ON ZECKENDORF RELATED PARTITIONS USING THE LUCAS SEQUENCE

HÙNG VIỆT CHU, DAVID C. LUO, AND STEVEN J. MILLER

Abstract. Zeckendorf proved that every natural number has a unique partition as a sum of non-consecutive Fibonacci numbers. Similarly, every natural number can be partitioned into a sum of non-consecutive terms of the Lucas sequence, although such partitions need not be unique. In this paper, we
1) prove that a natural number can have at most two distinct non-consecutive partitions in the Lucas sequence,
2) find all positive integers with a fixed term in their partition, and
3) calculate the limiting value of the proportion of natural numbers that are not uniquely partitioned into the sum of non-consecutive terms in the Lucas sequence.

1. Introduction

The Fibonacci numbers have fascinated mathematicians for centuries with many interesting properties. By convention, the Fibonacci sequence \( \{F_n\}_{n=0}^\infty \) is defined as follows: let \( F_0 = 0 \), \( F_1 = 1 \), and \( F_n = F_{n-1} + F_{n-2} \), for \( n \geq 2 \). A beautiful theorem of Zeckendorf \[31\] states that every natural number \( n \) can be uniquely written as a sum of non-consecutive Fibonacci numbers. This gives the so-called Zeckendorf partition of \( n \). A formal statement of Zeckendorf’s theorem is as follows:

**Theorem 1.1 (Zeckendorf).** For any \( n \in \mathbb{N} \), there exists a unique increasing sequence of positive integers \( \{c_1, c_2, \ldots, c_k\} \) such that \( c_1 \geq 2 \), \( c_i \geq c_{i-1} + 2 \) for \( i = 2, 3, \ldots, k \), and \( n = \sum_{i=1}^{k} F_{c_i} \).

Much work has been done to understand the structure of Zeckendorf partitions and their applications (see \[1, 2, 6, 8, 16, 18, 19, 20, 25, 29, 30\]) and to generalize them (see \[10, 12, 13, 14, 15, 17, 22, 23, 26, 27, 28\]). In this paper, we study the partition of natural numbers into Lucas numbers. The Lucas sequence \( \{L_n\}_{n=0}^\infty \) is defined as follows: let \( L_0 = 2 \), \( L_1 = 1 \), and \( L_n = L_{n-1} + L_{n-2} \), for \( n \geq 2 \). As the Lucas sequence is closely related to the Fibonacci sequence, it is not surprising that we can also partition natural numbers using Lucas numbers.

**Theorem 1.2 (Zeckendorf).** Every natural number can be partitioned into the sum of non-consecutive terms of the Lucas sequence.

Note that the distinction between Theorems 1.1 and 1.2 lies in the uniqueness property of such partitions of natural numbers in the Fibonacci and Lucas sequences. Although 5 is uniquely partitioned into \( F_5 = 5 \) in \( \{F_2, F_3, \ldots\} \), its partition is not unique in the Lucas sequence as \( 5 = L_0 + L_2 = 2 + 3 \) and \( 5 = L_1 + L_3 = 1 + 4 \). In \[7\], Brown shows various ways to have a unique partition using Lucas sequence. In this paper, we prove the following results.

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\[\text{1For more on Brown’s criteria, see [3] [4].}\]
Theorem 1.3. If we allow $L_0$ and $L_2$ to appear simultaneously in a partition, each natural number can have at most two distinct non-consecutive partitions in the Lucas sequence.

