GENERALIZATIONS OF THE M&M GAME

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Abstract. The M&M Game involves two players who begin with I_1 and I_2 M&M's. During each round, each player tosses a fair coin: if the coin lands heads, that player eats one M&M, and if it lands tails, the player does not eat. If, at the end of a round, one player still has M&M's while the other has none, then the player with M&M's remaining is declared the winner. If both players eat their last M&M in the same round, the game is said to end in a tie. In [BHM+17], the authors studied the probability of a tie in the M&M Game and derived a simple closed-form expression in the special case where both players start with the same number of M&M's. We generalize the M&M Game in several directions, including allowing players to toss multiple coins per round and modifying the probability distributions of the coin flips. We use the technique of generating functions, Monte Carlo methods, and non-linear curve fitting to study the generalized M&M Game.

Contents

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1. Introduction

1.1. **Background of the M&M Game.** The M&M Game arose from attempts to model life expectancy, inspired by a question asked by Cameron and Kayla Miller (aged 4 and 2): *If two people are born on the same day, do they die on the same day*? To explore this question, their father, Steven Miller, who happened to have bags of M&M's with him, reformulated it as a two-player game: *if two players start with the same number of M&M's, what is the probability that they eat their last M&M's at the same time*? Later, they and Badinski, Huffaker, McCue, and Stone [BHM$^+$17] explored several different approaches to analyze the game, which turned out to be a great springboard to see many powerful mathematical techniques: hypergeometric functions, memoryless processes, and walks on graphs to name a few. It is these connections which inspired the first paper, and this sequel which introduces generating functions to analyze generalizations.

From [BHM$^+$17], the rules of the M&M Game are as follows. Player A and Player B begin with I_1 and I_2 M&M's, respectively. We assume that both players start with k M&M's, so that $I_1 = I_2 = k$ for some positive integer k. During each round, each player flips a fair coin independently and simultaneously.[1] If a player's coin lands heads, that player eats one M&M; if it lands tails, that player does not eat an M&M. The game continues until at least one player has no M&M's remaining. If, at the end of a round, one player still has M&M's while the other has none, then the player with M&M's remaining is declared the winner. If both players run out of M&M's in the same round, the game ends in a tie. In [BHM$^+$17], the authors study the probability of a tie when both players start with k M&M's, and we briefly summarize their work below.

Let $\mathbb{P}(k,k)$ denote the probability of a tie when both players start with k M&M's. In [BHM$^+$17], the authors first obtained

$$\mathbb{P}(k,k) \;=\; \sum_{n=k}^{\infty} \binom{n-1}{k-1}^2 \left(\frac{1}{2}\right)^{2n}. \tag{1.1}$$

Their argument is as follows. First, they decompose the tie probability according to the round on which the game ends:

$$\mathbb{P}(k,k) \;=\; \sum_{n=k}^{\infty} \mathbb{P}_n(k,k), \tag{1.2}$$

[1] We retain the assumption that both players flip fair coins simultaneously only to remain faithful to the original description of the M&M Game in [BHM$^+$17]. In fact, the order of the flips is irrelevant; all that matters is that the players toss their coins independently.

where $\mathbb{P}_n(k,k)$ denotes the probability that the game ends in a tie after exactly n rounds, given that both players start with k M&M's. The sum starts at $n = k$ because *at least* k rounds are required for the game to end in a tie. It remains to compute $\mathbb{P}_n(k,k)$. For the game to end in a tie on the n^{th} round, each player must eat exactly $k-1$ M&M's during the first $n-1$ rounds, and then eat their last M&M on round n. As each player tosses only one coin in each round, this means that each player must obtain $k-1$ heads in the first $n-1$ tosses, followed by a head on the n^{th} toss. Thus, there are $\binom{n-1}{k-1}\binom{1}{1}$ ways to accomplish this. Since the coin is fair and the two players flip independently, $\mathbb{P}_n(k,k)$ is the product of the probability that each player having k^{th} head at the n^{th} toss, and this is given by

$$\mathbb{P}_n(k,k) \;=\; \binom{n-1}{k-1}\left(\frac{1}{2}\right)^n \binom{n-1}{k-1}\left(\frac{1}{2}\right)^n. \tag{1.3}$$

Substituting (1.3) into (1.2) gives (1.1), as desired.

Although (1.1) gives the probability of a tie for the game, the formula is somewhat unsatisfactory: it is expressed as an infinite sum, which is difficult to evaluate, and it is hard to see how the probability of a tie depends on the starting number of M&M's k. By adopting a different perspective, they reduced the probability of a tie to a finite, tractable sum. The key insight in [BHM+17] is that the M&M Game is a *memoryless process*: the probability of a tie only depends on how many M&M's each player currently possesses, not on how many rounds it takes for them to get there. By exploiting this memoryless property, they obtained the following result.

Theorem 1.1. [BHM+17] *The probability that the M&M Game ends in a tie when two players each start with k M&M's is*

$$\mathbb{P}(k,k) \;=\; \sum_{n=0}^{k-1}\binom{2k-n-2}{n}\binom{2k-2n-2}{k-n-1}\left(\frac{1}{3}\right)^{2k-n-1}. \tag{1.4}$$

To gain further insight into the game, it is helpful to view the M&M Game as a random walk on the two-dimensional integer lattice. Formally, we represent the state of the game after each round by lattice position (m,n), where m is the number of M&M's held by Player A and n is the number held by Player B. We will use this notation throughout the rest of the paper. Under this interpretation, each possible outcome of a round in the M&M Game corresponds to one of the following moves on the lattice.

- *Both Players Eat*: $(-1,-1)$, sending (m,n) to $(m-1,n-1)$.
- *Only Player A Eats*: $(-1,0)$, sending (m,n) to $(m-1,n)$.
- *Only Player B Eats*: $(0,-1)$, sending (m,n) to $(m,n-1)$.
- *No Players Eat*: $(0,0)$, leaving (m,n) unchanged.

Since each player independently eats one M&M with probability $1/2$ and does not eat with probability $1/2$, each of these four lattice moves occurs with probability $1/4$.

Thus, the tie probability in the M&M Game can be reformulated as follows: starting from (I_1, I_2), what is the probability of reaching the origin $(0,0)$ using the four moves above?

Because the number of rounds is irrelevant when studying the tie probability, we may ignore the move $(0,0)$, since it leaves the lattice position unchanged. After removing this move, the remaining three moves are equally likely with probability $1/3$. If we let $F(m,n)$ denote the probability that the walk eventually reaches $(0,0)$ when the current position is (m,n), then $F(m,n)$ satisfies the recurrence relation[2]

$$F(m,n) \;=\; \frac{1}{3}F(m-1,n)+\frac{1}{3}F(m,n-1)+\frac{1}{3}F(m-1,n-1). \tag{1.5}$$

In this recurrence, we define $F(0,0)=1$, since the probability of reaching $(0,0)$ when we are already at $(0,0)$ is 1. We also define $F(m,0)=F(0,n)=0$ for all positive integers m and n, since once one player has no M&M's remaining while the other still has some left, the game can no longer end in a tie.

1.2. **Generalizing the M&M Game.** The goal of this paper is to generalize the M&M Game. The rest of this paper is organized as follows. In Section 2, we investigate the M&M Game with modified game rules. In the original paper [BHM+17], each player flips a single fair coin at each turn. To generalize the game, we consider the following two methods for each player.

- **Method 1.** The player tosses 1 fair coin in each round. If it is a head, then eat d of the M&M's, if it is a tail eat none.
- **Method 2.** The player tosses 2 fair coins in each round. If the first coin is a head, then eat d_1 of the M&M's, if the second coin is a head eat d_2 of the M&M's, and eat no M&M's for tails.

In both methods, if the player is ever supposed to eat more than she has left, then the player eats all the remaining M&M's. We focus on the M&M Game for two players with arbitrary starting values I_1 and I_2, respectively. Using these two methods, we create three possible games for two players.

- **Game 1.** Both players play Method 1; that is, each eats d of their M&M's if a head is flipped, else eat nothing.
- **Game 2.** Player A plays Method 1 and Player B plays Method 2. Thus, Player A eats d M&M's if a head is flipped, while Player B eats d_1 M&M's if the first coin is a head and eats d_2 M&M's if the second coin is a head.
- **Game 3.** Both players play Method 2; that is, one eats d_1 M&M's if the first coin is a head, and one eats d_2 M&M's if the second coin is a head.

Following the rules of the M&M Game in [BHM+17], we say that the game ends in a tie if both players exhaust their M&M's in the same round. We formalize this using the lattice representation. Let (m,n) denote the numbers of M&M's currently possessed by Player A and Player B, respectively. Suppose that in a given round Player A is supposed to eat a

[2]Without using the memoryless nature of the problem, our recurrence relation would be

$$F(m,n) \;=\frac{1}{4}F(m-1,n)+\frac{1}{4}F(m,n-1)+\frac{1}{4}F(m-1,n-1)+\frac{1}{4}F(m,n).$$

Performing simple algebra would result in (1.5). Thus, we can arrive at the desired recurrence relation even without using the memoryless nature of the M&M Game.

M&M's and Player B is supposed to eat b M&M's. If $m \leq a$ and $n \leq b$, then we say the game ends in a tie. If, on the other hand, one player exhausts all of her M&M's while the other still has some M&M's remaining, then we declare the player with M&M's remaining to be the winner.

We can now ask, and fortunately answer in Section 2, several natural questions. What is the tie probability of each game? How does the tie probability depend on d_1 and d_2? The game become far more interesting and complicated now. To illustrate, we consider Game 2, in which Player A eats $d = 5$ M&M's whenever a head is flipped, while Player B eats $d_1 = 2$ M&M's if the first coin lands heads and eats $d_2 = 3$ if the second coin lands heads. Suppose both players start with $(4, 4)$. Then even though the average number of M&M's removed on each turn is the same – namely, 2.5 – for both methods, we see that the first player goes out in one turn with probability 1/2, while the second does that only with probability 1/4. Conversely, if we started with $(2, 2)$, then while the first player still goes out in one turn with probability 1/2, now the second player does that with probability 3/4. We now see that the two methods for playing the game may influence the outcome of the game in a nontrivial way, a complication that we haven't seen in [BHM$^+$17].

