Irrationality measure and lower bounds for $\pi(x)$

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Abstract

In this note we show how the irrationality measure of $\zeta(s) = \pi^2/6$ can be used to obtain explicit lower bounds for $\pi(x)$. We analyze the key ingredients of the proof of the finiteness of the irrationality measure, and show how to obtain good lower bounds for $\pi(x)$ from these arguments as well. While versions of some of the results here have been carried out by other authors, our arguments are more elementary and yield a lower bound of order $x/\log x$ as a natural boundary.

1 Introduction

One of the most important functions in number theory is $\pi(x)$, the number of primes at most x. Many of the proofs of the infinitude of primes fall naturally into one of two categories. First, there are those proofs which provide a lower bound for $\pi(x)$. A classic example of this is Chebyshev's proof (see [Da, MT-B]) that there is a positive constant c such that $cx/\log x \leq \pi(x)$. Another method of proof is to deduce a contradiction from assuming there are only finitely many primes. One of the nicest such arguments is due to Furstenberg (see Chapter 1 of [AZ]), who gives a topological proof of the infinitude of primes. As is often the case with arguments along these lines, we obtain no information about how rapidly $\pi(x)$ grows.

Sometimes proofs which at first appear to belong to one category in fact belong to another. For example, in one of the most famous proofs in mathematics (and probably the first proof many of us saw of this result), Euclid proved there are infinitely many primes by noting the following: if not, and if $p_1, \dots, p_N$ is a complete enumeration of the primes, then $p_1 \cdots p_N + 1$ is a new prime or it is divisible by a prime not in our list. When presented in courses it often appears to be in the second class, as many classes just aim on proving

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there are infinitely many primes and stop. However, a little thought shows this proof belongs to the first class, as it yields there are at least k primes at most 2^{2^k}, thus $\pi(x) \geq \log_2 \log_2(x)$. For more results and questions related to Euclid's argument, see Appendix A.

As $\pi(x)$ is approximately $x/\log x$, it seems Euclid's argument is quite poor, as it only gives a lower bound of size $\log_2 \log_2 x$. The main purpose of this note is to show that this bound is much better than it first appears, and to investigate what is needed to (elementarily) get close to the truth.[1] We examine a standard 'special value' proof; see [MT-B] for proofs of all the claims below. Consider the Riemann zeta function

$$\zeta(s) \ := \ \sum_{n=1}^{\infty} \frac{1}{n^s} \ = \ \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1}, \tag{1.1}$$

which converges for $\Re s > 1$; the product representation follows from the unique factorization properties of the integers. One can show $\zeta(2) = \pi^2/6$. As π^2 is irrational, there must be infinitely many primes; if not, the product over primes at $s = 2$ would be rational. While at first this argument may appear to belong to the second class (proving $\pi(x)$ tends to infinity without an estimate of its growth), we show below that it belongs to the first class and we obtain an explicit, though *very* weak, lower bound for $\pi(x)$ for all x. We deliberately do not attempt to obtain the optimal bounds attainable through this method, but rather concentrate on proving the easiest possible results which best highlight the idea. After a decent amount of work we see that our results are not as good as what we can get from Euclid's argument, hopefully gaining a newfound appreciation for an argument from over two thousand years ago. We then show how our weak estimates can be fed back into the argument to obtain (infinitely often) massive improvement over the original bounds; our best results here are almost as good as the estimates from Euclid's argument. In the final part, we open up some of the technical machinery and surpass Euclid's result infinitely often, getting arbitrarily close to $x/\log x$; we describe this in greater detail below.

