ when averaged over SO(even) and SO(odd). For small support the moments agree with those of the Gaussian; for larger support, however, the moments differ.

5.1. **Introduction.** Let $U$ be an $M\times M$ unitary matrix with eigenvalues $e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_M}$. For a real, even integrable function $\phi$ which decays sufficiently rapidly, define

$$\begin{aligned}
F_M(\theta) \ &= \ \sum_{j=-\infty}^{\infty}\phi\left(\frac{M}{2\pi}(\theta+2\pi j)\right) \\
&= \ \frac{1}{M}\sum_{k=-\infty}^{\infty}\widehat{\phi}\left(\frac{k}{M}\right)e^{\mathrm{i}k\theta},
\end{aligned} \tag{5.1}$$

which is a $2\pi$-periodic function emphasizing points near $0 \mod 2\pi$. Define

$$Z_\phi(U) \ = \ \sum_{n=1}^{M} F_M(\theta_n). \tag{5.2}$$

This is the random matrix equivalent of $D(f;\phi)$. More precisely, moments of $D(f,\phi)$ averaged over $f \in H_k^+(N)$ as $N \to \infty$ should correspond to moments of $Z_\phi(U)$ when averaged with respect to Haar measure over SO$(M)$ matrices as $M$ tends to infinity through *even* integers, while moments of $D(f,\phi)$ averaged over $f \in H_k^-(N)$ as $N\to\infty$ should correspond to moments of $Z_\phi(U)$ when averaged with respect to Haar measure over SO$(M)$ matrices as $M$ tends to infinity through *odd* integers.

**Remark 5.1.** If we restrict the eigenangles such that $-\pi < \theta_n \leq \pi$, then

$$Z_\phi(U) \ \sim \ \sum_{n=1}^{M} \phi\left(\frac{M}{2\pi}\theta_n\right). \tag{5.3}$$

However, using $F_M(\theta)$ in the definition of $Z_\phi(U)$ is more natural because the eigenangles of orthogonal matrices are $2\pi$-periodic.

Much of the work required to calculate the moments of $Z_\phi(U)$ was done in the paper of Hughes and Rudnick [HR1] (building on work of Soshnikov [Sosh]), and we simply quote the results we need to show Theorem 1.7. The novelty here is desymmetrizing the integrals to handle the combinatorics in the non-trivial range. This is necessary in order to write the formulas in such a way as to facilitate comparisons with number theory.

Theorem 5 of [HR1], when applied to the case $Z_\phi(U)$, shows that the means over SO(even) and SO(odd) are

$$C_1^{\mathrm{SO(even)}} \ := \ \lim_{\substack{M\to\infty \\ M \mathrm{even}}} \mathbb{E}_{\mathrm{SO}(M)}\left[Z_\phi(U)\right] \ = \ \widehat{\phi}(0) + \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(y)\ \mathrm{d}y \tag{5.4}$$

$$C_1^{\mathrm{SO(odd)}} \ := \ \lim_{\substack{M\to\infty \\ M \mathrm{odd}}} \mathbb{E}_{\mathrm{SO}(M)}\left[Z_\phi(U)\right] \ = \ \widehat{\phi}(0) + \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(u)\ \mathrm{d}u + \int_{|y|\geq 1} \widehat{\phi}(u)\ \mathrm{d}u, \tag{5.5}$$

respectively, where $\mathbb{E}_{\mathrm{SO}(M)}$ denotes expectation with respect to Haar measure over the classical compact group of $M \times M$ special orthogonal matrices. Furthermore, that theorem states that the variance of $Z_\phi(U)$ over SO(even) is

$$\begin{aligned} C_2^{\mathrm{SO(even)}} \ &:= \ \lim_{\substack{M\to\infty \\ M \mathrm{even}}} \mathbb{E}_{\mathrm{SO}(M)}\left[\left(Z_\phi(U) - C_1^{\mathrm{SO(even)}}\right)^2\right] \\ &= \ 2\int_{-\infty}^{\infty} \min(|y|,1)\widehat{\phi}(y)^2\ \mathrm{d}y + 2\int_{y=-1/2}^{1/2}\int_{|x|\geq 1/2} \widehat{\phi}(x+y)\widehat{\phi}(x-y)\ \mathrm{d}x\ \mathrm{d}y, \end{aligned} \tag{5.6}$$

and over SO(odd) is

$$\begin{aligned} C_2^{\mathrm{SO(odd)}} \ &:= \ \lim_{\substack{M\to\infty \\ M \mathrm{odd}}} \mathbb{E}_{\mathrm{SO}(M)}\left[\left(Z_\phi(U) - C_1^{\mathrm{SO(odd)}}\right)^2\right] \\ &= \ 2\int_{-\infty}^{\infty} \min(|y|,1)\widehat{\phi}(y)^2\ \mathrm{d}y - 2\int_{y=-1/2}^{1/2}\int_{|x|\geq 1/2} \widehat{\phi}(x+y)\widehat{\phi}(x-y)\ \mathrm{d}x\ \mathrm{d}y. \end{aligned} \tag{5.7}$$

Changing variables to $u = x + y$ and $v = x - y$ we see that

$$\int_{y=-1/2}^{1/2}\int_{|x|\geq 1/2} \widehat{\phi}(x+y)\widehat{\phi}(x-y)\ \mathrm{d}x\ \mathrm{d}y \ = \ \frac{1}{2}\iint_A \widehat{\phi}(u)\widehat{\phi}(v)\ \mathrm{d}u\ \mathrm{d}v, \tag{5.8}$$

where

$$A \ = \ \{|u+v| \geq 1\} \cap \{|u-v| \leq 1\}. \tag{5.9}$$

Note that if $|u| \leq 1$ and $|v| \leq 1$, then whenever $|u + v| \geq 1$ we have $\{|u - v| \leq 1\}$. Therefore if $\mathrm{supp}(\widehat{\phi}) \subseteq [-1, 1]$, we have

$$\begin{aligned} \frac{1}{2}\iint_A \widehat{\phi}(u)\widehat{\phi}(v)\ \mathrm{d}u\ \mathrm{d}v \ &= \ \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} \widehat{\phi}(u)\widehat{\phi}(v)\mathbb{1}_{\{|u+v|\geq 1\}}\ \mathrm{d}u\ \mathrm{d}v \\ &= \ -\left(\int_{-\infty}^{\infty} \phi(x)^2 \frac{\sin 2\pi x}{2\pi x}\ \mathrm{d}x - \frac{1}{2}\phi(0)^2\right). \end{aligned} \tag{5.10}$$

the last line following from the Fourier transform identity (see Lemma D.1 for a proof)

$$\begin{aligned}
&\int_{-\infty}^{\infty} \phi(x)^2 \frac{\sin 2\pi x}{2\pi x}\, \mathrm{d}x - \frac{1}{2}\phi(0)^2 \\
&= \frac{1}{2}\iint \widehat{\phi}(u)\widehat{\phi}(v)\mathbb{1}_{\{|u+v|\le 1\}}\, \mathrm{d}u\, \mathrm{d}v - \frac{1}{2}\iint \widehat{\phi}(u)\widehat{\phi}(v)\, \mathrm{d}u\, \mathrm{d}v \\
&= -\frac{1}{2}\iint \widehat{\phi}(u)\widehat{\phi}(v)\mathbb{1}_{\{|u+v|>1\}}\, \mathrm{d}u\, \mathrm{d}v. \qquad (5.11)
\end{aligned}$$

Furthermore, note that if either $|u|>1$ or $|v|>1$ then $|u+v|\ge 1$ does not necessarily imply $|u-v|\le 1$, and so (5.10) does not hold if the support of $\widehat{\phi}$ exceeds $[-1,1]$.

This proves Theorem 1.7 in the case $n=1$ and $n=2$. While the higher moments of $Z_\phi(U)$ can be calculated using Weyl's explicit representation of Haar measure for even and odd orthogonal groups, as in [HR1] we deal with its cumulants. Denote

$$\sum_{\ell=1}^{\infty} C_\ell^{\mathrm{SO(even)}} \frac{\lambda^\ell}{\ell!} = \lim_{\substack{M\to\infty \\ M \mathrm{even}}} \log \mathbb{E}_{\mathrm{SO}(M)}\left[\exp\left(\lambda Z_\phi(U)\right)\right] \qquad (5.12)$$

and

$$\sum_{\ell=1}^{\infty} C_\ell^{\mathrm{SO(odd)}} \frac{\lambda^\ell}{\ell!} = \lim_{\substack{M\to\infty \\ M \mathrm{odd}}} \log \mathbb{E}_{\mathrm{SO}(M)}\left[\exp\left(\lambda Z_\phi(U)\right)\right]. \qquad (5.13)$$

