LOW-LYING ZEROES OF MAASS FORM L-FUNCTIONS

LEVENT ALPOGE AND STEVEN J. MILLER

ABSTRACT. The Katz-Sarnak density conjecture states that the scaling limits of the distributions of zeroes of families of automorphic L-functions agree with the scaling limits of eigenvalue distributions of classical subgroups of the unitary groups $U(N)$. This conjecture is often tested by way of computing particular statistics, such as the one-level density, which evaluates a test function with compactly supported Fourier transform at normalized zeroes near the central point. Iwaniec, Luo, and Sarnak [ILS] studied the one-level densities of cuspidal newforms of weight k and level N. They showed in the limit as $kN \to \infty$ that these families have one-level densities agreeing with orthogonal type for test functions with Fourier transform supported in $(-2, 2)$. Exceeding $(-1, 1)$ is important as the three orthogonal groups are indistinguishable for support up to $(-1, 1)$ but are distinguishable for any larger support. We study the other family of GL_2 automorphic forms over $\mathbb{Q}$: Maass forms. To facilitate the analysis, we use smooth weight functions in the Kuznetsov formula which, among other restrictions, vanish to order M at the origin. For test functions with Fourier transform supported inside $\left(-2 + \frac{3}{2(M+1)}, 2 - \frac{3}{2(M+1)}\right)$, we unconditionally prove the one-level density of the low-lying zeros of level 1 Maass forms, as the eigenvalues tend to infinity, agrees only with that of the scaling limit of orthogonal matrices.

CONTENTS

1. INTRODUCTION

The zeros of L-functions, especially those near the central point, encode important arithmetic information. Understanding their distribution has numerous applications, ranging

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from bounds on the size of the class numbers of imaginary quadratic fields [CI, Go, GZ] to the size of the Mordell-Weil groups of elliptic curves [BSD1, BSD2]. We concentrate on the one-level density, which allows us to deduce many results about these low-lying zeros.

Definition 1.1. *Let $L(s,f)$ be an L-function with zeros in the critical strip $\rho_f = 1/2+i\gamma_f$ (note $\gamma_f \in \mathbb{R}$ if and only if the Grand Riemann Hypothesis holds for f), and let ϕ be an even Schwartz function whose Fourier transform has compact support. The **one-level density** is*

$$D_1(f;\phi,R) \ := \ \sum_{\rho_f} \phi\left(\frac{\log R}{2\pi}\gamma_f\right), \tag{1.1}$$

*where R is a scaling parameter. Given a family $\mathcal{F}$ of L-functions and a weight function w of rapid decay, we define the **averaged one-level density** of the family by*

$$\mathcal{D}_1(\mathcal{F};\phi) \ := \ \lim_{R\to\infty} \frac{1}{W(\mathcal{F},R)} \sum_{f\in\mathcal{F}} w(C_f/R) D_1(f;\phi,R), \tag{1.2}$$

with

$$W(\mathcal{F},R) \ := \ \sum_{f\in\mathcal{F}} w(C_f/R), \tag{1.3}$$

and C_f some normalization constant associated to the form f (typically it is related to the analytic conductor c_f, e.g. $C_f = c_f$ or $c_f^{1/2}$, etc.).

The Katz-Sarnak density conjecture [KaSa1, KaSa2] states that the scaling limits of eigenvalues of classical compact groups near 1 correctly model the behavior of these zeros in families of L-functions as the conductors tend to infinity. Specifically, if the symmetry group is $\mathcal{G}$, then for an appropriate choice of the normalization R we expect

$$\mathcal{D}_1(\mathcal{F};\phi) \ = \ \int_{-\infty}^{\infty} \phi(x) W_{1,\mathcal{G}}(x)dx \ = \ \int_{-\infty}^{\infty} \widehat{\phi}(t)\widehat{W_{1,\mathcal{G}}}(t)dt, \tag{1.4}$$

where $K(y) = \frac{\sin \pi y}{\pi y}$, $K_\epsilon(x,y) = K(x-y) + \epsilon K(x+y)$ for $\epsilon = 0, \pm 1$, and

$$\begin{aligned} W_{1,\mathrm{SO(even)}}(x) &= K_1(x,x) \\ W_{1,\mathrm{SO(odd)}}(x) &= K_{-1}(x,x) + \delta_0(x) \\ W_{1,\mathrm{O}}(x) &= \frac{1}{2}W_{1,\mathrm{SO(even)}}(x) + \frac{1}{2}W_{1,\mathrm{SO(odd)}}(x) \\ W_{1,\mathrm{U}}(x) &= K_0(x,x) \\ W_{1,\mathrm{Sp}}(x) &= K_{-1}(x,x). \end{aligned} \tag{1.5}$$

Note the Fourier transforms of the densities of the three orthogonal groups all equal $\delta_0(y)+1/2$ in the interval $(-1,1)$ but are mutually distinguishable for larger support (and are distinguishable from the unitary and symplectic cases for any support). Thus if the underlying symmetry type is believed to be orthogonal then it is necessary to obtain results for test functions ϕ with $\mathrm{supp}(\widehat{\phi})$ exceeding $(-1,1)$ in order to have a unique agreement.

The one-level density has been computed for many families for suitably restricted test functions, and has always agreed with a random matrix ensemble. Simple families of L-functions include Dirichlet L-functions, elliptic curves, cuspidal newforms, number field L-functions, and symmetric powers of GL_2 automorphic representations [DM1, FiMi, FI, Gao, GK, Gü, HM, HR, ILS, IMT, KaSa1, KaSa2, Mil, MilPe, OS1, OS2, RR, Ro, Rub1, Rub2, ShTe, Ya, Yo]. Dueñez and Miller [DM1, DM2] handled some compound families, and recently Shin and Templier [ShTe] determined the symmetry type of many families of automorphic forms on GL_n over $\mathbb{Q}$. The goal of this paper is to provide additional

evidence for these conjectures for the family of level 1 Maass forms for as large support of the test function as possible.

1.1. **Background and Notation.** By $A \ll B$ we mean that $|A| \leq c|B|$ for some positive constant c, and by $A \asymp B$ we mean that $A \ll B$ and $B \ll A$. We set

$$e(x) \ := \ \exp(2\pi i x) \tag{1.6}$$

and use the following convention for the Fourier transform:

$$\widehat{f}(\xi) \ := \ \int_{-\infty}^{\infty} f(x) e(-x\xi) dx. \tag{1.7}$$

We quickly review some properties of Maass forms; see [Iw1, IK, KL, Liu, LiuYe1, LiuYe2] for a detailed exposition and a derivation of the Kuznetsov trace formula, which will be a key ingredient in our analysis below.

Let u be a cuspidal (Hecke-Maass-Fricke) eigenform on $\mathrm{SL}_2(\mathbb{Z})$ with Laplace eigenvalue $\lambda_u =: \frac{1}{4} + t_u^2, t_u \in \mathbb{C}$. By work of Selberg we may take $t_u \geq 0$. We may write the Fourier expansion of u as

$$u(z) \ = \ y^{1/2} \sum_{n \neq 0} a_n(u) K_{s-1/2}(2\pi |n| y) e(ny). \tag{1.8}$$

Let

$$\lambda_n(u) \ := \ \frac{a_n(u)}{\cosh(t)^{1/2}}. \tag{1.9}$$

Changing u by a non-zero constant if necessary, by the relevant Hecke theory on this space without loss of generality we may take $\lambda_1 = 1$. This normalization is convenient in applying the Kuznetsov trace formula to convert sums over the Fourier coefficients of u to weighted sums over prime powers.