The key ingredient is the fact that the irrationality measure of $\pi^2/6$ is bounded. An upper bound on the irrationality measure of an irrational α is a number u such that there are only finitely many integer pairs p and q with

$$\left|\alpha - \frac{p}{q}\right| \ < \ \frac{1}{q^u}. \tag{1.2}$$

The irrationality measure $\mu_{\text{irr}}(\alpha)$ is defined to be the infimum of the numbers u and need not itself be a number for which (1.2) has at most finitely many solutions. Liouville constructed transcendental numbers by studying numbers

[1] An elementary proof of the Prime Number Theorem (PNT), due to Erdös and Selberg, exists; see [Gol] for the history and [Ed, EE] for an exposition. Elementary is *not* a synonym for easy. The first proof of the PNT used many results from complex analysis, and in fact was a huge motivation for the development of much of the theory; it took approximately another 50 years before an elementary proof was given.

with infinite irrationality measure, and Roth proved the irrationality measure of an algebraic number is 2. Currently the best known bound for $\zeta(2)$ is due to Rhin and Viola [RV2], who give 5.45 as a bound on its irrationality measure. Unfortunately, the published proofs of these bounds use good upper and lower bounds for $d_n := \text{lcm}(1, \dots, n)$. These upper and lower bounds are obtained by appealing to the Prime Number Theorem (or Chebyshev type bounds); this is a problem for us, as we are trying to prove a weaker version of the Prime Number Theorem (which we are thus subtly assuming in one of our steps).[2]

In the arguments below we first examine consequences of the finiteness of the irrationality measure of $\pi^2/6$, deriving lower bounds for $\pi(x)$ in §2. Our best elementary result is Theorem 2.3, where we show $\mu_{\text{irr}}(\pi^2/6) < \infty$ implies that there is an M such that $\pi(x) \geq \frac{\log\log x}{2\log\log\log x} - M$ infinitely often. We conclude in §3 by describing how we may modify the standard irrationality measure proofs to yield weaker irrationality bounds which do not require stronger input on d_n than we are assuming. Theorems 2.2 and 2.3 are unconditional (explicitly, we may remove the assumption that the irrationality measure of $\pi^2/6$ is finite through a slightly more involved argument). Theorem 3.1 requires results from Rhin and Viola's [RV2] proof of the irrationality measure, though it only needs weaker results that are independent of the Prime Number Theorem.

For our last result we need little-oh notation; by $f(x) = o(g(x))$ we mean that $\lim_{x\to\infty} f(x)/g(x) = 0$. Let $g(x)$ be any non-decreasing function such that $g(x) = o(x/\log x)$. In Theorem 3.1 we show that the irrationality measure arguments yield the existence of a positive c such that $\pi(x) \geq cg(x)$ for infinitely many integers x. Thus, as expected, we see that $x/\log x$ is a natural boundary for these methods and we are able to get arbitrarily close to the truth infinitely often.

2 Lower bounds for $\pi(x)$

We introduce some notation needed for analysis.

Define $T(x,k)$ by $T(x,k) = x\hat{}(x\hat{}(x\hat{}(\cdots\hat{}\, x)\cdots))$, with x occurring k times.

Theorem 2.1. *As $\mu_{\text{irr}}(\pi^2/6) < 5.45$, there exists an N_0 so that, for all k sufficiently large,*

$$\pi(T(N_0, 2k)) \ \geq \ k. \tag{2.1}$$

Proof: For any integer N let p_N and q_N be the relatively prime integers

[2] For another example along these lines, see Kowalski [K]. He proves $\pi(x) \gg \log\log x$ by combining the irrationality measure bounds of $\zeta(2)$ with deep results on the distribution of the least prime in arithmetic progressions. Our goal here is to see how far elementary methods can be pushed; in particular, we are trying to see how far one can get without using input about the distribution of primes in progressions. See also [S], where Sondow proves that $p_{n+1} \leq (p_1 \cdots p_n)^{2\mu_{\text{irr}}(1/\zeta(2))}$.

satisfying

$$\frac{p_N}{q_N} = \prod_{\substack{p \le N \\ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right)^{-1} = \prod_{\substack{p \le N \\ p \text{ prime}}} \left(1 + \frac{1}{p^2 - 1}\right). \tag{2.2}$$