Knowing the first $n$ cumulants is equivalent to knowing the first $n$ moments, which is evident from the identity

$$\mathbb{E}_{\mathrm{SO}(M)}\left[(Z_\phi(U))^n\right] = \sum \left(\frac{C_1}{1!}\right)^{k_1}\left(\frac{C_2}{2!}\right)^{k_2}\cdots\left(\frac{C_n}{n!}\right)^{k_n} \frac{n!}{k_1!\, k_2! \cdots k_n!}, \qquad (5.14)$$

where the sum runs over all non-negative values of $k_j$ $(j=1,\ldots,n)$ such that $\sum_{j=1}^n jk_j = n$. Theorem 1.4 implies that if $j\ge 3$ and $\mathrm{supp}(\widehat{\phi}) \subseteq [-\frac{1}{j},\frac{1}{j}]$, then both $C_j^{\mathrm{SO(even)}}=0$ and $C_j^{\mathrm{SO(odd)}}=0$. Therefore, by (5.14), if $\mathrm{supp}(\widehat{\phi})\subseteq[-\frac{1}{n-1},\frac{1}{n-1}]$ for $n\ge 3$, the only terms which contribute to the $n^{\mathrm{th}}$ moment are $C_1$, $C_2$ and $C_n$, and we have

$$\lim_{\substack{M\to\infty \\ M \mathrm{even}}} \left[\left(Z_\phi(U) - C_1^{\mathrm{SO(even)}}\right)^n\right] = \begin{cases} C_{2k}^{\mathrm{SO(even)}} + \left(C_2^{\mathrm{SO(even)}}\right)^k \dfrac{(2k)!}{2^k k!} & n=2k \\ C_{2k+1}^{\mathrm{SO(even)}} & n=2k+1, \end{cases} \qquad (5.15)$$

and similarly for SO(odd). If $\mathrm{supp}(\widehat{\phi})\subseteq[-\frac{1}{2},\frac{1}{2}]$, (5.6) and (5.7) yield

$$C_2^{\mathrm{SO(even)}} = C_2^{\mathrm{SO(odd)}} = 2\int_{-1/2}^{1/2} |y|\widehat{\phi}(y)^2\, \mathrm{d}y = \sigma_\phi^2, \qquad (5.16)$$

where $\sigma_\phi^2$ is given by (1.17). Therefore, by (5.15), Theorem 1.7 will follow from showing

$$C_n^{\mathrm{SO(even)}} = (-1)^{n-1}2^{n-1}\left[\int_{-\infty}^{\infty}\phi(x)^n \frac{\sin 2\pi x}{2\pi x}\, \mathrm{d}x - \frac{1}{2}\phi(0)^n\right] \qquad (5.17)$$

and

$$C_n^{\mathrm{SO(odd)}} = (-1)^{n}2^{n-1}\left[\int_{-\infty}^{\infty}\phi(x)^n \frac{\sin 2\pi x}{2\pi x}\, \mathrm{d}x - \frac{1}{2}\phi(0)^n\right] \qquad (5.18)$$

for $n\ge 3$ and $\mathrm{supp}(\widehat{\phi})\subseteq[-\frac{1}{n-1},\frac{1}{n-1}]$.

Let

$$\begin{aligned}
Q_n(\phi) := 2^{n-1}\sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j\ge 1}} &\frac{(-1)^{m+1}}{m}\frac{n!}{\lambda_1!\cdots\lambda_m!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi(x_1)^{\lambda_1}\cdots\phi(x_m)^{\lambda_m} \\
&\times S(x_1-x_2)S(x_2-x_3)\cdots S(x_{m-1}-x_m)S(x_m+x_1)\, \mathrm{d}x_1\cdots \mathrm{d}x_m, \qquad (5.19)
\end{aligned}$$

where

$$S(x) \;=\; \frac{\sin(\pi x)}{\pi x} \;=\; \int_{-\infty}^{\infty} \mathbb{1}_{\{|u|\le 1/2\}} e^{2\pi i x u}\, \mathrm{d}u. \tag{5.20}$$

To prove (5.17, 5.18), we again use results from [HR1] (Section 2.1, Lemma 6 and Theorem 7), where it was shown that if $\mathrm{supp}(\widehat{\phi}) \subseteq [-\frac{2}{n}, \frac{2}{n}]$, then $C_n^{\mathrm{SO(even)}} = Q_n(\phi)$ and $C_n^{\mathrm{SO(odd)}} = -Q_n(\phi)$. Therefore we must show that whenever $\mathrm{supp}(\widehat{\phi}) \subseteq [-\frac{1}{n-1}, \frac{1}{n-1}]$, $Q_n(\phi)$, defined in (5.19), can also be written as

$$\begin{aligned} Q_n(\phi) \;&=\; (-1)^n 2^{n-2} \int \cdots \int_{-\infty}^{\infty} \widehat{\phi}(u_1)\cdots\widehat{\phi}(u_n) \mathbb{1}_{\{|u_1+\cdots+u_n|\ge 1\}}\, \mathrm{d}u_1\cdots\mathrm{d}u_n \\ &=\; (-1)^{n-1} 2^{n-1} \left[ \int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x}\, \mathrm{d}x - \frac{1}{2}\phi(0)^n \right]; \end{aligned} \tag{5.21}$$

the two expressions for $Q_n(\phi)$ in (5.21) are equal by Lemma D.2.

To prove (5.21), we use Plancherel's identity in (5.19), and write the test function $\phi$ in terms of its Fourier transform $\widehat{\phi}$, and $S(x)$ in terms of its Fourier transform, obtaining

$$\begin{aligned} Q_n(\phi) \;=\; 2^{n-1} \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m = n \\ \lambda_j \ge 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(y_1)\cdots\widehat{\phi}(y_n) \\ e^{2\pi i x_1 \left(u_1+u_m+y_1+\cdots+y_{\lambda_1}\right)} e^{2\pi i x_2 \left(u_2-u_1+y_{\lambda_1+1}+\cdots+y_{\lambda_1+\lambda_2}\right)} \cdots e^{2\pi i x_m \left(u_m-u_{m-1}+y_{\lambda_1+\cdots+\lambda_{m-1}+1}+\cdots+y_n\right)} \\ \times\, \mathbb{1}_{\{|u_1|\le 1/2\}} \cdots \mathbb{1}_{\{|u_m|\le 1/2\}}\, \mathrm{d}u_1\cdots\mathrm{d}u_m\, \mathrm{d}y_1\cdots\mathrm{d}y_n\, \mathrm{d}x_1\cdots\mathrm{d}x_m. \end{aligned} \tag{5.22}$$

For simplicity write

$$\begin{aligned} Y_1 \;&:=\; y_1+\cdots+y_{\lambda_1} \\ Y_2 \;&:=\; y_{\lambda_1+1}+\cdots+y_{\lambda_1+\lambda_2} \\ &\;\vdots \\ Y_m \;&:=\; y_{\lambda_1+\cdots+\lambda_{m-1}+1}+\cdots+y_n. \end{aligned} \tag{5.23}$$

Integrating over $x_1$ to $x_m$ converts the exponentials to delta functionals, and we get

$$\begin{aligned} Q_n(\phi) \;&=\; 2^{n-1} \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m = n \\ \lambda_j \ge 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(y_1)\cdots\widehat{\phi}(y_n) \\ &\quad \times\, \delta\left(u_1+u_m+Y_1\right) \delta\left(u_2-u_1+Y_2\right) \cdots \delta\left(u_m-u_{m-1}+Y_m\right) \\ &\quad \times\, \mathbb{1}_{\{|u_1|\le 1/2\}} \cdots \mathbb{1}_{\{|u_m|\le 1/2\}}\, \mathrm{d}u_1\cdots\mathrm{d}u_m\, \mathrm{d}y_1\cdots\mathrm{d}y_n. \end{aligned} \tag{5.24}$$

Changing variables to

$$\begin{aligned} v_1 \;&:=\; u_1+u_m & u_1 \;&=\; \frac{1}{2}(v_1-v_2-\cdots-v_m) \\ v_2 \;&:=\; u_2-u_1 & u_2 \;&=\; \frac{1}{2}(v_1+v_2-v_3-\cdots-v_m) \\ &\;\vdots & &\;\vdots \\ v_m \;&:=\; u_m-u_{m-1} & v_m \;&=\; \frac{1}{2}(v_1+v_2+\cdots+v_m) \end{aligned} \tag{5.25}$$

(the Jacobian from this transformation is $\frac{1}{2}$) leads to

$$Q_n(\phi) = 2^{n-1} \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j\geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \widehat{\phi}(y_1)\cdots\widehat{\phi}(y_n)$$
$$\times \frac{1}{2}\mathbb{1}_{\{|Y_1-Y_2-Y_3-\cdots-Y_m|\leq 1\}}\mathbb{1}_{\{|Y_1+Y_2-Y_3-\cdots-Y_m|\leq 1\}}\cdots\mathbb{1}_{\{|Y_1+Y_2+Y_3+\cdots+Y_m|\leq 1\}}\,\mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{5.26}$$

Making use of the fact that $\widehat{\phi}$ is an even function, we desymmetrize this by writing

$$Q_n(\phi) \;=\; 2^{n-2}\int_0^\infty\cdots\int_0^\infty \widehat{\phi}(y_1)\cdots\widehat{\phi}(y_n)K(y_1,\dots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n, \tag{5.27}$$

where

$$K(y_1,\dots,y_n) = \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j\geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!} \sum_{\epsilon_1=\pm1,\dots,\epsilon_n=\pm1} \prod_{\ell=1}^{m} \mathbb{1}_{\{|\sum_{j=1}^n \eta(\ell,j)\epsilon_j y_j|\leq 1\}} \tag{5.28}$$

with

$$\eta(\ell,j) \;=\; \begin{cases} +1 & \text{if } j\leq \sum_{k=1}^{\ell}\lambda_k \\ -1 & \text{if } j> \sum_{k=1}^{\ell}\lambda_k. \end{cases} \tag{5.29}$$

If $0\leq y_j\leq \frac{1}{n}$ for all $j$, then

$$\prod_{\ell=1}^{m} \mathbb{1}_{\{|\sum_{j=1}^n \eta(\ell,j)\epsilon_j y_j|\leq 1\}} \;=\; 1 \tag{5.30}$$

for all choices of $\epsilon_j=\pm1$. There are $2^n$ choices of possible $n$-tuples $(\epsilon_1,\dots,\epsilon_n)$, and so if $n\geq 2$,

$$K(y_1,\dots,y_n) \;=\; \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j\geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!} 2^n \;=\; 0, \tag{5.31}$$

which follows from a trick of Soshnikov [Sosh] obtained by evaluating the generating series

$$z \;=\; \log(1+(e^z-1)) \;=\; \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j\geq 1}} \frac{(-1)^{m+1}}{m} \frac{1}{\lambda_1!\cdots\lambda_m!}. \tag{5.32}$$

If $0\leq y_j\leq \frac{1}{n-1}$ for all $j$, and $y_1+\cdots+y_n\geq 1$ (so at least one $y_j>\frac{1}{n}$), then

$$\mathbb{1}_{\{|\sum_{j=1}^n \eta(\ell,j)\epsilon_j y_j|\leq 1\}} \;=\; 0 \tag{5.33}$$

if and only if either $\eta(\ell,j)\epsilon_j=+1$ for all $j$, or $\eta(\ell,j)\epsilon_j=-1$ for all $j$. Note there are exactly $2m$ choices for the $n$-tuple $(\epsilon_1,\dots,\epsilon_n)$ which yield

$$\prod_{\ell=1}^{m} \mathbb{1}_{\{|\sum_{j=1}^n \eta(\ell,j)\epsilon_j y_j|\leq 1\}} \;=\; 0, \tag{5.34}$$

and the remaining $2^n-2m$ choices yield the product equals 1. This follows from the $\eta(\ell,j)$ change signs so that no choice of $(\epsilon_1,\dots,\epsilon_n)$ makes two terms in the product vanish. There are $m$ factors in the product, and each factor is zero for exactly two choices of $(\epsilon_1,\dots,\epsilon_n)$.

