STABILITY OF MATRIX RECURRENCE RELATIONS

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Abstract. Motivated by the rich properties and various applications of recurrence relations, we consider the extension of traditional recurrence relations to matrices, where we use matrix multiplication and the Kronecker product to construct matrix sequences. We provide a sharp condition, which when satisfied, guarantees that any fixed-depth matrix recurrence relation defined over a product (with respect to matrix multiplication) will converge to the zero matrix. We also show that the same statement applies to matrix recurrence relations defined over a Kronecker product. Lastly, we show that the dual of this condition, which remains sharp, guarantees the divergence of matrix recurrence relations defined over a consecutive Kronecker product. These results completely determine the stability of nontrivial fixed-depth complex-valued recurrence relations defined over a consecutive product.

1. Introduction

Presented with the recurrence relation $a_n = a_{n-1}a_{n-2}$ with $a_0, a_1 \in \mathbb{C}$, the standard approach to solving for a_n involves defining an auxiliary sequence

$$g_n \;=\; \log a_n, \tag{1}$$

which yields $g_n = g_{n-1} + g_{n-2}$, with $g_0 = \log a_0$ and $g_1 = \log a_1$. After some calculation, we obtain

$$\sum_{n\geq 0} g_n x^n \;=\; \frac{\log a_0 + (\log a_1 - \log a_0)x}{1 - x - x^2}, \tag{2}$$

and consequently that $g_n = \log a_0 \cdot F_{n-1} + \log a_1 \cdot F_n$, where $\{F_n\}$ is the Fibonacci sequence with $F_0 = 0$ and $F_1 = 1$. Thus,

$$a_n \;=\; a_0^{F_{n-1}} a_1^{F_n} \tag{3}$$

for any $n \geq 1$. With this formula, determining the stability of a_n is simple. This calculation is, in principle, straightforward; however, for more complicated recurrence relations of arbitrarily large depth, it can become quite arduous.

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Matrix recurrences, the natural generalization of nonlinear complex-valued recurrences, are considered due to their added complexity. Indeed, in fields of applied mathematics such as population modeling, taking the initial conditions as matrices allows for nuance in population dynamics unencodable by complex numbers.

Throughout this paper, we use submultiplicative matrix norms; a submultiplicative matrix norm $||\cdot||$ is a norm on a vector space V of matrices such that $||X_1X_2|| \leq ||X_1||\,||X_2||$ for any $X_1, X_2 \in V$. Examples include the Frobenius norm $||\cdot||_F$ and the operator norm $||\cdot||_{\text{op}}$ defined by

$$\begin{aligned} ||X||_F &:= \sqrt{\text{Tr}(XX^H)} \\ ||X||_{\text{op}} &:= \sqrt{\max(\text{Eigenvalues}(XX^T))} \end{aligned} \tag{4}$$

for any $X \in V$, where V may be any vector space of matrices. Note X^T and X^H denote the transpose and conjugate transpose of X respectively.

In this article, we determine the stability of matrix recurrence relations under matrix multiplication and the Kronecker product through the lens of "multiplicity" rather than the conventional auxiliary logarithmic sequence approach. In particular, we have the following result.

Proposition 1.1. *Let $S \subseteq \mathbb{Z}^+$ be finite such that $|S| \geq 2$. Let m be the largest integer such that $S \subseteq m\mathbb{Z}$ and let $j := \max(S)$. Define the sequence of matrices $\{A_n\}$ by the recurrence relation*

$$A_n \ = \ B\prod_{k\in S} A_{n-k}, \tag{5}$$

with $A_0, A_1, \ldots, A_{j-1}$ as fixed square matrices of the same size, B as a scalar or square matrix, and the product taken in any desired order. Let

$$\lambda \ = \ \begin{cases} 0 & ||B|| \geq 1 \\ m-1 & ||B|| < 1. \end{cases} \tag{6}$$

Suppose

$$\frac{\varphi_{S/m}^{-(j+\lambda)/m+1}}{\varphi_{S/m}-1}\log||B|| + \sum_{k=0}^{j-1}\log||A_k|| \sum_{j-k\leq \ell\in S}\varphi_{S/m}^{(-k-\ell)/m} \ < \ 0, \tag{7}$$

where $||\cdot||$ is some submultiplicative matrix norm and $\varphi_{S/m} \in \mathbb{R}^+$ uniquely satisfies

$$\sum_{\ell\in\{l/m: l\in S\}} \varphi_{S/m}^{-\ell} \ = \ 1. \tag{8}$$

Then $\{A_n\}$ converges to the zero matrix.

The relevance of m in Proposition 1.1 is not easily understood at first glance. To elucidate such, consider the class of recurrences

$$M_n \ = \ M_{n-2}M_{n-4} \tag{9}$$

where $M_0, \ldots, M_3$ fixed. Note that $m = 2$ for M_n. We have

$$\begin{aligned} M_4 &= M_2 M_0 \\ M_5 &= M_3 M_1 \\ M_6 &= M_2 M_0 M_2 \\ M_7 &= M_3 M_1 M_3 \\ M_8 &= M_2 M_0 M_2^2 M_0 \\ M_9 &= M_3 M_1 M_3^2 M_1 \\ M_{10} &= M_2 M_0 M_2^2 M_0 M_2 M_0 M_2. \end{aligned} \tag{10}$$

Now consider the class of recurrences

$$A_n = A_{n-1} A_{n-2}, \tag{11}$$

with A_0, A_1 fixed. We have

$$\begin{aligned} A_2 &= A_1 A_0 \\ A_3 &= A_1 A_0 A_1 \\ A_4 &= A_1 A_0 A_1^2 A_0 \\ A_5 &= A_1 A_0 A_1^2 A_0 A_1 A_0 A_1 \\ A_6 &= A_1 A_0 A_1^2 A_0 A_1 A_0 A_1^2 A_0 A_1^2 A_0. \end{aligned} \tag{12}$$

We observe that $M_{2n} = A_n$ with $A_1 = M_2$ and $A_0 = M_0$, and $M_{2n+1} = A_n$ with $A_1 = M_3$ and $A_0 = M_1$. From this observation, can say that M_n contains $m = 2$ different sequences which alternate depending on the parity of n. When considering asymptotics, each sequence contained in M_n grows $m = 2$ times as slow as A_n. Condition (7) is ultimately about bounding the growth of $||A_n||$; hence the ubiquitousness of m in Proposition 1.1.