Assume there are no primes $p \in (N, f(N)]$, where $f(x)$ is some rapidly growing function to be determined later. If $f(N)$ is too large relative to N, we will find that p_N/q_N is too good of a rational approximation to $\pi^2/6$, and thus there must be at least one prime between N and $f(N)$. Under our assumption, we find

$$\left|\frac{p_N}{q_N} - \frac{\pi^2}{6}\right| = \frac{p_N}{q_N}\left|1 - \prod_{p > f(N)} \left(1 + \frac{1}{p^2 - 1}\right)\right|. \tag{2.3}$$

Clearly $p_N/q_N \le \pi^2/6$, and

$$\begin{aligned} \prod_{p > f(N)} \left(1 + \frac{1}{p^2 - 1}\right) &= \exp\left(\log \prod_{p > f(N)} \left(1 + \frac{1}{p^2 - 1}\right)\right) \\ &\le \exp\left(\sum_{n > f(N)} \log\left(1 + \frac{1}{(n-1)^2}\right)\right) \\ &\le \exp\left(\sum_{n > f(N)} \frac{1}{(n-1)^2}\right) \\ &\le \exp\left(\frac{1}{f(N)^2} + \frac{1}{f(N)}\right) \end{aligned} \tag{2.4}$$

(the last inequality follows by replacing the sum over $n \ge f(N) + 2$ with an integral). Standard properties of the exponential function yield

$$\left|\frac{p_N}{q_N} - \frac{\pi^2}{6}\right| \le \frac{\pi^2}{6}\left|1 - \exp\left(\frac{1}{f(N)^2} + \frac{1}{f(N)}\right)\right| \le \frac{10}{f(N)}. \tag{2.5}$$

The largest q_N can be is $N!^2$, which happens only if all integers at most N are prime. We can greatly reduce this bound, as the only even prime is 2; however, our purpose is to highlight the method by using the most elementary arguments possible. If we take $f(x) = (x!)^{14}$, we find (for N sufficiently large) that

$$\left|\frac{\pi^2}{6} - \frac{p_N}{q_N}\right| \le \frac{10}{f(N)} < \frac{1}{q_N^6}; \tag{2.6}$$

however, this contradicts Rhin and Viola's bound on the irrationality measure of $\pi^2/6$ ($\mu_{\text{irr}}(\pi^2/6) < 5.45$). Thus there must be a prime between N and $f(N)$. Note $f(N) \le N^{14N} \le (14N)^{14N}$. Letting $f^{(k)}(N)$ denote the result of applying f a total of k times to N, for N_0 sufficiently large we see for large k that there are at least k primes at most $T(14N_0, 2k)$. QED

The inverse of the function $T(N, -)$ is called the $\log^*$ function to base N. It is the number of times one can iterate the logarithm without the number becoming non-positive and leaving the domain of the logarithm. It is this extremely slowly growing function that the above theorem yields as a lower bound for $\pi(x)$. The base was determined by the irrationality bound and the unspecified (but constructive) bound on the size of the finite number of approximations violating the irrationality bound.

Of course, this bound arises from assuming that all the numbers at most x are prime (as well as some weak estimation). If all the numbers at most x are prime then do not need to search for a prime between N and $f(N)$. This interplay suggests that a more careful argument should yield a significantly better estimate on $\pi(x)$, if not for all x then at least infinitely often. We will use an upper bound on $\pi(x)$ with the inequality $q_N \leq \prod_{p \leq N}(p^2 - 1) \leq N^{2\pi(N)}$. While isolating the true order of magnitude of our bound is difficult, we can easily prove the following.