Hence for $n\geq 2$,

$$K(y_1,\dots,y_n) \;=\; \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j\geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}(2^n-2m) \;=\; 2(-1)^n, \tag{5.35}$$

which comes from evaluating the coefficient of $z^n$ in (5.32) and in the generating series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n = e^{-z} \ = \ \frac{1}{1+(e^z-1)} \ = \ \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j \geq 1}} (-1)^m \frac{1}{\lambda_1!\cdots\lambda_m!}. \tag{5.36}$$

Combining (5.31) and (5.35), we see that if $\text{supp}(\widehat{\phi}) \subseteq [-\frac{1}{n-1}, \frac{1}{n-1}]$, then

$$Q_n(\phi) \ = \ (-1)^n 2^{n-1} \int_0^{\frac{1}{n-1}} \cdots \int_0^{\frac{1}{n-1}} \widehat{\phi}(y_1)\cdots\widehat{\phi}(y_n) \mathbb{1}_{\{y_1+\cdots+y_n \geq 1\}} \, dy_1 \cdots dy_n. \tag{5.37}$$

The final step in the proof of Theorem 1.7 is the observation (see Lemma D.1) that

$$\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} \, dx - \frac{1}{2}\phi(0)^n$$
$$= \ \frac{1}{2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(y_1)\cdots\widehat{\phi}(y_n) \left(\mathbb{1}_{\{|y_1+\cdots+y_n| \leq 1\}} - 1\right) \, dy_1 \cdots \, dy_n. \tag{5.38}$$

Furthermore, if we assume that $\text{supp}(\widehat{\phi}) \subseteq [-\frac{1}{n-1}, \frac{1}{n-1}]$, then $\left(\mathbb{1}_{\{|y_1+\cdots+y_n| \leq 1\}} - 1\right)$ equals zero if the $y_j$ are not all of the same sign. Under this assumption, we therefore may write

$$\begin{aligned} &\frac{1}{2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(y_1)\cdots\widehat{\phi}(y_n) \left(\mathbb{1}_{\{|y_1+\cdots+y_n| \leq 1\}} - 1\right) \, dy_1 \cdots \, dy_n \\ &= \int_0^{\frac{1}{n-1}} \cdots \int_0^{\frac{1}{n-1}} \widehat{\phi}(y_1)\cdots\widehat{\phi}(y_n) \left(\mathbb{1}_{\{y_1+\cdots+y_n \leq 1\}} - 1\right) \, dy_1 \cdots \, dy_n \\ &= -\int_0^{\frac{1}{n-1}} \cdots \int_0^{\frac{1}{n-1}} \widehat{\phi}(y_1)\cdots\widehat{\phi}(y_n) \mathbb{1}_{\{y_1+\cdots+y_n > 1\}} \, dy_1 \cdots dy_n. \end{aligned} \tag{5.39}$$

We therefore conclude that if $\text{supp}(\widehat{\phi}) \subseteq [-\frac{1}{n-1}, \frac{1}{n-1}]$, then

$$Q_n(\phi) \ = \ (-1)^{n-1} 2^{n-1} \left( \int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} \, dx - \frac{1}{2}\phi(0)^n \right) \tag{5.40}$$

as required.

**Remark 5.2.** Note that $[-\frac{1}{n-1}, \frac{1}{n-1}]$ is a natural boundary. We crucially used each $y_i \leq \frac{1}{n-1}$ in showing there are exactly $2m$ choices for the $\epsilon$-tuples which make (5.34) vanish. Indeed, beyond this point the kernel does not have the shape of (5.40), indicating the presence of additional terms. On the number theory side, these terms will arise from a more detailed study of the prime powers in Lemma 4.2. The new terms cannot arise from the integral in (4.51), as this hold for $\widehat{\phi}$ supported up to $(-\frac{2}{n}, \frac{2}{n})$.

## 6. The order of vanishing of $L$-functions at the critical point

We show how Corollary 1.9 follows from Theorem 1.6. We need to assume GRH for $L(s, f)$, which means that all non-trivial zeros are on the critical line; this allows us to obtain bounds for the number of zeros at the central point by using non-negative test functions. Note this rate of decay is significantly better than previous estimates.

We compare our results to the bounds obtained in Section 1 of [ILS]. We consider weight $k$ cuspidal newforms of prime level $N$ and odd functional equation. We use Theorem 1.6 with $n = 2$ and

$$\phi(x) \ = \ \left(\frac{\sin \pi\sigma x}{\pi\sigma x}\right)^2, \quad \widehat{\phi}(y) \ = \ \begin{cases} \frac{1}{\sigma} - \frac{|y|}{\sigma} & \text{if } |y| < \sigma \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

Arguing as in Section 1 of [ILS], we find that as $N \to \infty$ the probability that the zero at the central point is of order exactly one is at least $\frac{23}{27} - \epsilon \approx .8519 - \epsilon$, which is worse than the lower bound of $\frac{15}{16} - \epsilon = .9375 - \epsilon$ of [ILS]. To see this, take $\sigma = 1$ in (6.1); while we need $\sigma < 1$, we can take the limit as $\sigma$ approaches 1 (or better yet, introduce a factor of $\epsilon$ in the arguments). Let $p_r(N)$ be the percent of odd cuspidal newforms of weight $k$ and prime level $N$ that have exactly $r$ zeros at the central point. As our forms are odd, only odd values of $r$ are non-zero. We have

$$\sum_{j=0}^{\infty} p_{2j+1}(N) \ = \ 1, \qquad \sigma_\phi^2 - R_2(\phi) \ \le \ \frac{1}{3} + \epsilon. \tag{6.2}$$

Consider the terms $D(f;\phi) - \langle D(f;\phi)\rangle_-$ in Theorem 1.6. As $\phi(0) = \widehat{\phi}(0) = 1$ and $\phi$ is non-negative, we see that if there are $r \ge 3$ zeros at the central point, then

$$\begin{array}{rcccl} D(f;\phi) - \langle D(f;\phi)\rangle_- & \ge & r\phi(0) - \langle D(f;\phi)\rangle_- & \ge & 0 \\ & \ge & r\phi(0) - \left(\widehat{\phi}(0) + \frac{1}{2}\phi(0) + \epsilon\right) & \ge & 0 \\ & = & r - \frac{3}{2} - \epsilon & \ge & 0. \end{array}$$

If there is exactly one zero at the central point then $\langle D(f;\phi)\rangle_-$ might exceed $D(f;\phi)$, and the difference could be negative; if the difference is negative, when we square we could reverse the inequality. Thus

$$\begin{aligned} \frac{1}{3} + \epsilon \quad & \ge \quad \sigma_\phi^2 - R_2(\phi) \\ & \ge \quad \sum_{j=0}^{\infty} p_{2j+1}(N)\left(D(f;\phi) - \langle D(f;\phi)\rangle_-\right)^2 \\ & \ge \quad \sum_{j=1}^{\infty} p_{2j+1}(N)\left(2j+1-\frac{3}{2}-\epsilon\right)^2 \ \ge \ \left(\frac{9}{4}-\epsilon'\right)\sum_{j=1}^{\infty} p_{2j+1}(N). \end{aligned} \tag{6.3}$$

Therefore $\sum_{j\ge 1} p_{2j+1}(N) \le \frac{4}{27} + \epsilon''$, or $p_1(N) \ge \frac{23}{27} - \epsilon''$.

Our results are better as the order of the zeros increase. A similar analysis shows the probability that the zero at the central point is of order at least 5 is at most $\frac{4}{147} + \epsilon \approx .02721 + \epsilon$, which is better than the upper bound of $\frac{1}{32} + \epsilon = .03125 + \epsilon$ implicit in [ILS].

**Remark 6.1.** In order to obtain bounds on the order of vanishing at the central point, it is necessary to weigh each cusp form equally (by $c_k N^{-1}$). While the harmonic weights $\omega_N(f) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}(f,f)_N}$ are almost constant, by [I1, HL] they can fluctuate within the family as

$$N^{-1-\epsilon} \ \ll_k \ \omega_N(f) \ \ll_k \ N^{-1+\epsilon}; \tag{6.4}$$

if we allow ineffective constants we can replace $N^\epsilon$ with $\log N$ for $N$ large. The difficulty with using harmonic weights is that the larger weights could all be associated to $f$'s with large (or small) vanishing at the central point. This is one of the main reasons we chose to use uniform weights; see also Remark 2.11.