Section 2 is organized as follows. In Section 2.1, we introduce the technique of generating functions and provide an overview of our method for analyzing the tie probabilities in the three proposed games. In Section 2.2, we present formulas for the tie probability of each game. We then give the derivation for Game 1 in Section 2.3; since the arguments for Games 2 and 3 are similar, we refer the reader to Appendix B for the detailed proofs. Finally, we present numerical results illustrating how the number of coins a player may toss affects the tie probabilities.

In Section 3, we study how the tie probability changes when the coin-flip distribution is altered. We begin by giving a straightforward generalization of Theorem 1.1 to the case of a biased coin that lands heads with probability p, and we examine how the tie probability depends on p. This extension assumes that coin flips are independent across rounds, with a fixed probability of landing heads. It is also natural to ask how the game changes when this probability depends on the history of previous rounds. For this reason, we introduce a state-dependent version of the M&M Game in which the coin's probability of landing heads depends on the number of M&M's remaining. We then study this game through extensive Monte Carlo simulations and find that the tie probabilities are well approximated by a Gompertz curve.

Finally, in Section 4, we propose several directions for future work, including extensions to games with more complicated rules, the study of the expected number of turns, and more complex interactions between coin outcomes and M&M counts.

2. Comparison of Three Games

2.1. **Problem Approach.**

2.1.1. *Recurrence Formulation for the M&M Game.* We seek to study the tie probabilities for the three games proposed in Section 1.2. Since both players begin with a finite number of M&M's, and since in each turn each player may consume only a nonnegative number of M&M's, the game must terminate after finitely many turns. This allows us to reduce the problem to a finite one. As in [BHM$^+$17], we formulate the tie probability in terms of a recurrence relation, though the analysis here is technically more involved because the number of M&M's removed in a turn may vary.

We first introduce some notation. Let $F(m,n)$ denote the probability of a tie when Player A and Player B currently have m and n M&M's, respectively. In addition, let d_i denote that a player eats d_i M&M's when her i^{th} coin lands heads. When a player flips only one coin, we simply use d to denote this. As described earlier, each player follows either Method 1 or Method 2. Let $\mathcal{A}$ denote the set of possible moves available to Player A under her chosen method. If Player A uses Method 1, then on a given round Player A either eats d M&M's or eats none, depending on the outcome of Player A's coin flip; thus $\mathcal{A} = \{0, d\}$. If Player A plays Method 2, then Player A could eat d_1 M&M's, d_2 M&M's, $d_1 + d_2$ M&M's, or no M&M's, depending on the outcomes of Player A's two coin flips; thus, $\mathcal{A} = \{0, d_1, d_2, d_1 + d_2\}$. We define the set $\mathcal{B}$ analogously for Player B.

Let $\mathcal{S}$ denote the set of moves that at least one of the two players consume some M&M's in each round. Formally, $\mathcal{S} = (\mathcal{A} \times \mathcal{B}) \setminus \{(0,0)\}$, where the move $(0,0)$ is omitted because, by the memoryless property discussed in Section 1.1, it leaves the state unchanged and therefore does not affect the tie probability. From the perspective of random walks on the integer lattice, $\mathcal{S}$ may be interpreted as the set of admissible steps from one lattice point to another.

Finally, let $P(a,b)$ denote the probability that Player A eats a M&M's and Player B eats b M&M's. In the three games considered in this paper, we may take $P(a,b) = \frac{1}{|\mathcal{S}|}$, though this notation is also useful for possible future extensions involving non-uniform probabilities for each move.

Following [BHM$^+$17], we can write the tie probability of each game as a recurrence relation of the form

$$F(m,n) \;=\; \begin{cases} \sum_{(a,b)\in\mathcal{S}} P(a,b)\, F(m-a, n-b) & \text{if } m \geq 1 \text{ and } n \geq 1 \\ 0 & \text{if exactly one of } m,n \text{ is } \leq 0 \\ 1 & \text{if } m \leq 0 \text{ and } n \leq 0, \end{cases} \tag{2.1}$$

where the latter two cases are the initial conditions for the recurrence. We now explain each initial condition. If exactly one of m and n is less than or equal to 0, then one player has already exhausted her M&M's while the other has not, so a tie is impossible. Thus, $F(m,n) = 0$ if exactly one of $m, n \leq 0$. If $m = n = 0$, then both players have exhausted their M&M's at the same round, and the game ends in a tie. In our version of the game, it is possible for both players to consume more M&M's than they have remaining, and we still treat such case as a tie. Thus, we set $F(m,n) = 1$ whenever $m \leq 0$ and $n \leq 0$.

2.1.2. *Sketch of the Argument.* We now sketch our argument and present the details for Game 1 in Section 2.3. The arguments for Games 2 and 3 follow a similar approach and are given in Appendix B.

For each game, we first define a step set $\mathcal{S}$ based on the methods used by the two players. Substituting $\mathcal{S}$ into (2.1) yields the corresponding recurrence for the game under consideration. We then associate this recurrence with the generating function $A(x,y) := \sum_{m,n\geq 0} F(m,n)x^m y^n$.

Our goal is to use the technique of generating functions to derive an explicit finite formula for $F(m,n)$; see [Wil05] for background on generating functions. Accordingly, we write $[x^m y^n]A(x,y)$ for the coefficient of $x^m y^n$ in $A(x,y)$. We then apply the following results to simplify $A(x,y)$.

Lemma 2.1. *Under the recurrence relation* (2.1)*, we have*

$$A(x,y) \;=\; 1 + \sum_{m,n\geq 1} F(m,n)x^m y^n. \tag{2.2}$$

Lemma 2.2. *Let $a,b \geq 1$. Under the recurrence relation* (2.1)*, we have*

$$\sum_{m,n\geq 1} F(m-a,n-b)x^m y^n \;=\; \sum_{M=1}^{a}\sum_{N=1}^{b} x^M y^N + x^a y^b \left(A(x,y)-1\right). \tag{2.3}$$

Also, we have

$$\sum_{m\geq 1}\sum_{n\geq 1} F(m-a,n)x^m y^n \;=\; x^a \left(A(x,y)-1\right) \tag{2.4}$$

and

$$\sum_{m\geq 1}\sum_{n\geq 1} F(m,n-b)x^m y^n \;=\; y^b \left(A(x,y)-1\right). \tag{2.5}$$

The proofs for Lemmas 2.1 and 2.2 are algebraic manipulations according to the recurrence relation and initial conditions. The detailed proofs can be found in Appendix A.

Using Lemmas 2.1 and 2.2, we rewrite the generating function as

$$A(x,y) \;=\; 1 + tB(\alpha,\beta,t) \sum_{(D_1,D_2)\in\mathcal{D}} \sum_{M=1}^{D_1}\sum_{N=1}^{D_2} x^M y^N, \tag{2.6}$$

where $t < 1$ is a constant weight and

$$\mathcal{D} = \{(a,b) : (a,b) \in \mathcal{S} \text{ if } a > 0 \text{ and } b > 0\}. \tag{2.7}$$

For example, if we consider Game 1, then $\mathcal{D} = \{(d,d)\}$. If we consider Game 3, then $\mathcal{D} = \{d_1, d_2, d_1+d_2\} \times \{d_1, d_2, d_1+d_2\}$. Thus, $\mathcal{D}$ consists of those moves in which both players consume at least one M&M. In addition, we observe that the term in (2.6)

$$B(\alpha,\beta,t) \;=\; \frac{1}{1-t\left(\alpha+\beta+\alpha\beta\right)} \tag{2.8}$$

is exactly the same as the generating function for Delannoy Numbers (for a discussion on Delannoy Numbers, see [BS05] and `https://oeis.org/A008288`). We then rewrite (2.6) in terms of x and y by making a change of variables to $B(\alpha,\beta,t)$, which allows us to extract coefficients corresponding to specific powers of x and y.

2.2. **Results.** We first introduce some notations. Let $\mathbb{P}^{(i)}$ denote the tie probability for Game i. Let $\mathbb{P}^{(1)}_{(d)}(I_1, I_2)$ denote the tie probability in Game 1, where the players start with I_1 and I_2 M&M's, respectively, and a player eats d M&M's upon flipping a head. Let $\mathbb{P}^{(2)}_{(d,d_1,d_2)}(I_1, I_2)$ denote the tie probability in Game 2, where Player A starts with I_1 M&M's and eats d M&M's upon flipping a head, while Player B starts with I_2 M&M's and eats d_1 M&M's if the first coin lands heads and d_2 M&M's if the second coin lands heads. Finally, let $\mathbb{P}^{(3)}_{(d_1,d_2)}(I_1, I_2)$ denote the tie probability in Game 3, where both players start with I_1 and I_2 M&M's, respectively, and a player eats d_1 M&M's if the first coin lands heads and d_2 M&M's if the second coin lands heads.

We have the following results for our three games introduced in Section 1.2.