Theorem 2.2. *The finiteness of the irrationality measure of $\pi^2/6$ implies the existence of an $M > 0$ such that for infinitely many integers x we have $\pi(x) \geq \log\log\log(x) - M$.*

Proof: We choose our constants below to simplify the exposition, and not to obtain the sharpest results. Let b be a bound on the irrationality measure of $\pi^2/6$. The theorem trivially follows if $\pi(x) \geq (\log x)^{e-1}/4b$ infinitely often, so we may assume that $\pi(x) < (\log x)^{e-1}/4b$ for all x sufficiently large. Thus the denominator q_N in our rational approximation in equation (2.6), when we consider primes at most N for N sufficiently large, has the bound

$$q_N^b \;\leq\; N^{2b\pi(N)} \;=\; \exp(2b\pi(N)\log N) \;<\; \exp\left(\frac{(\log N)^e}{2}\right) \;\leq\; \frac{\exp\left(\log(N)^e\right)}{10}. \tag{2.7}$$

Thus, if $f(N) = \exp(\log(N)^e)$, we have checked the right-hand inequality of equation (2.6), which in this case is that $10/f(N) < 1/q_N^b < 10/\exp(\log N)^e$. This cannot hold for N sufficiently large without violating our bound b on the irrationality measure, unless of course there is a prime between N and $f(N)$. Thus there must be a prime between N and $f(N)$ for all N large. Define x_n by $x_0 = e^e$ and iterating by applying f, so that $x_{n+1} = f(x_n) = \exp((\log x_n)^e)$. Then $\log x_{n+1} = (\log x_n)^e$, so $\log x_n = (\log x_0)^{e^n} = \exp e^n$ or $x_n = \exp(\exp e^n)$. Once x_M is sufficiently large so that the above argument applies, there is a prime between every pair of x_i, so there are at least $n - M$ primes less than x_n. QED

The simple argument above illustrates how our result can improve itself (at least for an increasing sequence of x's). Namely, the lower bound we obtain is better the fewer primes there are, and if there are many primes we can afford to wait awhile before finding another prime. By more involved arguments, one can show that $\pi(x) \geq h(x)$ infinitely often for many choices of $h(x)$. Sadly, however, none of these arguments allow us to take $h(x) = \log\log x$. Our attempts at obtaining such a weak bound gave us a new appreciation of the estimate from Euclid's argument. Our best result along these lines is the following.

Theorem 2.3. *The finiteness of the irrationality measure of* $\pi^2/6$ *implies the existence of an* $M > 0$ *such that for infinitely many integers* x *we have* $\pi(x) \geq \frac{\log\log x}{2\log\log\log x} - M$.

Proof: The proof is similar to that in Theorem 2.2. As before, let b be a bound on the irrationality measure of $\pi^2/6$. We assume that $\pi(x) \leq (\log\log x)/4b$ for all sufficiently large x, as otherwise the claim trivially follows. We show that there is a prime between x_n and x_{n+1}, where $x_n = \exp(\exp a_n)$ and the sequence a_n is defined by $a_{n+1} = a_n + \log a_n$. It is easy to show that a_n grows like $n\log n$; from there the growth of x_n proves the theorem. Consider $h(x) = \log\log\log x/\log\log x$. Note $\log^{h(x)} x = \log\log x$, so our assumption can be rewritten as $\pi(x) \leq (\log^{h(x)} x)/4b$ for large x. Therefore, if N is sufficiently large we have the bound

$$q_N^b \leq N^{2b\pi(N)} \;=\; \exp(2b\pi(N)\log N) \;\leq\; \exp\left(\frac{\log^{h(N)+1} N}{2}\right) \;\leq\; \frac{\exp\left(\log^{h(N)+1} N\right)}{10}. \tag{2.8}$$

Setting $f(N) = \exp(\log^{h(N)+1} N)$, we see that for large N there must be a prime between N and $f(N)$. We define x_n by iterating f (so $x_{n+1} = f(x_n)$), starting at $x_2 = \exp(\exp(e))$. The recursion can be rewritten as $\log\log x_{n+1} = (h(x_n) + 1)\log\log x_n$. In terms of $a_n = \log\log x_n$, this is $a_{n+1} = \left(\frac{\log a_n}{a_n} + 1\right) a_n = a_n + \log a_n$. For an upper bound, we have $a_n \leq 2n\log n$. We prove this by induction. For the base case, $a_2 = e < 4\log 2$. If $a_n \leq 2n\log n$ with $n \geq 2$, then

$$a_{n+1} \;\leq\; 2n\log n + \log(2n\log n) \;<\; (2n+1)\log n + \log n \;<\; (2n+2)\log(n+1). \tag{2.9}$$