## Appendix A. Handling the Complementary Sum

**Lemma A.1.** *Assume GRH for $L(s,f)$ for $f \in H_k^*(1) \cup H_k^*(N)$. In the notation of Lemma 2.9, if $W = 1$ or $N$, $X = N-1$ or $N^\epsilon$, and $Y = N^\epsilon$, then*

$$\begin{aligned} & \frac{1}{|H_k^{\pm}(N)|} \sum_{\substack{q_1\nmid N,\dots,q_\ell\nmid N \\ q_j \text{ distinct}}} \prod_{j=1}^{\ell} \widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j} \left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j} \ \Delta_{k,N}^{\infty}(Wq_1^{m_1}\cdots q_\ell^{m_\ell}) \\ & \quad \ll \ O(N^{-\epsilon''}W^{-1}) \end{aligned} \tag{A.1}$$

*for some $\epsilon'' > 0$.*

*Proof.* From (2.10) we have that $H_k^{\pm}(N) \sim \frac{(k-1)N}{24}$. It suffices to show

$$\sum_{(q,N)=1} \lambda_f(q)a_q \ \ll \ (kN)^{\epsilon'}, \tag{A.2}$$

where

$$a_q \ = \ \begin{cases} \prod_{j=1}^{\ell}\left(\widehat{\phi}\left(\frac{\log q_j}{\log R}\right)\frac{\log q_j}{\sqrt{q_j}\log R}\right)^{n_j} & q = q_1^{m_1}\cdots q_\ell^{m_\ell}, q_j \le R^\alpha \text{ distinct primes, } q_j \nmid N \\ 0 & \text{otherwise.}\end{cases} \tag{A.3}$$

This is because if (A.2) holds, Lemma 2.9 implies that

$$\frac{1}{|H_k^\pm(N)|}\sum_{\substack{(q,WN)=1\\ \log q \ll \log N}} \Delta_{k,N}^\infty(Wq)a_q \ \ll \ N^{-1}\cdot N^{1-\epsilon''}W^{-1} \ \ll \ N^{-\epsilon''}W^{-1}. \tag{A.4}$$

Without loss of generality, we may relabel so that $q_1 > \cdots > q_\ell$. Up to combinatorial factors, our sum (A.1) becomes

$$\sum_{q_\ell=2}^{R^\alpha}\lambda_f(q_\ell^{m_\ell})\left(\widehat{\phi}\left(\frac{\log q_\ell}{\log R}\right)\frac{\log q_\ell}{\sqrt{q_\ell}\log R}\right)^{n_\ell}\sum_{q_{\ell-1}=q_\ell+1}^{R^\alpha}\cdots\sum_{q_1=q_2+1}^{R^\alpha}\lambda_f(q_1^{m_1})\left(\widehat{\phi}\left(\frac{\log q_1}{\log R}\right)\frac{\log q_1}{\sqrt{q_1}\log R}\right)^{n_1}. \tag{A.5}$$

The Generalized Riemann Hypothesis for $L(s,f)$ yields

$$\sum_{p\le P}\frac{\lambda_f(p)\log p}{\sqrt{p}} \ \ll \ (kN)^{\epsilon'/n} \tag{A.6}$$

if $\log P \ll \log kN$ (see equations 2.65–2.66 of [ILS]). Therefore by partial summation

$$\sum_{p\le P}\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{\lambda_f(p)\log p}{\sqrt{p}} \ \ll \ (kN)^{\epsilon'/n} \tag{A.7}$$

Thus all the sums with $n_j = 1$ are $\ll (kN)^{\epsilon'}$. For factors with $n_j > 1$, the Ramanujan bound for $\lambda_f(p)$ gives $|\lambda_f(q_j^{m_j})| \le \tau(q_j^{m_j}) \le n+1$ (as $m_j \le n$), and these prime sums are then at most

$$\sum_{p\le P}\frac{\log^{n_j} p}{p\log^{n_j} R} \ \ll \ \frac{\log^{n_j+1} P}{\log^{n_j} R} \ \ll \ (kN)^{\epsilon'/n}. \tag{A.8}$$

Thus $\sum_q \lambda_f(q)a_q \ll (kN)^{\epsilon'}$, and by Lemma 2.9 (and the remarks immediately following it), this completes the proof. □

## Appendix B. Handling the Error Terms in the Moment Expansion

In order to prove (2.29), we must show that if $\text{supp}(\widehat{\phi}) \in [-1,1]$ then

$$\left\langle\left(-P(f;\phi) + \ O\left(\frac{\log\log R}{\log R}\right)\right)^n\right\rangle_\sigma \ = \ (-1)^n\left\langle P(f;\phi)^n\right\rangle_\sigma \ + \ O\left(\frac{\log\log R}{\log R}\right), \tag{B.1}$$

where $\sigma \in \{+,-,*\}$ and where $n$ is an integer $\ge 1$. Note the $O$-term on the left hand side of (B.1) is independent of $f$. Let $P = P(f;\phi)$ and $E = O\left(\frac{\log\log R}{\log R}\right)$. Assume we know that if $\text{supp}(\widehat{\phi}) \in [-1,1]$ then

$$\left\langle P^{2m}\right\rangle_\sigma \ = \ O(1) \tag{B.2}$$

for all $0 \le m \le \frac{n}{2}$. In general, in investigations of the $n^{\text{th}}$ centered moments one has $m < \frac{n}{2}$ by induction, and handling $m = \frac{n}{2}$ is possible – in fact, this is the expected main term that we evaluate in §4. Expanding, we find

$$\left\langle(-P+E)^n\right\rangle_\sigma \ = \ (-1)^n\left\langle P^n\right\rangle_\sigma \ + \ \sum_{j=1}^{n}\binom{n}{j}\left\langle(-P)^{n-j}E^j\right\rangle_\sigma, \tag{B.3}$$

where $E = O\left(\frac{\log\log R}{\log R}\right)$ is independent of $f$. We show for all $j = 1,\dots,n$ that

$$\left\langle(-P)^{n-j}E^j\right\rangle_\sigma \ = \ O(E^j). \tag{B.4}$$

If $n-j$ is even then

$$\left\langle(-P)^{n-j}E^j\right\rangle_\sigma \ \le \ \left\langle P^{n-j}\right\rangle_\sigma O(E^j) \ = \ O(E^j) \tag{B.5}$$

since we assumed that $\langle P^{n-j}\rangle_\sigma = O(1)$ and that $E$ is independent of $f$ (so it can be taken out of the average). If $n-j$ is odd, we use the following form of Hölder's inequality: if $f, g, \mu$ are positive functions then for $0<\theta<1$,

$$\int f(x)g(x)\mu(x)\,\mathrm{d}x \;\le\; \left(\int f(x)^{1/\theta}\mu(x)\,\mathrm{d}x\right)^{\theta}\left(\int g(x)^{1/(1-\theta)}\mu(x)\,\mathrm{d}x\right)^{1-\theta}. \tag{B.6}$$

Hence

$$\begin{aligned}\left\langle(-P)^{n-j}E^j\right\rangle_\sigma &\le \left\langle|P|^{n-j}E^j\right\rangle_\sigma\\ &\le \left\langle|P|^{(n-j)/\theta}\right\rangle_\sigma^{\theta}\left\langle E^{j/(1-\theta)}\right\rangle_\sigma^{1-\theta}\\ &= \left\langle|P|^{(n-j)/\theta}\right\rangle_\sigma^{\theta} E^j.\end{aligned} \tag{B.7}$$

Now choose $\theta=(n-j)/(n-j+1)<1$, which means $(n-j)/\theta = n-j+1$. This will be even since $n-j$ is odd, and is clearly less than or equal to $n$ (as $j\ge 1$). Hence

$$\left\langle(-P)^{n-j}E^j\right\rangle_\sigma \;\le\; \left\langle P^{n-j+1}\right\rangle_\sigma^{(n-j)/(n-j+1)} E^j \;=\; O(E^j) \tag{B.8}$$

since we assumed that $\langle |P|^{n-j+1}\rangle_\sigma = O(1)$. This completes the proof of (2.29).

## Appendix C. Kloosterman Sum Expansion

As remarked in [ILS], it is advantageous to employ characters to a smaller modulus (to modulus $b$ rather than $Nb$) in expanding the Kloosterman terms.

**Lemma C.1.** *If $(P,b)=1$ and $(N,b)=1$, then*

$$S(m^2, PN; Nb) \;=\; \frac{-1}{\varphi(b)}\sum_{\chi(\mathrm{mod}\, b)}\chi(N)G_\chi(m^2)G_\chi(1)\overline{\chi}(P). \tag{C.1}$$

*Proof.* By the orthogonality relation for characters, since $(P,b)=1$ and $S(m^2,PN;Nb)$ is periodic in $P$ modulo $b$, we may write

$$\begin{aligned}S(m^2,PN;Nb) &= \frac{1}{\varphi(b)}\sum_{\chi(\mathrm{mod}\, b)}\ \sideset{}{^*}\sum_{a \bmod b}\chi(a)S(m^2,aN;Nb)\overline{\chi}(P)\\ &= \frac{1}{\varphi(b)}\sum_{\chi(\mathrm{mod}\, b)}\ \sideset{}{^*}\sum_{a \bmod b}\chi(a)\sideset{}{^*}\sum_{d \bmod Nb}e(m^2d/Nb)e(aN\overline{d}/Nb)\overline{\chi}(P)\\ &= \frac{1}{\varphi(b)}\sum_{\chi(\mathrm{mod}\, b)}\ \sideset{}{^*}\sum_{d \bmod Nb}\chi(d)e(m^2d/Nb)\sideset{}{^*}\sum_{a \bmod b}\chi(a)e(a/b)\overline{\chi}(P)\\ &= \frac{1}{\varphi(b)}\sum_{\chi(\mathrm{mod}\, b)}\ \sideset{}{^*}\sum_{d \bmod Nb}\chi(d)e(m^2d/Nb)G_\chi(1)\overline{\chi}(P).\end{aligned} \tag{C.2}$$

Since $(N,b)=1$ we may replace the sum over $d$ relatively prime to $Nb$ with $d=u_1N+u_2b$, with $u_1$ mod $b$ relatively prime to $b$ and $u_2$ mod $N$ relatively prime to $N$. As $\chi$ is a character modulo $b$, $\chi(u_1N+u_2b)=\chi(u_1N)$. Thus

$$\begin{aligned}\sideset{}{^*}\sum_{d \bmod Nb}\chi(d)e(m^2d/Nb) &= \sideset{}{^*}\sum_{u_1 \bmod b}\chi(u_1N)e(m^2u_1/b)\sideset{}{^*}\sum_{u_2 \bmod N}e(m^2u_2/N)\\ &= \chi(N)G_\chi(m^2)\cdot\left[-1+\sum_{u_2=0}^{N-1}e(m^2u_2/N)\right]\\ &= -\chi(N)G_\chi(m^2),\end{aligned} \tag{C.3}$$

because the $u_1$-sum is $G_\chi(m^2)$ and $N$ is prime. Substituting back yields the lemma. □

The reason for using characters with smaller moduli is that we obtain a savings in estimating the contributions from the non-principal characters.