Theorem 2.3 (Tie Probability for Game 1)**.** *In Game 1, Player A and Player B start with I_1 and I_2 M&M's, respectively. In each round, each player tosses one fair coin and a player eats d M&M's if that player's coin lands heads and eats none if it lands tails. The probability of a tie in Game 1 is*

$$\mathbb{P}^{(1)}_{(d)}(I_1, I_2) \ = \ \frac{1}{3} \sum_{M=1}^{\min(I_1,d)} \sum_{N=1}^{\min(I_2,d)} \left[x^{I_1-M} y^{I_2-N}\right] B\left(x, y, \frac{1}{3}\right), \tag{2.9}$$

where

$$B\left(x, y, \frac{1}{3}\right) \ = \ \sum_{J_1, J_2 \geq 0} \mathbb{I}\{d \mid J_1 \text{ and } d \mid J_2\} x^{J_1/d} y^{J_2/d} D(J_1/d, J_2/d), \tag{2.10}$$

and

$$D(m, n) \ = \ \sum_{\ell=0}^{\min(m,n)} \binom{m+n-\ell}{m-\ell, n-\ell, \ell} \left(\frac{1}{3}\right)^{m+n-\ell}. \tag{2.11}$$

Theorem 2.4 (Tie Probability for Game 2)**.** *In Game 2, Player A and Player B start with I_1 and I_2 M&M's, respectively. In each round, Player A tosses one fair coin and eats d M&M's if the coin lands heads. Player B tosses two fair coins, eating d_1 M&M's if the first coin lands heads and d_2 M&M's if the second coin lands heads. The probability of a tie in Game 2 is*

$$\mathbb{P}^{(2)}_{(d,d_1,d_2)}(I_1, I_2) \ = \ \frac{1}{7} \sum_{(D_1,D_2)\in\mathcal{D}} \sum_{M=1}^{\min(D_1,I_1)} \sum_{N=1}^{\min(D_2,I_2)} \left[x^{I_1-M} y^{I_2-N}\right] B\left(x, y, \frac{1}{7}\right), \tag{2.12}$$

where $\mathcal{D} = \{d\} \times \{d_1, d_2, d_1 + d_2\}$ and

$$B\left(x, y, \frac{1}{7}\right) \ = \ \sum_{J_1, J_2 \geq 0} \mathbb{I}\left\{d \mid J_1\right\} x^{J_1} y^{J_2} \sum_{\substack{i,j,k \geq 0 \\ d_1(i+k)+d_2(j+k)=J_2}} \binom{i+j+k}{i, j, k} D(J_1/d, i+j+k), \tag{2.13}$$

and

$$D(m, n) \ = \ \sum_{\ell=0}^{\min(m,n)} \binom{m+n-\ell}{m-\ell, n-\ell, \ell} \left(\frac{1}{7}\right)^{m+n-\ell}. \tag{2.14}$$

Theorem 2.5 (Tie Probability for Game 3)**.** *In Game 3, Player A and Player B start with I_1 and I_2 M&M's, respectively. In each round, each player tosses two fair coins. If the first coin lands heads, that player eats d_1 M&M's, and if the second coin lands heads, that player eats d_2 M&M's. The probability of a tie in Game 3 is*

$$\mathbb{P}^{(3)}_{(d_1,d_2)}(I_1, I_2) = \frac{1}{15} \sum_{(D_1,D_2)\in\mathcal{D}} \sum_{M=1}^{\min(D_1,I_1)} \sum_{N=1}^{\min(D_2,I_2)} \left[x^{I_1-M} y^{I_2-N}\right] B\left(x, y, \frac{1}{15}\right), \tag{2.15}$$

where $\mathcal{D} = \{d_1, d_2, d_1 + d_2\} \times \{d_1, d_2, d_1 + d_2\}$ *and*

$$B\left(x, y, \frac{1}{15}\right) = \sum_{J_1,J_2\geq 0} x^{J_1} y^{J_2} \Bigg[\sum_{\substack{r,s,t\geq 0\\ J_1=d_1(r+t)+d_2(s+t)}} \binom{r+s+t}{r,s,t} \sum_{\substack{i,j,k\geq 0\\ J_2=d_1(i+k)+d_2(j+k)}} \binom{i+j+k}{i,j,k} D(r+s+t,\, i+j+k)\Bigg] \tag{2.16}$$

and

$$D(m,n) = \sum_{\ell=0}^{\min(m,n)} \binom{m+n-\ell}{m-\ell, n-\ell, \ell} \left(\frac{1}{15}\right)^{m+n-\ell}. \tag{2.17}$$

As an illustration, we computed tie probabilities in the case when each player starts with I M&M's and eats exactly one M&M whenever a head is flipped; that is, $d = d_1 = d_2 = 1$. For each game, we report both the exact value and its decimal approximation. These results are displayed in Table 1 and Figure 1.

	Game 1		Game 2		Game 3	
(I_1, I_2)	Exact	Decimal	Exact	Decimal	Exact	Decimal
$(1,1)$	1/3	0.3333	3/7	0.4285	3/5	0.6000
$(2,2)$	5/27	0.1851	61/343	0.1778	113/375	0.3013
$(3,3)$	11/81	0.1358	1759/16807	0.1046	22361/84375	0.2650

Table 1. Comparison of Tie Probabilities For the Three Games.

2.3. **Derivation of Game 1.** We now use Game 1 to illustrate our technique. Suppose a player eats $d > 0$ number of M&M's if a head is flipped. Inserting the step set $\mathcal{S} = \{0, d\} \times \{0, d\} \setminus \{(0, 0)\}$ into recurrence relation (2.1), we have

$$F(m,n) = \begin{cases} \frac{1}{3}\Big[F(m-d,n) + F(m,n-d) + F(m-d,n-d)\Big] & \text{if } m \geq 1 \text{ and } n \geq 1 \\ 0 & \text{if exactly one of } m,n \text{ is } \leq 0 \\ 1 & \text{if } m \leq 0 \text{ and } n \leq 0. \end{cases}$$

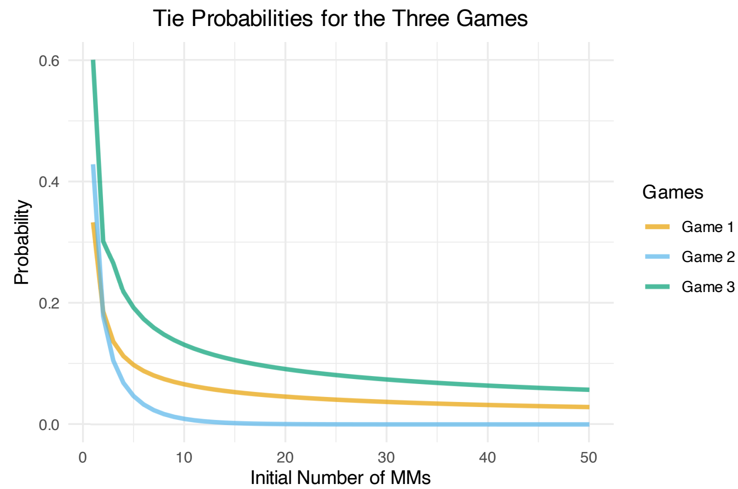

FIGURE 1. Tie Probabilities For Three Games.

Define the generating function $A(x,y) = \sum_{m,n\geq 0} F(m,n)x^m y^n$. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned} A(x,y) &= 1 + \sum_{m\geq 1}\sum_{n\geq 1} F(m,n)x^m y^n \\ &= 1 + \frac{1}{3}\Bigg[\sum_{m\geq 1}\sum_{n\geq 1} F(m-d,n)x^m y^n \\ &\quad + \sum_{m\geq 1}\sum_{n\geq 1} F(m,n-d)x^m y^n + \sum_{m\geq 1}\sum_{n\geq 1} F(m-d,n-d)x^m y^n\Bigg] \\ &= 1 + \frac{1}{3}\Big(x^d\big(A(x,y)-1\big) + y^d\big(A(x,y)-1\big) \\ &\quad + \sum_{M=1}^{d}\sum_{N=1}^{d} x^M y^N + x^d y^d\big(A(x,y)-1\big)\Big). \end{aligned} \tag{2.18}$$

For notational convenience, let $\alpha = x^d$ and $\beta = y^d$. After simple algebra, we find

$$\begin{aligned} A(x,y) &= 1 + \frac{\frac{1}{3}\sum_{M=1}^{d}\sum_{N=1}^{d} x^M y^N}{1 - \frac{1}{3}(\alpha+\beta+\alpha\beta)} \\ &= 1 + \frac{1}{3}\sum_{M=1}^{d}\sum_{N=1}^{d} x^M y^N B\left(\alpha,\beta,\frac{1}{3}\right), \end{aligned} \tag{2.19}$$

where

$$B\left(\alpha,\beta,\frac{1}{3}\right) = \frac{1}{1-\frac{1}{3}(\alpha+\beta+\alpha\beta)}. \tag{2.20}$$

We first study the expansion of $B\left(\alpha, \beta, \frac{1}{3}\right)$. After a simple application of the geometric series formula and multinomial expansion, we see that

$$B\left(\alpha, \beta, \frac{1}{3}\right) = \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^{k} \sum_{\substack{i,j,\ell\geq 0\\ i+j+\ell=k}} \frac{k!}{i!j!\ell!}\alpha^{i+\ell}\beta^{j+\ell}. \tag{2.21}$$

Let $m = i+\ell$ and $n = j+\ell$, we then have

$$\begin{aligned} B\left(\alpha, \beta, \frac{1}{3}\right) &= \sum_{m,n\geq 0} \alpha^{m}\beta^{n} \sum_{\ell=0}^{\min(m,n)} \binom{m+n-\ell}{m-\ell, n-\ell, \ell}\left(\frac{1}{3}\right)^{m+n-\ell} \\ &= \sum_{m,n\geq 0} \alpha^{m}\beta^{n} D\left(m,n\right), \end{aligned} \tag{2.22}$$

where

$$D\left(m,n\right) = \sum_{\ell=0}^{\min(m,n)} \binom{m+n-\ell}{m-\ell, n-\ell, \ell}\left(\frac{1}{3}\right)^{m+n-\ell}. \tag{2.23}$$

Now, we re-parametrize $B\left(\alpha, \beta, \frac{1}{3}\right)$ in terms of x and y, and we see that

$$\begin{aligned} B\left(x, y, \frac{1}{3}\right) &= \sum_{m,n\geq 0} x^{dm}y^{dn}D(m,n) \\ &= \sum_{J_1,J_2\geq 0} \mathbb{I}\{d \mid J_1 \text{ and } d \mid J_2\} x^{J_1/d}y^{J_2/d}D\left(J_1/d, J_2/d\right), \end{aligned} \tag{2.24}$$

where we performed a change of variables by letting $J_1 = dm$ and $J_2 = dn$. Note that we assumed J_1 and J_2 are multiples of d, so we let the sum vanishes when they are not a multiple of d. This is the reason for including the indicator function $\mathbb{I}$.

We have already computed $B\left(x, y, \frac{1}{3}\right)$. We can then insert the result (2.24) into (2.19), yielding $A(x,y)$ equals

$$1 + \frac{1}{3}\sum_{M=1}^{d}\sum_{N=1}^{d} x^{M}y^{N}\left(\sum_{J_1,J_2\geq 0} \mathbb{I}\{d \mid J_1 \text{ and } d \mid J_2\} x^{J_1/d}y^{J_2/d}D(J_1/d, J_2/d)\right). \tag{2.25}$$

To extract the coefficient $\left[x^{I_1}y^{I_2}\right] A(x,y)$, we use the following result.