The L-function associated to u is

$$L(s,u) \ := \ \sum_{n \geq 1} \lambda_n n^{-s}. \tag{1.10}$$

By the work of Kim and Sarnak [K, KSa] the L-function is absolutely convergent in the right half-plane $\mathfrak{Re}(s) > 71/64$ (it is believed to converge for $\mathfrak{Re}(s) > 1$). These L-functions analytically continue to entire functions of the complex plane, satisfying the functional equation

$$\Lambda(s,u) \ = \ (-1)^{\epsilon} \Lambda(1-s,u), \tag{1.11}$$

with

$$\Lambda(s,u) \ := \ \pi^{-s} \Gamma\left(\frac{s+\epsilon+it}{2}\right) \Gamma\left(\frac{s+\epsilon-it}{2}\right) L(s,u). \tag{1.12}$$

Factoring

$$1 - \lambda_p X + X^2 \ =: \ (1-\alpha_p X)(1-\beta_p X) \tag{1.13}$$

at each prime (the α_p, β_p are the Satake parameters at p), we get an Euler product

$$L(s,u) \ = \ \prod_p (1-\alpha_p p^{-s})^{-1} (1-\beta_p p^{-s})^{-1}, \tag{1.14}$$

which again converges for $\mathfrak{Re}(s)$ sufficiently large.

We let $\mathcal{M}_1$ denote an orthonormal basis of Maass eigenforms, which we fix for the remainder of the paper. In what follows $\mathrm{Avg}(A;B)$ will denote the average value of A over our orthonormal basis of level 1 Maass forms weighted by B. That is to say,

$$\mathrm{Avg}(A;B) \ := \ \frac{\sum_{u\in\mathcal{M}_1} A(u)B(u)}{\sum_{u\in\mathcal{M}_1} B(u)}. \tag{1.15}$$

1.2. **Main result.** Before stating our main result we first describe the weight function used in the one-level density for the family of level 1 Maass forms. The weight function we consider is not as general as other ones investigated (see the arguments for other families of Maass forms in [AAILMZ]), but leads to a significantly simpler analysis and much greater support. In this sense our work is similar to analyses in other problems where the weight function is chosen to facilitate the application of a summation formula (for example, the use of harmonic weights for the Petersson formula). As previous work on Maass forms could not deal with test functions whose Fourier transforms are supported outside $(-1,1,)$, these calculations were insufficient to determine the underlying symmetry. As extending this support is the primary motivation for this work, we thus chose a weight function which is ideally suited for using the Kuznetsov trace formula.

As we will see below, some type of weighting is necessary in order to restrict to conductors of comparable size. While our choice does not include say the characteristic function of $[T,2T]$, we are able to localize for the most part to conductors near T, with polynomial decay before T and exponential decay beyond. By choosing such weight functions, we are able to unconditionally obtain support in $(-2,2)$. Note this equals the best unconditional results for any family of L-functions, that of Dirichlet L-functions (support this large is known for cuspidal newforms, but only by assuming GRH for Dirichlet L-functions to expand the Kloosterman sums).

Let $h\in C^\infty(\mathbb{R})$ be an even smooth function with an even smooth square-root of Paley-Wiener class such that $\hat{h}\in C^\infty((-1/4,1/4))$ and h has a zero of order at least 8 at 0. In fact, the higher the order of the zero of h at 0, the better the support we are able to obtain: this will be made precise below.

By the ideas that go into the proof of the Paley-Wiener theorem, since $\widehat{h}$ is compactly supported we have that h extends to an entire holomorphic function, with the estimate

$$h(x+iy) \ \ll \ \exp\left(\frac{\pi|y|}{2}\right). \tag{1.16}$$

Note also that, by exhibiting h as the square of a real-valued even smooth function on the real line (that also extends to an entire holomorphic function by Paley-Wiener), by the Schwarz reflection principle we have that h takes non-negative real values along the imaginary axis as well.

Throughout this paper T will be a large positive odd integer tending to infinity.

Let

$$h_T(r) \ := \ \frac{\left(\frac{r}{T}\right)h\left(\frac{ir}{T}\right)}{\sinh\left(\frac{\pi r}{T}\right)}. \tag{1.17}$$

For $r\in\mathbb{R}$ we have

$$h_T(r) \ \ll \ \exp\left(-\frac{\pi|r|}{4T}\right). \tag{1.18}$$

Further, h_T extends to an entire meromorphic function, with poles exactly at the non-zero integral multiples of iT. Figure 1 shows a plot of $h_{101}(r)$ on $[0, 1000]$ for one choice of h. The point is that $h_T(r)$ is order 1 for r on the order of T, decays exponentially at infinity, and decays polynomially at zero (like e.g. the Maxwell-Boltzmann distribution, and many other well-known distributions).

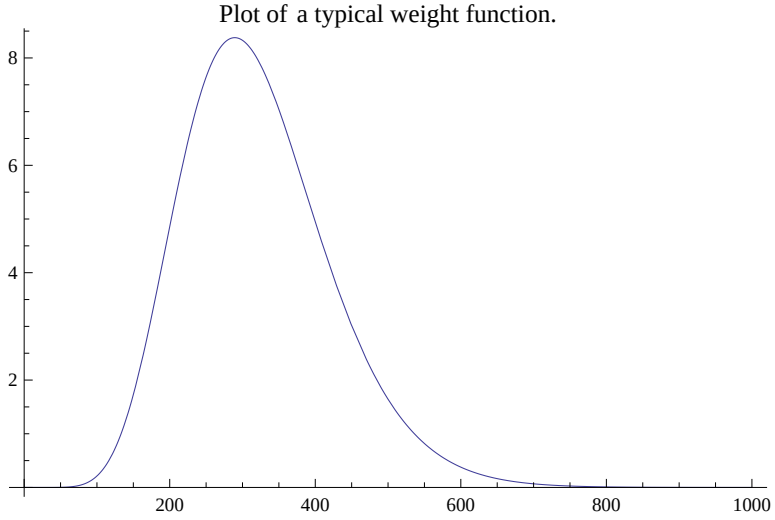

FIGURE 1. A plot of h_{101}. Here $b(t) = \exp(-1/(1/100 - t^2))$ if $|t| \leq 1/10$ and 0 otherwise, $h(\xi) = \xi^8 \widehat{b}(\xi)$ (where $\widehat{b}$ is the Fourier transform of b), and $h_T(r) = (r/T)h(ir/T)/\sinh(\pi r/T)$.

In our one-level calculations we take our test function ϕ to be an even Schwartz function such that $\mathrm{supp}(\widehat{\phi}) \subset (-\eta, \eta)$ for some $\eta > 0$. The goal of course is to prove results for the largest η possible. We suppress any dependence of constants on h or η or ϕ as these are fixed, but not on T as that tends to infinity.

In computing the one-level density for the family $\mathcal{M}_1$, we have some freedom in the choice of weight function. We choose to weight u by $h_T(t_u)/||u||^2$, where $t_u^2 + 1/4$ is the Laplace eigenvalue of u, and $||u|| = ||u||_{L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h})}$ is the L^2 norm of u. We may write the averaged one-level density as (we will see that $R \asymp T^2$ is forced)

$$\begin{aligned} \mathcal{D}_1(\mathcal{M}_1; \phi) &= \lim_{\substack{T\to\infty \\ T \text{ odd}}} \frac{1}{\sum_{u\in\mathcal{M}_1} h_T(t_u)/||u||^2} \sum_{u\in\mathcal{M}_1} D_1(u; \phi, T^2) \frac{h_T(t_u)}{||u||^2} \\ &= \lim_{\substack{T\to\infty \\ T \text{ odd}}} \mathrm{Avg}\left(D_1(u; \phi, T^2); \frac{h_T(t_u)}{||u||^2} \right). \end{aligned} \tag{1.19}$$

Based on results from [AAILMZ] and [ShTe], which determined the one-level density for support contained in $(-1, 1)$, we believe the following conjecture.