In addition to matrix multiplication, we consider the stability of matrix recurrences defined over a Kronecker product. The Kronecker product between an $n_1 \times m_1$ matrix X and an $n_2 \times m_2$ matrix Y, denoted $X \otimes Y$, is the $n_1 n_2 \times m_1 m_2$ block matrix

$$X \otimes Y := \begin{bmatrix} x_{1,1}Y & \cdots & x_{1,m_1}Y \\ \vdots & \ddots & \vdots \\ x_{n_1,1}Y & \cdots & x_{n_1,m_1}Y \end{bmatrix}, \tag{13}$$

where $x_{i,j}$ is the $(i,j)^{\text{th}}$ entry of X.

Since the Kronecker product behaves nicely under submultiplicative matrix norms, we essentially have an identical result to Proposition 1.1 for matrix recurrences of the form

$$A_n = B \otimes \bigotimes_{k \in S} A_{n-k}. \tag{14}$$

Lastly, we determine when recurrences defined over a consecutive Kronecker product diverge to infinity in norm; that is, all recurrences of the form

$$A_n \;=\; B \otimes \bigotimes_{1\le k\le j} A_{n-k}. \tag{15}$$

Proposition 1.2. *Let $j > 1$ be an integer. Define the sequence of matrices $\{A_n\}$ by the recurrence relation*

$$A_n \;=\; B \otimes \bigotimes_{1\le k\le j} A_{n-k}, \tag{16}$$

with $B, A_0, A_1, \ldots, A_{j-1}$ fixed and the Kronecker product taken in any desired order. Suppose that

$$||B|| \prod_{k=0}^{j-1} ||A_k||^{1-\varphi_j^{-k-1}} \;>\; 1, \tag{17}$$

where $||\cdot||$ is any submultiplicative matrix norm and $\varphi_j \in \mathbb{R}^+$ uniquely satisfies

$$\sum_{k=1}^{j} \varphi_j^{-k} \;=\; 1. \tag{18}$$

Then $\{A_n\}$ diverges to infinity in norm.

Note that φ_j is also the positive real root of

$$x^j - \sum_{k=0}^{j-1} x^k,$$

the characteristic polynomial of the j-nacci sequence. This is a natural generalization of the golden ratio, which is equal to φ_2.

By considering 1×1 matrices, we can apply these results to completely determine the stability of nonlinear complex-valued nontrivial recurrence relations defined over a finite consecutive product. By nontrivial, we mean that the recurrence is not of the form

$$a_n \;=\; b a_{n-1}, \tag{19}$$

which easily lends the formula

$$a_n \;=\; a_0 b^n. \tag{20}$$

2. Multiplicity

To yield our theorems on stability, we first find the "multiplicity" of each initial value matrix in the matrix sequences. Once again, consider the recurrence

$$A_n \;=\; A_{n-1} A_{n-2} \tag{21}$$

with A_0, A_1 fixed. By (12), the multiplicity of A_1, or colloquially, the number of times A_1 is multiplied in A_n, is $0, 1, 1, 2, 3, 5, 8$ for $n = 0, 1, \ldots, 6$ respectively. For the sake of the curious reader, this sequence is indeed the Fibonacci sequence, arising as a special case of Theorem 2.1.

To this end, we introduce a generalization of the Fibonacci sequence and determine its generating function.

Definition 2.1 (S-nacci sequence and S-nacci constant). *Let $\varnothing \neq S \subseteq \mathbb{Z}^+$ be a finite set and let $j := \max(S)$. Define the sequence of integers $\{F_n^{(S)}\}$, which we call the S-nacci sequence, by the recurrence relation*

$$F_n^{(S)} \;=\; \sum_{\ell \in S} F_{n-\ell}^{(S)}, \tag{22}$$

with $F_0^{(S)}, F_1^{(S)}, \ldots, F_{j-2}^{(S)} = 0$ and $F_{j-1}^{(S)} = 1$. The S-nacci constant, denoted φ_S, is defined as the positive real number satisfying

$$\sum_{\ell \in S} \varphi_S^{-\ell} \;=\; 1. \tag{23}$$

These definitions coincide with the definitions of the k-nacci sequence and k-nacci constant when $S = \{1, 2, 3, \ldots, k-1, k\}$. Note if S is not a singleton, then $\varphi_S > 1$.

Lemma 2.1. *Let $\varnothing \neq S \subseteq \mathbb{Z}^+$ be a finite set and $j := \max(S)$. Then*

$$\sum_{k \geq 0} F_k^{(S)} z^k \;=\; \frac{z^{j-1}}{1 - \sum_{\ell \in S} z^\ell}. \tag{24}$$

Proof. To determine the ordinary generating function of $\{F_n^{(S)}\}$, we use the standard ansatz (see [3] for example) that this function is rational with denominator $1 - \sum_{\ell \in S} z^\ell$. Indeed, we have

$$\begin{aligned}
\left(1 - \sum_{\ell \in S} z^\ell\right) \sum_{k \geq 0} F_k^{(S)} z^k &= \sum_{k \geq 0} F_k^{(S)} z^k - \sum_{\ell \in S} \sum_{k \geq 0} F_k^{(S)} z^{k+\ell} \\
&= \sum_{k \geq 0} F_k^{(S)} z^k - \sum_{\ell \in S} \sum_{k \geq \ell} F_{k-\ell}^{(S)} z^k \\
&= F_{j-1}^{(S)} z^{j-1} + \sum_{k \geq j} \left(F_k^{(S)} - \sum_{\ell \in S} F_{k-\ell}^{(S)}\right) z^k \qquad (*) \\
&= F_{j-1}^{(S)} z^{j-1} \\
&= z^{j-1},
\end{aligned} \tag{25}$$

where $(*)$ follows from $F_k^{(S)} = 0$ if $k \leq j-2$. So

$$\sum_{k \geq 0} F_k^{(S)} z^k \;=\; \frac{z^{j-1}}{1 - \sum_{\ell \in S} z^\ell}, \tag{26}$$

as desired. □

Definition 2.2 (Indicator Function)**.** *For $S \subseteq \mathbb{Z}^+$, let $\mathbf{1}_S : \mathbb{Z}^+ \to \{0,1\}$ be defined for all $n \in \mathbb{Z}^+$ by*

$$\mathbf{1}_S(n) \;=\; \begin{cases} 1 & n \in S \\ 0 & n \notin S. \end{cases} \tag{27}$$