Theorem 1.4. Suppose that we do not allow $L_0$ and $L_2$ to appear simultaneously in a partition. The set of all positive integers having the summand $L_k$ in their partition is given by

$$Z(k) = \begin{cases} 
2 + 3n + \left\lfloor \frac{n+1}{2} \right\rfloor & \text{if } k = 0, \\
3n + \left\lfloor \frac{n+\Phi^2}{2} \right\rfloor & \text{if } k = 1, \\
L_k \left\lfloor \frac{n+\Phi^2}{2} \right\rfloor + nL_{k+1} + j : n \geq 0 \text{ and } 0 \leq j \leq L_{k-1} - 1 & \text{if } k \geq 2.
\end{cases}$$

Theorem [1.4] is an analogue of [13, Theorem 3.4]. For $k \geq 0$, we find all positive integers having the summand $L_k$ in their partition. We have a different formula when $k = 0$ instead of one formula for all values of $k$ as in [13, Theorem 3.4].

Our next result is predicted by [9, Theorem 1], which deals with general recurrence relations; however, in the case of Lucas numbers, we can relate Lucas partitions to the golden string.

Theorem 1.5. If we allow $L_0$ and $L_2$ to appear simultaneously in a partition, the proportion of natural numbers that are not uniquely partitioned into the sum of non-consecutive terms of the Lucas sequence converges to $\frac{1}{\Phi^3-1}$, where $\Phi$ is the golden ratio.

2. Preliminaries

2.1. Definitions.

Definition 2.1. Let $A = \{a_0, a_1, \ldots, a_m\}$ be the set consisting of the first $m+1$ terms of the sequence $\{a_k\}_{k=0}^\infty$. We say a proper subset $B$ of $A$ is a non-consecutive subset of $A$ if the elements of $B$ are pairwise non-consecutive in $\{a_k\}_{k=0}^\infty$. Furthermore, we say a sum $S$ is a non-consecutive sum of $A$ if $S$ is the sum of distinct elements of $A$ that are pairwise non-consecutive in $\{a_k\}_{k=0}^\infty$.

Definition 2.2. Let $A_m = \{L_0, L_1, \ldots, L_m\}$ denote the set consisting of the first $m+1$ terms of the Lucas sequence.

2.2. The Golden string. The golden string $S = BABBABABBABBA\ldots$ is defined to be the infinite string of $A$’s and $B$’s constructed recursively as follows. Let $S_1 = A$ and $S_2 = B$, and then, for $k \geq 3$, $S_k$ is the concatenation of $S_{k-1}$ and $S_{k-2}$, which we denote by $S_{k-1} \circ S_{k-2}$. For example, $S_3 = S_2 \circ S_1 = a_2 \circ a_1 = BA$, $S_4 = S_3 \circ S_2 = a_2a_1 \circ a_2 = BAB$, $S_5 = S_4 \circ S_3 = BABBAB$, and so on. Interestingly, the golden string is highly connected to the Zeckendorf partition [19]. As we will see later, the string is also closely related to the partitions of natural numbers into Lucas numbers.

Remark 2.3. We mention two properties of the golden string that we will use in due course.

1. For $j \geq 1$, the $(F_{2j})$th character of $S$ is $B$ and the $(F_{2j+1})$th character of $S$ is $A$. This can be easily proved using induction.
2. The number of $B$’s amongst the first $n$ characters of $S$ is given by $\left\lfloor \frac{n+1}{\Phi^2} \right\rfloor$, where $\Phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. For a proof of this result, see [19, Lemma 3.3].

3. At Most Two Partitions

In this section, we present our results that determine the maximum number of non-consecutive partitions that a natural number can have in the Lucas sequence, the proofs of which are adapted from [21]. Before we prove Theorem 1.3, we introduce the following preliminary lemmas. For the proofs of Lemmas 3.1 and 3.2, see Appendix B.
Lemma 3.1. Let S be any non-consecutive sum of $A_m$. Then

(1) if $m$ is odd, $S$ assumes all values from 0 to $L_{m+1} - 1$ inclusive, and
(2) if $m$ is even, then $S$ assumes all values from 0 to $L_{m+1} + 1$ inclusive, excluding $L_{m+1}$.

Lemma 3.2. If $m \geq 0$, then $L_{2m+1} + 1$ has exactly two non-consecutive partitions in the Lucas sequence.