**Lemma C.2.** *We have*

$$\frac{1}{\varphi(b)}\sum_{\chi(\text{mod}\, b)}\left|G_\chi(m^2)G_\chi(1)\right| \ \ll\ \varphi(b)\ \ll\ b; \tag{C.4}$$

*Proof.* From the orthogonality of the characters we have

$$\sum_{\chi(\text{mod}\, b)}|G_\chi(n)|^2 \ =\ \varphi(b)^2, \tag{C.5}$$

and (C.4) follows from this bound and the Cauchy-Schwartz inequality. □

Note that if we used characters of modulus $Nb$ then the bound $b$ would be replaced with $Nb$.

**Lemma C.3.** *Assume* $(Q,N)=(N,b)=1$*, and set* $r=(Q,b)$*,* $b'=b/r$ *and* $Q'=Q/r$*. If additionally* $(r,b')=1$ *then*

$$S(m^2,Q;Nb) \ =\ \frac{1}{\varphi(Nb/r)}\sum_{\chi(\text{mod}\, Nb/r)}\overline{\chi}(Q/r)\chi(r)R(m^2,r)G_\chi(m^2)G_\chi(1). \tag{C.6}$$

*Proof.* From the definition of $r$ we have $(Q',b')=1$ (if not, $r$ is not the greatest common divisor of $Q$ and $b$). By the orthogonality relation for characters, since $(Q',Nb')=1$ we may write

$$\begin{aligned}
S(m^2,Q;Nb) \ &=\ S(m^2,Q'r;Nb'r)\\
&=\ \frac{1}{\varphi(Nb')}\sum_{\chi(\text{mod}\, Nb')}\ \sideset{}{^*}\sum_{a \bmod Nb'}\chi(a)\overline{\chi}(Q')S(m^2,ar;Nb'r)\\
&=\ \frac{1}{\varphi(Nb')}\sum_{\chi(\text{mod}\, Nb')}\ \sideset{}{^*}\sum_{a \bmod Nb'}\chi(a)\overline{\chi}(Q')\ \sideset{}{^*}\sum_{d \bmod Nb'r}e(m^2d/Nb'r)e(ar\overline{d}/Nb'r)\\
&=\ \frac{1}{\varphi(Nb')}\sum_{\chi(\text{mod}\, Nb')}\ \sideset{}{^*}\sum_{d \bmod Nb'r}\overline{\chi}(Q')\chi(d)e(m^2d/Nb'r)\ \sideset{}{^*}\sum_{a \bmod Nb'}\chi(a)e(a/Nb')\\
&=\ \frac{1}{\varphi(Nb')}\sum_{\chi(\text{mod}\, Nb')}\ \sideset{}{^*}\sum_{d \bmod Nb'r}\overline{\chi}(Q')\chi(d)e(m^2d/Nb'r)G_\chi(1).
\end{aligned} \tag{C.7}$$

As $r|Q$ and $(Q,N)=1$, $(r,N)=1$. Thus $(Nb',r)=1$, and we may replace the sum over $d$ relatively prime to $Nb'r$ with $d=u_1Nb'+u_2r$, with $u_1 \bmod r$ relatively prime to $r$ and $u_2 \bmod Nb'$ relatively prime to $Nb'$. As $\chi$ is a character modulo $Nb'$, $\chi(u_1Nb'+u_2r)=\chi(u_2r)$. Thus

$$\begin{aligned}
\sideset{}{^*}\sum_{d \bmod Nb'r}\chi(d)e(m^2d/Nb'r) \ &=\ \sideset{}{^*}\sum_{u_1 \bmod r}e(m^2u_1Nb'/Nb'r)\ \sideset{}{^*}\sum_{u_2 \bmod Nb'}\chi(u_2r)e(m^2u_2r/Nb'r)\\
&=\ \sideset{}{^*}\sum_{u_1 \bmod r}e(m^2u_1/r)\cdot\chi(r)\cdot\ \sideset{}{^*}\sum_{u_2 \bmod Nb'}\chi(u_2)e(m^2u_2/Nb')\\
&=\ \chi(r)R(m^2,r)G_\chi(m^2),
\end{aligned} \tag{C.8}$$

because by (2.2) the $u_1$-sum is $R(m^2,r)$ and by (2.1) the $u_2$-sum is $G_\chi(m^2)$. Substituting back yields the lemma. □

## Appendix D. Fourier Transform Identities

Let $\mathbb{1}_{\{|u|\le 1\}}$ be the characteristic function of $[-1,1]$. Let $S(x)=\frac{\sin\pi x}{\pi x}$. Note that

$$S(2x) \ =\ \int_{-\infty}^{\infty}\frac{1}{2}\mathbb{1}_{\{|u|\le 1\}}\ e^{2\pi i x u}\,\mathrm{d}u, \tag{D.1}$$

so $S(2x)$ and $\frac12\mathbb{1}_{\{|u|\le 1\}}$ are a Fourier transform pair. All test functions below will be even Schwartz functions whose Fourier transforms have finite support. We have made much use of a certain Fourier transform identity; we give the proof here for completeness.

**Lemma D.1.** *We have*

$$\int_{-\infty}^{\infty} \phi(x)^n S(2x)\ \mathrm{d}x - \frac{1}{2}\phi(0)^n = -\frac{1}{2}\int\cdots\int_{-\infty}^{\infty} \widehat{\phi}(u_1)\cdots\widehat{\phi}(u_n)\mathbb{1}_{\{|u_1+\cdots+u_n|>1\}}\ \mathrm{d}u_1\cdots\ \mathrm{d}u_n. \quad \text{(D.2)}$$

*Proof.* This lemma follows from Plancherel's identity, which states that if $f$ and $g$ are Schwartz functions (in fact it is true for a much larger class of functions) then

$$\int f(x)g(x)\ \mathrm{d}x = \int \widehat{f}(u)\widehat{g}(u)\ \mathrm{d}u. \quad \text{(D.3)}$$

In this particular case it is more complicated since we are integrating $n+1$ functions. We obtain

$$\phi(0)^n = \int\cdots\int_{-\infty}^{\infty} \widehat{\phi}(u_1)\cdots\widehat{\phi}(u_n)\ \mathrm{d}u_1\cdots\ \mathrm{d}u_n \quad \text{(D.4)}$$

and

$$\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x}\ \mathrm{d}x = \frac{1}{2}\int\cdots\int_{-\infty}^{\infty} \widehat{\phi}(u_1)\cdots\widehat{\phi}(u_n)\mathbb{1}_{\{|u_1+\cdots+u_n|\le 1\}}\ \mathrm{d}u_1\cdots\ \mathrm{d}u_n, \quad \text{(D.5)}$$

where we have used (D.1) and (repeatedly)

$$\widehat{fg}(u) = \int_{-\infty}^{\infty} \widehat{f}(v)\widehat{g}(u-v)\ \mathrm{d}v. \quad \text{(D.6)}$$

Combining (D.4) and (D.5) yields

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(x)^n S(2x)\ \mathrm{d}x - \frac{1}{2}\phi(0)^n &= \frac{1}{2}\int\cdots\int_{-\infty}^{\infty} \widehat{\phi}(u_1)\cdots\widehat{\phi}(u_n)\left(\mathbb{1}_{\{|u_1+\cdots+u_n|\le 1\}} - 1\right)\ \mathrm{d}u_1\cdots\ \mathrm{d}u_n \\ &= -\frac{1}{2}\int\cdots\int_{-\infty}^{\infty} \widehat{\phi}(u_1)\cdots\widehat{\phi}(u_n)\mathbb{1}_{\{|u_1+\cdots+u_n|>1\}}\ \mathrm{d}u_1\cdots\ \mathrm{d}u_n, \quad \text{(D.7)}\end{aligned}$$

which is (D.2). □

## Appendix E. Increasing the Support in Theorem 1.1

As it stands, Theorem 1.1 holds for $\mathrm{supp}(\widehat{\phi}) \subset \left(-\frac{2}{n}\left(1-\frac{1}{2k}\right), \frac{2}{n}\left(1-\frac{1}{2k}\right)\right)$. We show how a more careful book-keeping and using GRH for Dirichlet $L$-functions allows us to remove the factors of $\frac{1}{2k}$ for $n \ge 2$ and $2k \ge n$. In particular, we prove

**Theorem E.1.** *Assume GRH for Dirichlet L-functions. If $2k \ge n$ then Theorem 1.1 holds for even Schwartz test functions supported in $(-\frac{2}{n}, \frac{2}{n})$.*

In proving Lemma 3.1 (which is equivalent to Theorem 1.1) we showed, without any restriction on the support of $\widehat{\phi}$, that for $S_1^{(n)}$ defined as in (2.34), then under GRH for $L(s,f)$

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} S_1^{(n)} = \begin{cases} \frac{(2m)!}{2^m m!}\left(2\int_{-\infty}^{\infty}|y|\widehat{\phi}(y)^2\ \mathrm{d}y\right)^{2m} + E(n) & \text{if } n=2m \text{ is even} \\ E(n) & \text{if } n \text{ is odd,}\end{cases} \quad \text{(E.1)}$$

where $E(n)$ is made up of a linear combination of terms like

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \sum_{\substack{q_1,\dots,q_\ell \\ q_j \text{ distinct primes}}} \prod_{j=1}^{\ell} \widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j}\left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j} \times \sum_{m\le N^\epsilon} \frac{2\pi \mathrm{i}^k}{m}\sum_{b=1}^{\infty} \frac{S(m^2, q_1^{m_1}\cdots q_\ell^{m_\ell}, Nb)}{Nb} J_{k-1}\left(4\pi m\frac{\sqrt{q_1^{m_1}\cdots q_\ell^{m_\ell}}}{Nb}\right), \quad \text{(E.2)}$$

where $n_j \geq 1$ with $n_1+\cdots+n_\ell = n$, and $m_j \leq n_j$ with $m_j \equiv n_j \bmod 2$. Lemma 3.1 followed from this by showing $E(n) = 0$ if $\text{supp}(\widehat{\phi}) \subset (-\frac{2k-1}{kn}, \frac{2k-1}{kn})$, via the bound (2.4) on the Kloosterman sum, and the bound from Lemma 2.6(3) on the Bessel function. We prove Theorem E.1 by showing, in a sequence of lemmas, that whenever $2k \geq n$ then $E(n) = 0$ for $\text{supp}(\widehat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$.