Lemma 2.6. *Define*

$$f(x,y) := \sum_{J_1,J_2\geq 0} x^{J_1}y^{J_2}C(J_1,J_2),$$

where $C(J_1,J_2)$ *is the coefficient for* $x^{J_1}y^{J_2}$ *and thus a function of* J_1 *and* J_2*. We then have*

$$\left[x^{I_1}y^{I_2}\right]\sum_{M=1}^{a}\sum_{N=1}^{b} x^{M}y^{N}f(x,y) = \sum_{M=1}^{\min(a,I_1)}\sum_{N=1}^{\min(b,I_2)} [x^{I_1-M}y^{I_2-N}]f(x,y). \tag{2.26}$$

The proof of Lemma 2.6 is in Appendix A, and a simple application of it to (2.25) gives the tie probability for Game 1.

2.4. **Numerical Observations.** We first performed extensive computations to verify the correctness of our formulas. For each game, we computed the exact tie probability using both the recurrence (2.1) and the corresponding formula from Section 2.2. For Game 1, we computed $\mathbb{P}^{(1)}_{(d)}(I_1, I_2)$ for all $(I_1, I_2, d) \in [100] \times [100] \times [20]$, where $[N] = \{1, \ldots, N\}$ for a positive integer N. For Game 2, we computed $\mathbb{P}^{(2)}_{(d,d_1,d_2)}(I_1, I_2)$ for all $(I_1, I_2, d, d_1, d_2) \in [100] \times [100] \times [20] \times [20] \times [20]$. For Game 3, we computed $\mathbb{P}^{(3)}_{(d_1,d_2)}(I_1, I_2)$ for all $(I_1, I_2, d_1, d_2) \in [100] \times [100] \times [20] \times [20]$. In every case, the values obtained from the two approaches were identical. Since the full output is too large to include in the paper, we put the results to the code repository linked in Section 5.

We now discuss two numerical observations. First, when both players begin with I initial M&M's, the number of coins each player tosses per round affects the behavior of the tie probability. In the original M&M Game studied by [BHM$^+$17], as well as in our result for Game 1, the tie probability decreases with I. This decreasing behavior, however, does not hold when players are allowed to toss multiple coins.

In panel (a) of Figure 2, we compare $\mathbb{P}^{(1)}_{(5)}(I, I)$, $\mathbb{P}^{(2)}_{(5,2,3)}(I, I)$, and $\mathbb{P}^{(3)}_{(2,3)}(I, I)$ for $I \in [30]$, and make the following observation. While the tie probabilities in Games 1 and 2 are monotonically decreasing in I, the tie probability for Game 3 is no longer monotonically decreasing and exhibits small upward jumps at certain values of I. We circle one such region for illustration. For example, $\mathbb{P}^{(3)}_{(2,3)}(5, 5) = 0.262$, whereas $\mathbb{P}^{(3)}_{(2,3)}(6, 6) = 0.285$. In our simulations, we observed the same behavior for other choices of (d_1, d_2). We also carried out Monte Carlo simulations for a three-coin version of the game and again found similar jumps. Panel (b) of Figure 2 illustrates this, where we simulated the game in which a player eats $d_1 = 3$, $d_2 = 5$, and $d_3 = 10$ M&M's when the corresponding coins land heads. These jumps suggest that multi-coin versions of the M&M Game possess additional structural features that deserve further study.

Second, we found that when two players use Method 1 and Method 2, respectively, the probability distributions of the game outcomes change. In Figure 3, we present Monte Carlo simulations for the three games when the starting number of M&M's is $I = 2$ and $I = 20$, and plot the corresponding outcome distributions. On the horizontal axis, *Player A* denotes the probability that Player A wins, *Player B* denotes the probability that Player B wins, and *Tie* denotes the probability of a tie. In these simulations, Game 1 uses $d = 5$ for both players, Game 2 uses $d = 5$ for Player A and $(d_1, d_2) = (2, 3)$ for Player B, and Game 3 uses $(d_1, d_2) = (2, 3)$ for both players. We find that the winning probabilities for Player A and Player B are symmetric in Games 1 and 3, but become asymmetric in Game 2. This asymmetry reinforces the phenomenon discussed earlier in Section 1.1: even when two methods have the same expected number of M&M's removed per turn, they can still produce different probabilities of winning or losing, depending on which method each player uses.

3. The M&M Game Under Different Probabilities

3.1. **Extensions to Biased Coins.** The work done by [BHM$^+$17] and our previous extensions in this paper only focused on the game using fair coins. It is of interest to see what would

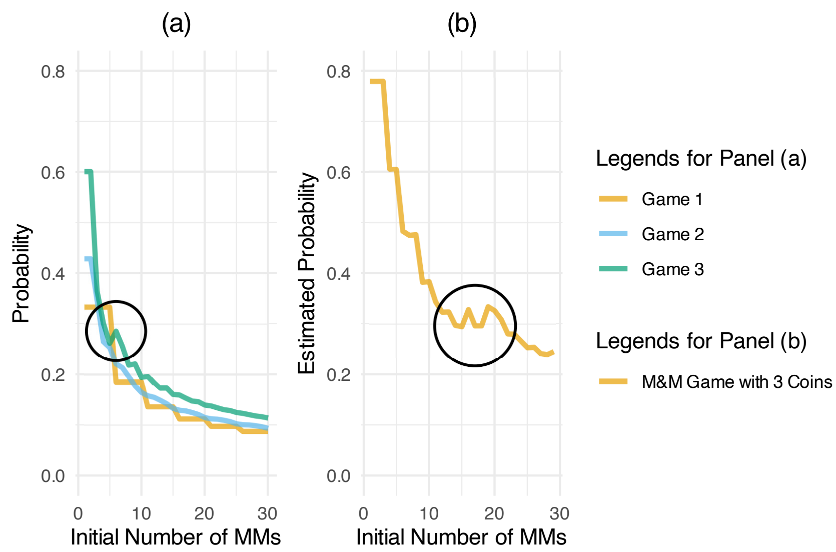

FIGURE 2. Effect of the Number of Coins on Tie Probabilities.

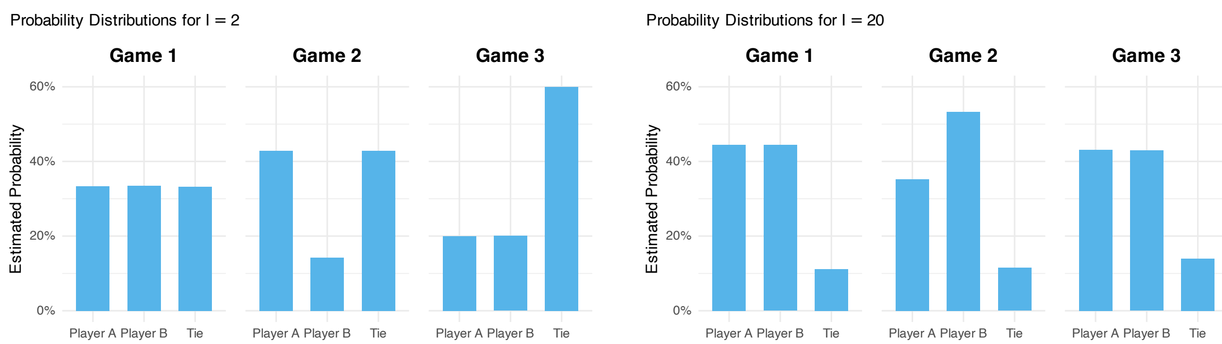

FIGURE 3. Probability Distributions of Three Games for $I = 2$ and $I = 20$ Based on Monte Carlo Simulations.

happen if we let two players use a biased coin in the game. We focus on the M&M's Game proposed by [BHM+17] with the modification that now each player uses a biased coin.

By generalizing Theorem 1.1, we easily have the following result.

Theorem 3.1. *Consider two players who each start with k M&M's and flip one coin per round. On each round, a player eats one M&M if their coin lands heads and eats none if it lands tails. Suppose each coin is biased, landing heads with probability p and tails with probability $1-p$. Then the probability of a tie is given by*

$$\mathbb{P}(k,k) \;=\; \sum_{n=0}^{k-1} \binom{2k-n-2}{n} \binom{2k-2n-2}{k-n-1} \left(\frac{p}{2-p}\right)^{n+1} \left(\frac{1-p}{2-p}\right)^{2k-2n-2}.$$

Proof. We omit the proof for the formula, as the argument is the same as the one provided by [BHM+17]. □

Using Theorem 3.1, we computed the tie probabilities for the game when each player starts with n initial number of M&M's and flips a biased coin landing heads with probability $p \in \{0, 0.1, \ldots, 1\}$. The results are displayed in Figure 4. As the figure shows, the tie probability is an increasing function of p, with the increase becoming particularly steep when $p \geq 0.9$. This behavior is intuitive: as p increases, both players are more likely to eat an M&M in each round, which raises the chance that they finish their last M&M in the same round.

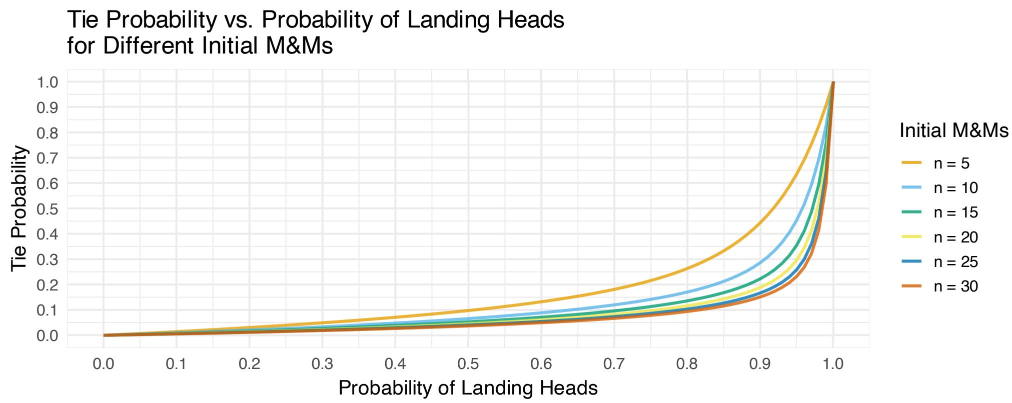

FIGURE 4. Tie Probability as a Function of p for Different Initial Numbers of M&M's.