Theorem 2.1 (Multiplicity Theorem)**.** *Let $S \subseteq \mathbb{Z}^+$ be a finite set and $j := \max(S)$. Define the sequence of matrices $\{A_n\}$ by the recurrence relation*

$$A_n \;=\; \prod_{k\in S} A_{n-k}, \tag{28}$$

with $A_0, A_1, \ldots, A_{j-1}$ as fixed square matrices of the same size, and the product taken in any desired order. Then for all $n \geq j$, A_n is a product of $A_0, A_1, \ldots, A_{j-1}$'s where each A_k with $0 \leq k \leq j-1$ has multiplicity

$$\sum_{j-k\leq \ell \in S} F^{(S)}_{n+j-1-k-\ell}. \tag{29}$$

Proof. Let $S \subseteq \mathbb{Z}^+$ be a finite set and $j := \max(S)$. We have that

$$Q^{(S)} \;:=\; \begin{bmatrix} \mathbf{1}_S(1) & \mathbf{1}_S(2) & \cdots & \mathbf{1}_S(j-1) & \mathbf{1}_S(j) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{30}$$

is the $j \times j$ companion matrix of the S-nacci sequence. This matrix encodes the recurrence relation for the S-nacci sequence such that

$$\begin{bmatrix} F^{(S)}_{n+1} \\ F^{(S)}_{n} \\ \vdots \\ F^{(S)}_{n-j+2} \end{bmatrix} \;=\; Q^{(S)} \begin{bmatrix} F^{(S)}_{n} \\ F^{(S)}_{n-1} \\ \vdots \\ F^{(S)}_{n-j+1} \end{bmatrix}. \tag{31}$$

For $0 \leq k \leq j-1$, let $\#A_{k,n}$ denote the multiplicity of A_k in A_n. From the recurrence relation

$$A_n \;=\; \prod_{k\in S} A_{n-k}, \tag{32}$$

we can deduce the recurrence relation

$$\#A_{k,n} \;=\; \sum_{\ell\in S} \#A_{k,n-\ell}. \tag{33}$$

This relation is identical to that of the S-nacci sequence; thus,

$$\begin{bmatrix} \#A_{k,n+1} \\ \#A_{k,n} \\ \vdots \\ \#A_{k,n-j+2} \end{bmatrix} = Q^{(S)} \begin{bmatrix} \#A_{k,n} \\ \#A_{k,n-1} \\ \vdots \\ \#A_{k,n-j+1} \end{bmatrix}. \tag{34}$$

Given initial conditions, we can think about applying the $Q^{(S)}$ matrix n times to recover the n^{th} iteration of the vector sequence. Specifically,

$$\begin{bmatrix} \#A_{k,n+j-1} \\ \#A_{k,n+j-2} \\ \vdots \\ \#A_{k,n} \end{bmatrix} = \left(Q^{(S)}\right)^n \begin{bmatrix} \#A_{k,j-1} \\ \#A_{k,j-2} \\ \vdots \\ \#A_{k,0} \end{bmatrix}. \tag{35}$$

We now focus on the left-hand side vector's last entry, $\#A_{k,n}$.

Observe that when looking at the multiplicity of a specific initial matrix A_k, the initial conditions vector on the right-hand side which we multiply by $\left(Q^{(S)}\right)^n$ consists of all zeros except for a 1 in the k^{th} entry (with entries numbered bottom-up starting from 0). For example, when considering the multiplicity of A_{j-1}, we have that

$$\begin{bmatrix} \#A_{j-1,n+j-1} \\ \#A_{j-1,n+j-2} \\ \vdots \\ \#A_{j-1,n} \end{bmatrix} = \left(Q^{(S)}\right)^n \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{36}$$

Note that this recovers precisely the S-nacci sequence, which is defined with initial conditions $F_k^{(S)} = 0$ for $0 \le k \le j-2$ and $F_{j-1}^{(S)} = 1$.

We claim that $\left(Q^{(S)}\right)^n$ equals

$$\begin{bmatrix} \sum_{\ell\in S} F^{(S)}_{n+j-1-\ell} & \sum_{2\le\ell\in S} F^{(S)}_{n+j-\ell} & \cdots & \sum_{j-1\le\ell\in S} F^{(S)}_{n+2j-3-\ell} & \sum_{j\le\ell\in S} F^{(S)}_{n+2j-2-\ell} \\ \sum_{\ell\in S} F^{(S)}_{n+j-2-\ell} & \sum_{2\le\ell\in S} F^{(S)}_{n+j-1-\ell} & \cdots & \sum_{j-1\le\ell\in S} F^{(S)}_{n+2j-4-\ell} & \sum_{j\le\ell\in S} F^{(S)}_{n+2j-3-\ell} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sum_{\ell\in S} F^{(S)}_{n+1-\ell} & \sum_{2\le\ell\in S} F^{(S)}_{n+2-\ell} & \cdots & \sum_{j-1\le\ell\in S} F^{(S)}_{n+j-1-\ell} & \sum_{j\le\ell\in S} F^{(S)}_{n+j-\ell} \\ \sum_{\ell\in S} F^{(S)}_{n-\ell} & \sum_{2\le\ell\in S} F^{(S)}_{n+1-\ell} & \cdots & \sum_{j-1\le\ell\in S} F^{(S)}_{n+j-2-\ell} & \sum_{j\le\ell\in S} F^{(S)}_{n+j-1-\ell} \end{bmatrix}$$

for any integer $n \ge j$. Indeed, let $q_{a,b}^{(n)}$ denote the $(a,b)^{\text{th}}$ entry of $(Q^{(S)})^n$. By [1, Theorem 3.2],

$$\sum_{n\ge 0} q_{a,b}^{(n)} z^n = \frac{z^{a-b}\left(1 - \sum_{\ell\in S}^{b-1} z^\ell\right)}{1 - \sum_{\ell\in S} z^\ell} \tag{37}$$

if $a \geq b$, and

$$\sum_{n\geq 0} q_{a,b}^{(n)} z^n \;=\; \frac{\sum_{b\leq \ell\in S} z^{a-b+\ell}}{1-\sum_{\ell\in S} z^{\ell}} \tag{38}$$

if $a < b$. Lemma 2.1 gives that

$$\sum_{n\geq 0} F_n^{(S)} z^k \;=\; \frac{z^{j-1}}{1-\sum_{\ell\in S} z^{\ell}}. \tag{39}$$

From these formulas, we see that

$$q_{a,b}^{(n)} \;=\; F_{n+j-1+b-a}^{(S)} - \sum_{\ell\in S}^{b-1} F_{n+j-1+b-a-\ell} \;=\; \sum_{b\leq \ell\in S} F_{n+j-1+b-a-\ell} \tag{40}$$

for any a, b, verifying the formula for $(Q^{(S)})^n$.