For a lower bound, note that $\log a_k \geq 1$ so $a_n \geq n$. This improves to $a_{n+1} - a_n = \log a_n \geq \log n$. Therefore $a_{n+1} \geq \sum_{k=1}^n \log k > \int_1^n \log x\, dx = n\log n - n + 1$. Thus $n\log n - n < a_n \leq 2n\log n$. Therefore $\pi(x_n) \geq n - M$, where x_M is large enough that the assumed bound on $\pi(x_M)$ applies. To derive our asymptotic conclusions, we need to know the inverse of the sequence x_n. For n large there are at least $n - M$ primes that are at most $x_n = \exp(\exp a_n) \leq \exp(\exp(2n\log n))$. Letting $x = \exp(\exp(2n\log n))$, we find n is at least $\log\log x/2\log\log\log x$. Therefore, for infinitely many integers x we have $\pi(x) \geq \log\log x/2\log\log\log x - M$ (where we subtract M for the same reasons as in Theorem 2.2). QED

Remark 2.4. The lower bound from Theorem 2.3 is slightly weaker than the one from Euclid's argument, namely that $\pi(x) \geq \log_2\log_2 x$. It is possible to obtain slightly better results by assuming instead that $\pi(x) \leq (\log\log x)^{c(x)}$ / b; a good choice is to take $c(x) = \log g(x)$ / $\log(g(x)\log g(x))$ with $g(x) = \log\log x/\log\log\log x$. The sequence $a_{n+1} = a_n + \log a_n$ which arises in our proof is interesting, as the Prime Number Theorem states the leading term in the average spacing between primes of size x for large x is $\log x$. Thus

a_n is approximately the n^{th} prime p_n; for example, $a_{1000000} \sim 15479041$ and $p_{1000000} = 15485863$, which differ by about .044%.

3 Bounds for the irrationality measure of $\pi^2/6$

We briefly describe how to modify standard arguments on the irrationality measure of $\zeta(2) = \pi^2/6$ to make Theorems 2.2 and 2.3 unconditional. As always, we merely highlight the ideas and do not attempt to prove optimal results. We follow the argument in [RV1], and recall by $A(x) = o(B(x))$ we mean $\lim_{x\to\infty} A(x)/B(x) = 0$. With $d_n = \text{lcm}(1,\dots,n)$, they show the existence of sequences $\{a_n\}$, $\{b_n\}$ such that

$$a_n - b_n\zeta(2) \ = \ d_n^2 \int_0^1\int_0^1 \frac{H_n(x+y,xy)dxdy}{(1-xy)^{n+1}} \ =: \ d_n^2 I_n \tag{3.1}$$

for a sequence of polynomials $H_n(u,v)$ with integer coefficients, with $\rho, \sigma > 0$ such that

(RV1) $\limsup_{n\to\infty} \frac{\log|b_n|}{n} \le \rho$, and

(RV2) $\lim_{n\to\infty} \frac{\log|a_n - b_n\zeta(2)|}{n} = -\sigma$.

Then $\mu_{\text{irr}}(\zeta(2)) \le 1 + \frac{\rho}{\sigma}$ (this is their Lemma 4, and is a special case of Lemma 3.5 in [C]). Unfortunately (for us), they use the Prime Number Theorem to prove that $d_n = \exp(n + o(n))$. From this they deduce that there exist constants a and b such that for any $\epsilon > 0$, (i) $\exp((a+2-\epsilon)n) \le d_n^2 I_n \le \exp((a+2+\epsilon)n)$ and (ii) $|b_n| \le \exp((b+2+\epsilon)n)$. Note (i) and (ii) imply (RV1) and (RV2) for our sequences $\{a_n\}$ and $\{b_n\}$ with $\rho = b+2$ and $\sigma = 2-a$, which gives $\mu_{\text{irr}}(\zeta(2)) \le (a-b)/(a+2)$. It is very important that the upper and lower bounds of d_n are close, as the limit in (RV2) needs to exist. We now show how to make Theorems 2.2 and 2.3 independent of the Prime Number Theorem (i.e., we do not assume the irrationality measure of $\zeta(2)$ is finite, as the published proofs we know use the Prime Number Theorem). *Assume $\pi(x) \le \log x$ for all x sufficiently large*; if not, then $\pi(x) > \log x$ infinitely often and Theorems 2.2 and 2.3 are thus trivial. Under this assumption, we have $1 \le d_n \le \exp(\log^2 n)$. The lower bound is clear. For the upper bound, note the largest power of a prime $p \le n$ that is needed is $\lfloor \log_p n \rfloor \le \log n/\log p$. Thus