3.2. **The M&M Game with Evolving Coin Probabilities.** We propose a state-dependent version of the M&M Game in which the probability that a coin lands heads evolves according to an exponential model. In this version of the game, the probability that a player's coin lands heads depends on the number of M&M's that player has remaining: in particular, the fewer M&M's a player has left, the greater the probability that the player's coin lands heads. This reflects the idea that the closer a player is to depletion (that is, death), the more likely the player is to consume another M&M. Because the probability of landing heads changes over the course of the game, we call this version of the M&M Game the *Evolving Coin Probabilities Game*.

Modifying the setup of the M&M Game in [BHM+17], we define the Evolving Coin Probabilities Game as follows. Two players each begin with n M&M's, and in each round each player independently flips a coin. A player consumes one M&M if the coin lands heads and consumes none if it lands tails. The probability that a player's coin lands heads in a given round is

$$\mathbb{P}(\text{head}) \;=\; 1 - e^{-\lambda(n-m+1)}, \tag{3.1}$$

where $\lambda > 0$ is the rate parameter controlling the rate of increase, and m is the number of M&M's currently remaining for that player. The rules determining when a player wins and when the game ends in a tie are the same as the rules given in Section 1.1.

Given the initial number of M&M's n and the rate parameter λ, we see from (3.1) that the probability of landing heads depends on the difference $n - m$, which is the number of M&M's the player has already consumed. As a player consumes more M&M's, the probability in (3.1) increases toward 1. We study the tie probability for the Evolving Coin Probabilities Game. Let $\mathbb{P}(\text{tie}; \lambda, n)$ denote the probability of a tie in this game with rate parameter λ

and initial number of M&M's n. In what follows, we conduct extensive simulations to study $\mathbb{P}(\text{tie}; \lambda, n)$.

3.2.1. *Simulation Results.* We performed extensive Monte Carlo simulations to examine the influence of parameters λ and n on $\mathbb{P}(\text{tie}; \lambda, n)$. Specifically, we simulated the game for all parameter pairs (λ, n) with $\lambda \in \{0.1, 0.2, \ldots, 5\}$ and $n \in \{1, 2, \ldots, 400\}$, using step size 0.1 in λ. For each choice of (λ, n), we performed 100,000 simulation trials. The results are displayed in Figure 6. In the figure, the yellow curve corresponds to the case $n = 1$, the black curve corresponds to $n = 2$, and the collection of blue curves corresponds to $n \in \{3, \ldots, 400\}$. To facilitate the discussion, we use $\mathbb{P}_T^{(\text{mc})}(\text{tie}; \lambda, n)$ to denote the Monte Carlo estimate of the tie probability for a fixed rate parameter λ and an initial number of M&M's n, and the subscript T is used to denote that the Monte Carlo estimate is done in T number of simulation trials.

To evaluate the quality of our Monte Carlo simulations, we computed the 99% confidence intervals of $\mathbb{P}_T^{(\text{mc})}(\text{tie}; \lambda, n)$ for different numbers of trials T over the range $[10, 100,000]$. The 99% confidence interval is computed according to the binomial proportion confidence interval:

$$\mathbb{P}_T^{(\text{mc})}(\text{tie}; \lambda, n) \;\pm\; \frac{2.58}{\sqrt{T}} \sqrt{\mathbb{P}_T^{(\text{mc})}(\text{tie}; \lambda, n) \left(1 - \mathbb{P}_T^{(\text{mc})}(\text{tie}; \lambda, n)\right)}, \tag{3.2}$$

where T denotes the number of simulation trials and 2.58 is the z-score corresponding to a 99% confidence level for the standard normal distribution. For a detailed overview on the construction of confidence intervals for Monte Carlo simulations, see [Owe13]. Selected examples are shown in Figure 5, where the blue curve represents the Monte Carlo estimates of the tie probabilities and the shaded region represents the corresponding confidence intervals. As the number of simulation trials T increases, the confidence intervals steadily narrow and become negligible once the number of simulation trials reaches $100,000$. This indicates that the estimate $\mathbb{P}_T^{(\text{mc})}(\text{tie}; \lambda, n)$ converges sufficiently to the true tie probability $\mathbb{P}(\text{tie}; \lambda, n)$.

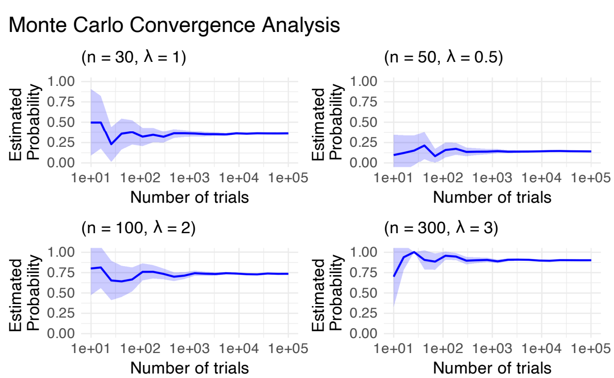

FIGURE 5. Monte Carlo Convergence Analysis for Selected Values of (λ, n).

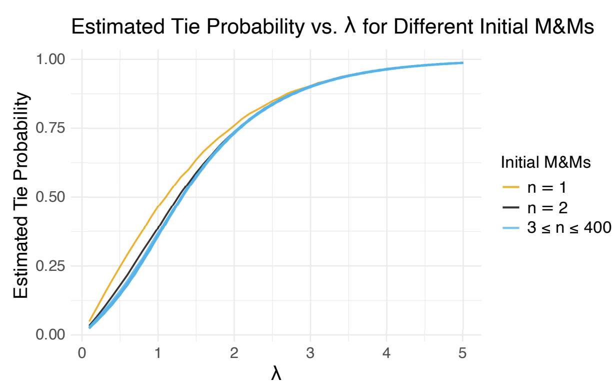

FIGURE 6. Estimated Tie Probabilities as a Function of the Rate Parameter λ.

Based on the simulation results, we make two key observations about the Evolving Coin Probabilities Game. First, the tie probability appears to be increasing in λ: as λ grows, the probability of a tie also increases. This is because larger values of λ cause the probability of flipping heads to rise more rapidly over the course of the game, thereby increasing the likelihood that both players consume their M&M's in the same round and hence eat their last M&M's in the same round. By contrast, smaller values of λ lead to a lower probability of flipping heads, which reduces the likelihood that both players consume their M&M's in the same round and thereby lowers the tie probability.

Second, the tie probabilities for different values of n appear to align remarkably well. As illustrated in Figure 6, the curves for different initial numbers of M&M's n begin to cluster together once $n \geq 3$. This indicates that, for moderate to large values of n, the tie probability is nearly independent of n. This behavior stands in sharp contrast to the M&M Game with biased coins (see Figure 4), where for a fixed probability of landing heads p, the tie probability decreases as n increases. In that setting, the initial number of M&M's n plays a more visible role in determining the game outcome. In the Evolving Coin Probabilities Game, however, the number of initial M&M's n exerts comparatively little influence on the tie probability once n is moderately large. Instead, the rate parameter λ emerges as the primary factor governing the tie probabilities.

We used statistical analysis to support our observations. In particular, we performed a Pearson correlation analysis [Ric06]. The results show a strong positive correlation between the rate parameter λ and the tie probability, with a correlation coefficient of 0.9251. In contrast, the correlation between the initial number of M&M's n and the tie probability is negligible, with a correlation coefficient of -0.0006. Since the correlation between the tie probability and λ is close to 1, while the correlation between the tie probability and n is close to 0, these findings provide further empirical support for our observation that λ is the dominant factor governing the tie probability, whereas the initial number of M&M's has minimal influence on the outcome of the game.

3.2.2. *Modeling Tie Probabilities with a Gompertz Curve.* In Figure 6, we observed that the simulated tie probabilities $\mathbb{P}_T^{(\mathrm{mc})}(\mathrm{tie}; \lambda, n)$ nearly collapse onto a single curve for all $n \geq 3$, which suggests that the influence of n on the tie probability becomes negligible for moderately large n. The correlation analysis similarly supports the conclusion that the tie probability is nearly independent of n. Based on these observations, it is natural to ask whether a probability model depending only on λ, denoted $G(\lambda)$, could approximate the tie probability $\mathbb{P}(\mathrm{tie}; \lambda, n)$ well for moderately large values of n. Since the simulated probability curves exhibit slow initial growth, followed by a phase of more rapid increase, and then level off toward an asymptote, a Gompertz curve appears to be an appropriate choice for modeling this behavior.

We now present a brief overview of the Gompertz curve. The Gompertz curve is commonly used to model growth processes characterized by an initial slow phase, followed by a phase of rapid increase, and eventual saturation [DB18]. Let $G(\lambda)$ denote the tie probability for the Evolving Coin Probabilities Game as predicted by the Gompertz model. The Gompertz model is given by

$$G(\lambda) \;=\; L \exp\left(-h \exp(-\lambda_0 \lambda)\right). \tag{3.3}$$

In (3.3), the parameter L is the upper asymptote, representing the limiting value of the growth process. Clearly, we see that $G(\lambda) = L$ as $\lambda \to \infty$. The parameter h determines the horizontal placement of the curve. Finally, λ_0 is the growth rate, determining how quickly the curve rises toward its asymptote.

We now seek to fit a Gompertz model to the simulated tie probabilities. Because the parameters (L, h, λ_0) are unknown, we estimate them by minimizing the sum of squared errors between the simulated probabilities and the Gompertz model given in (3.3). Since the resulting optimization problem is nonlinear, a closed-form solution for the optimal parameters is generally unavailable. Accordingly, we employ a numerical least-squares procedure, specifically the Levenberg-Marquardt algorithm, to obtain estimates of (L, h, λ_0) [Mar63]. The fitting procedure is summarized in Algorithm 1.

