ON ZECKENDORF RELATED PARTITIONS USING THE LUCAS SEQUENCE

HÙNG VIỆT CHU, DAVID C. LUO, AND STEVEN J. MILLER

Abstract. Zeckendorf proved that every natural number has a unique partition as a sum of non-consecutive Fibonacci numbers. Similarly, every natural number can be partitioned into a sum of non-consecutive terms of the Lucas sequence, although such partitions need not be unique. In this paper, we

(1) prove that a natural number can have at most two distinct non-consecutive partitions in the Lucas sequence,
(2) find all positive integers with a fixed term in their partition, and
(3) calculate the limiting value of the proportion of natural numbers that are not uniquely partitioned into the sum of non-consecutive terms in the Lucas sequence.

1. Introduction

The Fibonacci numbers have fascinated mathematicians for centuries with many interesting properties. By convention, the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined as follows: let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. A beautiful theorem of Zeckendorf [31] states that every natural number n can be uniquely written as a sum of non-consecutive Fibonacci numbers. This gives the so-called Zeckendorf partition of n. A formal statement of Zeckendorf's theorem is as follows:

Theorem 1.1 (Zeckendorf). *For any $n \in \mathbb{N}$, there exists a unique increasing sequence of positive integers $\{c_1, c_2, \ldots, c_k\}$ such that $c_1 \geq 2$, $c_i \geq c_{i-1} + 2$ for $i = 2, 3, \ldots, k$, and $n = \sum_{i=1}^{k} F_{c_i}$.*

Much work has been done to understand the structure of Zeckendorf partitions and their applications (see [1, 2, 6, 8, 16, 18, 19, 20, 25, 29, 30]) and to generalize them (see [10, 12, 13, 14, 15, 17, 22, 23, 26, 27, 28]). In this paper, we study the partition of natural numbers into Lucas numbers. The Lucas sequence $\{L_n\}_{n=0}^{\infty}$ is defined as follows: let $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$. As the Lucas sequence is closely related to the Fibonacci sequence, it is not surprising that we can also partition natural numbers using Lucas numbers.

Theorem 1.2 (Zeckendorf). *Every natural number can be partitioned into the sum of non-consecutive terms of the Lucas sequence.*

Note that the distinction between Theorems 1.1 and 1.2 lies in the *uniqueness* property of such partitions of natural numbers in the Fibonacci and Lucas sequences. Although 5 is uniquely partitioned into $F_5 = 5$ in $\{F_2, F_3, \ldots\}$, its partition is not unique in the Lucas sequence as $5 = L_0 + L_2 = 2 + 3$ and $5 = L_1 + L_3 = 1 + 4$. In [7], Brown shows various ways to have a unique partition using Lucas sequence. [1] In this paper, we prove the following results.

Date: February 23, 2021.

The authors thank Curtis D. Herink and David Zureick-Brown for helpful conversations, Jeffrey Shallit for pointing out a gap in reasoning in an earlier version, the anonymous referee for useful comments, and Elvin Gu for coding support. The third author was supported by NSF grants DMS1561945.

[1] For more on Brown's criteria, see [3, 4].

Theorem 1.3. *If we allow L_0 and L_2 to appear simultaneously in a partition, each natural number can have at most two distinct non-consecutive partitions in the Lucas sequence.*

Theorem 1.4. *Suppose that we do not allow L_0 and L_2 to appear simultaneously in a partition. The set of all positive integers having the summand L_k in their partition is given by*

$$Z(k) \;=\; \begin{cases} \left\{2+3n+\left\lfloor \frac{n+1}{\Phi}\right\rfloor \,:\, n\geq 0\right\}, & \text{if } k=0,\\ \left\{3n+\left\lfloor \frac{n+\Phi^2}{\Phi}\right\rfloor \,:\, n\geq 0\right\}, & \text{if } k=1,\\ \left\{L_k\left\lfloor \frac{n+\Phi^2}{\Phi}\right\rfloor + nL_{k+1}+j \,:\, n\geq 0 \text{ and } 0\leq j\leq L_{k-1}-1\right\}, & \text{if } k\geq 2.\end{cases}$$

Theorem 1.4 is an analogue of [18, Theorem 3.4]. For $k \geq 0$, we find all positive integers having the summand L_k in their partition. We have a different formula when $k = 0$ instead of one formula for all values of k as in [18, Theorem 3.4].