Conjecture 1.2. *Let h_T be as defined in* (1.17) *and ϕ an even Schwartz function with $\widehat{\phi}$ of compact support. Then*

$$\mathcal{D}_1(\mathcal{M}_1; \phi) = \int_{-\infty}^{\infty} \phi(t) W_{1,\mathrm{O}}(t) dt, \tag{1.20}$$

with $W_{1,\mathrm{O}}(t) = 1 + \frac{1}{2}\delta_0$. In other words, the symmetry group associated to the family of level 1 cuspidal Maass forms is orthogonal.

Unfortunately, the previous one-level calculations are insufficient to distinguish which of the three orthogonal candidates is the correct corresponding symmetry type, as they all agree in the regime calculated. There are two solutions to this issue. The first is to compute the two-level density, which is able to distinguish the three candidates for arbitrarily small support (see [Mil]). The second is to compute the one-level density in a range exceeding $(-1, 1)$, which we do here. Our main result is the following.

Theorem 1.3. *Let $T > 1$ be an odd integer and ϕ an even Schwartz function with $\mathrm{supp}(\widehat{\phi}) \subset (-\eta, \eta)$. Let $h \in C^\infty(\mathbb{R})$ be an even smooth function with an even smooth square-root of Paley-Wiener class such that $\widehat{h} \in C^\infty((-1/4, 1/4))$ and h has a zero of order at least 8 at 0. Let h_T be as defined in* (1.17)*. Then, for all $\eta < 5/4$, we have that*

$$\mathcal{D}_1(\mathcal{M}_1; \phi) \ = \ \int_{-\infty}^{\infty} \phi(t) W_{1,\mathrm{O}}(t) dt, \tag{1.21}$$

the density corresponding to the orthogonal group, O. That is to say, the symmetry group associated to the family of level 1 cuspidal Maass forms is orthogonal.

As mentioned, padding the weight function with more zeroes at 0 allows us to increase the support with the same methods. By further restricting our weight functions, we may take the support of $\widehat{\phi}$ to be $(-2 + \epsilon, 2 - \epsilon)$ for any $\epsilon > 0$; note that we do not assume GRH. This equals the best support obtainable either unconditionally or under just GRH for any family of L-functions (such as Dirichlet L-functions [FiMi, Gao, HR, OS1, OS2] and cuspidal newforms not split by sign [ILS]), and thus provides strong evidence for the Katz-Sarnak density conjecture for this family. Specifically, we have

Theorem 1.4. *Let $T > 1$ be an odd integer and ϕ an even Schwartz function with $\mathrm{supp}(\widehat{\phi}) \subset (-\eta, \eta)$. Let $h \in C^\infty(\mathbb{R})$ be an even smooth function with an even smooth square-root of Paley-Wiener class such that $\widehat{h} \in C^\infty((-1/4, 1/4))$ and h has a zero of order at least $M \geq 8$ at 0. Let h_T be as defined in* (1.17)*. Then, for all $\eta < 2 - \frac{3}{2(M+1)}$, we have that*

$$\mathcal{D}_1(\mathcal{M}_1; \phi) \ = \ \int_{-\infty}^{\infty} \phi(t) W_{1,\mathrm{O}}(t) dt, \tag{1.22}$$

the density corresponding to the orthogonal group, O. That is to say, the symmetry group associated to the family of level 1 cuspidal Maass forms is orthogonal.

The only difference in the proof is that we are allowed to integrate by parts M times in Section 3.3 (see the proof of Proposition 3.2).

1.3. **Outline of proof.** We give a quick outline of the argument. We carefully follow the seminal work of Iwaniec-Luo-Sarnak [ILS] in our preliminaries. Namely, we first write down the explicit formula to convert the relevant sums over zeroes to sums over Hecke eigenvalues. We then average and apply the Kuznetsov trace formula to leave ourselves with calculating various integrals, which we then sum. To be slightly more specific, we reduce the difficulty to bounding an integral of shape

$$\int_{-\infty}^{\infty} J_{2ir}(X) \frac{r h_T(r)}{\cosh(\pi r)} dr, \tag{1.23}$$

where these J are Bessel functions, and h_T is as in Theorem 1.3. We break into cases: X "small" and X "large". For X small, we move the line of integration from $\mathbb{R}$ down to

$\mathbb{R} - iR$ and take $R \to +\infty$, converting the integral to a sum over residues. The difficulty then lies in bounding a sum of residues of shape

$$T \sum_{k \geq 0} (-1)^k J_{2k+1}(X) \frac{P\left(\frac{2k+1}{2T}\right)}{\sin\left(\frac{2k+1}{2T}\pi\right)}, \tag{1.24}$$

where P is closely related to h. To do this (after a few tricks), we apply an integral formula for these Bessel functions, switch summation and integration, apply Poisson summation, apply Fourier inversion, and then *apply Poisson summation again*. The result is a sum of Fourier coefficients, to which we apply the stationary phase method one by one. This yields the bound for X small.

To handle X large, we use a precise asymptotic for the $J_{2ir}(X)$ term from Dunster [Du] (as found in [ST]). In fact, for X large it is enough to simply use the oscillation of $J_{2ir}(X)$ to get cancelation. It is worth noting that the same considerations would also be enough for the case of X small were the asymptotic expansion convergent.

2. Calculating the averaged one-level density

The starting point is to use the explicit formula to convert weighted averages of the Fourier coefficients to weighted sums over prime powers. The calculation is standard and easily modified from [RS] (see also Lemma 2.8 of [AAILMZ]).

Lemma 2.1 (Explicit formula)**.** *Let h_T be as in Theorem 1.3. Then*

$$\begin{aligned} \mathrm{Avg}\left(D_1(u;\phi,R); \frac{h_T(t_u)}{||u||^2}\right) &= \frac{\phi(0)}{2} + \hat{\phi}(0)\frac{\mathrm{Avg}\left(\log(1+t_u^2); \frac{h_T(t_u)}{||u||^2}\right)}{\log R} \\ &\quad - \sum_p \frac{2\log p}{p^{1/2}\log T}\hat{\phi}\left(\frac{\log p}{2\log T}\right)\mathrm{Avg}\left(\lambda_p(u); \frac{h_T(t_u)}{||u||^2}\right) \\ &\quad - \sum_p \frac{2\log p}{p\log T}\hat{\phi}\left(\frac{\log p}{\log T}\right)\mathrm{Avg}\left(\lambda_{p^2}(u); \frac{h_T(t_u)}{||u||^2}\right) \\ &\quad + O\left(\frac{\log\log T}{\log T}\right). \end{aligned} \tag{2.1}$$

To prove Theorem 1.3, it therefore suffices to show the following.