Algorithm 1 Parameter Estimation Procedure for Gompertz Model

1: **Input:** $\Lambda = \{\lambda_i\}_{i=1}^{50}$ for $\lambda_i = i/10$, number of Monte Carlo trials $T = 100{,}000$, and initial numbers of M&M's $n \in \{3, \ldots, 400\}$.
2: **Output:** Estimated parameters for the Gompertz model (3.3).
3: Create an empty list χ.
4: **for** each λ_i in Λ **do**
5: Simulate $\mathbb{P}_T^{(\mathrm{mc})}(\mathrm{tie}; \lambda_i, n)$ for all values of $n \in \{3, \ldots, 400\}$.
6: Average the values of $\mathbb{P}_T^{(\mathrm{mc})}(\mathrm{tie}; \lambda_i, n)$ over n to obtain

$$\overline{\mathbb{P}_T^{(\mathrm{mc})}}(\lambda_i) \;=\; \frac{1}{398}\sum_{n=3}^{400} \mathbb{P}_T^{(\mathrm{mc})}(\mathrm{tie}; \lambda_i, n).$$

7: Append the value $\left(\lambda_i, \overline{\mathbb{P}_T^{(\mathrm{mc})}}(\lambda_i)\right)$ to χ.
8: **end for**
9: Use Levenberg-Marquardt Algorithm to solve the least-squares problem

$$\min_{L,h,b} \sum_{\left(\lambda_i, \overline{\mathbb{P}_T^{(\mathrm{mc})}}(\lambda_i)\right) \in \chi} \left(\overline{\mathbb{P}_T^{(\mathrm{mc})}}(\lambda_i) - L\exp(-he^{-b\lambda_i})\right)^2,$$

and obtain the estimated parameters $(\widehat{L}, \widehat{h}, \widehat{b})$ that minimize the quantity above.
10: **Return:** Estimated parameters $(\widehat{L}, \widehat{h}, \widehat{b})$.

By applying Algorithm 1, we obtain the estimated parameters $\widehat{L} \approx 0.986, \widehat{h} \approx 3.525, \widehat{b} \approx 1.241$. Our final model is then given by

$$\widehat{G}(\lambda) \;\approx\; 0.986\exp(-3.525\exp(-1.241\lambda)). \tag{3.4}$$

Our model aligns with our expectations. In particular, the estimated parameter $\widehat{L} = 0.986$ is very close to 1, which suggests that the fitted Gompertz curve $\widehat{G}(\lambda)$ has a horizontal asymptote near 1. This agrees with our numerical observation that the tie probabilities approach 1 as λ becomes sufficiently large. In Figure 7, we plot the fitted Gompertz model $\widehat{G}(\lambda)$ together with the simulated curves shown earlier in Figure 6. The tie probabilities predicted by (3.4), represented by red circles in Figure 7, capture the overall trend of the tie probabilities and agree well with $\mathbb{P}_T^{(\mathrm{mc})}(\mathrm{tie}; \lambda, n)$ for all $n \geq 3$. As expected, the Gompertz model does not fully predict the cases $n = 1$ and $n = 2$, since for such small initial values the tie probability still depends on n.

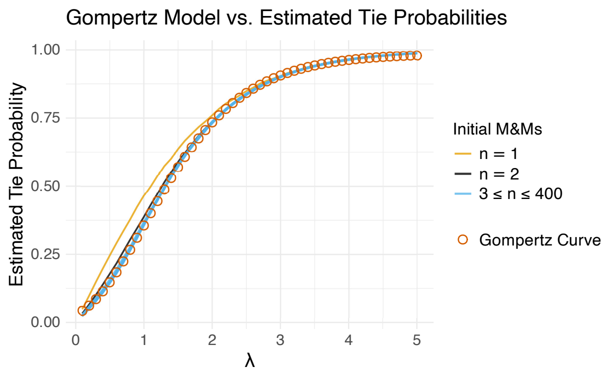

FIGURE 7. Modeling Tie Probabilties via Gompertz Model.

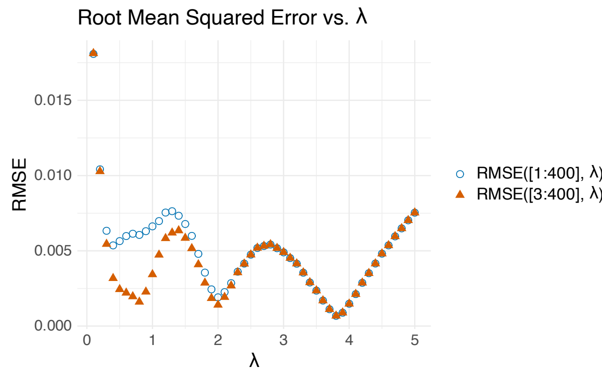

FIGURE 8. RMSE of the Fitted Gompertz Model.

For each value of λ, we also assessed the goodness of fit of the Gompertz model to the simulated probability curves $\mathbb{P}_T^{(\mathrm{mc})}(\mathrm{tie}; \lambda, n)$. We used the root mean squared error (RMSE) as our measure of fit. One advantage of RMSE is that it is reported on the same scale as the original quantity being modeled. In our setting, for example, an RMSE of 0.001 indicates that the fitted model typically differs from the simulated tie probability by about 0.001. Specifically, we computed two RMSE values: $\mathrm{RMSE}([1:400], \lambda)$, based on all simulated tie probabilities from Section 3.2.1, and $\mathrm{RMSE}([3:400], \lambda)$, based only on the probabilities for $3 \leq n \leq 400$, where the simulated curves effectively collapse onto a single curve. These quantities are computed as follows:

$$\mathrm{RMSE}([1:400], \lambda) \;=\; \sqrt{\frac{1}{400} \sum_{n \in [1:400]} \left(\mathbb{P}_T^{(\mathrm{mc})}(\mathrm{tie}; \lambda, n) - \widehat{G}(\lambda) \right)^2}, \tag{3.5}$$

and

$$\mathrm{RMSE}([3:400], \lambda) \;=\; \sqrt{\frac{1}{398} \sum_{n \in [3:400]} \left(\mathbb{P}_T^{(\mathrm{mc})}(\mathrm{tie}; \lambda, n) - \widehat{G}(\lambda) \right)^2}. \tag{3.6}$$

We plot these quantities in Figure 8. We observe that $\mathrm{RMSE}([1:400], \lambda)$ is larger than $\mathrm{RMSE}([3:400], \lambda)$ for $\lambda \in [0.5, 2]$. This is consistent with our earlier observation in Figure 7 that the probability curves for $n = 1$ and $n = 2$ do not align closely with the Gompertz model in this interval. Outside this interval, however, the two RMSE curves agree closely. We also observe that the RMSE is largest for $\lambda < 0.5$, suggesting a greater discrepancy between our fitted model and the tie probability in this region. One possible explanation is that, for small values of λ, the growth rate predicted by the Gompertz model does not match the true rate at which the tie probability changes with λ. Nevertheless, the Gompertz model still approximates the tie probability accurately to two decimal places, indicating that it remains a useful model for approximating tie probabilities.

While the Gompertz model captures the overall trend in the tie probabilities reasonably well, it still has several limitations. First, our fitted model only accounts for tie probabilities when $n \geq 3$. Second, the Gompertz curve imposes a particular asymmetric form: slow initial growth, followed by more rapid increase, and finally gradual saturation. Although our simulated tie probabilities are broadly consistent with this pattern, the rate of growth is not equally well captured across all values of λ. This likely contributes to the wave-shaped behavior of the RMSE curve. For these reasons, it may be worthwhile to explore alternative models that better capture these features, and we leave this for future exploration.

4. Conclusion and Future Work

We investigated several extensions of the M&M Game, focusing on how modified game rules and probability distributions of coin flips affect the tie probability. We now discuss future work for the M&M Game. It is likely some of these and related questions will be explored in Polymath Jr 2026; if you are interested in joining such efforts email sjm1@williams.edu.

In Section 2, we significantly generalized the M&M Game, though many further generalizations remain possible. First, it would be worthwhile to study the M&M Game in the more general setting of P players rather than only two players. Second, we studied a generalized version of the M&M Game in which a player tosses two coins in each round, consuming d_1 M&M's if the first coin lands heads and d_2 M&M's if the second coin lands heads. It is interesting to study the tie probability when a player can flip r number of coins in a given round. In particular, what is the tie probability when, on each turn, a player may consume $d_1, d_2, \ldots, d_r$ M&M's, where the probability of consuming d_i M&M's is p_i? We leave these more general cases for future investigation.

Besides studying the tie probability for generalized versions of the M&M Game, another unexplored question in both [BHM+17] and this paper concerns the length of the game: what is the expected number of turns for the game to terminate? During the course of writing this paper, we examined this question, and we believe this problem could be similarly solved via recurrence relation. We briefly sketch the idea. Let $F(m, n)$ denote the expected number of turns for the game to terminate when Player A has m M&M's and Player B has n M&M's. Previously, we defined $\mathcal{S} = (\mathcal{A} \times \mathcal{B}) \setminus \{(0, 0)\}$, omitting $(0, 0)$ because we were concerned only with game turns in which the numbers of M&M's possessed by the two players change. When computing the expected number of turns, however, we must also account for the possibility that neither player eats an M&M in a given round. Therefore, we need to consider $\mathcal{S} \cup \{(0, 0)\}$, the set of all possible moves in a round. Using $\mathcal{S} \cup \{(0, 0)\}$, we can obtain the following recurrence relation:

$$\begin{aligned} F(m, n) &= 1 + \frac{1}{(|\mathcal{S}| + 1)} \left(\sum_{(a,b) \in \mathcal{S} \cup (0,0)} F(m, n) \right) \\ &= 1 + \frac{1}{|S| + 1} \left(\sum_{(a,b) \in \mathcal{S}} F(m - a, n - b) + F(m, n) \right) \\ &= \frac{|\mathcal{S}| + 1}{|\mathcal{S}|} + \frac{1}{|\mathcal{S}|} \sum_{(a,b) \in \mathcal{S}} F(m - a, n - b). \end{aligned} \tag{4.1}$$

We impose the initial conditions $F(0,0) = 0$, since the expected number of turns for the game to terminate is 0 when the game is already at $(0,0)$. We also set $F(m,0) = F(0,n) = 0$ for $m,n > 0$, since the game terminates immediately once one player has exhausted all of her M&M's.