Our next result is predicted by [9, Theorem 1], which deals with general recurrence relations; however, in the case of Lucas numbers, we can relate Lucas partitions to the golden string.

Theorem 1.5. *If we allow L_0 and L_2 to appear simultaneously in a partition, the proportion of natural numbers that are not uniquely partitioned into the sum of non-consecutive terms of the Lucas sequence converges to $\frac{1}{3\Phi+1}$, where Φ is the golden ratio.*

2. Preliminaries

2.1. Definitions.

Definition 2.1. *Let $A = \{a_0, a_1, \dots, a_m\}$ be the set consisting of the first $m+1$ terms of the sequence $\{a_k\}_{k=0}^{\infty}$. We say a proper subset B of A is a non-consecutive subset of A if the elements of B are pairwise non-consecutive in $\{a_k\}_{k=0}^{\infty}$. Furthermore, we say a sum S is a non-consecutive sum of A if S is the sum of distinct elements of A that are pairwise non-consecutive in $\{a_k\}_{k=0}^{\infty}$.*

Definition 2.2. *Let $A_m = \{L_0, L_1, \dots, L_m\}$ denote the set consisting of the first $m+1$ terms of the Lucas sequence.*

2.2. **The golden string.** The golden string $S = BABBABABBABBA\dots$ is defined to be the infinite string of A's and B's constructed recursively as follows. Let $S_1 = A$ and $S_2 = B$, and then, for $k \geq 3$, S_k is the concatenation of S_{k-1} and S_{k-2}, which we denote by $S_{k-1} \circ S_{k-2}$. For example, $S_3 = S_2 \circ S_1 = a_2 \circ a_1 = BA$, $S_4 = S_3 \circ S_2 = a_2a_1 \circ a_2 = BAB$, $S_5 = S_4 \circ S_3 = BABBA$, and so on. Interestingly, the golden string is highly connected to the Zeckendorf partition [19]. As we will see later, the string is also closely related to the partitions of natural numbers into Lucas numbers.

Remark 2.3. We mention two properties of the golden string that we will use in due course.

1. For $j \geq 1$, the (F_{2j})th character of S is B and the (F_{2j+1})th character of S is A. This can be easily proved using induction.
2. The number of B's amongst the first n characters of S is given by $\left\lfloor \frac{n+1}{\Phi}\right\rfloor$, where $\Phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. For a proof of this result, see [19, Lemma 3.3].

3. At Most Two Partitions

In this section, we present our results that determine the maximum number of non-consecutive partitions that a natural number can have in the Lucas sequence, the proofs of which are adapted from [21]. Before we prove Theorem 1.3, we introduce the following preliminary lemmas. For the proofs of Lemmas 3.1 and 3.2, see Appendix B.

Lemma 3.1. *Let S be any non-consecutive sum of A_m. Then*

1. *if m is odd, S assumes all values from 0 to $L_{m+1} - 1$ inclusive, and*
2. *if m is even, then S assumes all values from 0 to $L_{m+1} + 1$ inclusive, excluding L_{m+1}.*

Lemma 3.2. *If $m \geq 0$, then $L_{2m+1} + 1$ has exactly two non-consecutive partitions in the Lucas sequence.*

Proof of Theorem 1.3. It suffices to show that for every non-negative integer m, there is no natural number that is equal to three or more distinct non-consecutive sums of A_m. We proceed by strong induction. No natural is equal to three or more distinct non-consecutive sums of A_0 and A_1. This shows the base case. Assume Theorem 1.3 holds for all non-negative integers less than or equal to $m = k$. In our first case, suppose that k is odd. From Lemma 3.1, the non-consecutive sums that we can form from A_k are the values from 0 to $L_{k+1} - 1$ inclusive. Hence, when we add the term L_{k+1} to A_k, all new non-consecutive sums that can be formed must be at least L_{k+1}. This implies there is no possible way in which we can form a third distinct non-consecutive sum of A_{k+1} for any natural number because there is no intersection between the non-consecutive sums in which we can form before and after the addition of the term L_{k+1}. When $k \geq 2$ is even, we have from Lemma 3.1 that all non-consecutive sums we can form from A_k are the values from 0 to $L_{k+1} + 1$ inclusive, excluding L_{k+1}. When we add the term L_{k+1} to A_k, all new non-consecutive sums that can be formed are at least L_{k+1} with $L_{k+1} + 1$ being the only non-consecutive sum formed again, namely $L_{k+1} + L_1$. By Lemma 3.2, we know that $L_{k+1} + 1$ has exactly two distinct non-consecutive partitions in the Lucas sequence. Therefore, there is no possible way in which we can form a third distinct non-consecutive sum of A_{k+1} for any natural number. This completes the inductive step. □