Proof of Theorem 1.3. It suffices to show that for every non-negative integer $m$, there is no natural number that is equal to three or more distinct non-consecutive sums of $A_m$. We proceed by strong induction. No natural is equal to three or more distinct non-consecutive sums of $A_0$ and $A_1$. This shows the base case. Assume Theorem 1.3 holds for all non-negative integers less than or equal to $m = k$. In our first case, suppose that $k$ is odd. From Lemma 3.1, the non-consecutive sums that we can form from $A_k$ are the values from 0 to $L_{k+1} - 1$ inclusive. Hence, when we add the term $L_{k+1}$ to $A_k$, all new non-consecutive sums that can be formed must be at least $L_{k+1}$. This implies there is no possible way in which we can form a third distinct non-consecutive sum of $A_{k+1}$ for any natural number because there is no intersection between the non-consecutive sums in which we can form before and after the addition of the term $L_{k+1}$. When $k \geq 2$ is even, we have from Lemma 3.1 that all non-consecutive sums we can form from $A_k$ are the values from 0 to $L_{k+1} + 1$ inclusive, excluding $L_{k+1}$. When we add the term $L_{k+1}$ to $A_k$, all new non-consecutive sums that can be formed are at least $L_{k+1}$ with $L_{k+1} + 1$ being the only non-consecutive sum formed again, namely $L_{k+1} + L_1$. By Lemma 3.2 we know that $L_{k+1} + 1$ has exactly two distinct non-consecutive partitions in the Lucas sequence. Therefore, there is no possible way in which we can form a third distinct non-consecutive sum of $A_{k+1}$ for any natural number. This completes the inductive step. □

4. Partitions with a Fixed Term

Let $\mathcal{X}_k$ denote the set of all positive integers having $L_k$ as the smallest summand in their partition. Let $Q_k = (q_k(j))_{j \geq 1}$ be the strictly increasing sequence obtained by rearranging the elements of $\mathcal{X}_k$ into ascending numerical order. We consider the cases $k = 0$ and $k \geq 1$ separately.

4.1. When $k = 0$. Table 1 replaces each term $q_k(j)$ in $Q_k$ with an ordered list of the summands in its partition.

<table>
<thead>
<tr>
<th>Row</th>
<th>$L_0$</th>
<th>$L_3$</th>
<th>$L_4$</th>
<th>$L_5$</th>
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<td>5</td>
<td>$L_0$</td>
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<td>$L_4$</td>
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<tr>
<td>6</td>
<td>$L_0$</td>
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<td>$L_6$</td>
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<tr>
<td>7</td>
<td>$L_0$</td>
<td>$L_3$</td>
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<tr>
<td>8</td>
<td>$L_0$</td>
<td></td>
<td>$L_4$</td>
<td></td>
<td>$L_6$</td>
</tr>
</tbody>
</table>

Table 1. The partitions of the positive integers having $L_0$ as their smallest summand.

Lemma 4.1. For $j \geq 3$, the rows of Table 1 for which $L_j$ is the largest summand are those numbered from $F_{j-1} + 1$ to $F_j$ inclusive.

Proof. We prove by induction. Base cases: it is easy to check that the statement of the lemma is true for $j = 3$ and $j = 4$. Inductive hypothesis: assume that it is true for all $j$ such that
3 \leq j \leq m \text{ for some } m \geq 4. \text{ By the inductive hypothesis, the number of rows such that their largest summands are no greater than } L_{m-1} \text{ is}

\[ 1 + \sum_{j=3}^{m-1}(F_j - F_{j-1}) = F_{m-1}, \]

which is also the number of rows whose largest summand is \( L_{m+1} \). Due to the inductive hypothesis, the rows whose largest summand is \( L_m \) are numbered from \( F_{m-1}+1 \) to \( F_m \) inclusive. Therefore, the rows whose largest summand is \( L_{m+1} \) are numbered from \( F_m + 1 \) to \( F_{m+1} \), as desired. This completes our proof.