In Section 3, we studied the M&M Game under modified coin-flip probability distributions. In our study, we have provided only empirical evidence that λ is the dominant parameter governing the tie probability and that, to a large extent, the tie probability is nearly independent of n. It would be valuable to develop a rigorous theoretical explanation for this phenomenon, as well as to obtain a closed-form expression for the tie probability in the Evolving Coin Probabilities Game. In addition, it would be interesting to study the M&M Game under alternative coin-flip distributions.

5. Codes

All simulations in the paper are conducted via Python. For readers interested in replicating our studies, all codes for our simulations can be accessed at `https://github.com/MatthewYilong/MMGame`.

6. Acknowledgments

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Appendix A. Proof of Lemmas

Proof of Lemma 2.1. Simple computation and application of recurrence relation (2.1) yield

$$\begin{aligned} A(x,y) &= \sum_{m\geq 0}\sum_{n\geq 0} F(m,n)x^m y^n \\ &= F(0,0) + \sum_{n>0} F(0,n)y^n + \sum_{m>0} F(m,0)x^m + \sum_{m>0}\sum_{n>0} F(m,n)x^m y^n \\ &= 1+0+0+\sum_{m>0}\sum_{n>0} F(m,n)x^m y^n \\ &= 1+\sum_{m\geq 1}\sum_{n\geq 1} F(m,n)x^m y^n, \end{aligned} \tag{A.1}$$

where $\sum_{n>0} F(0,n)y^n$ and $\sum_{m>0} F(m,0)x^m$ vanish as a result of recurrence relation (2.1). ☐

Proof of Lemma 2.2. We prove the first formula. Consider $\sum_{m,n\geq 1} F(m-a,n-b)x^m y^n$ for $a,b\geq 1$. If we let $i=m-a$ and $j=n-b$, then we have the following:

$$\begin{aligned} \sum_{m,n\geq 1} F(m-a,n-b)x^m y^n &= \sum_{i+a\geq 1}\sum_{j+b\geq 1} F(i,j)x^{i+a}y^{j+b} \\ &= x^a y^b \sum_{i\geq -(a-1)}\sum_{j\geq -(b-1)} F(i,j)x^i y^j \\ &= x^a y^b \left(\sum_{i=-(a-1)}^{0}\sum_{j=-(b-1)}^{0} F(i,j)x^i y^j + \sum_{i,j\geq 1} F(i,j)x^i y^j\right) \\ &= \sum_{i=-(a-1)}^{0}\sum_{j=-(b-1)}^{0} F(i,j)x^{i+a}y^{j+b} + x^a y^b \sum_{i,j\geq 1} F(i,j)x^i y^j \\ &= \sum_{i=-(a-1)}^{0}\sum_{j=-(b-1)}^{0} x^{i+a}y^{j+b} + x^a y^b \sum_{i,j\geq 1} F(i,j)x^i y^j \\ &= \sum_{M=1}^{a}\sum_{N=1}^{b} x^M y^N + x^a y^b \left(A(x,y)-1\right). \end{aligned} \tag{A.2}$$

In the penultimate step, we see that $F(i,j)=1$ if both i and $j\leq 0$, so we have that $\sum_{i=-(a-1)}^{0}\sum_{j=-(b-1)}^{0} F(i,j)x^{i+a}y^{j+b}$ equals to $\sum_{i=-(a-1)}^{0}\sum_{j=-(b-1)}^{0} x^{i+a}y^{j+b}$. In the last step, we let $M=i+a$ and $N=j+b$ for term $\sum_{i=-(a-1)}^{0}\sum_{j=-(b-1)}^{0} x^{i+a}y^{j+b}$, and we applied Lemma 2.1 to the term $\sum_{i,j\geq 1} F(i,j)x^i y^j$.

We next prove $\sum_{m,n\geq 1} F(m-a,n)x^m y^n$. Let $i=m-a$. We then have

$$\begin{aligned}
\sum_{m,n\geq 1} F(m-a,n)x^m y^n &= \sum_{i+a\geq 1}\sum_{n\geq 1} F(i,n)x^{i+a}y^n \\
&= x^a \sum_{i\geq -(a-1)}\sum_{n\geq 1} F(i,n)x^i y^n \\
&= x^a \left\{ \sum_{i=-(a-1)}^{0}\sum_{n\geq 1} F(i,n)x^i y^n + \sum_{i\geq 1}\sum_{n\geq 1} F(i,n)x^i y^n \right\} \\
&= x^a \left(A(x,y)-1\right), \qquad \text{(A.3)}
\end{aligned}$$

where the first sum vanishes due to our boundary conditions in recurrence relation (2.1) and we apply Lemma 2.1 to the second term. The proof for $\sum_{m,n\geq 1} F(m,n-b)x^m y^n$ is exactly the same, so we omit it here. □

Proof of Lemma 2.6. Simple expansion yields

$$\begin{aligned}
\left[x^{I_1}y^{I_2}\right] \sum_{M=1}^{a}\sum_{N=1}^{b} x^M y^N f(x,y) &= \left[x^{I_1}y^{I_2}\right] \sum_{M=1}^{a}\sum_{N=1}^{b} x^M y^N \left(\sum_{J_1,J_2\geq 0} x^{J_1}y^{J_2} C(J_1,J_2) \right) \\
&= \sum_{M=1}^{a}\sum_{N=1}^{b} \left[x^{I_1}y^{I_2}\right] \sum_{J_1,J_2\geq 0} x^{J_1+M}y^{J_2+N} C(J_1,J_2) \\
&= \sum_{M=1}^{\min(a,I_1)} \sum_{N=1}^{\min(b,I_2)} C(I_1-M, I_2-N), \qquad \text{(A.4)}
\end{aligned}$$

where we let $J_1 = I_1 - M$ and $J_2 = I_2 - N$ for $M \in [a]$ and $N \in [b]$. Since $J_1, J_2 \geq 0$, we need the upper bound to be $M \in [\min(a, I_1)]$ and $N \in [\min(b, I_2)]$. Note that

$$\sum_{M=1}^{\min(a,I_1)} \sum_{N=1}^{\min(b,I_2)} C(I_1-M, I_2-N) = \sum_{M=1}^{\min(a,I_1)} \sum_{N=1}^{\min(b,I_2)} \left[x^{I_1-M}y^{I_2-N}\right] f(x,y),$$

giving the desired result. □

Appendix B. Proof of Results for Game 2 and Game 3

Proof of Theorem 2.4. We study Game 2. Let $d, d_1, d_2 \geq 1$, so we have the following recurrence relation

$$F(m,n) = \begin{cases} \frac{1}{7}\Big[F(m-d,n-d_1) + F(m-d,n-d_2) + F(m-d,n-d_1-d_2) & \\ \quad + F(m-d,n) + F(m,n-d_1) + F(m,n-d_2) & \text{if } m,n\geq 1, \\ \quad + F(m,n-d_1-d_2)\Big] & \\ 0 & \text{if } m=0, n\geq 1 \\ & \text{or } n=0, m\geq 1 \\ 1 & \text{if } m,n\leq 0. \end{cases}$$

Define the generating function $A(x,y) = \sum_{m,n\geq 0} F(m,n)x^m y^n$. We have the following

$$\begin{aligned} A(x,y) &= 1 + \frac{1}{7}\sum_{m\geq 1}\sum_{n\geq 1} F(m-d, n-d_1)x^m y^n + \frac{1}{7}\sum_{m\geq 1}\sum_{n\geq 1} F(m-d, n-d_2)x^m y^n \\ &\quad + \frac{1}{7}\sum_{m\geq 1}\sum_{n\geq 1} F(m-d, n-d_1-d_2)x^m y^n + \frac{1}{7}\sum_{m\geq 1}\sum_{n\geq 1} F(m-d, n)x^m y^n \\ &\quad + \frac{1}{7}\sum_{m\geq 1}\sum_{n\geq 1} F(m, n-d_1)x^m y^n + \frac{1}{7}\sum_{m\geq 1}\sum_{n\geq 1} F(m, n-d_2)x^m y^n \\ &\quad + \frac{1}{7}\sum_{m\geq 1}\sum_{n\geq 1} F(m, n-d_1-d_2)x^m y^n. \end{aligned} \tag{B.1}$$

By Lemma 2.2, we can rewrite $A(x,y)$ as

$$\begin{aligned} A(x,y) &= 1 + \frac{1}{7}\Bigg[y^{d_1}(A(x,y)-1) + y^{d_2}(A(x,y)-1) + y^{d_1+d_2}(A(x,y)-1) + x^d(A(x,y)-1) \\ &\quad + \sum_{(D_1,D_2)\in\mathcal{D}}\left(\sum_{M=1}^{D_1}\sum_{N=1}^{D_2} x^M y^N + x^{D_1}y^{D_2}(A(x,y)-1)\right)\Bigg], \end{aligned} \tag{B.2}$$

where $\mathcal{D} = \{d\} \times \{d_1, d_2, d_1+d_2\}$. We rearrange all terms containing $A(x,y)$ to the LHS and the rest of terms to the RHS. We see the LHS equals

$$A(x,y)\left(1 - \frac{1}{7}\left(x^d y^{d_1} + x^d y^{d_2} + x^d y^{d_1+d_2} + y^{d_1} + y^{d_2} + y^{d_1+d_2} + x^d\right)\right),$$

and RHS equals

$$\left(1 - \frac{1}{7}\left(x^d y^{d_1} + x^d y^{d_2} + x^d y^{d_1+d_2} + y^{d_1} + y^{d_2} + y^{d_1+d_2} + x^d\right)\right) + \frac{1}{7}\sum_{(D_1,D_2)\in\mathcal{D}}\sum_{M=1}^{D_1}\sum_{N=1}^{D_2} x^M y^N.$$

Let $\alpha = x^d$ and $\beta = y^{d_1} + y^{d_2} + y^{d_1+d_2}$, we then see that

$$\begin{aligned} A(x,y)\left(1 - \frac{1}{7}(\alpha+\beta+\alpha\beta)\right) &= \left(1 - \frac{1}{7}(\alpha+\beta+\alpha\beta)\right) + \frac{1}{7}\sum_{(D_1,D_2)\in\mathcal{D}}\sum_{M=1}^{D_1}\sum_{N=1}^{D_2} x^M y^N \\ &= 1 + \frac{\frac{1}{7}\sum_{(D_1,D_2)\in\mathcal{D}}\sum_{M=1}^{D_1}\sum_{N=1}^{D_2} x^M y^N}{1 - \frac{1}{7}(\alpha+\beta+\alpha\beta)}. \end{aligned} \tag{B.3}$$