4. Partitions with a Fixed Term

Let $\mathcal{X}_k$ denote the set of all positive integers having L_k as the smallest summand in their partition. Let $\mathcal{Q}_k = (q_k(j))_{j \geq 1}$ be the strictly increasing sequence obtained by rearranging the elements of $\mathcal{X}_k$ into ascending numerical order. We consider the cases $k = 0$ and $k \geq 1$ separately.

4.1. When $k = 0$.

Table 1 replaces each term $q_k(j)$ in $\mathcal{Q}_k$ with an ordered list of the summands in its partition.

Row					
1	L_0				
2	L_0	L_3			
3	L_0		L_4		
4	L_0			L_5	
5	L_0	L_3		L_5	
6	L_0				L_6
7	L_0	L_3			L_6
8	L_0		L_4		L_6

Table 1. The partitions of the positive integers having L_0 as their smallest summand.

Lemma 4.1. *For $j \geq 3$, the rows of Table 1 for which L_j is the largest summand are those numbered from $F_{j-1} + 1$ to F_j inclusive.*

Proof. We prove by induction. *Base cases:* it is easy to check that the statement of the lemma is true for $j = 3$ and $j = 4$. *Inductive hypothesis:* assume that it is true for all j such that

$3 \le j \le m$ for some $m \ge 4$. By the inductive hypothesis, the number of rows such that their largest summands are no greater than L_{m-1} is

$$1 + \sum_{j=3}^{m-1} (F_j - F_{j-1}) \ = \ F_{m-1},$$

which is also the number of rows whose largest summand is L_{m+1}. Due to the inductive hypothesis, the rows whose largest summand is L_m are numbered from $F_{m-1}+1$ to F_m inclusive. Therefore, the rows whose largest summand is L_{m+1} are numbered from $F_m + 1$ to F_{m+1}, as desired. This completes our proof. □

Lemma 4.2. *For $j \ge 1$, we have*

$$q_k(j+1) - q_k(j) \ = \ \begin{cases} L_2, & \text{if } A \text{ is the } j\text{th character of } S, \\ L_3, & \text{if } B \text{ is the } j\text{th character of } S. \end{cases}$$

Proof. We prove by induction. *Base cases:* it is easy to check that the statement of the lemma is true for $1 \le j \le F_4 - 1$. *Inductive hypothesis:* suppose that it is true for $1 \le j \le F_m - 1$ for some $m \ge 4$. By Lemma 4.1, the number of rows in Table 1 whose largest summand is no greater than L_{m-1} is

$$1 + \sum_{j=3}^{m-1} (F_j - F_{j-1}) \ = \ F_{m-1},$$

which is also the number of rows whose largest summand is L_{m+1}. Furthermore, the rows for which L_{m+1} is the largest summand are numbered from $F_m + 1$ to F_{m+1} inclusive. Therefore, the ordering of the rows in Table 1 implies that $q_k(i+F_m) \ = \ q_k(i)+L_{m+1}$, for $1 \le i \le F_{m-1}$. Hence, for $1 \le i \le F_{m-1} - 1$, we have

$$q_k(i+1+F_m) - q_k(i+F_m) \ = \ (q_k(i+1) + L_{m+1}) - (q_k(i) + L_{m+1}) \ = \ q_k(i+1) - q_k(i).$$