Lemma 4.2. For \( j \geq 1 \), we have

\[ q_k(j + 1) - q_k(j) = \begin{cases} L_2, & \text{if } A \text{ is the } j\text{th character of } S, \\ L_3, & \text{if } B \text{ is the } j\text{th character of } S. \end{cases} \]

Proof. We prove by induction. Base cases: it is easy to check that the statement of the lemma is true for \( 1 \leq j \leq F_4 - 1 \). Inductive hypothesis: suppose that it is true for \( 1 \leq j \leq F_m - 1 \) for some \( m \geq 4 \). By Lemma 4.1 the number of rows in Table 1 whose largest summand is no greater than \( L_{m-1} \) is

\[ 1 + \sum_{j=3}^{m-1}(F_j - F_{j-1}) = F_{m-1}, \]

which is also the number of rows whose largest summand is \( L_{m+1} \). Furthermore, the rows for which \( L_{m+1} \) is the largest summand are numbered from \( F_m + 1 \) to \( F_{m+1} \) inclusive. Therefore, the ordering of the rows in Table 1 implies that \( q_k(i + F_m) = q_k(i) + L_{m+1} \), for \( 1 \leq i \leq F_{m-1} \). Hence, for \( 1 \leq i \leq F_{m-1} - 1 \), we have

\[ q_k(i + 1 + F_m) - q_k(i + F_m) = (q_k(i + 1) + L_{m+1}) - (q_k(i) + L_{m+1}) = q_k(i + 1) - q_k(i). \]

By the construction of \( S \), the substring comprising of its first \( F_{m-1} \) characters is identical to the substring of its characters numbered from \( F_m + 1 \) to \( F_{m+1} \) inclusive. Thus the lemma is true for \( F_m + 1 \leq j \leq F_{m+1} - 1 \). It remains to show that it is true for \( j = F_m \). We have

\[ q_k(F_m + 1) - q_k(F_m) = \begin{cases} L_{m+1} - (L_m + L_{m-2} + \cdots + L_4) = L_3, & \text{if } m \text{ is even}, \\ L_{m+1} - (L_m + L_{m-2} + \cdots + L_3) = L_2, & \text{if } m \text{ is odd}. \end{cases} \]

By Remark 2.3 item (1), we know that the lemma is true for \( j = F_m \), completing the proof. \( \square \)

4.2. When \( k \geq 1 \). Table 2 replaces each term \( q_k(j) \) in \( Q_k \) with an ordered list of the summands in its partition.

<table>
<thead>
<tr>
<th>Row</th>
<th>( L_k )</th>
<th>( L_{k+2} )</th>
<th>( L_{k+3} )</th>
<th>( L_{k+4} )</th>
<th>( L_{k+5} )</th>
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<tbody>
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<td>1</td>
<td>( L_k )</td>
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<tr>
<td>2</td>
<td>( L_k )</td>
<td>( L_{k+2} )</td>
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<tr>
<td>3</td>
<td>( L_k )</td>
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<td>( L_{k+4} )</td>
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<tr>
<td>4</td>
<td>( L_k )</td>
<td>( L_{k+2} )</td>
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<td>( L_{k+4} )</td>
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<tr>
<td>5</td>
<td>( L_k )</td>
<td>( L_{k+2} )</td>
<td>( L_{k+3} )</td>
<td>( L_{k+4} )</td>
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<td>6</td>
<td>( L_k )</td>
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<td>( L_{k+4} )</td>
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<td>7</td>
<td>( L_k )</td>
<td>( L_{k+2} )</td>
<td>( L_{k+3} )</td>
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<tr>
<td>8</td>
<td>( L_k )</td>
<td>( L_{k+2} )</td>
<td>( L_{k+3} )</td>
<td>( L_{k+4} )</td>
<td>( L_{k+5} )</td>
</tr>
</tbody>
</table>

Table 2. The partitions of the positive integers having \( L_k \) as their smallest summand.