For simplicity we write $Q = q_1^{m_1} \dots q_\ell^{m_\ell}$ and $\text{supp}(\widehat{\phi}) \subset [-\sigma, \sigma] \subset (-\frac{2}{n}, \frac{2}{n})$. As $n \geq 2$, $(Q, N) = 1$. Set $r = (Q, Nb)$, $Q' = Q/r$ and $b' = b/r$. If additionally $(r, b/r) = 1$ then Lemma C.3 yields

$$S\left(m^2, \frac{Q}{r}r, \frac{Nb}{r}r\right) = \frac{1}{\varphi(Nb/r)} \sum_{\chi (\text{mod } Nb/r)} \overline{\chi}(Q/r)\chi(r)R(m^2, r)G_\chi(m^2)G_\chi(1). \tag{E.3}$$

We sketch the proof of Theorem E.1. In §E.2 we handle the terms in $E(n)$ with $(r, b/r) > 1$ (and thus the expansion of Lemma C.3 is unavailable), thereby reducing the proof to an analysis of the terms with $(r, b/r) = 1$. In §E.2 we show we may truncate the $b$-sum at $N$; this is useful as some later terms are $\sum_b b^{-1}$. We then show in §E.3 that we may assume $r = 1$, and then use Lemma C.3 to expand the Kloosterman sums. The proof is completed in §E.4 where we bound the contributions from the character sums arising from the Kloosterman expansions; it is here that we must assume GRH for Dirichlet $L$-functions.

E.1. **Bounding the terms with** $(r, b/r) > 1$**.**

**Lemma E.2.** *Notation as above, the contribution to* $E(n)$ *from terms with* $(r, b/r) > 1$ *is negligible for* $\text{supp}(\widehat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$*, provided that* $8k - 2 \geq n$.

*Proof.* As $Q = q_1^{m_1} \cdots q_\ell^{m_\ell}$ is a product over distinct primes, if $(r, b/r) > 1$ then the square of some $q_j$ divides $b$. Without loss of generality we may assume $b = q_1^2 v$. For $\text{supp}(\widehat{\phi})$ and $k$ as in the lemma, we show the contribution to $E(n)$ (equation (E.2)) from each tuple $(n_1, \dots, m_\ell)$ is negligible. This proves the lemma as the number of such tuples depends only on $n$ and not $N$.

We use $J_{k-1}(x) \ll x^{k-1}$. This is the best available Bessel bound for our purposes, as our argument is at most $N^\epsilon$. We use (2.4) to bound the Kloosterman sum. Thus the contribution from such a tuple is

$$\begin{aligned} E(\overrightarrow{n}, \overrightarrow{m}) &\ll \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{v=1}^{\infty} \frac{(Nq_1^2 v/q_1)^{\frac{1}{2}+\epsilon}}{Nq_1^2 v} \sum_{q_1, \dots, q_\ell = 2}^{N^\sigma} (q_1 \cdots q_\ell)^{-\frac{n_j}{2}} \left(\frac{m\sqrt{q_1 \cdots q_\ell}}{Nq_1^2 v}\right)^{k-1} \\ &\ll N^{\frac{1}{2}-k+\epsilon'} \sum_{m \leq N^\epsilon} m^{k-2} \sum_{v=1}^{\infty} \frac{1}{v^{k-\frac{1}{2}}} \sum_{q_1=2}^{N^\sigma} q_1^{-\frac{n_1}{2}+\frac{m_1}{2}(k-1)-2k+\frac{1}{2}} \prod_{j=2}^{\ell} \sum_{q_j=2}^{N^\sigma} q_j^{-\frac{n_j}{2}+\frac{m_j}{2}(k-1)}. \end{aligned} \tag{E.4}$$

The worst case is when $m_j = n_j$. As the $v$-sum is $O(1)$, the $m$-sum is $O(N^{(k-1)\epsilon})$, and $\sum n_j = n$,

$$E(\overrightarrow{n}, \overrightarrow{m}) \ll N^{\frac{1}{2}-k+\epsilon''} N^{(\frac{nk}{2}-n+\ell-2k+\frac{1}{2})\sigma}. \tag{E.5}$$

The worst case is when $\ell = n$, and we find

$$E(\overrightarrow{n}, \overrightarrow{m}) \ll N^{\frac{1}{2}((nk-4k+1)\sigma-(2k-1))+\epsilon'''}. \tag{E.6}$$

This is negligible provided that $\sigma < \frac{2k-1}{nk-4k+1}$. If $8k - 2 \geq n$, simple algebra shows $\frac{2k-1}{nk-4k+1} \geq \frac{2}{n}$. □

E.2. **Restricting the** $b$**-sum.**

**Lemma E.3.** *If* $a \geq \frac{1}{2k-3}$*, then the contribution to* $E(n)$ *(equation* (E.2)*) from the* $b \geq N^a$ *terms is negligible for any admissible* $(m, n)$*-tuple.*

*Proof.* Using $J_{k-1}(x) \ll x^{k-1}$ and $S(m^2, Q, bN) \ll (bN)^{\frac{1}{2}} N^{\epsilon}$, the contribution to (E.2) from terms with $b \geq N^a$ is

$$\begin{aligned} &\ll \quad N^{-1} \sum_{m \leq N^{\epsilon}} m^{k-2} \prod_{j=1}^{\ell} \left( \sum_{q_j=2}^{N^{\sigma}} q_j^{\frac{m_j}{2}(k-1) - \frac{n_j}{2}} \right) \sum_{b=N^a}^{\infty} \frac{b^{\frac{1}{2}} N^{\frac{1}{2}+\epsilon}}{b \cdot b^{k-1}} \frac{1}{N^{k-1}} \\ &\ll \quad N^{-k+\frac{1}{2}+\epsilon'} N^{\frac{nk\sigma}{2}} N^{\left(-k+\frac{3}{2}\right)a}, \end{aligned} \tag{E.7}$$

because the worst case is when $\ell = n$ and $m_j = n_j$. If $\sigma < \frac{2}{n}$ then for $\epsilon'$ sufficiently small there is no contribution provided

$$\frac{1}{2} + \left(-k + \frac{3}{2}\right) a \ \leq \ 0, \tag{E.8}$$

which means $a \geq \frac{1}{2k-3}$. As $k \geq 2$, $a \leq 1$. □

E.3. **Restricting the $r$-sum and Expanding the Kloosterman Sum.** The proof of Theorem E.1 is therefore reduced to showing that there is no contribution from admissible $(m,n)$-tuples as $N \to \infty$ in

$$\begin{aligned} &E(\overrightarrow{n}, \overrightarrow{m}) \ = \\ &\sum_{\substack{q_1,\dots,q_\ell \\ q_j \text{distinct}}} \prod_{j=1}^{\ell} \widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j} \left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j} \sum_{m \leq N^{\epsilon}} \frac{2\pi \mathrm{i}^k}{m} \sum_{\substack{b=1 \\ (Q,b)=r,(r,b/r)=1}}^{N^{1/(2k-3)}} \frac{S(m^2,Q,Nb)}{bN} J_{k-1}\left(4\pi m \frac{\sqrt{Q}}{bN}\right); \end{aligned} \tag{E.9}$$

by an admissible $(m,n)$-tuple we mean $\sum n_j = n$, $m_j \leq n_j$ and $m_j \equiv n_j \bmod 2$. As $r = (Q,b)$ and $Q = q_1^{m_1} \cdots q_\ell^{m_\ell}$, we may write

$$r \ = \ q_1^{c_1} \cdots q_\ell^{c_\ell}, \quad c_j \in \{0, \dots, m_j\}. \tag{E.10}$$

For a given $m$-tuple $(m_1, \dots, m_\ell)$, the number of $c$-tuples is $\prod_{j=1}^{\ell}(m_j+1) \ll (2n)^n$. We show

**Lemma E.4.** *Notation as above, a $c$-tuple with $\sum c_j = c$ has a negligible contribution to $E(\overrightarrow{n}, \overrightarrow{m})$ for $\text{supp}(\widehat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$, provided that $4ck \geq n$. In particular, if $4k \geq n$ then to bound $E(\overrightarrow{n}, \overrightarrow{m})$ it suffices to consider only the contributions from the $c$-tuple $(0, \dots, 0)$ (i.e., $r = 1$).*

*Proof.* We use (2.4) to bound the Kloosterman sum, and $J_{k-1}(x) \ll x^{k-1}$. For a fixed $c$-tuple with $\sum c_j = c$ we have $b = b'r$. We insert absolute values and ignore the condition $(r, b/r) = 1$, as this only increases the sums below. We have

$$E(\overrightarrow{n}, \overrightarrow{m}, \overrightarrow{c}) \quad \ll \quad \sum_{m \leq N^{\epsilon}} \frac{1}{m} \sum_{q_1,\dots,q_\ell=2}^{N^{\sigma}} \sum_{b'=1}^{N} \frac{(Nb')^{\frac{1}{2}+\epsilon}}{Nb' q_1^{c_1} \cdots q_\ell^{c_\ell}} \prod_{j=1}^{\ell} q_j^{-\frac{n_j}{2}} \cdot \left( \frac{m\sqrt{q_1^{m_1} \cdots q_\ell^{m_\ell}}}{b' q_1^{c_1} \cdots q_\ell^{c_\ell} N} \right)^{k-1}. \tag{E.11}$$

The $m$-sum is $O(N^{(k-1)\epsilon})$, the $b'$-sum is $O(1)$, and the worst case is when each $m_j = n_j$. Thus

$$\begin{aligned} E(\overrightarrow{n}, \overrightarrow{m}, \overrightarrow{c}) \quad &\ll \quad N^{\frac{1}{2}-k+\epsilon'} \prod_{j=1}^{\ell} \sum_{q_j=2}^{N^{\sigma}} q_j^{\frac{n_j k}{2} - n_j - c_j k} \\ &\ll \quad N^{\frac{1}{2}-k+\epsilon'} N^{(\frac{nk}{2} - n + \ell - ck)\sigma}. \end{aligned} \tag{E.12}$$

As usual, the worst case is when $\ell = n$, and we find

$$E(\overrightarrow{n}, \overrightarrow{m}, \overrightarrow{c}) \ \ll \ N^{\frac{1}{2}((nk-2ck)\sigma - (2k-1)) + \epsilon''}; \tag{E.13}$$

this is negligible provided that $\sigma < \frac{2k-1}{nk-2ck}$. If $4ck \geq n$ then $\frac{2k-1}{nk-2ck} \geq \frac{2}{n}$. Thus all $c$-tuples with $\sum c_j \geq 1$ yield negligible contributions for $\text{supp}(\widehat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$, provided that $4k \geq n$. □

**Remark E.5.** While assuming $4k \geq n$ is more restrictive than $8k \geq n+2$ (the relation from Lemma E.2), this allows us to take $r = 1$ below, and greatly simplifies the arguments. At the cost of a more involved argument one could analyze the $c$-tuples where $\sum c_j \in \{1, 2\}$.