We can see that for an initial matrix A_k, the last entry, $\#A_{k,n}$, of the left-hand side vector in (35) is obtained by picking out the k^{th} entry (with entries numbered from right to left starting from 0) of the last row of $\left(Q^{(S)}\right)^n$. This gives that

$$\#A_{k,n} \;=\; \sum_{j-k\leq \ell\in S} F_{n+j-1-k-\ell}^{(S)}, \tag{41}$$

as desired. □

3. Stability

Now that we have Theorem 2.1, we need only give a few lemmas on asymptotics before finally proving the stability theorems.

Definition 3.1. *Let F be a complex function analytic at zero and let $\{a_n\}$ be the sequence such that*

$$F(z) = \sum_{n\geq 0} a_n z^n \tag{42}$$

for some neighborhood of zero. We define

$$[z^n]F(z) := a_n \tag{43}$$

for all $n \in \mathbb{N}$.

Lemma 3.1 (Asymptotics of a supercritical sequence, [2, page 294])**.** *Let G be a generating function with non-negative coefficients that is analytic at zero with $G(0) = 0$. Let r be the radius of convergence of G. Suppose*

(i) $1 < G(r) \leq \infty$, and
(ii) there does not exist an integer $d \geq 2$ and h analytic at zero such that $G(z) = h(z^d)$.

Let $F(z) = 1/(1 - G(z))$. Then there is some $|q| < 1$ such that

$$[z^n]F(z) \;=\; \frac{1}{\sigma G'(\sigma)} \cdot \sigma^{-n}(1 + O(q^n)), \tag{44}$$

for $\sigma \in (0, r)$ with $G(\sigma) = 1$.

Definition 3.2. *As in* [2, page 294]*, conditions (i) and (ii) in the above lemma are hereafter referred to as supercriticality and strong aperiodicity respectively.*

We now apply Lemma 3.1 to the generating function of $\{F_n^{(S)}\}$ to yield the asymptotics of $\{F_n^{(S)}\}$. This is not quite as straightforward as it may first seem, as these generating functions can fail the strong aperiodicity condition for certain $S \subseteq \mathbb{Z}^+$. Nevertheless, we work around this inconvenience without great difficulty by relating S-nacci sequences by their greatest common divisors.

Definition 3.3. *Given two functions $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{N} \to \mathbb{C}$, we write that*

$$f(n) \;\sim\; g(n) \tag{45}$$

if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1. \tag{46}$$

Definition 3.4. *Let $S \subseteq \mathbb{Z}^+$ be a finite set. We define*

$$S/m \;:=\; \{\ell/m : \ell \in S\}, \tag{47}$$

where $m \in \mathbb{Z}^+$.

Definition 3.5 (S-nacci coefficient)**.** *Let $S \subseteq \mathbb{Z}^+$ be a finite set such that $S \not\subseteq m\mathbb{Z}$ for any integer $m \geq 2$. Then the S-nacci coefficient, denoted c_S, is defined as the unique positive real number satisfying*

$$F_n^{(S)} \;\sim\; c_S \varphi_S^n. \tag{48}$$

This asymptotic relation is proven via a special case of the following lemma.

Lemma 3.2. *Let $S \subseteq \mathbb{Z}^+$ be finite, $j := \max(S)$, and m be the largest integer such that $S \subseteq m\mathbb{Z}$. Then for any $k \in \mathbb{N}$ such that $m \nmid k + 1$,*

$$F_k^{(S)} \;=\; 0. \tag{49}$$

We also have the relation

$$F_{nm-1}^{(S)} \;=\; F_{n-1}^{(S/m)} \;\sim\; c_{S/m} \varphi_{S/m}^{n-1} \tag{50}$$

for all $n \in \mathbb{Z}^+$.

Proof. By Lemma 2.1,

$$\sum_{k\geq 0} F_k^{(S)} z^k \;=\; \frac{z^{j-1}}{1 - \sum_{\ell \in S} z^\ell}. \tag{51}$$

Consequently,

$$F_{n+j-1}^{(S)} \;=\; [z^n]\frac{1}{1-\sum_{\ell\in S} z^\ell}. \tag{52}$$

We now split the proof into two cases: first, suppose that $S \not\subseteq m\mathbb{Z}$ for any integer $m \geq 2$. Let $G(z) := \sum_{\ell \in S} z^\ell$.

As G is a polynomial, its radius of convergence is infinite. As the leading coefficient of G is positive, the supercriticality condition is satisfied. To see that G satisfies the strong aperiodicity condition, note that for any $d \geq 2$, $h_d(z) := G(z^{1/d})$ necessarily has a fractional power term since $S \not\subseteq d\mathbb{Z}$; thus no such h_d can be analytic at zero. Therefore, applying Lemma 3.1 gives that

$$F_{n+j-1}^{(S)} \;=\; [z^n]\frac{1}{1-\sum_{\ell\in S} z^\ell} \;=\; \frac{1}{\sigma G'(\sigma)}\cdot\sigma^{-n}(1+O(q^n)), \tag{53}$$

where $\sigma \in \mathbb{R}^+$ satisfies $G(\sigma) = 1$ and $|q| < 1$. By the definition of φ_S, we have that $\sigma = 1/\varphi_S$, and so

$$F_{n+j-1}^{(S)} \;=\; \frac{\varphi_S}{G'(1/\varphi_S)}\cdot\varphi_S^n(1+O(q^n)), \tag{54}$$

and

$$F_n^{(S)} \;\sim\; \frac{\varphi_S^{2-j}}{G'(1/\varphi_S)}\varphi_S^n \;\equiv\; c_S\varphi_S^n. \tag{55}$$

Note that since $G'(x) > 1$ when $x > 0$, we have that $c_S \equiv \varphi_S^{2-j}/G'(1/\varphi_S) > 0$.

Now suppose $S \subseteq b\mathbb{Z}$ for some integer $b \geq 2$. Let m be the largest integer such that $S \subseteq m\mathbb{Z}$. We have

$$\sum_{k\geq 0} F_k^{(S/m)} z^k \;=\; \frac{z^{\max(S/m)-1}}{1-\sum_{\ell\in S/m} z^\ell} \;=\; \frac{z^{j/m-1}}{1-\sum_{\ell\in S/m} z^\ell}. \tag{56}$$

Thus,

$$\sum_{k\geq 0} F_k^{(S/m)} z^{mk} = \frac{z^{j-m}}{1-\sum_{\ell\in S/m} z^{m\ell}} = \frac{z^{1-m}\cdot z^{j-1}}{1-\sum_{\ell\in S} z^\ell} = \sum_{k\geq 0} F_k^{(S)} z^{k-m+1} = \sum_{k\geq 0} F_{k+m-1}^{(S)} z^k, \tag{57}$$

where the last equality in (57) follows from $m-1 \leq j-1$. Expanding gives

$$F_0^{(S/m)} + F_1^{(S/m)} z^m + F_2^{(S/m)} z^{2m} + \cdots = F_{m-1}^{(S)} + \cdots + F_{2m-1}^{(S)} z^m + \cdots + F_{3m-1}^{(S)} z^{2m} + \cdots,$$

which shows that

$$F_{nm-1}^{(S)} \;=\; F_{n-1}^{(S/m)} \tag{58}$$

for all $n \in \mathbb{Z}^+$, and $F_n^{(S)} = 0$ if $m \nmid n+1$. Lastly, note that there is no integer $N \geq 2$ such that $S/m \subseteq N\mathbb{Z}$; if such an N existed, then Nm would divide each element of S, contradicting the definition of m. So

$$F_{n-1}^{(S/m)} \;\sim\; c_{S/m}\varphi_{S/m}^{n-1}, \tag{59}$$

as desired. ☐

Lastly, we introduce the following lemma to allow a constant matrix (or scalar) to be appended to the recurrence relation.