Table 2 is similar to Table 1 in [13]. The next lemma follows from [13] Lemma 3.1.
ON ZECKENDORF RELATED PARTITIONS USING THE LUCAS SEQUENCE

Lemma 4.3. For $j \geq 2$, the rows of Table 2 for which $L_{k+j}$ is the largest summand are those numbered from $F_j + 1$ to $F_{j+1}$ inclusive.

Lemma 4.4. For $j \geq 1$, we have

$$q_k(j + 1) - q_k(j) = \begin{cases} L_{k+1}, & \text{if } A \text{ is the } j \text{th character of } S, \\ L_{k+2}, & \text{if } B \text{ is the } j \text{th character of } S. \end{cases}$$

Proof. We prove by induction. **Base cases:** it is easy to check that the statement of the lemma is true for $j$ such that $1 \leq j \leq F_4 - 1$. **Inductive hypothesis:** assume that it is true for $1 \leq j \leq F_m - 1$ for some $m \geq 4$. From Lemma 4.3, the first $F_m - 1$ rows of Table 2 are those for which the largest summand is no greater than $L_{k+m-2}$. Also, the rows for which $L_{k+m}$ is the largest summand are those numbered from $F_m + 1$ to $F_{m+1}$ inclusive. Therefore, the ordering of the rows implies that $q_k(i + F_m) = q_k(i) + L_{k+m}$, for $i = 1, 2, \ldots, F_{m-1}$. Hence, for $i = 1, 2, \ldots, F_{m-1} - 1$, we have

$$q_k(i + 1 + F_m) - q_k(i + F_m) = (q_k(i + 1) + L_{k+m}) - (q_k(i) + L_{k+m}) = q_k(i + 1) - q_k(i).$$

By the construction of $S$, the substring comprising its first $F_m - 1$ characters is identical to the substring of its characters numbered from $F_m + 1$ to $F_{m+1}$ inclusive. Thus, the lemma is true for $F_m + 1 \leq j \leq F_{m+1} - 1$. It remains to show that the lemma is true for $j = F_m$. We have

$$q_k(F_m + 1) - q_k(F_m) = \begin{cases} L_{k+m} - (L_{k+m-1} + L_{k+m-3} + \cdots + L_{k+3}) = L_{k+2}, & \text{if } m \text{ is even,} \\ L_{k+m} - (L_{k+m-1} + L_{k+m-3} + \cdots + L_{k+2}) = L_{k+1}, & \text{if } m \text{ is odd.} \end{cases}$$

By Remark 2.3 item (1), we know that the lemma is true for $j = F_m$, completing the proof. □

We are ready to prove Theorem 1.4

Proof of Theorem 1.4 We consider three cases.

Case 1: $k = 0$. By Lemma 4.2, we have $X_0 = \{2 + a(n)L_2 + b(n)L_3 : n \geq 0\}$, where $a(n)$ and $b(n)$ denote the number of $A$'s and $B$'s, respectively, amongst the first $n$ characters in the golden string. Using Remark 2.3 item (2), we have

$$X_0 = \left\{ 2 + 3\left(n - \left\lfloor \frac{n + 1}{\Phi} \right\rfloor \right) + 4\left\lfloor \frac{n + 1}{\Phi} \right\rfloor : n \geq 0 \right\} = \left\{ 2 + 3n + \left\lfloor \frac{n + 1}{\Phi} \right\rfloor : n \geq 0 \right\}.$$ 

It is clear that $Z(0) = X_0$; hence, the statement of the lemma is true when $k = 0$.