$$d_n \ \le \ \prod_{p\le n} p^{\log n/\log p} \ = \ \exp\left(\sum_{p\le n} \frac{\log n}{\log p}\cdot \log p\right) \ = \ \exp(\pi(n)\log n); \tag{3.2}$$

the claimed upper bound follows from our assumption that $\pi(x) \le \log x$. We now find for any $\epsilon > 0$ that (i') $\exp((a-\epsilon)n) \le d_n^2 I_n \le \exp((a+\epsilon)n + 2\log^2 n)$ and (ii') $|b_n| \le \exp((b+\epsilon)n + 2\log^2 n)$. We again find that (RV1) and (RV2) hold, and $\mu_{\text{irr}}(\zeta(2)) \le (a-b)/a$.

Using the values of a and b from their paper, we obtain (under the assumption that $\pi(x) \le \log x$) that $\mu_{\rm irr}(\zeta(2))$ is finite. Thus Theorems 2.2 and 2.3 are independent of the Prime Number Theorem. Using the values of a and b in [RV1], we can prove that $\pi(x)$ is quite large infinitely often.

Theorem 3.1. *Let $g(x)$ be any function satisfying $g(x) = o(x/\log x)$. Then there is a $c > 0$ such that infinitely often $\pi(x) \ge cg(x)$. In particular, for any $\epsilon > 0$ we have $\pi(x) \ge x/\log^{1+\epsilon} x$ infinitely often.*

PROOF: We assume $\pi(x) \le g(x)$ for all x sufficiently large, as otherwise the claim is trivial. In [RV1] numerous admissible values of a and b are given (*and the determination of these bounds does not use any estimates on the number of primes*); we use $a = -2.55306095\dots$ and $b = 1.70036709\dots$ (page 102). From (3.2) we have $1 \le d_n \le \exp(\pi(n)\log n)$. Using $\pi(x) \le g(x)$ we find (i") $\exp((a-\epsilon)n) \le d_n^2 I_n \le \exp((a+\epsilon)n + 2g(n)\log n)$ and (ii') $|b_n| \le \exp((b+\epsilon)n + 2g(n)\log n)$. We again find (RV1) and (RV2) hold, with the same values of a and b. For example, to see that (RV2) holds we need to show $\lim_{n\to\infty}(1/n)\log|a_n - b_n\zeta(2)| = -\sigma$. As $a_n - b_n\zeta(2) = d_n^2 I_n$, we have for any $\epsilon > 0$ that

$$\lim_{n\to\infty} \frac{(a-\epsilon)n}{n} \ \le\ \lim_{n\to\infty} \frac{\log|a_n - b_n\zeta(2)|}{n} \ \le\ \lim_{n\to\infty} \frac{(a+\epsilon)n + 2g(n)\log n}{n}. \tag{3.3}$$

Our assumption on $g(x)$ implies that $\lim_{n\to\infty} \frac{g(n)\log n}{n} = 0$, and thus the limit exists as before. We find we may take $\rho = b$ and $\sigma = -a$, which yields $\mu_{\rm irr}(\zeta(2)) \le 1 - \frac{b}{a} = 1.666\ldots < 2$. As the irrationality exponent of an irrational number is at least 2 (see [MT-B] for a proof of this and a proof of the irrationality of π^2), this is a contradiction. Thus $\pi(x)$ cannot be less than $g(x)$ for all x sufficiently large (and thus infinitely often we beat Euclid by an enormous amount). QED