Lemma 3.3. *Let $S \subseteq \mathbb{Z}^+$ be a finite set. Let m be the largest integer such that $S \subseteq m\mathbb{Z}$ and let $j := \max(S)$. Define the sequence of matrices $\{A_n\}$ by the recurrence relation*

$$A_n \;=\; B \prod_{k\in S} A_{n-k}, \tag{60}$$

with $A_0, A_1, \ldots, A_{j-1}$ as fixed square matrices of the same size, B as a scalar or square matrix, and the product taken in any desired order. Let $\#B_n$ denote the multiplicity of B in A_n. Then

$$\#B_n \;=\; \sum_{k=1}^{n} F_{n-k}^{(S)} \;=\; \frac{c_{S/m}}{\varphi_{S/m}-1}\varphi_{S/m}^{\lfloor n/m\rfloor} + R + O(\phi^{n/m}), \tag{61}$$

for some $R \in \mathbb{R}$, where ϕ is the maximal characteristic root[1] in modulus of $F_n^{(S/m)}$ such that $\phi \neq \varphi_{S/m}$. Moreover, $|\phi| < |\varphi_{S/m}|$.

Proof. Note that for $n \geq j$, $\#B_n$ is represented by the recurrence relation

$$\#B_n \;=\; 1 + \sum_{\ell\in S} \#B_{n-\ell} \tag{62}$$

with $\#B_0, \#B_1, \ldots, \#B_{j-1} = 0$. Using this relation, we determine the ordinary generating function of $\#B_n$. By similar computations as those given in the proof of Lemma 2.1,

$$\left(1 - \sum_{\ell\in S} z^\ell\right)\sum_{k\geq 0} \#B_k z^k \;=\; \sum_{k\geq j}\left(\#B_k - \sum_{\ell\in S}\#B_{k-\ell}\right) z^k \;=\; \sum_{k\geq j} z^k = \frac{z^j}{1-z}. \tag{63}$$

Thus,

$$\sum_{k\geq 0} \#B_k z^k \;=\; \frac{z^j}{(1-z)\left(1-\sum_{\ell\in S} z^\ell\right)} \;=\; \frac{z}{1-z}\cdot\frac{z^{j-1}}{1-\sum_{\ell\in S} z^\ell}. \tag{64}$$

So

$$\#B_n = \sum_{k=1}^{n} F_{n-k}^{(S)} = \sum_{k=0}^{n-1} F_{n-k-1}^{(S)} = \sum_{\substack{k=0\\ m\mid n-k}}^{n-1} F_{(n-k)/m-1}^{(S/m)} = \sum_{\substack{k=0\\ m\mid n-k}}^{n-1} c_{S/m}\varphi_{S/m}^{(n-k)/m-1} + R_0 + O(\phi^{n/m})$$

for some $R_0 \in \mathbb{R}$, where $\phi \neq \varphi_{S/m}$ is the maximal characteristic root in modulus of $F_n^{(S/m)}$. This last equivalence follows since $F_n^{(S/m)}$ is a linear combination of powers of its characteristic roots. Moreover, we have $|\phi| < |\varphi_{S/m}|$; otherwise,

$$F_n^{(S/m)} \;\not\sim\; c_{S/m}\varphi_{S/m}^n, \tag{65}$$

[1]A *characteristic root* of a sequence defined by a recurrence relation is a root of the characteristic polynomial of the recurrence relation.

contradicting Lemma 3.2. Now simplifying the sum, we find that

$$\begin{aligned}
\sum_{\substack{k=0\\ m\mid n-k}}^{n-1} c_{S/m}\varphi_{S/m}^{(n-k)/m-1} &= c_{S/m}\varphi_{S/m}^{n/m-1}\sum_{\substack{k=0\\ m\mid n-k}}^{n-1}\varphi_{S/m}^{-k/m} \\
&= c_{S/m}\varphi_{S/m}^{n/m-1}\sum_{k=0}^{(n-1-(n \bmod m))/m}\varphi_{S/m}^{-(mk+(n \bmod m))/m} \\
&= c_{S/m}\varphi_{S/m}^{(n-(n \bmod m))/m-1}\sum_{k=0}^{(n-1-(n \bmod m))/m}\varphi_{S/m}^{-k} \\
&= c_{S/m}\varphi_{S/m}^{(n-(n \bmod m))/m-1}\sum_{k=0}^{\lfloor n/m\rfloor-1}\varphi_{S/m}^{-k} \\
&= c_{S/m}\frac{\varphi_{S/m}^{n/m-\lfloor n/m\rfloor}\left(\varphi_{S/m}^{\lfloor n/m\rfloor}-1\right)}{\varphi_{S/m}^{(n \bmod m)/m}(\varphi_{S/m}-1)} \\
&= c_{S/m}\frac{\varphi_{S/m}^{\lfloor n/m\rfloor}-1}{\varphi_{S/m}-1}. \qquad (66)
\end{aligned}$$

Thus, for some $R \in \mathbb{R}$,

$$\#B_n \;=\; \sum_{k=1}^{n} F_{n-k}^{(S)} \;=\; \frac{c_{S/m}}{\varphi_{S/m}-1}\varphi_{S/m}^{\lfloor n/m\rfloor} + R + O(\phi^{n/m}). \qquad (67)$$

□

Theorem 3.1 (Stability Theorem)**.** *Let $S \subseteq \mathbb{Z}^+$ be finite such that $|S| \geq 2$. Let m be the largest integer such that $S \subseteq m\mathbb{Z}$ and let $j := \max(S)$. Define the sequence of matrices $\{A_n\}$ by the recurrence relation*

$$A_n \;=\; B\prod_{k\in S} A_{n-k}, \qquad (68)$$

with $A_0, A_1, \dots, A_{j-1}$ as fixed square matrices of the same size, B as a scalar or square matrix, and the product taken in any desired order. Let

$$\lambda \;=\; \begin{cases} 0 & ||B|| \geq 1 \\ m-1 & ||B|| < 1. \end{cases} \qquad (69)$$

Suppose

$$\frac{\varphi_{S/m}^{-(j+\lambda)/m+1}}{\varphi_{S/m}-1}\log||B|| + \sum_{k=0}^{j-1}\log||A_k||\sum_{j-k\leq \ell\in S}\varphi_{S/m}^{(-k-\ell)/m} \;<\; 0, \qquad (70)$$

where $||\cdot||$ is some submultiplicative matrix norm. Then $\{A_n\}$ converges to the zero matrix.