By the construction of S, the substring comprising of its first F_{m-1} characters is identical to the substring of its characters numbered from $F_m + 1$ to F_{m+1} inclusive. Thus the lemma is true for $F_m + 1 \le j \le F_{m+1} - 1$. It remains to show that it is true for $j = F_m$. We have

$$q_k(F_m+1) - q_k(F_m) \ = \begin{cases} L_{m+1} - (L_m + L_{m-2} + \cdots + L_4) \ = \ L_3, & \text{if } m \text{ is even}, \\ L_{m+1} - (L_m + L_{m-2} + \cdots + L_3) \ = \ L_2, & \text{if } m \text{ is odd}. \end{cases}$$

By Remark 2.3 item (1), we know that the lemma is true for $j = F_m$, completing the proof. □

4.2. **When $k \ge 1$.** Table 2 replaces each term $q_k(j)$ in $\mathcal{Q}_k$ with an ordered list of the summands in its partition.

Row					
1	L_k				
2	L_k	L_{k+2}			
3	L_k		L_{k+3}		
4	L_k			L_{k+4}	
5	L_k	L_{k+2}		L_{k+4}	
6	L_k				L_{k+5}
7	L_k	L_{k+2}			L_{k+5}
8	L_k		L_{k+3}		L_{k+5}

Table 2. The partitions of the positive integers having L_k as their smallest summand.

Table 2 is similar to Table 1 in [18]. The next lemma follows from [18, Lemma 3.1].

Lemma 4.3. *For $j \geq 2$, the rows of Table 2 for which L_{k+j} is the largest summand are those numbered from $F_j + 1$ to F_{j+1} inclusive.*

Lemma 4.4. *For $j \geq 1$, we have*

$$q_k(j+1) - q_k(j) \;=\; \begin{cases} L_{k+1}, & \text{if } A \text{ is the } j\text{th character of } S, \\ L_{k+2}, & \text{if } B \text{ is the } j\text{th character of } S. \end{cases}$$

Proof. We prove by induction. *Base cases:* it is easy to check that the statement of the lemma is true for j such that $1 \leq j \leq F_4 - 1$. *Inductive hypothesis:* assume that it is true for $1 \leq j \leq F_m - 1$ for some $m \geq 4$. From Lemma 4.3, the first F_{m-1} rows of Table 2 are those for which the largest summand is no greater than L_{k+m-2}. Also, the rows for which L_{k+m} is the largest summand are those numbered from $F_m + 1$ to F_{m+1} inclusive. Therefore, the ordering of the rows implies that $q_k(i + F_m) = q_k(i) + L_{k+m}$, for $i = 1, 2, \ldots, F_{m-1}$. Hence, for $i = 1, 2, \ldots, F_{m-1} - 1$, we have

$$q_k(i+1+F_m) - q_k(i+F_m) \;=\; (q_k(i+1) + L_{k+m}) - (q_k(i) + L_{k+m}) \;=\; q_k(i+1) - q_k(i).$$

By the construction of S, the substring comprising its first F_{m-1} characters is identical to the substring of its characters numbered from $F_m + 1$ to F_{m+1} inclusive. Thus, the lemma is true for $F_m + 1 \leq j \leq F_{m+1} - 1$. It remains to show that the lemma is true for $j = F_m$. We have

$$q_k(F_m+1) - q_k(F_m) \;=\; \begin{cases} L_{k+m} - (L_{k+m-1} + L_{k+m-3} + \cdots + L_{k+3}) \;=\; L_{k+2}, & \text{if } m \text{ is even}, \\ L_{k+m} - (L_{k+m-1} + L_{k+m-3} + \cdots + L_{k+2}) \;=\; L_{k+1}, & \text{if } m \text{ is odd}. \end{cases}$$

By Remark 2.3 item (1), we know that the lemma is true for $j = F_m$, completing the proof. $\square$

We are ready to prove Theorem 1.4.

Proof of Theorem 1.4. We consider three cases.

Case 1: $k = 0$. By Lemma 4.2, we have $\mathcal{X}_0 = \{2 + a(n)L_2 + b(n)L_3 : n \geq 0\}$, where $a(n)$ and $b(n)$ denote the number of A's and B's, respectively, amongst the first n characters in the golden string. Using Remark 2.3 item (2), we have

$$\mathcal{X}_0 \;=\; \left\{2 + 3\left(n - \left\lfloor \frac{n+1}{\Phi} \right\rfloor\right) + 4\left\lfloor \frac{n+1}{\Phi} \right\rfloor \;:\; n \geq 0\right\} \;=\; \left\{2 + 3n + \left\lfloor \frac{n+1}{\Phi} \right\rfloor \;:\; n \geq 0\right\}.$$

It is clear that $Z(0) = \mathcal{X}_0$; hence, the statement of the lemma is true when $k = 0$.