Lemma 2.2. *Let h_T be as in Theorem 1.3. Then as $T \to \infty$ through the odd integers we have*

$$\begin{aligned} &(1) \quad \mathrm{Avg}\left(\log(1+t_u^2); \frac{h_T(t_u)}{||u||^2}\right) = \log(T^2) + O\left(\log\log T\right) \\ &(2) \quad \sum_p \frac{\log p}{p^{1/2}\log T}\hat{\phi}\left(\frac{\log p}{2\log T}\right)\mathrm{Avg}\left(\lambda_p(u); \frac{h_T(t_u)}{||u||^2}\right) \to 0 \\ &(3) \quad \sum_p \frac{\log p}{p\log T}\hat{\phi}\left(\frac{\log p}{\log T}\right)\mathrm{Avg}\left(\lambda_{p^2}(u); \frac{h_T(t_u)}{||u||^2}\right) \to 0. \end{aligned} \tag{2.2}$$

The first determines the correct scale to normalize the zeros, $R \asymp T^2$ (see [Mil] for comments on normalizing each form's zeros by a local factor and not a global factor such as T^2 here; briefly if only the one-level density is being studied then either is fine). The third is far easier than the second. Each will be handled via the Kuznetsov trace formula (see for example [IK, KL, LiuYe2]), which we now state.

Theorem 2.3 (Kuznetsov trace formula)**.** *Let $m, n \in \mathbb{Z}^+$. Let H be an even holomorphic function on the strip $\{x+iy \,|\, |y| < \frac{1}{2}+\epsilon\}$ (for some $\epsilon > 0$) such that $H(z) \ll \frac{1}{1+y^2}$. Then*

$$\sum_{u \in \mathcal{M}_1} \frac{H(t_u)}{||u||^2} \lambda_m(u)\overline{\lambda_n(u)} = \frac{\delta_{m,n}}{\pi^2}\int_{-\infty}^{\infty} rH(r)\tanh(\pi r)dr - \frac{1}{\pi}\int_{-\infty}^{\infty} m^{ir}\sigma_{ir}(m)n^{-ir}\sigma_{-ir}(n)\frac{H(r)}{|\zeta(1+2ir)|^2}dr + \frac{2i}{\pi}\sum_{c \geq 1}\frac{S(m,n;c)}{c}\int_{-\infty}^{\infty} J_{2ir}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{rH(r)}{\cosh(\pi r)}dr, \tag{2.3}$$

the sum taken over an orthonormal basis of Hecke-Maass-Fricke eigenforms on $\mathrm{SL}_2(\mathbb{Z})$, with S the usual Kloosterman sum, σ the extended divisor function and $\delta_{m,n}$ Kronecker's delta.

Observe that our weight function h_T satisfies the hypotheses of the above theorem once $T > 1$, since the sine function has a simple zero at 0.

Our first application of the Kuznetsov trace formula is to determine the total mass (i.e., the normalizing factor in our averaging).

Lemma 2.4. *Let h_T be as in Theorem 1.3. Then*

$$\sum_{u \in \mathcal{M}_1} \frac{h_T(t_u)}{||u||^2} \asymp T^2. \tag{2.4}$$

Proof. We apply Theorem 2.3 to h_T, with $m = n = 1$. We obtain

$$\sum_{u \in \mathcal{M}_1} \frac{h_T(t_u)}{||u||^2} = \frac{1}{\pi^2}\int_{-\infty}^{\infty} rh_T(r)\tanh(\pi r)dr - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{h_T(r)}{|\zeta(1+2ir)|^2}dr + \frac{2i}{\pi}\sum_{c \geq 1}\frac{S(1,1;c)}{c}\int_{-\infty}^{\infty} J_{2ir}\left(\frac{4\pi}{c}\right)\frac{rh_T(r)}{\cosh(\pi r)}dr. \tag{2.5}$$

It is rather easy to see that the first term is $\asymp T^2$, since h_T is non-negative and essentially supported on $r \asymp T$. Similarly, using $|\zeta(1+2ir)| \gg 1/\log(2+|r|)$ (see for example [Liu]), the second term is readily seen to be

$$\ll \frac{T\log T}{(mn)^{1/2}}. \tag{2.6}$$

Applying the Weil bound, it certainly suffices to show that

$$\int_{-\infty}^{\infty} J_{2ir}\left(\frac{4\pi}{c}\right)\frac{rh_T(r)}{\cosh(\pi r)}dr \ll c^{-1}. \tag{2.7}$$

But this follows from Proposition 3.3 and the bound

$$J_n(x) \ll \frac{\left(\frac{x}{2}\right)^n}{n!}, \tag{2.8}$$

completing the proof. □

We can now prove the first part of the main lemma needed to prove Theorem 1.3.

Proof of Lemma 2.2, part (1). We cut the sum above at $T\log T$ and below at $\frac{T}{\log T}$ and apply the previous lemma along with the fact that $||u|| \asymp 1$ under our normalizations (see [Smi]). □

We are thus left with the last two parts of Lemma 2.2.

3. HANDLING THE BESSEL INTEGRALS

In this section we analyze the Bessel terms. Crucial in our analysis is the fact that our weight function h_T is holomorphic with nice properties; this allows us to shift contours and convert our integral to a sum over residues. The goal of the next few subsections is to prove the following two propositions, which handle X small and large.

Proposition 3.1. *Let h_T be as in* (1.17)*. Suppose $X \leq T$. Then*

$$\int_{-\infty}^{\infty} J_{2ir}(X)\frac{rh_T(r)}{\cosh(\pi r)}dr \ \ll \ \frac{X}{T}. \tag{3.1}$$

Proposition 3.2. *Let h_T be as in* (1.17)*. Suppose $X \geq \frac{T}{8}$. Then*

$$\int_{-\infty}^{\infty} J_{2ir}(X)\frac{rh_T(r)}{\cosh(\pi r)}dr \ \ll \ \frac{X}{T^{1/2}}. \tag{3.2}$$

3.1. **Calculating the Bessel integral.** We begin our analysis of the Bessel terms, which will eventually culminate in a proof of Proposition 3.1.

Proposition 3.3. *Let h_T be as in* (1.17)*. Then*

$$\begin{aligned}\int_{-\infty}^{\infty} J_{2ir}(X)\frac{rh_T(r)}{\cosh(\pi r)}dr &= c_1\sum_{k\geq 0}(-1)^k J_{2k+1}(X)(2k+1)h_T\left(\left(k+\frac{1}{2}\right)i\right)\\ &\quad + c_2 T\sum_{k\geq 1}(-1)^k J_{2kT}(X)k^2 h(k)\\ &= c_1\sum_{k\geq 0}(-1)^k J_{2k+1}(X)(2k+1)h_T\left(\left(k+\frac{1}{2}\right)i\right)\\ &\quad + O\left(Xe^{-c_3T}\right),\end{aligned} \tag{3.3}$$

where c_1, c_2, and c_3 are some constants independent of X and T.

Proof of Proposition 3.3. The idea here is to move the contour from $\mathbb{R}$ down to $\mathbb{R} - i\infty$, picking up poles at all the half-integers multiplied by i (poles arising from the $\cosh(\pi r)$ in the denominator) and integer multiples of iT (poles arising from the $\sinh\left(\frac{\pi r}{T}\right)$ hidden in h_T) that are passed. Indeed, the first sum is precisely the sum of the former residues, while the second is the sum of the latter. The final point is that $J_\alpha(z)$ decays extremely rapidly as $\mathfrak{Re}\,\alpha \to \infty$, with z fixed. One way to see this decay is to use the expansion

$$J_\alpha(2z) \ = \ \sum_{n\geq 0}\frac{(-1)^n z^{2n+\alpha}}{\Gamma(n+1)\Gamma(n+\alpha+1)}, \tag{3.4}$$

switch the sum and integral, and use Stirling's formula to do the relevant calculations, switching sums and integrals back at the end to consolidate the form into the above. The details will not be given here, as the bounds already given on h_T, as well as Stirling's bounds on Γ (and the outline above), reduce this to a routine computation.