Proof. Let $j := \max(S)$. Let $\varepsilon > 0$ and for each $0 \le k \le j-1$, define

$$\varepsilon_k \ := \ \begin{cases} \varepsilon & ||A_k|| \ge 1 \\ -\varepsilon & ||A_k|| < 1. \end{cases} \tag{71}$$

Let

$$\varepsilon_B \ := \ \begin{cases} \varepsilon & ||B|| \ge 1 \\ -\varepsilon & ||B|| < 1. \end{cases} \tag{72}$$

Lastly, define

$$\lambda \ := \ \begin{cases} 0 & ||B|| \ge 1 \\ m-1 & ||B|| < 1. \end{cases} \tag{73}$$

Via Lemma 3.3, we deduce

$$\begin{aligned} ||B||^{\#B_n} \ &= \ O\left(||B||^{(c_{S/m}\varphi_{S/m}^{-\lambda/m}/(\varphi_{S/m}-1)+\varepsilon_B)\varphi_{S/m}^{n/m}}\right) \\ = O&\left(\left(\left(||B||^{\varphi_{S/m}^{-(j+\lambda)/m+1}/(\varphi_{S/m}-1)}\right)^{c_{S/m}}\left(\max(||B||,||B||^{-1})^{\varphi_{S/m}^{-j/m+1}}\right)^{\varepsilon}\right)^{\varphi_{S/m}^{(n+j)/m-1}}\right) \end{aligned} \tag{74}$$

since

$$\#B_n - (c_{S/m}\varphi_{S/m}^{-\lambda/m}/(\varphi_{S/m}-1)+\varepsilon_B)\varphi_{S/m}^{n/m} \tag{75}$$

tends to $+\infty$ if $||B|| < 1$ and $-\infty$ if $||B|| \ge 1$. This relation is used to justify the last statement of (76).

By the submultiplicity of the given matrix norm, Theorem 2.1, and Lemma 3.2, we have that for any $n \ge j$,

$$\begin{aligned} ||A_n|| \ &= \ \left\|B\prod_{k\in S}A_{n-k}\right\| \ \le \ ||B||^{\#B_n}\prod_{k=0}^{j-1}||A_k||^{\sum_{j-k\le\ell\in S}F_{n+j-1-k-\ell}^{(S)}} \\ &= \ ||B||^{\#B_n}\prod_{k=0}^{j-1}\exp\left(\log(||A_k||)\left(\sum_{\substack{j-k\le\ell\in S\\ m\mid n+j-k-\ell}}F_{(n+j-k-\ell)/m-1}^{(S/m)}\right)\right) \\ &= \ O\left(||B||^{\#B_n}\prod_{k=0}^{j-1}\exp\left(\log(||A_k||)\left(\sum_{\substack{j-k\le\ell\in S\\ m\mid n+j-k-\ell}}(c_{S/m}+\varepsilon_k)\varphi_{S/m}^{(n+j-k-\ell)/m-1}\right)\right)\right) \\ &= \ O\left(||B||^{\#B_n}\prod_{k=0}^{j-1}\exp\left(\log(||A_k||)\left(\sum_{j-k\le\ell\in S}(c_{S/m}+\varepsilon_k)\varphi_{S/m}^{(n+j-k-\ell)/m-1}\right)\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= O\left(||B||^{\#B_n}\prod_{k=0}^{j-1}\exp\left((c_{S/m}+\varepsilon_k)\varphi_{S/m}^{(n+j)/m-1}\log(||A_k||)\left(\sum_{j-k\le\ell\in S}\varphi_{S/m}^{(-k-\ell)/m}\right)\right)\right)\\
&= O\left(||B||^{\#B_n}\left(\prod_{\substack{k=0\\||A_k||\ge 1}}^{j-1}||A_k||^{\sum_{j-k\le\ell\in S}\varphi_{S/m}^{(-k-\ell)/m}}\right)^{(c_{S/m}+\varepsilon)\varphi_{S/m}^{(n+j)/m-1}}\right.\\
&\qquad\left.\times\left(\prod_{\substack{k=0\\||A_k||<1}}^{j-1}||A_k||^{\sum_{j-k\le\ell\in S}\varphi_{S/m}^{(-k-\ell)/m}}\right)^{(c_{S/m}-\varepsilon)\varphi_{S/m}^{(n+j)/m-1}}\right)\\
&= O\left(\left(\left(||B||^{\varphi_{S/m}^{-(j+\lambda)/m+1}/(\varphi_{S/m}-1)}\prod_{k=0}^{j-1}||A_k||^{\sum_{j-k\le\ell\in S}\varphi_{S/m}^{(-k-\ell)/m}}\right)^{c_{S/m}}\right.\right.\\
&\quad\left.\left.\times\left(\max(||B||,||B||^{-1})^{\varphi_{S/m}^{-j/m+1}}\frac{\prod_{k=0,\ ||A_k||\ge 1}^{j-1}||A_k||^{\sum_{j-k\le\ell\in S}\varphi_{S/m}^{(-k-\ell)/m}}}{\prod_{k=0,\ ||A_k||<1}^{j-1}||A_k||^{\sum_{j-k\le\ell\in S}\varphi_{S/m}^{(-k-\ell)/m}}}\right)^{\varepsilon}\ \right)^{\varphi_{S/m}^{(n+j)/m-1}}\right).
\end{aligned}
\tag{76}
$$

Note that

$$
\frac{\varphi_{S/m}^{-(j+\lambda)/m+1}}{\varphi_{S/m}-1}\log||B||+\sum_{k=0}^{j-1}\log||A_k||\sum_{j-k\le\ell\in S}\varphi_{S/m}^{(-k-\ell)/m}\ <\ 0 \tag{77}
$$

if and only if

$$
||B||^{\varphi_{S/m}^{-(j+\lambda)/m+1}/(\varphi_{S/m}-1)}\prod_{k=0}^{j-1}||A_k||^{\sum_{j-k\le\ell\in S}\varphi_{S/m}^{(-k-\ell)/m}}\ <\ 1. \tag{78}
$$