Case 2: $k = 1$. Using a similar reasoning as above, we have

$$\begin{aligned} \mathcal{X}_1 \;&=\; \left\{1 + L_2\left(n - \left\lfloor \frac{n+1}{\Phi} \right\rfloor\right) + L_3\left\lfloor \frac{n+1}{\Phi} \right\rfloor \;:\; n \geq 0\right\} \\ &=\; \left\{1 + 3\left(n - \left\lfloor \frac{n+1}{\Phi} \right\rfloor\right) + 4\left\lfloor \frac{n+1}{\Phi} \right\rfloor \;:\; n \geq 0\right\} \;=\; \left\{3n + \left\lfloor \frac{n+\Phi^2}{\Phi} \right\rfloor \;:\; n \geq 0\right\}. \end{aligned}$$

It is clear that $Z(1) = \mathcal{X}_1$; hence, the statement of the lemma is true when $k = 1$.

Case 3: $k \geq 2$. Using a similar reasoning as above, we have

$$\begin{aligned} \mathcal{X}_k \;&=\; \left\{L_k + L_{k+1}\left(n - \left\lfloor \frac{n+1}{\Phi} \right\rfloor\right) + L_{k+2}\left\lfloor \frac{n+1}{\Phi} \right\rfloor \;:\; n \geq 0\right\} \\ &=\; \left\{L_k\left(1 + \left\lfloor \frac{n+1}{\Phi} \right\rfloor\right) + nL_{k+1} \;:\; n \geq 0\right\} \;=\; \left\{L_k\left\lfloor \frac{n+\Phi^2}{\Phi} \right\rfloor + nL_{k+1} \;:\; n \geq 0\right\}. \end{aligned}$$

If $k \geq 3$, the numbers in $\{L_0, L_1, \ldots, L_{k-2}\}$ are used to obtain the partitions of all integers for which the largest summand is no greater than L_{k-2}. In particular, such partitions generate all integers from 1 to $L_{k-1} - 1$ inclusive. Furthermore, such partitions can be appended to any partition having L_k as its smallest summand to produce another partition. Therefore,

$$Z(k) \;=\; \left\{ L_k \left\lfloor \frac{n + \Phi^2}{\Phi} \right\rfloor + nL_{k+1} + j \;:\; n \geq 0 \text{ and } 0 \leq j \leq L_{k-1} - 1 \right\},$$

as desired. It is easy to check that this formula is also true for $k = 2$. □

5. Proportion of Nonunique Partitions

Let $c(N)$ count the number of numbers that are not uniquely represented in the Lucas sequence and are at most N. We want to show that $\lim_{N\to\infty} \frac{c(N)}{N} = \frac{1}{1+3\Phi}$, where $\Phi = (1+\sqrt{5})/2$ is the golden ratio. Note that [7, Lemma 3] says we can make the Lucas partition unique by requiring that not both L_0 and L_2 appear in the partition. Therefore, if a number has two partitions, then one of the partition starts with $L_0 + L_2$. If we can characterize all of these numbers and find a formula for $c(N)$ in terms of N, we are done. Call the set of these numbers K. We form the following table listing all of such numbers in increasing order. Let $q_k(j)$ be the jth smallest number in K.

Row					
1	$L_0 + L_2$				
2	$L_0 + L_2$	L_4			
3	$L_0 + L_2$		L_5		
4	$L_0 + L_2$			L_6	
5	$L_0 + L_2$	L_4		L_6	
6	$L_0 + L_2$				L_7
7	$L_0 + L_2$	L_4			L_7
8	$L_0 + L_2$		L_5		L_7

Table 3. The partitions of the positive integers having L_0 and L_2 as their smallest summands.

Observe that Table 3 has the same structure as Table 1. Therefore, Lemma 4.2 applies with a change of index. In particular, we have the following.