The claimed bound on the error term follows by trivially bounding by using (for $0 \leq x \leq 1$, n a positive integer)

$$|J_n(nx)| \leq \left(\frac{xe^{\sqrt{1-x^2}}}{1+\sqrt{1-x^2}}\right)^n, \tag{3.5}$$

which can be found in [AS]. □

3.2. **Averaging Bessel functions of integer order for small primes.** Iwaniec-Luo-Sarnak, in proving the Katz-Sarnak density conjecture for $\widehat{\phi}$ supported in $(-2,2)$ for holomorphic cusp forms of weight at most K, demonstrate a crucial lemma pertaining to averages of Bessel functions. In some sense our analogous work here moving this to the Kuznetsov setting requires only one more conceptual leap, which is to apply Poisson summation a second time to a resulting weighted exponential sum. The original argument can be found in Iwaniec's book ([Iw2]), which we basically reproduce as a first step in handling the remaining sum from above.

Remark 3.4. *We will use the fact that $J_{-n}(x) = (-1)^n J_n(x)$ several times in what follows. Moreover, we introduce the notation*

$$\tilde{h}(x) \ := \ xh(x), \tag{3.6}$$

and similarly for iterated tildes.

Thus (in this notation) to prove Proposition 3.1 it suffices to show the following.

Proposition 3.5. *Let h_T be as in* (1.17)*. Suppose $X \leq T$. Then*

$$S_J(X) \ := \ T\sum_{k\geq 0}(-1)^k J_{2k+1}(X)\frac{\tilde{\tilde{h}}\left(\frac{2k+1}{2T}\right)}{\sin\left(\frac{2k+1}{2T}\pi\right)} \ \ll \ \frac{X}{T}. \tag{3.7}$$

Proof. Observe that $k \mapsto \sin\left(\frac{\pi k}{2}\right)$ is supported only on the odd integers, and maps $2k+1$ to $(-1)^k$. Hence, rewriting gives

$$S_J(X) \ = \ T\sum_{\substack{k\geq 0\\ k\notin 2T\mathbb{Z}}} J_k(X)\tilde{\tilde{h}}\left(\frac{k}{2T}\right)\frac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{2T}\right)}. \tag{3.8}$$

As

$$\frac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{2T}\right)} \ = \ \frac{e^{\frac{\pi i k}{2}} - e^{-\frac{\pi i k}{2}}}{e^{\frac{\pi i k}{2T}} - e^{\frac{\pi i k}{2T}}} \ = \ \sum_{\alpha=-\left(\frac{T-1}{2}\right)}^{\frac{T-1}{2}} e^{\frac{\pi i k\alpha}{T}} \tag{3.9}$$

when k is not a multiple of $2T$, we find that

$$S_J(X) \ = \ T\sum_{|\alpha|<\frac{T}{2}}\sum_{\substack{k\geq 0\\ k\notin 2T\mathbb{Z}}} e\left(\frac{k\alpha}{2T}\right)J_k(X)\tilde{\tilde{h}}\left(\frac{k}{2T}\right). \tag{3.10}$$

Observe that, since the sum over α is invariant under $\alpha \mapsto -\alpha$ (and it is non-zero only for k odd!), we may extend the sum over k to the entirety of $\mathbb{Z}$ at the cost of a factor of 2 and of replacing h by

$$g(x) \ := \ \mathrm{sgn}(x)h(x). \tag{3.11}$$

Note that g is as differentiable as h has zeroes at 0, less one. That is to say, $\widehat{g}$ decays like the reciprocal of a degree $\mathrm{ord}_{z=0}\, h(z) - 1$ polynomial at ∞. This will be crucial in what follows.

Next, we add back on the $2T\mathbb{Z}$ terms and obtain

$$\begin{aligned}\frac{1}{2}S_J(X) &= T\sum_{|\alpha|<\frac{T}{2}}\sum_{k\in\mathbb{Z}} e\left(\frac{k\alpha}{2T}\right) J_k(X)\tilde{\tilde{g}}\left(\frac{k}{2T}\right) - T^2\sum_{k\in\mathbb{Z}} J_{2kT}(X)k^2h(k)\\ &= T\sum_{|\alpha|<\frac{T}{2}}\sum_{k\in\mathbb{Z}} e\left(\frac{k\alpha}{2T}\right) J_k(X)\tilde{\tilde{g}}\left(\frac{k}{2T}\right) + O\left(Xe^{-c_4T}\right)\\ &=: V_J(X) + O\left(Xe^{-c_4T}\right), \end{aligned} \tag{3.12}$$

by the same argument as the last step of Proposition 3.3 (since the sign was immaterial).

Now we move to apply Poisson summation. Write $X =: 2\pi Y$. We apply the integral formula (for $k \in \mathbb{Z}$)

$$J_k(2\pi x) \;=\; \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(kt - x\sin(2\pi t)\right) dt \tag{3.13}$$

and interchange sum and integral (via rapid decay of g) to get that

$$V_J(X) \;=\; T\sum_{|\alpha|<\frac{T}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}} e\left(\frac{k\alpha}{2T} + kt\right)\tilde{\tilde{g}}\left(\frac{k}{2T}\right)\right) e\left(-Y\sin(2\pi t)\right) dt. \tag{3.14}$$

By Poisson summation, (3.14) is just (interchanging sum and integral once more)

$$\begin{aligned} V_J(X) &= T^2\sum_{|\alpha|<\frac{T}{2}}\sum_{k\in\mathbb{Z}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{g}''\left(2T(t-k)+\alpha\right) e\left(-Y\sin(2\pi t)\right) dt\\ &= c_5T\sum_{|\alpha|<\frac{T}{2}}\int_{-\infty}^{\infty} \hat{g}''(t) e\left(Y\sin\left(\frac{\pi t}{T}+\frac{\pi\alpha}{T}\right)\right) dt\\ &=: c_5W_g(X). \end{aligned} \tag{3.15}$$

As

$$\sin\left(\frac{\pi t}{T}+\frac{\pi\alpha}{T}\right) \;=\; \sin\left(\frac{\pi\alpha}{T}\right) + \frac{\pi t}{T}\cos\left(\frac{\pi\alpha}{T}\right) - \frac{\pi^2t^2}{T^2}\sin\left(\frac{\pi\alpha}{T}\right) + O\left(\frac{t^3}{T^3}\right), \tag{3.16}$$

we see that

$$\begin{aligned} W_g(X) = c_6T\sum_{|\alpha|<\frac{T}{2}} & e\left(Y\sin\left(\frac{\pi\alpha}{T}\right)\right)\int_{-\infty}^{\infty}\hat{g}''(t)e\left(\frac{\pi Yt}{T}\cos\left(\frac{\pi\alpha}{T}\right)\right)dt\\ & - c_7\frac{\pi^2Y}{T}\sum_{|\alpha|<\frac{T}{2}} e\left(Y\sin\left(\frac{\pi\alpha}{T}\right)\right)\sin\left(\frac{\pi\alpha}{T}\right)\int_{-\infty}^{\infty} t^2\hat{g}''(t)e\left(\frac{\pi Yt}{T}\cos\left(\frac{\pi\alpha}{T}\right)\right)dt\\ & + O\left(\frac{Y}{T}+\frac{Y^2}{T^2}\right) \end{aligned} \tag{3.17}$$

$$\begin{aligned} = c_8T\sum_{|\alpha|<\frac{T}{2}} & e\left(Y\sin\left(\frac{\pi\alpha}{T}\right)\right)\tilde{\tilde{g}}\left(\frac{\pi Y}{T}\cos\left(\frac{\pi\alpha}{T}\right)\right)\\ & - c_9\frac{Y}{T}\sum_{|\alpha|<\frac{T}{2}} e\left(Y\sin\left(\frac{\pi\alpha}{T}\right)\right)\sin\left(\frac{\pi\alpha}{T}\right)\tilde{\tilde{g}}''\left(\frac{\pi Y}{T}\cos\left(\frac{\pi\alpha}{T}\right)\right)\\ & + O\left(\frac{Y}{T}+\frac{Y^2}{T^2}\right). \end{aligned} \tag{3.18}$$