Remark 3.2. We have proved the above in the case of $g(x) = o(x/\log x)$. Now, suppose we wanted to get $\pi(x) \sim cx/\log x$ for some x. Then, following the calculations above, we would have $b_n \le (b+\epsilon)n + 2g(n\log(n)) = (b+\epsilon)n + 2cn$, so then taking the limit sup as above gives $\rho = b + 2c$. However, if we attempt to take the limit for σ, we get $\exp((a-\epsilon)n) \le d^2 I_n \le \exp((a+\epsilon)n + 2cn)$, and then we can find $a \le \lim_{n\to\infty}(\log|a_n - b_n\zeta(2)|)/n \le a + 2c$. Notably, the upper and lower bounds are not equal, so we do not know if the limit exists; to show this we would need to have a non-trivial lower bound on d_n, which requires the Prime Number Theorem. However, if we had the limit equal to the upper bound, we would have $-\sigma = a + 2c$, and then the irrationality of π^2 implies $\mu_{\rm irr}(\zeta(2)) \ge 1 - \frac{b+2c}{a+2c}$, which would give us that $c < 0.213$. So, this is possible to show if a lower bound for d_n can be found independent of the Prime Number Theorem.

Remark 3.3. It was essential that the limit in (RV2) exist in the above argument. By $f(x) \gg g(x)$ we mean there is a positive constant C such that for all x sufficiently large $f(x) \ge Cg(x)$. If $\pi(x) \gg x/\log x$ infinitely often and $\pi(x) \ll x/\log^{1+\epsilon} x$ infinitely often then our limit might not exist and we cannot use Lemma 4 of [RV1]. Kowalksi [K] notes[3] that knowledge of $\zeta(s)$ as $s \to 1$ yields $\pi(x) \gg x/\log^{1+\epsilon} x$ infinitely often, which is significantly better than his proof using knowledge of $\zeta(2)$ and Linnik's theorem on the least prime in arithmetic progressions to get $\pi(x) \gg \log\log x$. We may interpret our arguments as correcting this imbalance, as now an analysis of $\zeta(2)$ gives a comparable order of magnitude estimate. It is interesting that the correct growth rate of $\pi(x)$, namely $x/\log x$, surfaces in these arguments as a natural boundary.

We conclude by improving Theorem 3.1 to show that not only are we infinitely often close to the true order of growth, but when we are close we are close for large stretches of integers. For notational simplicity we work with logarithms below, but one can easily modify the argument to $o(x/\log x)$.

Corollary 3.4. *For any $\widetilde{\epsilon} > 0$ there exists an increasing sequence of numbers $X_{n,\widetilde{\epsilon}}$ tending to infinity such that for each n, $\pi(x) \geq x/\log^{1+\widetilde{\epsilon}} x$ for almost all $x \leq X_{n,\widetilde{\epsilon}}$ (in other words, if $F_{n,\widetilde{\epsilon}}$ denotes the number of $x \leq X_{n,\widetilde{\epsilon}}$ such that $\pi(x) < x/\log^{1+\widetilde{\epsilon}} x$, then $F_{n,\widetilde{\epsilon}}|/X_{n,wepsilon} \to 0$).*

Proof: Let $Y_{n,\widetilde{\epsilon}}$ be an increasing sequence tending to infinity so that the result of Theorem 3.1 holds with exponent $\epsilon = \widetilde{\epsilon}/2$; thus $\pi(Y_{n,\widetilde{\epsilon}}) > Y_{n,\widetilde{\epsilon}}/\log^{1+\widetilde{\epsilon}/2} Y_{n,\widetilde{\epsilon}}$.