Lemma 5.1. *For $j \geq 1$, we have*

$$q_k(j+1) - q_k(j) \;=\; \begin{cases} L_3, & \text{if } A \text{ is the } jth \text{ character of } S, \\ L_4, & \text{if } B \text{ is the } jth \text{ character of } S. \end{cases}$$

Therefore, we can write

$$K \;=\; \{L_0 + L_2 + a(n)L_3 + b(n)L_4 : n \geq 0\},$$

where $a(n)$ and $b(n)$ denote the number of A's and B's, respectively, amongst the first n characters in the golden string. Hence,

$$K \;=\; \{5 + 4(n - \lfloor (n+1)/\Phi \rfloor) + 7\lfloor (n+1)/\Phi \rfloor \;:\; n \geq 0\} \;=\; \{5 + 4n + 3\lfloor (n+1)/\Phi \rfloor \;:\; n \geq 0\}.$$

Now, we are ready to compute the limit.

Proof of Theorem 1.5. The number of integers with two partitions up to a number N is exactly $\#\{n \geq 0 \,|\, 5 + 4n + 3\lfloor (n+1)/\Phi \rfloor \leq N\}$. The number is found to be $\frac{N-1}{4+3/\Phi}$ within an error of at most 1. Therefore, as claimed, the limit is

$$\lim_{N\to\infty} \frac{1}{N}\frac{N-1}{4+3/\Phi} = \frac{1}{4+3/\Phi} = \frac{1}{1+3\Phi}.$$

□

Among the first N natural numbers, we see how $\alpha = \frac{1}{3\Phi+1} \approx 0.17082$ estimates the proportion of natural numbers within this range that do not have unique non-consecutive partitions in the Lucas sequence. The data we collect is shown in Table 4.

N	$c(N)$	$\beta(N)$
10	1	10.000 %
100	17	17.000%
1,000	171	17.100%
10,000	1,708	17.080%
10^5	17,082	17.082%
10^6	170,820	17.082%

Table 4. Proportion $\beta(N)$ of the first N natural numbers that do not have unique non-consecutive partitions in the Lucas sequence.

Appendix A. Java Code

The following is our Java code for calculating non-consecutive partitions of natural numbers in any infinite integer sequence given by a second-order linear recurrence. It is available on github at `https://github.com/dluo6745/Zeckendorf-Partitions/blob/master/ZP.java`. For each natural number n from 1 to N, the code returns the non-consecutive partition(s) of n as a list of integers that correspond to the indices of the terms in the second-order linear recurrence sequence we are enumerating. Furthermore, the code also returns the number of natural numbers from 1 to N that do not have unique non-consecutive partitions.

Appendix B. Proofs of Lemmas

Proof of Lemma 3.1. We proceed by strong induction. The non-consecutive sums that we can form from A_0 are 0 and $L_1 + 1$ because the empty set results in a sum of 0 and the non-consecutive sums that we can form from A_1 are 0, L_1, and $L_2 - 1$. This shows the base case. Assume Lemma 3.1 holds for all non-negative integers less than or equal to $m = k$. Without loss of generality, suppose that k is odd. To find the range of non-consecutive sums that we can form from A_{k+1}, we consider the subset $A_{k+1} - \{L_k\}$. From our inductive hypothesis, the non-consecutive sums that we can form from A_{k-1} are the values from 0 to $L_k + 1$ inclusive, excluding L_k. By adding L_{k+1} to these values, we have the following non-consecutive sums that we can form from A_{k+1} range from 0 to $L_{k+2} + 1$ inclusive.

To show that L_{k+2} cannot be formed as a non-consecutive sum of A_{k+1}, we first prove a general result. Let B be a non-consecutive subset of A_{2j}, where j is a non-negative integer such that $2j < k$. For sake of contradiction, suppose that the sum of the elements of B is equal to L_{2j+1}. In our first case, suppose that L_{2j} is not in B. This implies B is a non-consecutive subset of A_{2j-1} and that the sum of the elements of B is less than or equal to $L_{2j+1} - 1$ from our inductive hypothesis. Hence, we have a contradiction which implies B contains the term L_{2j}. Consider the set $B' = B - \{L_{2j}\}$, which is a non-consecutive subset of A_{2j-2}. Because the sum of the elements of B' is equal to the difference between the sum of the elements of B and L_{2j}, this implies that the sum of the elements of B' is equal to L_{2j-1}, which cannot be formed as a non-consecutive sum of A_{2j-2} by our inductive hypothesis. Therefore, we have a contradiction and L_{2j+1} cannot be formed as a non-consecutive sum of A_{2j}.