Case 2: $k = 1$. Using a similar reasoning as above, we have

$$X_1 = \left\{ 1 + L_2\left(n - \left\lfloor \frac{n + 1}{\Phi} \right\rfloor \right) + L_3\left\lfloor \frac{n + 1}{\Phi} \right\rfloor : n \geq 0 \right\} = \left\{ 1 + 3\left(n - \left\lfloor \frac{n + 1}{\Phi} \right\rfloor \right) + 4\left\lfloor \frac{n + 1}{\Phi} \right\rfloor : n \geq 0 \right\} = \left\{ 3n + \left\lfloor \frac{n + \Phi^2}{\Phi} \right\rfloor : n \geq 0 \right\}.$$ 

It is clear that $Z(1) = X_1$; hence, the statement of the lemma is true when $k = 1$.

Case 3: $k \geq 2$. Using a similar reasoning as above, we have

$$X_k = \left\{ L_k + L_{k+1}\left(n - \left\lfloor \frac{n + 1}{\Phi} \right\rfloor \right) + L_{k+2}\left\lfloor \frac{n + 1}{\Phi} \right\rfloor : n \geq 0 \right\} = \left\{ L_k \left(1 + \left\lfloor \frac{n + 1}{\Phi} \right\rfloor \right) + nL_{k+1} : n \geq 0 \right\} = \left\{ L_k\left\lfloor \frac{n + \Phi^2}{\Phi} \right\rfloor + nL_{k+1} : n \geq 0 \right\}.$$
If \( k \geq 3 \), the numbers in \( \{L_0, L_1, \ldots, L_{k-2}\} \) are used to obtain the partitions of all integers for which the largest summand is no greater than \( L_{k-2} \). In particular, such partitions generate all integers from 1 to \( L_{k-1} - 1 \) inclusive. Furthermore, such partitions can be appended to any partition having \( L_k \) as its smallest summand to produce another partition. Therefore,

\[
Z(k) = \left\{ L_k \left\lfloor \frac{n + \Phi^2}{\Phi} \right\rfloor + nL_{k+1} + j : n \geq 0 \text{ and } 0 \leq j \leq L_{k-1} - 1 \right\},
\]
as desired. It is easy to check that this formula is also true for \( k = 2 \). \( \square \)

5. Proportion of Nonunique Partitions

Let \( c(N) \) count the number of numbers that are not uniquely represented in the Lucas sequence and are at most \( N \). We want to show that \( \lim_{N \to \infty} \frac{c(N)}{N} = \frac{1}{1 + 3\Phi} \), where \( \Phi = (1 + \sqrt{5})/2 \) is the golden ratio. Note that [Lemma 3] says we can make the Lucas partition unique by requiring that not both \( L_0 \) and \( L_2 \) appear in the partition. Therefore, if a number has two partitions, then one of the partition starts with \( L_0 + L_2 \). If we can characterize all of these numbers and find a formula for \( c(N) \) in terms of \( N \), we are done. Call the set of these numbers \( K \). We form the following table listing all of such numbers in increasing order. Let \( q_k(j) \) be the \( j \)th smallest number in \( K \).

<table>
<thead>
<tr>
<th>Row</th>
<th>( L_0 + L_2 )</th>
<th>( L_1 + L_3 )</th>
<th>( L_2 + L_4 )</th>
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<th>( L_4 + L_6 )</th>
<th>( L_5 + L_7 )</th>
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<tr>
<td>8</td>
<td>( L_0 + L_2 )</td>
<td>( L_1 + L_3 )</td>
<td>( L_2 + L_4 )</td>
<td>( L_3 + L_5 )</td>
<td>( L_4 + L_6 )</td>
<td>( L_5 + L_7 )</td>
<td>( L_6 + L_8 )</td>
</tr>
</tbody>
</table>

Table 3. The partitions of the positive integers having \( L_0 \) and \( L_2 \) as their smallest summands.

Observe that Table 3 has the same structure as Table 1. Therefore, Lemma \[4.2\] applies with a change of index. In particular, we have the following.