Let $\varepsilon > 0$ be sufficiently small such that

$$
\begin{aligned}
&\left(||B||^{\varphi_{S/m}^{-(j+\lambda)/m+1}/(\varphi_{S/m}-1)}\prod_{k=0}^{j-1}||A_k||^{\sum_{j-k\le\ell\in S}\varphi_{S/m}^{(-k-\ell)/m}}\right)^{c_{S/m}}\\
&\times\left(\max(||B||,||B||^{-1})^{\varphi_{S/m}^{-j/m+1}}\frac{\prod_{k=0,\ ||A_k||\ge 1}^{j-1}||A_k||^{\sum_{j-k\le\ell\in S}\varphi_{S/m}^{(-k-\ell)/m}}}{\prod_{k=0,\ ||A_k||<1}^{j-1}||A_k||^{\sum_{j-k\le\ell\in S}\varphi_{S/m}^{(-k-\ell)/m}}}\right)^{\varepsilon}\ <\ 1.
\end{aligned}
\tag{79}
$$

We know that such an ε exists since the first factor in (79) is less than 1. Thus for some $|q| < 1$,

$$
||A_n||\ =\ O\left(q^{\varphi_{S/m}^{n/m}}\right). \tag{80}
$$

Since $\varphi_{S/m} > 1$, it follows that $\lim_{n\to\infty} ||A_n|| = 0$. Thus, A_n converges to the zero matrix. □

Consider one of the simplest nontrivial matrix recurrence relations under matrix multiplication:

$$A_n \;=\; BA_{n-1}A_{n-2} \tag{81}$$

with B, A_0, and A_1 as fixed square matrices of the same size. Theorem 3.1 states that A_n is guaranteed to converge if for some submultiplicative matrix norm $||\cdot||$, we have

$$\begin{aligned}&\frac{\varphi^{-2+1}}{\varphi-1}\log||B|| + \sum_{k=0}^{2-1}\log||A_k|| \sum_{2-k\le\ell\in\{1,2\}} \varphi^{-k-\ell}\\ =\;& \log||B|| + \varphi^{-2}\log||A_0|| + (\varphi^{-2}+\varphi^{-3})\log||A_1|| < 0,\end{aligned} \tag{82}$$

where φ is the golden ratio. Equivalently,

$$||B||^{\varphi}||A_0||^{1/\varphi}||A_1|| \;<\; 1. \tag{83}$$

Using Theorem 2.1 and Lemma 3.3, we can verify this form of the condition without much difficulty. Indeed, with the aid of the identity

$$\#B_n \;=\; \sum_{k=1}^{n} F_{n-k} \;=\; \sum_{k=1}^{n-1} F_k \;=\; F_{n+1}-1 \tag{84}$$

as seen in [5, pg.4], observe that

$$\begin{aligned}||A_n|| \;&\le\; ||B||^{F_{n+1}-1}||A_0||^{F_{n-1}}||A_1||^{F_n} \;\sim\; ||B||^{-1+\varphi^{n+1}/\sqrt{5}}||A_0||^{\varphi^{n-1}/\sqrt{5}}||A_1||^{\varphi^n/\sqrt{5}}\\ &= ||B||^{-1}\left(||B||^{\varphi}||A_0||^{1/\varphi}||A_1||\right)^{\varphi^n/\sqrt{5}}.\end{aligned} \tag{85}$$

Even this result for such a simple recurrence is surprising, as it implies that the asymptotic contributions of B, A_1, and A_0 are quite different; the reader would be forgiven for expecting their asymptotic contribution to be equivalent.

Taking $S = \{1,2\}$ and $A_0, A_1, B \in \mathbb{C}$ (viewed as 1×1 matrices), the recurrence

$$A_n \;=\; B\prod_{k\in S} A_{n-k} \tag{86}$$

shows that Theorem 3.1 is sharp; this follows since

$$||A_n|| \;\sim\; ||B||^{-1}\left(||B||^{\varphi}||A_0||^{1/\varphi}||A_1||\right)^{\varphi^n/\sqrt{5}}, \tag{87}$$

which converges to zero if and only if

$$\log||B|| + \varphi^{-2}\log||A_0|| + (\varphi^{-2}+\varphi^{-3})\log||A_1|| \;<\; 0. \tag{88}$$

Since Kronecker products behave quite well under submultiplicative matrix norms, we can give an identical stability theorem for matrix recurrence relations defined over a Kronecker product.

Theorem 3.2 (Stability Theorem for the Kronecker Product)**.** *Let $S \subseteq \mathbb{Z}^+$ be finite such that $|S| \geq 2$. Let m be the largest integer such that $S \subseteq m\mathbb{Z}$ and let $j := \max(S)$. Define the sequence of matrices $\{A_n\}$ by the recurrence relation*

$$A_n \;=\; B \otimes \bigotimes_{k \in S} A_{n-k}, \tag{89}$$

with $B, A_0, A_1, \ldots, A_{j-1}$ fixed and the product taken in any desired order. Let

$$\lambda \;=\; \begin{cases} 0 & ||B|| \geq 1 \\ m-1 & ||B|| < 1. \end{cases} \tag{90}$$

Suppose

$$\frac{\varphi_{S/m}^{-(j+\lambda)/m+1}}{\varphi_{S/m}-1} \log ||B|| + \sum_{k=0}^{j-1} \log ||A_k|| \sum_{j-k \leq \ell \in S} \varphi_{S/m}^{(-k-\ell)/m} \;<\; 0, \tag{91}$$

where $||\cdot||$ is any submultiplicative matrix norm. Then A_n converges to zero in norm.[2]

Proof. By [4, Theorem 8], we have that $||X_1 \otimes X_2|| = ||X_1|| \cdot ||X_2||$. With Lemma 2.1 and Lemma 3.1, we have that for any $n \geq j$,

$$||A_n|| \;=\; \left\| B \bigotimes_{k \in S} A_{n-k} \right\| \;=\; ||B||^{\#B_n} \prod_{k=0}^{j_S-1} ||A_k||^{\sum_{j_S-k \leq \ell \in S} F^{(S)}_{n+j_S-1-k-\ell}}, \tag{92}$$

which converges to zero as established in the proof of Theorem 3.1. □

Definition 3.6. *For $j \in \mathbb{Z}^+$, let $[j] := \{1, 2, \ldots, j-1, j\}$.*

If the Kronecker product is consecutive (that is, the product is indexed over some $[j]$), then we can guarantee when the recurrence diverges to infinity in norm.