Applying this result to our inductive step, we have that L_k cannot be formed as a non-consecutive sum of A_{k-1}. This implies there is no possible way to form $L_{k+2} = L_k + L_{k+1}$ as a non-consecutive sum of $A_{k+1} - \{L_k\}$. From our inductive hypothesis, the maximum possible sum we can form from A_k is $L_{k+1} - 1$, which is less than L_{k+2}. Therefore, L_{k+2} cannot be formed as a non-consecutive sum of A_{k+1}, completing the inductive step. □

Proof of Lemma 3.2. It suffices to show that every natural number of the form $L_{2m+1} + 1$ is equal to only one non-consecutive sum of A_{2m}. We proceed by strong induction. Note that $L_3 + 1$ is equal to only one non-consecutive sum of A_2, and $L_5 + 1$ is equal to only one non-consecutive sum of A_4. This shows the base case. Assume Lemma 3.2 holds for all non-negative integers less than or equal to $m = k$. Let B be a non-consecutive subset of A_{2k+2}. For sake of contradiction, suppose that the sum of the elements of B is equal to $L_{2k+3} + 1$ and that B does not contain the term L_{2k+2}. From Lemma 3.1 the non-consecutive sums that we can form from A_{2k+2} are the values from 0 to $L_{2k+3} + 1$ inclusive, excluding L_{2k+2}. This implies B is a non-consecutive subset of A_{2k+1}. From Lemma 3.1 we have that the sum of the elements of B must be less than or equal to $L_{2k+2} - 1$. Hence we have a contradiction, which implies B contains the term L_{2k+2}. From our inductive hypothesis, we know that $L_{2k+1} + 1$ is equal to only one non-consecutive sum of A_{2k}. Because $L_{2k+3} + 1 = L_{2k+2} + (L_{2k+1} + 1)$ and B cannot contain both L_{2k+2} and L_{2k+1}, this implies $L_{2k+3} + 1$ is equal to only one non-consecutive sum of A_{2k+2}. This completes the inductive step. □

References

[1] A. Best, P. Dynes, X. Edelsbrunner, B. McDonald, S. J. Miller, C. Turnage-Butterbaugh, and M. Weinstein, Benford behavior of Zeckendorf's decompositions, *Fibonacci Quart.* **52** (2014), 35–46.

[2] A. Best, P. Dynes, X. Edelsbrunner, B. McDonald, S. J. Miller, C. Turnage-Butterbaugh, and M. Weinstein, Gaussian behavior of the number of summands in Zeckendorf's decompositions in small intervals, *Fibonacci Quart.* **52** (2014), 47–53.

[3] E. Boldyriew, J. Havilan, P. Lam, J. Lentfer, S. J. Miller, and F. T. Suarez, An introduction to completeness of positive linear recurrence sequences, preprint, 2020, https://arxiv.org/pdf/2010.04071.

[4] E. Boldyriew, J. Havilan, P. Lam, J. Lentfer, S. J. Miller, and F. T. Suarez, Completeness of positive linear recurrence sequences, *Fibonacci Quart.* **58** (2020), 77–90.

[5] J. Brown, Jr., Note on complete sequences of integers, *Am. Math. Mon.* **68** (1961), 557–560.

[6] J. Brown, Jr., Zeckendorf's theorem and some applications, *Fibonacci Quart.* **2** (1964), 163–168.

[7] J. Brown, Jr., Unique representation of integers as sums of distinct Lucas numbers, *Fibonacci Quart.* **7** (1969), 243–252.

[8] L. Carlitz, R. Scoville, and V. Hoggatt, Jr., Fibonacci representations, *Fibonacci Quart.* **10** (1972), 1–28.

[9] S. Chang, Average number of Zeckendorf integers, *J. Number Theory* **186** (2018), 452–472.