Thus the proof of Theorem E.1 is reduced to bounding $E(\vec{n}, \vec{m}, \vec{0})$. In this case $r = 1$, so $Q' = Q$ and $b' = b$. From (2.2) we have $R(m^2, 1) = 1$. By Lemma C.3, we are left with bounding

$$E(\vec{n}, \vec{m}, \vec{0}) \;=\; \sum_{\substack{q_1,\dots,q_\ell \\ q_j \text{distinct}}} \prod_{j=1}^{\ell} \widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j} \left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j} \sum_{m \leq N^\epsilon} \frac{2\pi i^k}{m}$$
$$\cdot \sum_{\substack{b=1 \\ (Q,b)=1}}^{N^{1/(2k-3)}} \sum_{\substack{\chi \bmod Nb \\ \chi^{m_j} \neq \chi_0}} \frac{\overline{\chi}(Q) G_\chi(m^2) G_\chi(1)}{Nb\varphi(Nb)} J_{k-1}\left(4\pi m \frac{\sqrt{Q}}{bN}\right). \quad \text{(E.14)}$$

E.4. **Using GRH for Dirichlet $L$-Functions.** We use GRH for Dirichlet $L$-functions to show (E.14) is negligible in the desired range. Note

$$\overline{\chi}(Q) \;=\; \overline{\chi^{m_1}}(q_1) \cdots \overline{\chi^{m_\ell}}(q_\ell). \quad \text{(E.15)}$$

There is a slight complication due to the fact that $\chi$ may not be the principal character, but a $\chi^{m_j}$ is the principal character. We say $\chi$ is a **bad character** if $\chi^{m_j}$ is the principal character for at least one non-zero $m_j$; otherwise $\chi$ is a **good character**. Fortunately, as $(N, b) = 1$ and $N$ is prime, for each admissible $(m, n)$-tuple the number of bad characters is $O_n(b)$; this follows from the structure theorem for finite abelian groups and the fact that our prime $N$ is relatively prime to $b$.

The proof of Theorem E.1 is completed in the following two lemmas (Lemmas E.6 and E.7), which combined show that for an admissible $(m, n)$-tuple, the contribution to $E(\vec{n}, \vec{m}, \vec{0})$ from both the bad and the good characters is negligible.

**Lemma E.6.** *For an admissible $(m, n)$-tuple, the contribution to $E(\vec{n}, \vec{m}, \vec{0})$ from the bad characters is negligible for* $\text{supp}(\widehat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$.

*Proof.* There are at most $O_n(b)$ bad characters. We insert absolute values in all sums below. Thus the worst case is when each $\chi^{m_j}$ is the principal character $\chi_0$. As $G_\chi(a)$ is a Gauss sum for a character of modulus $bN$, we have $G_\chi(a) \ll \sqrt{bN}$ if $\chi$ is not the principal character; otherwise by (2.2) and $m \leq N^\epsilon$ we find $G_{\chi_0}(m^2) G_{\chi_0}(1) \ll m^4 \cdot 1 \ll N^{4\epsilon}$. We use $J_{k-1}(x) \ll x^{k-1}$ and insert absolute values. Thus we only increase the sum when we remove the condition that $(Q, b) = 1$. The contribution to $E(\vec{n}, \vec{m}, \vec{0})$ from the bad characters is bounded by

$$E_{\text{bad}}(\vec{n}, \vec{m}, \vec{0}) \;\ll\; \sum_{m \leq N^\epsilon} \frac{m^{k-1}}{m} \sum_{b=1}^{N^{1/(2k-3)}} \frac{1}{bN} \sum_{\chi \text{ bad}} \frac{|G_\chi(m^2) G_\chi(1)|}{\varphi(bN)}$$
$$\cdot \prod_{j=1}^{\ell} \left( \sum_{q_j=2}^{N^\sigma} \chi_0(q_j) \log q_j \cdot q_j^{\frac{m_j(k-1)}{2} - \frac{n_j}{2}} \right) \cdot \frac{1}{b^{k-1} N^{k-1}}. \quad \text{(E.16)}$$

As usual, the worst case is when each $m_j = n_j$ and $\ell = n$. The product of the $q_j$-sums is at most $N^{\frac{nk}{2}\sigma}$. The sum over the bad characters is $\ll O(b)$ as each summand $\frac{|G_\chi(m^2) G_\chi(1)|}{\varphi(bN)}$ is $O(1)$. The $m$-sum is $O(N^{(k-1)\epsilon})$, and as $k \geq 2$, the $b$-sum is $\sum_{b \leq N} b^{-1} \ll \log N$. Thus

$$E_{\text{bad}}(\vec{n}, \vec{m}, \vec{0}) \;\ll\; N^{-k+\epsilon'} N^{\frac{nk}{2}\sigma}, \quad \text{(E.17)}$$

which is negligible provided that $\sigma < \frac{2}{n}$. □

**Lemma E.7.** *Assume GRH for all Dirichlet L-functions. For an admissible $(m, n)$-tuple and $2k \geq n$, the good characters' contribution to $E(\vec{n}, \vec{m}, \vec{0})$ is negligible for* $\text{supp}(\widehat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$.

*Proof.* A modification of the argument in Lemma C.2 gives

$$\frac{1}{\varphi(bN)} \sum_{\substack{\chi (\mathrm{mod}\, bN) \\ \chi \text{ good}}} \left|G_\chi(m^2)G_\chi(1)\right| \ \ll\ \varphi(bN) \ \ll\ bN. \tag{E.18}$$

We have

$$\begin{aligned} E_{\mathrm{good}}(\overrightarrow{n}, \overrightarrow{m}, \overrightarrow{0}) \quad &= \quad \sum_{\substack{q_1,\dots,q_\ell \\ q_j \text{distinct}}} \prod_{j=1}^{\ell} \widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j} \left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j} \sum_{m \le N^\epsilon} \frac{2\pi \mathrm{i}^k}{m} \\ &\quad \cdot \sum_{\substack{b=1 \\ (Q,b)=1}}^{N^{1/(2k-3)}} \sum_{\chi \bmod Nb} \frac{\overline{\chi}(Q) G_\chi(m^2) G_\chi(1)}{Nb\varphi(Nb)} J_{k-1}\left(4\pi m \frac{\sqrt{Q}}{bN}\right). \end{aligned} \tag{E.19}$$

As each $\chi$ is a character modulo $b$, if $(Q,b) > 1$ then $\overline{\chi}(Q) = 0$. Thus we may drop the condition that $(Q,b) = 1$.

We first show that we may assume each $m_j \neq 0$ if $2k \ge n$; note if an $m_j = 0$ then $\chi^{m_j}$ is the principal character for a trivial reason. Without loss of generality, assume $m_1 = 0$. Since $m_j \equiv n_j \bmod 2$, $n_1 \ge 2$ and $\ell \le n-1$. We use $J_{k-1}(x) \ll x^{k-1}$; as $m_1 = 0$ there are no factors of $q_1$ arising from the argument of the Bessel function. Thus the $q_1$-sum in $E_{\mathrm{good}}(\overrightarrow{n}, \overrightarrow{m}, \overrightarrow{0})$ is $O(\log N)$. The worst case is when each remaining $m_j = n_j$ and $\ell = n-1$ (thus $n_1 = 2$). By (E.18), $\sum_{\chi \bmod Nb} \frac{\overline{\chi}(Q)G_\chi(m^2)G_\chi(1)}{Nb\varphi(Nb)} \ll 1$. The $b$-sum is $O(1)$, the $m$-sum is $O(N^{(k-1)\epsilon})$, and we have a contribution bounded by

$$\ll\ N^{1-k+\epsilon'} N^{\frac{(n-2)k}{2}\sigma} \ =\ N^{\frac{1}{2}((n-2)k\sigma - (2k-2)) + \epsilon''}. \tag{E.20}$$

This is negligible for $\sigma < \frac{2k-2}{(n-2)k}$, and if $2k \ge n$ then $\frac{2k-2}{(n-2)k} \ge \frac{2}{n}$.