Theorem 3.3. *Let $j > 1$ be an integer. Define the sequence of matrices A_n by the recurrence relation*

$$A_n \;=\; B \otimes \bigotimes_{1 \leq k \leq j} A_{n-k}, \tag{93}$$

with $B, A_0, A_1, \cdots, A_{j-1}$ fixed and the Kronecker product taken in any desired order. Suppose that

$$||B|| \prod_{k=0}^{j-1} ||A_k||^{1-\varphi_j^{-k-1}} \;>\; 1, \tag{94}$$

where φ_j is the $[j]$-nacci constant and $||\cdot||$ is any submultiplicative matrix norm. Then A_n diverges to infinity in norm.

[2] Stating that A_n converges to *the* zero matrix is not quite precise as the size of the matrix A_n may be increasing with n.

Proof. Note that $[j] \not\subseteq m\mathbb{Z}$ for any integer $m \geq 2$. Let c_j and φ_j be the $[j]$-nacci coefficient and $[j]$-nacci constant respectively. By [4, Theorem 8], we have that $||X_1 \otimes X_2|| = ||X_1|| \cdot ||X_2||$. So for any $n \geq j$,

$$\begin{aligned} ||A_n|| &= \left\| B \otimes \bigotimes_{1\leq k\leq j} A_{n-k} \right\| = ||B||^{\#B_n} \prod_{k=0}^{j-1} ||A_k||^{\sum_{i=j-k}^{j} F^{([j])}_{n+j-1-k-i}} \\ &= ||B||^{\#B_n} \prod_{k=0}^{j-1} ||A_k||^{\sum_{i=0}^{k} F^{([j])}_{n-i-1}} = \Theta\left(||B||^{\frac{c_j\varphi_j^n}{\varphi_j-1}} \prod_{k=0}^{j-1} ||A_k||^{\sum_{i=0}^{k} c_j\varphi_j^{n-i-1}} \right) \\ &= \Theta\left(||B||^{\frac{c_j\varphi_j^n}{\varphi_j-1}} \prod_{k=0}^{j-1} ||A_k||^{c_j\varphi_j^n\left(\frac{\varphi_j^{k+1}-1}{\varphi_j^{k+1}(\varphi_j-1)}\right)} \right) = \Theta\left(\left(||B|| \prod_{k=0}^{j-1} ||A_k||^{1-\varphi_j^{-k-1}} \right)^{\frac{c_j\varphi_j^n}{\varphi_j-1}} \right). \end{aligned} \tag{95}$$

The second line of (95) follows since for some $\phi \in \mathbb{C} : |\phi| < 1$, set $\{q_k \in \mathbb{C} : |q_k| < 1, 0 \leq k \leq j-1\}$, and $R \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{||B||^{\#B_n} \prod_{k=0}^{j-1} ||A_k||^{\sum_{i=0}^{k} F^{([j])}_{n-i-1}}}{||B||^{\frac{c_j\varphi_j^n}{\varphi_j-1}} \prod_{k=0}^{j-1} ||A_k||^{\sum_{i=0}^{k} c_j\varphi_j^{n-i-1}}} = \lim_{n\to\infty} ||B||^{O(\phi^{n/m})+R} \prod_{k=0}^{j-1} ||A_k||^{O(q_k^n)} = ||B||^R. \tag{96}$$

By [6, pgs.747-748], the characteristic roots of $F^{([j])}_n$ are all less than one in modulus except φ_j; thus, we know that each $\sum_{i=0}^{k} F^{([j])}_{n-i-1} - c_j\varphi_j^{n-i-1}$ is $O(q_k^n)$ for some $|q_k| < 1$ since $F^{([j])}_n$ is a linear combination of powers of its characteristic roots. Furthermore, we deduce that

$$\#B_n - \frac{c_j}{\varphi_j - 1}\varphi_j^n = O(\phi^{n/m}) + R \tag{97}$$

for some $\phi \in \mathbb{C} : |\phi| < 1$ and $R \in \mathbb{R}$ as a consequence of Lemma 3.3.

Thus, since $\varphi_j > 1$, $c_j > 0$, and $||B|| \prod_{k=0}^{j-1} ||A_k||^{1-\varphi_j^{-k-1}} > 1$, it follows that $\lim_{n\to\infty} ||A_n|| = \infty$. Hence, A_n diverges to infinity. □

Note that the condition

$$||B|| \prod_{k=0}^{j-1} ||A_k||^{1-\varphi_j^{-k-1}} > 1 \tag{98}$$

is the dual of

$$\frac{\varphi_{S/m}^{-(j+\lambda)/m+1}}{\varphi_{S/m} - 1} \log ||B|| + \sum_{k=0}^{j-1} \log ||A_k|| \sum_{j-k\leq \ell\in S} \varphi_{S/m}^{(-k-\ell)/m} < 0 \tag{99}$$

since (99) is equivalent to

$$||B|| \prod_{k=0}^{j-1} ||A_k||^{1-\varphi_j^{-k-1}} \ < \ 1 \tag{100}$$

when $S = [j]$ (when $S = [j]$, note $m = 1$; consequently, $\lambda = 0$).

Taking $j = 2$, $A_0 \in \mathbb{C}$, $A_1 \in \mathbb{C}$, and $B \in \mathbb{C}$ (viewed as 1×1 matrices), the complex-valued recurrence

$$A_n \ = \ B \otimes \bigotimes_{k \in S} A_{n-k} \tag{101}$$

shows that Theorem 3.3 is sharp since

$$||A_n|| \sim ||B||^{-1} \left(||B||^{\varphi}||A_0||^{1/\varphi}||A_1||\right)^{\varphi^n/\sqrt{5}} = ||B||^{-1} \left(||B|| \ ||A_0||^{1-\varphi^{-1}}||A_1||^{1-\varphi^{-2}}\right)^{\varphi^{n+1}/\sqrt{5}}. \tag{102}$$

An immediate corollary of Theorem 3.1 and Theorem 3.3 gives that the stability of all recurrence relations of the form

$$a_n \ = \ b \prod_{k=1}^{j} a_{n-k} \tag{103}$$

is completely determined. We of course may apply Theorems 3.1 and 3.3 by simply considering $\{a_n\}$ as a sequence of complex 1×1 matrices.

4. Acknowledgements

We thank Joe Cooper for helpful conversations and for highlighting errors and typos in an earlier draft. This work was supported by the National Science Foundation grant *DMS-2241623*, Williams College, the Finnerty Fund, and the Winston Churchill Foundation. We also thank participants of the 21st International Fibonacci Conference for comments on an earlier version of this work.

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