**Lemma 5.1.** For \( j \geq 1 \), we have

\[
q_k(j + 1) - q_k(j) = \begin{cases} L_3, & \text{if } A \text{ is the } j \text{th character of } S, \\ L_4, & \text{if } B \text{ is the } j \text{th character of } S. \end{cases}
\]

Therefore, we can write

\[
K = \{L_0 + L_2 + a(n)L_3 + b(n)L_4 : n \geq 0\},
\]

where \( a(n) \) and \( b(n) \) denote the number of \( A \)'s and \( B \)'s, respectively, amongst the first \( n \) characters in the golden string. Hence,

\[
K = \{5 + 4(n - \lfloor (n+1)/\Phi \rfloor) + 7\lfloor (n+1)/\Phi \rfloor : n \geq 0\} = \{5 + 4n + 3\lfloor (n+1)/\Phi \rfloor : n \geq 0\}.
\]

Now, we are ready to compute the limit.

**Proof of Theorem 1.5** The number of integers with two partitions up to a number \( N \) is exactly \( \#\{n \geq 0 | 5 + 4n + 3\lfloor (n+1)/\Phi \rfloor \leq N\} \). The number is found to be \( \frac{N-1}{1+3\Phi} \) within an error of
at most 1. Therefore, as claimed, the limit is
\[
\lim_{N \to \infty} \frac{1}{N} \frac{N-1}{4+3/\Phi} = \frac{1}{4+3/\Phi} = \frac{1}{1+3/\Phi}.
\]
\[\square\]

Among the first \(N\) natural numbers, we see how \(\alpha = \frac{1}{3\Phi + 1} \approx 0.17082\) estimates the proportion of natural numbers within this range that do not have unique non-consecutive partitions in the Lucas sequence. The data we collect is shown in Table 4.

<table>
<thead>
<tr>
<th>(N)</th>
<th>(c(N))</th>
<th>(\beta(N))</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1</td>
<td>10.000%</td>
</tr>
<tr>
<td>100</td>
<td>17</td>
<td>17.000%</td>
</tr>
<tr>
<td>1,000</td>
<td>171</td>
<td>17.100%</td>
</tr>
<tr>
<td>10,000</td>
<td>1,708</td>
<td>17.080%</td>
</tr>
<tr>
<td>(10^6)</td>
<td>17,082</td>
<td>17.082%</td>
</tr>
<tr>
<td>(10^7)</td>
<td>170,820</td>
<td>17.082%</td>
</tr>
</tbody>
</table>

Table 4. Proportion \(\beta(N)\) of the first \(N\) natural numbers that do not have unique non-consecutive partitions in the Lucas sequence.

**Appendix A. Java Code**

The following is our Java code for calculating non-consecutive partitions of natural numbers in any infinite integer sequence given by a second-order linear recurrence. It is available on github at [https://github.com/dluo6745/Zeckendorf-Partitions/blob/master/ZP.java](https://github.com/dluo6745/Zeckendorf-Partitions/blob/master/ZP.java). For each natural number \(n\) from 1 to \(N\), the code returns the non-consecutive partition(s) of \(n\) as a list of integers that correspond to the indices of the terms in the second-order linear recurrence sequence we are enumerating. Furthermore, the code also returns the number of natural numbers from 1 to \(N\) that do not have unique non-consecutive partitions.

**Appendix B. Proofs of Lemmas**

**Proof of Lemma [3.1]**. We proceed by strong induction. The non-consecutive sums that we can form from \(A_0\) are 0 and \(L_1 + 1\) because the empty set results in a sum of 0 and the non-consecutive sums that we can form from \(A_1\) are 0, \(L_1\), and \(L_2 - 1\). This shows the base case. Assume Lemma [3.1] holds for all non-negative integers less than or equal to \(m = k\). Without loss of generality, suppose that \(k\) is odd. To find the range of non-consecutive sums that we can form from \(A_{k+1}\), we consider the subset \(A_{k+1} - \{L_k\}\). From our inductive hypothesis, the non-consecutive sums that we can form from \(A_{k-1}\) are the values from 0 to \(L_k + 1\) inclusive, excluding \(L_k\). By adding \(L_{k+1}\) to these values, we have the following non-consecutive sums that we can form from \(A_{k+1}\) range from 0 to \(L_{k+2} + 1\) inclusive.