[10] H. V. Chu, Generalized Zeckendorf decompositions and generalized golden strings, to appear in *Fibonacci Quart.*, https://arxiv.org/abs/2006.02966.

[11] H. V. Chu, D. C. Luo, and S. J. Miller, On Zeckendorf related partitions in the Lucas sequence, preprint, 2020, https://arxiv.org/abs/2004.08316.

[12] D. E. Daykin, Representation of natural numbers as sums of generalized Fibonacci numbers, *J. Lond. Math. Soc.* **35** (1960), 143–160.

[13] P. Demontigny, T. Do, A. Kulkarni, S. J. Miller, D. Moon, and U. Varma, Generalizing Zeckendorf's theorem to f-decompositions, *J. Number Theory* **141** (2014), 136–158.

[14] R. Dorward, P. Ford, E. Fourakis, P. Harris, S. J. Miller, E. Palsson, and H. Paugh, Individual gap measures from generalized Zeckendorf decompositions, *Unif. Distrib. Theory* **12** (2017), 27–36.

[15] P. Filipponi, P. J. Grabner, I. Nemes, A. Pethö, and R. F. Tichy, Corrigendum to: "Generalized Zeckendorf expansions", *Appl. Math. Lett.* **7** (1994), 25–26.

[16] A. S. Fraenkel, Systems of numeration, *Amer. Math. Monthly* **92** (1985), 105–114.

[17] P. J. Grabner, R. F. Tichy, I. Nemes, and A. Pethö, Generalized Zeckendorf expansions, *Appl. Math. Lett.* **7** (1994), 25–28.
[18] M. Griffiths, Fixed-term Zeckendorf representations, *Fibonacci Quart.* **52** (2014), 331–335.
[19] M. Griffiths, The golden string, Zeckendorf representations, and the sum of a series, *Amer. Math. Monthly* **118** (2011), 497–507.
[20] E. Hart and L. Sanchis, On the occurrence of F_n in the Zeckendorf decomposition of nF_n, *Fibonacci Quart.* **37** (1999), 21–33.
[21] C. Herink, personal communication (E-mail to David C. Luo), (2020).
[22] V. Hoggatt, Jr., Generalized Zeckendorf theorem, *Fibonacci Quart.* **10** (1972), 89–93.
[23] T. Keller, Generalizations of Zeckendorf's theorem, *Fibonacci Quart.* **10** (1972), 95–102.
[24] M. Kologlu, G. Kopp, S. J. Miller, and Y. Wang, On the number of summands in Zeckendorf decompositions, *Fibonacci Quart.* **49** (2011), 116–130.
[25] C. Lekkerkerker, Voorstelling van natuurlyke getallen door een som van getallen van Fibonacci, *Bull. Belg. Math. Soc. Simon Stevin* **29** (1951–1952), 190–195.
[26] R. Li and S. J. Miller, Central limit theorems for gaps of generalized Zeckendorf's decompositions, *Fibonacci Quart.* **57** (2019), 213–230.
[27] T. Martinez, C. Mizgerd, S. J. Miller, and C. Sun, Generalizing Zeckendorf's Theorem to Homogeneous Linear Recurrences, I, to appear in *Fibonacci Quart.*.
[28] T. Martinez, C. Mizgerd, S. J. Miller, J. Murphy, and C. Sun, Generalizing Zeckendorf's Theorem to Homogeneous Linear Recurrences, II, to appear in *Fibonacci Quart.*.
[29] S. Miller and Y. Wang, From Fibonacci numbers to central limit type theorems, *J. Combin. Theory Ser. A* **119** (2012), 1398–1413.
[30] S. Miller and Y. Wang, Gaussian behavior in generalized Zeckendorf decompositions, in M. B. Nathanson, ed., *Combinatorial and Additive Number Theory, CANT 2011 and 2012*, Springer Proceedings in Mathematics & Statistics, 2014, pp. 159–173.
[31] E. Zeckendorf, Representation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, *Bull. Soc. Roy. Sci. Liege* **41** (1972), 179–182.

MSC2010: 11B39

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61820
Email address: hungchu2@illinois.edu

Department of Mathematics, Emory University, 400 Dowman Dr., Atlanta, GA 30322
Email address: dluo6745@gmail.com

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267
Email address: sjm1@williams.edu