As the rest of the argument is a bit long, we isolate it in Lemma 3.6 immediately below. Its proof uses Poisson summation again. By (3.18), this finishes the proof of Proposition 3.7 (and hence that of Proposition 3.1 as well). □

Lemma 3.6. *Let g be as in* (3.11)*, and $Y \leq \frac{T}{2\pi}$. Then*

$$(1)\ A_g(Y) := T\sum_{|\alpha|<\frac{T}{2}} e\left(Y\sin\left(\frac{\pi\alpha}{T}\right)\right)\tilde{\tilde{g}}\left(\frac{\pi Y}{T}\cos\left(\frac{\pi\alpha}{T}\right)\right) \ll \frac{Y^4}{T^7}$$

$$(2)\ B_g(Y) := \frac{Y}{T}\sum_{|\alpha|<\frac{T}{2}} e\left(Y\sin\left(\frac{\pi\alpha}{T}\right)\right)\sin\left(\frac{\pi\alpha}{T}\right)\tilde{\tilde{g}}''\left(\frac{\pi Y}{T}\cos\left(\frac{\pi\alpha}{T}\right)\right) \ll \frac{Y^5}{T^9}. \tag{3.19}$$

Proof of Lemma 3.6. Let $p \in C^\infty\left(\left[-\frac{T}{2},\frac{T}{2}\right]\right)$ such that $p|_{\left[-\frac{T-1}{2},\frac{T-1}{2}\right]} = 1$. We view p as a Schwartz function on $\mathbb{R}$. Then

$$A_g(Y) = T\sum_{\alpha\in\mathbb{Z}} p(\alpha)e\left(Y\sin\left(\frac{\pi\alpha}{T}\right)\right)\tilde{\tilde{g}}\left(\frac{\pi Y}{T}\cos\left(\frac{\pi\alpha}{T}\right)\right). \tag{3.20}$$

Applying Poisson summation,

$$\begin{aligned} A_g(Y) &= T\sum_{n\in\mathbb{Z}}\int_{-\frac{T}{2}}^{\frac{T}{2}} p(t)\tilde{\tilde{g}}\left(\frac{\pi Y}{T}\cos\left(\frac{\pi t}{T}\right)\right)e\left(Y\sin\left(\frac{\pi t}{T}\right)-nt\right)dt \\ &=: T\sum_{n\in\mathbb{Z}} C_g(Y,n). \end{aligned} \tag{3.21}$$

For each n, the derivative of the phase in $C_g(Y,n)$ is

$$\frac{\pi Y}{T}\cos\left(\frac{\pi t}{T}\right) - n. \tag{3.22}$$

Here is where our hypothesis on Y (née X) comes in: for $Y \leq \frac{T}{2\pi}$ and $n \neq 0$, we have

$$\left|\frac{\pi Y}{T}\cos\left(\frac{\pi t}{T}\right) - n\right| \gg n. \tag{3.23}$$

Now we integrate by parts four times. There is nothing special about four other than the fact that g has far more than four zeroes at 0 and $\sum n^{-4}$ converges. Integrating by parts more times would give us no improvement in the end. First consider the $n = 0$ term of (3.21) — i.e., $C_g(Y,0)$ — where the phase *is* stationary (albeit at a boundary point of the integration region).

$$\begin{aligned} C_g(Y,0) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} p(t)\tilde{\tilde{g}}\left(\frac{\pi Y}{T}\cos\left(\frac{\pi t}{T}\right)\right)e\left(Y\sin\left(\frac{\pi t}{T}\right)\right)dt \\ &= -\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(p(t)\tilde{g}\left(\frac{\pi Y}{T}\cos\left(\frac{\pi t}{T}\right)\right)\right)' e\left(Y\sin\left(\frac{\pi t}{T}\right)\right)dt. \end{aligned} \tag{3.24}$$

Note that the g has lost one tilde because we have divided out by the derivative of the phase, and also that the boundary terms vanish thanks to the support condition on p.

We remark before we repeat this three more times that $p' = 0$ on $\left[-\frac{T-1}{2},\frac{T-1}{2}\right]$, and on $\pm\left[\frac{T-1}{2},\frac{T}{2}\right]$ we have that

$$\tilde{\tilde{g}}\left(\frac{\pi Y}{T}\cos\left(\frac{\pi t}{T}\right)\right) \ll \left(\frac{Y}{T^2}\right)^8, \tag{3.25}$$

for instance (since, again, g has a high order zero at 0). Further, differentiating the $\tilde{\tilde{g}}$ term picks up a factor of Y/T^2, and differentiating the denominator we absorbed earlier would again pick up a factor of Y/T^2. The point is that repeating this process three more times gives us a bound of the form

$$C_g(Y,0) \ll \left(\frac{Y}{T^2}\right)^4. \tag{3.26}$$

The exact same argument works for $n \neq 0$, except now we pick up at least one factor of n each time we integrate by parts (since the derivative of the phase is $\frac{\pi Y}{T}\cos\left(\frac{\pi t}{T}\right) - n$). The same process and reasoning leads us to a bound of shape:

$$A_g(Y) \ll T\left(\frac{Y}{T^2}\right)^4\left(1+\sum_{n\neq 0}\frac{1}{n^4}\right) \ll \frac{Y^4}{T^7}, \tag{3.27}$$

as desired. □

3.3. **Handling the remaining large primes.** The goal of this subsection is to prove Proposition 3.2. For this we apply the following asymptotic expansion, due to Dunster [Du] and (essentially) found in Sarnak-Tsimerman [ST].

Lemma 3.7. *Let $x, r > 0$. Then*

$$J_{2ir}(x) = \frac{c_{10}e^{2ir\xi\left(\frac{x}{2r}\right)}}{(4r^2+x^2)^{\frac{1}{4}}}e^{\pi r} + O\left(\frac{e^{\pi r}}{|r|\,(4r^2+x^2)^{\frac{1}{4}}} + \frac{e^{-\pi r}}{(4r^2+x^2)^{\frac{1}{4}}}\right), \tag{3.28}$$

where $\xi(z) := (1+z^2)^{\frac{1}{2}} + \log\left(\frac{z}{1+\sqrt{1+z^2}}\right)$. □

Proof of Proposition 3.2. Write

$$D_J(X) := \int_{-\infty}^{\infty} J_{2ir}(X)\frac{rh_T(r)}{\cosh(\pi r)}dr \tag{3.29}$$

for our integral.