Thus we may now assume each $m_j \neq 0$ and each $\chi^{m_j} \neq \chi_0$. We fix a $b$ and consider the $q_j$-sums. We use partial summation and GRH for Dirichlet $L$-functions to convert the $q_j$-sums to integrals. Since $q_j < N$ and $\chi^{m_j}$ is not the principal character, under GRH we have for $u_j \le N$ that

$$\sum_{q_j \le u_j} \overline{\chi}(q_j^{m_j}) \ =\ H(u_j) \ \ll\ u_j^{\frac{1}{2}} N^\epsilon, \tag{E.21}$$

where $H(u)$ is a non-differentiable function. This and the compact support of $\widehat{\phi}$ imply that

$$\begin{aligned} &\sum_{q_1=2}^{N^\sigma} \overline{\chi}(q_1^{m_1}) \cdot \widehat{\phi}\left(\frac{\log q_1}{\log R}\right)^{n_1} \left(\frac{2\log q_1}{\sqrt{q_1}\log R}\right)^{n_1} J_{k-1}\left(4\pi m \frac{\sqrt{q_1^{m_1} q_2^{m_2} \cdots q_\ell^{m_\ell}}}{bN}\right) \\ &= \int_1^{N^\sigma} H(u_1) \frac{\mathrm{d}}{\mathrm{d}u_1}\left[\left(\widehat{\phi}\left(\frac{\log u_1}{\log R}\right) \frac{2\log u_1}{\sqrt{u_1}\log R}\right)^{n_1} J_{k-1}\left(4\pi m \frac{\sqrt{u_1^{m_1} q_2^{m_2} \cdots q_\ell^{m_\ell}}}{bN}\right)\right] \mathrm{d}u_1. \end{aligned} \tag{E.22}$$

Proceeding in this manner we find

$$\begin{aligned} &\prod_{j=1}^{\ell} \sum_{q_j=2}^{N^\sigma} \overline{\chi}(q_j^{m_j}) \cdot \widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{n_j} \left(\frac{2\log q_j}{\sqrt{q_j}\log R}\right)^{n_j} J_{k-1}\left(4\pi m \frac{\sqrt{Q}}{bN}\right) \\ &= \int \cdots \int_1^{N^\sigma} H(u_1) \cdots H(u_\ell) \\ &\quad \cdot \frac{\mathrm{d}^\ell}{\mathrm{d}u_1 \cdots \mathrm{d}u_\ell} \prod_{j=1}^{\ell} \left[\left(\widehat{\phi}\left(\frac{\log u_j}{\log R}\right) \frac{2\log u_j}{\sqrt{u_j}\log R}\right)^{n_j} J_{k-1}\left(4\pi m \frac{\sqrt{u_1^{m_1} \cdots u_\ell^{m_\ell}}}{bN}\right)\right] \mathrm{d}u_1 \cdots \mathrm{d}u_\ell \\ &= I(m, b, N; \overrightarrow{n}, \overrightarrow{m}). \end{aligned} \tag{E.23}$$

Therefore

$$E_{\text{good}}(\vec{n}, \vec{m}, \vec{0}) \quad = \quad \sum_{m \le N^\epsilon} \frac{|2\pi \mathrm{i}^k|}{m} \sum_{\substack{b=1 \\ (Q,b)=1}}^{N^{1/(2k-3)}} \sum_{\chi \bmod Nb} \frac{|G_\chi(m^2) G_\chi(1)|}{Nb\varphi(Nb)} \; I(m, b, N; \vec{n}, \vec{m}). \tag{E.24}$$

As $\sigma < \frac{2}{n}$, the derivative in (E.23) is bounded by

$$\sum_{t=1}^{\ell} \left| \prod_{j=1}^{\ell} u_j^{-\frac{n_j}{2}-1} J_{k-1}^{(t)} \left( 4\pi m \frac{\sqrt{u_1^{m_1} \cdots u_\ell^{m_\ell}}}{bN} \right) \right|. \tag{E.25}$$

This follows because each derivative of a $\left(\widehat{\phi}\left(\frac{\log u_j}{\log R}\right) \frac{2\log u_j}{\sqrt{u_j}\log R}\right)^{n_j}$ with respect to $u_j$ decreases the exponent of $u_j$ by 1. If the differentiation hits the Bessel piece, we get the derivative of the Bessel function, as well as a factor of $\frac{4\pi m \cdot u_1^{m_1/2} \cdots u_\ell^{m_\ell/2}}{bN} \frac{1}{u_j}$. Because $\sigma < \frac{2}{n}$ and $m \ll N^\epsilon$, the first factor is at most $O(N^\epsilon)$ and thus this factor is bounded by $N^\epsilon u_j^{-1}$. If additional differentiations with respect to $u_h$ hit powers of $\frac{4\pi m \cdot u_1^{m_1/2} \cdots u_\ell^{m_\ell/2}}{bN}$, the effect is still just to reduce the exponent of $u_h$ by 1. Further, by Lemma 2.6(5), $2J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$. Using this relation $\ell$ times, as well as the bound $J_{k-1}(x) \ll x$, we find that $J_{k-1}^{(t)}(x) \ll x$ and thus the derivative term in (E.23) is

$$\ll \; (u_1 \cdots u_\ell)^{-1} \prod_{j=1}^{\ell} u_j^{-\frac{n_j}{2}} \cdot \frac{4\pi m \cdot u_1^{\frac{m_1}{2}} \cdots u_\ell^{\frac{m_\ell}{2}}}{bN} \; \ll \; \frac{m}{bN u_1 \cdots u_\ell}, \tag{E.26}$$

as the worst case is when $m_j = n_j$. Thus, inserting absolute values and approximating the Bessel function as above, the integral in (E.23) satisfies

$$I(m, b, N; \vec{n}, \vec{m}) \; \ll \; N^{\epsilon'} \int \cdots \int_1^{N^\sigma} \frac{m}{bN} \frac{\mathrm{d}u_1 \cdots \mathrm{d}u_\ell}{\sqrt{u_1 \cdots u_\ell}} \; \ll \; \frac{m}{bN} N^{\frac{\ell\sigma}{2}+\epsilon'}. \tag{E.27}$$

As usual, the worst case is when $\ell = n$, and $I(m, b, N; \vec{n}, \vec{m}) \ll N^{\frac{n\sigma}{2}-1+\epsilon'} m/b$. Substituting this bound into (E.24) yields

$$E_{\text{good}}(\vec{n}, \vec{m}, \vec{0}) \quad \ll \quad \sum_{m \le N^\epsilon} \sum_{\substack{b=1 \\ (Q,b)=1}}^{N^{1/(2k-3)}} \frac{1}{b} \left[ \sum_{\chi \bmod Nb} \frac{|G_\chi(m^2) G_\chi(1)|}{Nb\varphi(Nb)} \right] N^{\frac{n\sigma}{2}-1+\epsilon'}. \tag{E.28}$$

By (E.18) the bracketed character sum is $O(1)$. Thus the $b$-sum is $O(\log N)$ and the $m$-sum is $O(N^\epsilon)$. Thus

$$E_{\text{good}}(\vec{n}, \vec{m}, \vec{0}) \quad \ll \quad N^{\frac{n\sigma}{2}-1+\epsilon''}, \tag{E.29}$$

which is negligible for $\sigma < \frac{2}{n}$. □

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## References


[CFKRS] B. Conrey, D. Farmer, P. Keating, M. Rubinstein and N. Snaith, *Integral Moments of L-Functions*, Proc. Lond. Math. Soc. **91** (2005), 33–104.

[DM] E. Dueñez and S. J. Miller, *The Low Lying Zeros of a* GL(4) *and a* GL(6) *family of L-functions* (preprint) `arxiv.org/abs/math.NT/0506462`.

[FI] E. Fouvry and H. Iwaniec, *Low-lying zeros of dihedral L-functions*, Duke Math. J. **116** (2003), 189–217.

[Gao] P. Gao, *N-level density of the low-lying zeros of quadratic Dirichlet L-functions*, Ph. D thesis, University of Michigan, 2005.

[GR] I. Gradshteyn and I. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, New York, 1965.

[Gü] A. Güloğlu, *Low Lying Zeros of Symmetric Power L-Functions*, Internat. Math. Res. Notices (2005), no. 9, 517–550.

[Guy] R.K. Guy, *Catwalks, sandsteps and Pascal pyramids*, J. Integer Seq. **3** (2000), Article 00.1.6, `http://www.cs.uwaterloo.ca/journals/JIS/VOL3/GUY/catwalks.html`.

[Hej] D. Hejhal, *On the triple correlation of zeros of the zeta function*, Internat. Math. Res. Notices (1994), no. 7, 294–302.

[HL] J. Hoffstein and P. Lockhart, *Coefficients of Maass forms and the Siegel zero. With an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman*, Ann. of Math. (2) **140** (1994), no. 1, 161–181.

[HR1] C. Hughes and Z. Rudnick, *Mock Gaussian behaviour for linear statistics of classical compact groups*, J. Phys. A **36** (2003), 2919–2932.

[HR2] C. Hughes and Z. Rudnick, *Linear Statistics of Low-Lying Zeros of L-functions*, Quart. J. Math. Oxford **54** (2003), 309–333.

[I1] H. Iwaniec, *Small eigenvalues of Laplacian for* $\Gamma_0(N)$, Acta Arith. **56** (1990), no. 1, 65–82.

[I2] H. Iwaniec, *Topics in Classical Automorphic Forms*, Graduate Texts in Mathematics **17**, AMS, Providence, 1997.

[ILS] H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Études Sci. Publ. Math. **91** (2000), 55–131.

[KS1] N. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, AMS Colloquium Publications **45**, AMS, Providence, 1999.

[KS2] N. Katz and P. Sarnak, *Zeros of zeta functions and symmetries*, Bull. AMS **36** (1999), 1–26.

[KeSn] J. P. Keating and N. C. Snaith, *Random matrices and L-functions*, Random matrix theory, J. Phys. A **36** (2003), 2859–2881.

[Mil1] S. J. Miller, 1- *and* 2*-Level Densities for Families of lliptic Curves: Evidence for the Underlying Group Symmetries*, Ph.D. Thesis, Princeton University, 2002, http://www.math.princeton.edu/∼sjmiller/thesis/thesis.pdf.

[Mil2] S. J. Miller, 1- *and* 2*-Level Densities for Families of Elliptic Curves: Evidence for the Underlying Group Symmetries*, Compositio Mathematica **104** (2004), 952–992.

[Mon] H. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic Number Theory, Proc. Sympos. Pure Math. AMS **24** (1973), 181–193.

[Ro] E. Royer, *Petits zéros de fonctions L de formes modulaires*, Acta Arith. **99** (2001), 47–172.

[Ru] M. Rubinstein, *Low-lying zeros of L-functions and random matrix theory*, Duke Math. J. **109** (2001), 147–181.

[RS] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. **81** (1996), 269–322.

[T] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function* 2nd ed., revised by D. R. Heath-Brown, Oxford Science Publications, 1986.

[Sosh] A. Soshnikov, *Central limit theorem for local linear statistics in classical compact groups and related combinatorial identities*, Ann. Probab. **28** (2000), 1353–1370.

[Wat] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, second edition, Cambridge University Press, 1966.

[Yo] M. Young, *Low Lying Zeros of Families of Elliptic Curves*, J. Amer. Math. Soc. **19** (2006), no. 1, 205–250.



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