Let $X_{n,\widetilde{\epsilon}} = Y_{n,\widetilde{\epsilon}} \log^{\widetilde{\epsilon}/2} Y_{n,\widetilde{\epsilon}}$; we show the claim in the theorem holds for almost all $x \leq X_{n,\epsilon}$. We may assume $Y_{n,\widetilde{\epsilon}} \leq x \leq X_{n,\widetilde{\epsilon}}$, as the fraction of numbers less than $X_{n,\widetilde{\epsilon}}$ which are also less than $Y_{n,\widetilde{\epsilon}}$ tends to zero (the percentage is just $1/\log^{\widetilde{\epsilon}/2} Y_{n,\widetilde{\epsilon}}$).

The claim follows by showing $\pi(x) > x/\log^{1+\widetilde{\epsilon}} x$ for such x. We use the fact that $\pi(x)$ is non-decreasing, and by definition of $Y_{n,\widetilde{\epsilon}}$ there are at least $Y_{n,\widetilde{\epsilon}}/\log^{1+\widetilde{\epsilon}/2} Y_{n,\widetilde{\epsilon}}$ primes at most $X_{n,\widetilde{\epsilon}}$. The worst case for us would be that these are all the primes up to $X_{n,\widetilde{\epsilon}}$, but as the number of primes at most x is non-decreasing we have $\pi(x) \geq Y_{n,\widetilde{\epsilon}}/\log^{1+\widetilde{\epsilon}/2} Y_{n,\widetilde{\epsilon}}$ for all x under consideration; we now need to rewrite this in terms of x. We have

$$\pi(x) \;\geq\; \frac{Y_{n,\widetilde{\epsilon}}}{\log^{1+\widetilde{\epsilon}/2} Y_{n,\widetilde{\epsilon}}} \;\geq\; \frac{x \log^{-\widetilde{\epsilon}/2} Y_{n,\widetilde{\epsilon}}}{\log^{1+\widetilde{\epsilon}/2} Y_{n,\widetilde{\epsilon}}} \;=\; \frac{x}{\log^{1+\widetilde{\epsilon}} Y_{n,\widetilde{\epsilon}}} \;\geq\; \frac{x}{\log^{1+\widetilde{\epsilon}} x}; \qquad (3.4)$$

thus for x in the desired range we have $\pi(x) \geq x/\log^{1+\widetilde{\epsilon}} x$, completing the proof. QED

Appendix A Euclid's sequence

Euclid's argument leads to an interesting sequence: 2, 3, 7, 43, 13, 53, 5, 6221671, 38709183810571, 139, 2801, 11, 17, 5471, 52662739, 23003, 30693651606209, 37, 1741, 1313797957, 887, 71, 7127, 109, 23, 97, 159227, 643679794963466223081509857, 103, 1079990819, 9539, 3143065813, 29, 3847, 89, 19, 577, 223, 139703, 457,

[3] His note incorrectly mixed up a negation, and the claimed bound of $\pi(x) \gg x^{1-\epsilon}$ is wrong.

9649, 61, 4357.... This sequence is generated as follows. Let $a_1 = 2$, the first prime. We apply Euclid's argument and consider $2 + 1$; this is the prime 3 so we set $a_2 = 3$. We apply Euclid's argument and now have $2 \cdot 3 + 1 = 7$, which is prime, and set $a_3 = 7$. We apply Euclid's argument again and have $2 \cdot 3 \cdot 7 + 1 = 43$, which is prime and set $a_4 = 43$. Now things get interesting: we apply Euclid's argument and obtain $2 \cdot 3 \cdot 7 \cdot 43 + 1 = 1807 = 13 \cdot 139$, and set $a_5 = 13$. Thus a_n is the smallest prime not on our list generated by Euclid's argument at the n^{th} stage.

There are a plethora of unknown questions about this sequence, the biggest of course being whether or not it contains every prime. This is a great sequence to think about, but it is a computational nightmare to enumerate. These terms from the Online Encyclopedia of Integer Sequences (see http://oeis.org/A000945). You can enter the first few terms of an integer sequence, and it will list whatever sequences it knows that start this way, provide history, generating functions, connections to parts of mathematics, ... This is a great website to know if you want to continue in mathematics.

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