We write

$$A(x,y) = 1 + \frac{1}{7}B\left(\alpha,\beta,\frac{1}{7}\right)\sum_{(D_1,D_2)\in\mathcal{D}}\sum_{M=1}^{D_1}\sum_{N=1}^{D_2} x^M y^N, \tag{B.4}$$

where

$$B\left(\alpha,\beta,\frac{1}{7}\right) = \frac{1}{1 - \frac{1}{7}(\alpha+\beta+\alpha\beta)}.$$

We first study the expansion of $B\left(\alpha, \beta, \frac{1}{7}\right)$. After a simple application of the geometric series formula and multinomial expansion, we see that

$$
\begin{aligned}
B\left(\alpha, \beta, \frac{1}{7}\right) &= \sum_{m,n\geq 0} \alpha^m \beta^n \sum_{\ell=0}^{\min(m,n)} \binom{m+n-\ell}{m-\ell, n-\ell, \ell} \left(\frac{1}{7}\right)^{m+n-\ell} \\
&= \sum_{m,n\geq 0} \alpha^m \beta^n D(m,n), \qquad \text{(B.5)}
\end{aligned}
$$

where

$$
D(m,n) = \sum_{\ell=0}^{\min(m,n)} \binom{m+n-\ell}{m-\ell, n-\ell, \ell} \left(\frac{1}{7}\right)^{m+n-\ell}. \qquad \text{(B.6)}
$$

We now hope to rewrite $B\left(\alpha, \beta, \frac{1}{7}\right)$ as $B\left(x, y, \frac{1}{7}\right)$, where $\alpha = x^d$ and $\beta = y^{d_1} + y^{d_2} + y^{d_1+d_2}$. After an application of multinomial expansion and a change of variables, we find

$$
\begin{aligned}
B\left(x, y, \frac{1}{7}\right) &= \sum_{m,n\geq 0} x^{dm} \left(y^{d_1} + y^{d_2} + y^{d_1+d_2}\right)^n D(m,n) \\
&= \sum_{m,n\geq 0} x^{dm} \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \binom{i+j+k}{i,j,k} y^{d_1(i+k)+d_2(j+k)} D(m,n) \\
&= \sum_{J_1,J_2\geq 0} \mathbb{I}\left\{d \mid J_1\right\} x^{J_1} y^{J_2} \sum_{\substack{i,j,k\geq 0 \\ d_1(i+k)+d_2(j+k)=J_2}} \binom{i+j+k}{i,j,k} D(J_1/d, i+j+k), \qquad \text{(B.7)}
\end{aligned}
$$

where $J_1 = dm$ and $J_2 = d_1(i+j) + d_2(j+k)$. Applying Lemma 2.6 gives the desired result. □

Proof of Theorem 2.5. We now investigate into Game 3. Let $d_1, d_2 \geq 1$, so we have the following recurrence

$$
F(m,n) = \begin{cases}
\frac{1}{15}\Big[F(m-d_1-d_2, n-d_1-d_2) + F(m-d_1-d_2, n-d_1) \\
\quad + F(m-d_1-d_2, n-d_2) + F(m-d_1-d_2, n) \\
\quad + F(m-d_1, n-d_1-d_2) + F(m-d_1, n-d_1) \\
\quad + F(m-d_1, n-d_2) + F(m-d_1, n) + F(m-d_2, n-d_1-d_2) & \text{if } m,n \geq 1, \\
\quad + F(m-d_2, n-d_1) + F(m-d_2, n-d_2) \\
\quad + F(m-d_2, n) + F(m, n-d_1-d_2) + F(m, n-d_1) + F(m, n-d_2)\Big] \\
0 & \text{if } m = 0, n \geq 1 \\
 & \text{or } n = 0, m \geq 1, \\
1 & \text{if } m, n \leq 0.
\end{cases}
$$

Define the generating function $A(x,y) = \sum_{m,n\geq 0} F(m,n)x^m y^n$. We have the following

$$\begin{aligned} A(x,y) &= 1 + \frac{1}{15}\Bigg[y^{d_1+d_2}\left(A(x,y)-1\right) + y^{d_1}\left(A(x,y)-1\right) + y^{d_2}\left(A(x,y)-1\right) \\ &\quad + x^{d_1+d_2}\left(A(x,y)-1\right) + x^{d_1}\left(A(x,y)-1\right) + x^{d_2}\left(A(x,y)-1\right) \\ &\quad + \sum_{(D_1,D_2)\in\mathcal{D}}\left(\sum_{M=1}^{D_1}\sum_{N=1}^{D_2} x^M y^N + x^{D_1}y^{D_2}\left(A(x,y)-1\right)\right)\Bigg], \end{aligned} \tag{B.8}$$

where $\mathcal{D} = \{d_1, d_2, d_1+d_2\} \times \{d_1, d_2, d_1+d_2\}$. We rearrange all terms containing $A(x,y)$ to the LHS and the rest of terms to the RHS. We see the LHS equals

$$\begin{aligned} A(x,y)\bigg(1 - \frac{1}{15}\big(&y^{d_1} + y^{d_2} + y^{d_1+d_2} + x^{d_1} + x^{d_2} + x^{d_1+d_2} \\ &+ x^{d_1}y^{d_1} + x^{d_1}y^{d_2} + x^{d_1}y^{d_1+d_2} + x^{d_2}y^{d_1} + x^{d_2}y^{d_2} + x^{d_2}y^{d_1+d_2} \\ &+ x^{d_1+d_2}y^{d_1} + x^{d_1+d_2}y^{d_2} + x^{d_1+d_2}y^{d_1+d_2}\big)\bigg), \end{aligned} \tag{B.9}$$

and RHS equals

$$\begin{aligned} 1 - \frac{1}{15}\big(&y^{d_1} + y^{d_2} + y^{d_1+d_2} + x^{d_1} + x^{d_2} + x^{d_1+d_2} \\ &+ x^{d_1}y^{d_1} + x^{d_1}y^{d_2} + x^{d_1}y^{d_1+d_2} + x^{d_2}y^{d_1} + x^{d_2}y^{d_2} + x^{d_2}y^{d_1+d_2} \\ &+ x^{d_1+d_2}y^{d_1} + x^{d_1+d_2}y^{d_2} + x^{d_1+d_2}y^{d_1+d_2}\big) \\ &+ \frac{1}{15}\sum_{(D_1,D_2)\in\mathcal{D}}\sum_{M=1}^{D_1}\sum_{N=1}^{D_2} x^M y^N. \end{aligned} \tag{B.10}$$

Let $\alpha = x^{d_1} + x^{d_2} + x^{d_1+d_2}$ and $\beta = y^{d_1} + y^{d_2} + y^{d_1+d_2}$. We then find the following after isolating $A(x,y)$:

$$\begin{aligned} A(x,y) &= 1 + \frac{\frac{1}{15}\sum_{(D_1,D_2)\in\mathcal{D}}\sum_{M=1}^{D_1}\sum_{N=1}^{D_2} x^M y^N}{1 - \frac{1}{15}\left(\alpha+\beta+\alpha\beta\right)} \\ &= 1 + \frac{1}{15}B\left(\alpha,\beta,\frac{1}{15}\right)\sum_{(D_1,D_2)\in\mathcal{D}}\sum_{M=1}^{D_1}\sum_{N=1}^{D_2} x^M y^N, \end{aligned} \tag{B.11}$$

where

$$B\left(\alpha,\beta,\frac{1}{15}\right) = \frac{1}{1 - \frac{1}{15}\left(\alpha+\beta+\alpha\beta\right)}. \tag{B.12}$$

We now express $B(\alpha,\beta,\frac{1}{15})$ in terms of $B(x,y,\frac{1}{15})$. After a simple application of the geometric series formula and multinomial expansion, we first see that

$$B\left(\alpha,\beta,\frac{1}{15}\right) = \sum_{m,n\geq 0}\alpha^m\beta^n D(m,n), \tag{B.13}$$

where

$$D(m,n) \ = \ \sum_{\ell=0}^{\min(m,n)} \binom{m+n-\ell}{m-\ell, n-\ell, \ell} \left(\frac{1}{15}\right)^{m+n-\ell}. \tag{B.14}$$

Substituting $\alpha = x^{d_1} + x^{d_2} + x^{d_1+d_2}$ and $\beta = y^{d_1} + y^{d_2} + y^{d_1+d_2}$ into (B.13), we arrive at

$$\begin{aligned} B\left(x, y, \frac{1}{15}\right) &= \sum_{m,n\geq 0} \left(x^{d_1} + x^{d_2} + x^{d_1+d_2}\right)^m \left(y^{d_1} + y^{d_2} + y^{d_1+d_2}\right)^n D(m,n) \\ &= \sum_{m,n\geq 0} \Bigg[\sum_{\substack{r,s,t\geq 0 \\ r+s+t=m}} \binom{r+s+t}{r,s,t} x^{d_1(r+t)+d_2(s+t)} \\ &\qquad \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \binom{i+j+k}{i,j,k} y^{d_1(i+k)+d_2(j+k)} D(r+s+t, i+j+k) \Bigg] \\ &= \sum_{J_1,J_2\geq 0} x^{J_1} y^{J_2} \Bigg[\sum_{\substack{r,s,t\geq 0 \\ J_1=d_1(r+t)+d_2(s+t)}} \binom{r+s+t}{r,s,t} \\ &\qquad \sum_{\substack{i,j,k\geq 0 \\ J_2=d_1(i+k)+d_2(j+k)}} \binom{i+j+k}{i,j,k} D(r+s+t,\, i+j+k) \Bigg], \end{aligned} \tag{B.15}$$

where $J_1 = d_1(r+t) + d_2(s+t)$ and $J_2 = d_1(i+k) + d_2(j+k)$. Applying Lemma 2.6 to (B.11) gives the desired result. □

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