Observe that

$$\left(r\xi\left(\frac{X}{2r}\right)\right)' = -\log\left(\frac{2r}{X}+\sqrt{1+\frac{4r^2}{X^2}}\right). \tag{3.30}$$

Note that this logarithm is zero at 0, but its derivative is not (whence the quotient of r by this remains bounded near 0). These will both come in handy in a moment. Applying the asymptotic expansion of (3.28) (and using evenness), we see that

$$\begin{aligned} D_J(X) &\ll \int_{\mathbb{R}^+}\frac{e^{2ir\xi\left(\frac{X}{2r}\right)}r\tilde{h}\left(\frac{ir}{T}\right)}{(4r^2+X^2)^{\frac{1}{4}}\sinh\left(\frac{\pi r}{T}\right)}dr + O\left(\int_{\mathbb{R}^+}\frac{\left|\tilde{h}\left(\frac{ir}{T}\right)\right|}{(4r^2+X^2)^{\frac{1}{4}}\sinh\left(\frac{\pi r}{T}\right)}dr\right) \\ &=: N_J(X) + E_J(X). \end{aligned} \tag{3.31}$$

Using our hypothesis on X (and the exponential decay of h_T at ∞),

$$E_J(X) \ll T^{\frac{1}{2}}. \tag{3.32}$$

Thus it suffices to study the first term of (3.31) — i.e., $N_J(X)$. But, via $r \mapsto Tr$ and then an integration by parts, we see that

$$\begin{aligned} N_J(X) &\ll T^{\frac{3}{2}} \int_{\mathbb{R}^+} e\left(\frac{rT}{\pi}\xi\left(\frac{X}{2rT}\right)\right)\left(\frac{r\tilde{h}(ir)/\sinh(\pi r)}{\left(4r^2+\frac{X^2}{T^2}\right)^{\frac{1}{4}}\left(rT\xi\left(\frac{X}{2rT}\right)\right)'}\right)' dr \\ &\ll \frac{X}{T^{1/2}}\int_{\mathbb{R}^+}(1+r)e^{-c_6 r}dr \\ &\ll \frac{X}{T^{1/2}}, \end{aligned} \tag{3.33}$$

as desired. □

The following remark describes the changes needed in our argument to prove Theorem 1.4.

Remark 3.8. *Each integration by parts picks up a factor of X/T^2. Hence, so long as h has a zero of order at least M at 0, integrating by parts M times gives instead*

$$N_J(X) \ll X^M T^{\frac{3}{2}-2M}. \tag{3.34}$$

In fact pushing the asymptotic expansion further (again, see [Du]*) produces terms differing by a factor $O(1/r)$ smaller each time, and so pushing the expansion out to two more terms we get the bound*

$$D_J(X) \ll X^M T^{\frac{3}{2}-2M} + T^{-3/2}. \tag{3.35}$$

4. Proof of Theorem 1.3

We can now prove our main result.

Proof of Theorem 1.3. We prove part (2) of Lemma 2.2. Part (3) follows entirely analogously (in fact, we obtain better bounds in this case).

We have already seen that the total mass of the averages is on the order of T^2. So it suffices to give a bound of size $o(T^2)$ for

$$\sum_p \frac{\log p}{p^{1/2}\log T}\hat{\phi}\left(\frac{\log p}{2\log T}\right)\sum_{u\in\mathcal{M}_1}\frac{h_T(t_u)}{||u||^2}\lambda_p(u). \tag{4.1}$$

Applying the Kuznetsov trace formula and using the same arguments used for (2.6) gives us that

$$\sum_{u\in\mathcal{M}_1}\frac{h_T(t_u)}{||u||^2}\lambda_p(u) = c_{11}\sum_{c\geq 1}\frac{S(1,p;c)}{c}\int_{-\infty}^{\infty} J_{2ir}\left(\frac{4\pi\sqrt{p}}{c}\right)\frac{rh_T(r)}{\cosh(\pi r)}dr + O\left(\frac{T\log T}{p^{1/2}}\right). \tag{4.2}$$

Since ϕ has compact support, the sum of the error term over the primes is

$$\sum_p \frac{\log p}{p^{1/2}\log T}\hat{\phi}\left(\frac{\log p}{2\log T}\right)O\left(\frac{T\log T}{p^{1/2}}\right) \ll T\log T. \tag{4.3}$$

We split the remaining double sum into three parts as follows.

$$\begin{aligned}
&\sum_p \frac{\log p}{p^{1/2}\log T}\hat{\phi}\left(\frac{\log p}{2\log T}\right)\sum_{c\geq 1}\frac{S(1,p;c)}{c}\int_{-\infty}^{\infty}J_{2ir}\left(\frac{4\pi\sqrt{p}}{c}\right)\frac{rh_T(r)}{\cosh(\pi r)}dr\\
&=\sum_{\frac{T^2}{4\pi^2}\leq p\leq T^{2\eta}}\frac{\log p}{p^{1/2}\log T}\hat{\phi}\left(\frac{\log p}{2\log T}\right)\sum_{c\leq\frac{4\pi\sqrt{p}}{T}}\frac{S(1,p;c)}{c}\int_{-\infty}^{\infty}J_{2ir}\left(\frac{4\pi\sqrt{p}}{c}\right)\frac{rh_T(r)}{\cosh(\pi r)}dr\\
&\quad+\sum_{\frac{T^2}{4\pi^2}\leq p\leq T^{2\eta}}\frac{\log p}{p^{1/2}\log T}\hat{\phi}\left(\frac{\log p}{2\log T}\right)\sum_{c>\frac{4\pi\sqrt{p}}{T}}\frac{S(1,p;c)}{c}\int_{-\infty}^{\infty}J_{2ir}\left(\frac{4\pi\sqrt{p}}{c}\right)\frac{rh_T(r)}{\cosh(\pi r)}dr\\
&\quad+\sum_{p<\frac{T^2}{4\pi^2}}\frac{\log p}{p^{1/2}\log T}\hat{\phi}\left(\frac{\log p}{2\log T}\right)\sum_{c\geq 1}\frac{S(1,p;c)}{c}\int_{-\infty}^{\infty}J_{2ir}\left(\frac{4\pi\sqrt{p}}{c}\right)\frac{rh_T(r)}{\cosh(\pi r)}dr.
\end{aligned}\tag{4.4}$$

We apply the Weil bound for Kloosterman sums to each: $|S(1,p;c)|\ll c^{1/2+\epsilon}$. Moreover, we apply Proposition 3.2 to the integrals in the first sum of (4.4), and Proposition 3.1 to those in the second and third sums of (4.4). We get that (4.4) is bounded by

$$\begin{aligned}
&T^{-\frac{1}{2}}\sum_{\frac{T^2}{4\pi^2}\leq p\leq T^{2\eta}}\frac{\log p}{\log T}\hat{\phi}\left(\frac{\log p}{2\log T}\right)\sum_{c\leq\frac{4\pi\sqrt{p}}{T}}c^{-\frac{3}{2}+\epsilon}\\
+\;&T^{-1}\sum_{\frac{T^2}{4\pi^2}\leq p\leq T^{2\eta}}\frac{\log p}{\log T}\hat{\phi}\left(\frac{\log p}{2\log T}\right)\sum_{c>\frac{4\pi\sqrt{p}}{T}}c^{-\frac{3}{2}+\epsilon}\\
+\;&T^{-1}\sum_{p<\frac{T^2}{4\pi^2}}\frac{\log p}{\log T}\hat{\phi}\left(\frac{\log p}{2\log T}\right)\sum_{c\geq 1}c^{-\frac{3}{2}+\epsilon}.
\end{aligned}\tag{4.5}$$

Applying Chebyshev's prime number theorem estimates, (4.5) is

$$\ll\frac{T^{2\eta-\frac{1}{2}+\epsilon}}{\log T}+\frac{T^{1+\epsilon}}{\log T},\tag{4.6}$$

which is of the desired shape when $\eta<\frac{5}{4}$, completing the argument. □

Proof of Theorem 1.4. The proof of Theorem 1.4 follows similarly. We argue as above and apply (3.35) instead to the first term. □

REFERENCES

[AAILMZ] L. Alpoge, N. Amersi, G. Iyer, O. Lazarev, S. J. Miller and L. Zhang, *The low-lying zeros of cuspidal Maass forms on* $\mathrm{SL}_2(\mathbb{Z})$ (2013), preprint.