To show that \(L_{k+2}\) cannot be formed as a non-consecutive sum of \(A_{k+1}\), we first prove a general result. Let \(B\) be a non-consecutive subset of \(A_{2j}\), where \(j\) is a non-negative integer such that \(2j < k\). For sake of contradiction, suppose that the sum of the elements of \(B\) is equal to \(L_{2j+1}\). In our first case, suppose that \(L_{2j}\) is not in \(B\). This implies \(B\) is a non-consecutive subset of \(A_{2j-1}\) and that the sum of the elements of \(B\) is less than or equal to \(L_{2j+1} - 1\) from our inductive hypothesis. Hence, we have a contradiction which implies \(B\) contains the term \(L_{2j}\). Consider the set \(B' = B - \{L_{2j}\}\), which is a non-consecutive subset of \(A_{2j-2}\). Because the sum of the elements of \(B'\) is equal to the difference between the sum of the elements of \(B\) and \(L_{2j}\), this implies that the sum of the elements of \(B'\) is equal to \(L_{2j-1}\), which cannot be
formed as a non-consecutive sum of $A_{2j-2}$ by our inductive hypothesis. Therefore, we have a contradiction and $L_{2j+1}$ cannot be formed as a non-consecutive sum of $A_{2j}$.

Applying this result to our inductive step, we have that $L_k$ cannot be formed as a non-consecutive sum of $A_{k-1}$. This implies there is no possible way to form $L_{k+2} = L_k + L_{k+1}$ as a non-consecutive sum of $A_{k+1} - \{L_k\}$. From our inductive hypothesis, the maximum possible sum we can form from $A_k$ is $L_{k+1} - 1$, which is less than $L_{k+2}$. Therefore, $L_{k+2}$ cannot be formed as a non-consecutive sum of $A_{k+1}$, completing the inductive step.

Proof of Lemma 3.2. It suffices to show that every natural number of the form $L_{2m+1} + 1$ is equal to only one non-consecutive sum of $A_{2m}$. We proceed by strong induction. Note that $L_3 + 1$ is equal to only one non-consecutive sum of $A_2$, and $L_5 + 1$ is equal to only one non-consecutive sum of $A_4$. This shows the base case. Assume Lemma 3.2 holds for all non-negative integers less than or equal to $m = k$. Let $B$ be a non-consecutive subset of $A_{2k+2}$. For sake of contradiction, suppose that the sum of the elements of $B$ is equal to $L_{2k+3} + 1$ and that $B$ does not contain the term $L_{2k+2}$. From Lemma 3.1 the non-consecutive sums that we can form from $A_{2k+2}$ are the values from 0 to $L_{2k+3} + 1$ inclusive, excluding $L_{2k+2}$. This implies $B$ is a non-consecutive subset of $A_{2k+1}$. From Lemma 3.1 we have that the sum of the elements of $B$ must be less than or equal to $L_{2k+2} - 1$. Hence we have a contradiction, which implies $B$ contains the term $L_{2k+2}$. From our inductive hypothesis, we know that $L_{2k+1} + 1$ is equal to only one non-consecutive sum of $A_{2k}$. Because $L_{2k+3} + 1 = L_{2k+2} + (L_{2k+1} + 1)$ and $B$ cannot contain both $L_{2k+2}$ and $L_{2k+1}$, this implies $L_{2k+3} + 1$ is equal to only one non-consecutive sum of $A_{2k+2}$. This completes the inductive step.

**References**


[21] C. Herink, personal communication (E-mail to David C. Luo), (2020).


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