[AS] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing, New York: Dover, 1972.

[BSD1] B. Birch and H. Swinnerton-Dyer, *Notes on elliptic curves. I*, J. reine angew. Math. **212**, 1963, $7-25$.

[BSD2] B. Birch and H. Swinnerton-Dyer, *Notes on elliptic curves. II*, J. reine angew. Math. **218**, 1965, $79-108$.

[CI] J. B. Conrey and H. Iwaniec, *Spacing of Zeros of Hecke L-Functions and the Class Number Problem*, Acta Arith. **103** (2002) no. 3, 259–312.

[DM1] E. Dueñez and S. J. Miller, *The low lying zeros of a GL*(4) *and a GL*(6) *family of L-functions*, Compositio Mathematica **142** (2006), no. 6, 1403–1425.

[DM2] E. Dueñez and S. J. Miller, *The effect of convolving families of L-functions on the underlying group symmetries*, Proceedings of the London Mathematical Society, 2009; doi: 10.1112/plms/pdp018.

[Du] T.M. Dunster, *Bessel Functions of Purely Imaginary Order, with an Application to Second-Order Linear Differential Equations Having a Large Parameter*, SIAM Journal of Mathematical Analysis, **21** (1990), no. 4.

[FiMi] D. Fiorilli and S. J. Miller, *Surpassing the Ratios Conjecture in the 1-level density of Dirichlet L-functions* (2013), preprint. `http://arxiv.org/abs/1111.3896`.

[FI] E. Fouvry and H. Iwaniec, *Low-lying zeros of dihedral L-functions*, Duke Math. J. **116** (2003), no. 2, 189-217.

[Gao] P. Gao, *N-level density of the low-lying zeros of quadratic Dirichlet L-functions*, Ph. D thesis, University of Michigan, 2005.

[Go] D. Goldfeld, *The class number of quadratic fields and the conjectures of Birch and Swinnerton-Dyer*, Ann. Scuola Norm. Sup. Pisa (4) **3** (1976), 623–663.

[GK] D. Goldfeld and A. Kontorovich, *On the $\mathrm{GL}(3)$ Kuznetsov formula with applications to symmetry types of families of L-functions*, preprint.

[GZ] B. Gross and D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math **84** (1986), 225–320.

[Gü] A. Güloğlu, *Low-Lying Zeros of Symmetric Power L-Functions*, Internat. Math. Res. Notices 2005, no. 9, 517-550.

[HM] C. Hughes and S. J. Miller, *Low-lying zeros of L-functions with orthogonal symmtry*, Duke Math. J., **136** (2007), no. 1, 115–172.

[HR] C. Hughes and Z. Rudnick, *Linear Statistics of Low-Lying Zeros of L-functions*, Quart. J. Math. Oxford **54** (2003), 309–333.

[Iw1] H. Iwaniec, *Introduction to the Spectral Theory of Automorphic Forms*, Biblioteca de la Revista Matemática Iberoamericana, 1995.

[Iw2] H. Iwaniec, *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics, Vol. 17, AMS, Providence, RI, 1997.

[IK] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, AMS Colloquium Publications **53**, AMS, Providence, RI, 2004.

[ILS] H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes ï£¡tudes Sci. Publ. Math. **91**, 2000, 55–131.

[IMT] G. Iyer, S. J. Miller and N. Triantafillou, *Moment Formulas for Ensembles of Classical Compact Groups* (2013), preprint.

[KaSa1] N. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, AMS Colloquium Publications **45**, AMS, Providence, 1999.

[KaSa2] N. Katz and P. Sarnak, *Zeros of zeta functions and symmetries*, Bull. AMS **36**, 1999, $1 - 26$.

[K] H. Kim, *Functoriality for the exterior square of GL_2 and the symmetric fourth of GL_2*, Jour. AMS **16** (2003), no. 1, 139–183.

[KL] C. Li and A. Knightly, *Kuznetsov's trace formula and the Hecke eigenvalues of Maass forms*, Mem. Amer. Math. Soc., to appear.

[KSa] H. Kim and P. Sarnak, *Appendix: Refined estimates towards the Ramanujan and Selberg conjectures*, Appendix to [K].

[Liu] J. Liu, *Lectures On Maass Forms*, Postech, March 25–27, 2007. `http://www.prime.sdu.edu.cn/lectures/LiuMaassforms.pdf`.

[LiuYe1] J. Liu and Y. Ye, *Subconvexity for Rankin-Selberg L-Functions for Maass Forms*, Geom. funct. anal. **12** (2002), 1296–1323.

[LiuYe2] J. Liu and Y. Ye, *Petersson and Kuznetsov trace formulas*, in Lie groups and automorphic forms (Lizhen Ji, Jian-Shu Li, H. W. Xu and Shing-Tung Yau editors), AMS/IP Stud. Adv. Math. **37**, AMS, Providence, RI, 2006, pages 147–168.

[Mil] S. J. Miller, 1- *and* 2*-level densities for families of elliptic curves: evidence for the underlying group symmetries*, Compositio Mathematica **140** (2004), 952–992.

[MilPe] S. J. Miller and R. Peckner, *Low-lying zeros of number field L-functions*, Journal of Number Theory **132** (2012), 2866–2891.

[OS1] A. E. Özlük and C. Snyder, *Small zeros of quadratic L-functions*, Bull. Austral. Math. Soc. **47** (1993), no. 2, 307–319.

[OS2] A. E. Özlük and C. Snyder, *On the distribution of the nontrivial zeros of quadratic L-functions close to the real axis*, Acta Arith. **91** (1999), no. 3, 209–228.

[RR] G. Ricotta and E. Royer, *Statistics for low-lying zeros of symmetric power L-functions in the level aspect*, preprint, to appear in Forum Mathematicum.

[Ro] E. Royer, *Petits zéros de fonctions L de formes modulaires*, Acta Arith. **99** (2001), no. 2, 147-172.

[Rub1] M. Rubinstein, *Evidence for a spectral interpretation of the zeros of L-functions*, P.H.D. Thesis, Princeton University, 1998.

[Rub2] M. Rubinstein, *Low-lying zeros of L–functions and random matrix theory*, Duke Math. J. **109** (2001), 147–181.

[RS] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. **81**, 1996, $269 - 322$.

[ST] P. Sarnak and J. Tsimerman, *On Linnik and Selberg's Conjecture about Sums of Kloosterman Sums*, preprint.

[ShTe] S.-W. Shin and N. Templier, *Sato-Tate theorem for families and low-lying zeros of automorphic L-functions*, preprint.

[Smi] R.A. Smith, *The L^2 norm of Maass wave functions*, Proc. A.M.S. **82** (1981), no. 2, 179–182.

[Ya] A. Yang, *Low-lying zeros of Dedekind zeta functions attached to cubic number fields*, preprint.

[Yo] M. Young, *Low-lying zeros of families of elliptic curves*, J. Amer. Math. Soc. **19** (2006), no. 1, 205–250.

E-mail address: alpoge@college.harvard.edu

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138

E-mail address: sjm1@williams.edu, Steven.Miller.MC.96@aya.yale.edu

DEPARTMENT OF MATHEMATICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267