A relationship between Mersenne primes and perfect numbers

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Abstract

The goal of this brief outreach article is to show how a characterization of Mersenne primes through the Euclid-Euler theorem can be stated and proved, emphasizing different aspects than the proofs usually seen. We give easy proofs of this perspective with simple tools. The concepts and results presented are illustrated with examples to make reading simple and entertaining.

Keywords: Mersenne numbers, Mersenne primes, perfect numbers, Euclid-Euler theorem, abundant numbers, abundancy.

Mathematics Subject Classification: 11A41,11P32.

We dedicate this work to those who have researched this beautiful subject, in particular to all who have participated in the GIMPS project, which has yielded so many new prime numbers, of sizes unimaginable a few years ago. In particular, we would like to highlight the contributions of Curtis Cooper, who, together with Steven Boone, using GIMPS software found the 43rd and 44th Mersenne primes. Cooper also found the 48th and 49th Mersenne primes, the 14th and 15th primes for GIMPS, the last in 2016; no one on the GIMPS project has found more primes than he. [1]

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1 Introduction

Special sequences of integers have fascinated people for millenia. Our goal in this outreach paper is to describe connections between two of the best known: prime numbers (more precisely, the subset of Mersenne primes) and perfect numbers. After introducing these two sequences we discuss the well-known Euclid-Euler theorem, the most important result connecting the two. Interestingly, this theorem provides a characterization of the Mersenne primes that can be proved independently and with tools other than those usually used in proofs of the Euclid-Euler theorem. Specifically, we write the sum of the proper divisors of the n^{th} Mersenne numbers as itself plus an excess term, which vanishes only when our number is prime.

We give two simple, direct proofs of this characterization in Section 4, which have the virtue of explicitly stating the most important terms leading to the proofs. The explicit expressions are easy-to-use formulas that we then illustrate with examples. By using this characterization, we show a simple proof of the Euclid-Euler theorem.

Without going into all the definitions and details, we can give a high level overview of what follows. This paper was inspired by the following easy to state problem: given the sum

$$T \;:=\; 1 \,+\, 2 \,+\, \cdots \,+\, 2^n - 1 \tag{1.1}$$

where n is an integer, does it equal the sum of the proper divisors of T? Answering this question as simply as possible is one of the objectives of this work, and provides an opportunity to talk about the connections between Mersenne primes and perfect numbers. In Section 4 we describe two ways of seeing when the answer is yes. Both are based on decomposing T into two summands, one is the sum of the proper divisors of T, and if the other term is zero obviously the answer to our question is yes.

Therefore, our approach to the problem is directed towards the explicit calculation of this second summand. We show two expressions for this term that clearly show it is zero if, and only if, $2^n - 1$ is prime. This nice problem is, at its core, the well-known Euclid-Euler theorem, which in addition tells us that any even number with the property of being equal to the sum of its proper divisors has to be necessarily of the form (1.1) with $2^n - 1$ prime. It is quite likely that the expressions we find are implicitly given in one of the many existing proofs of the Euclid-Euler Theorem, but we think that viewing them separately, in addition to simplifying the theorem, may suggest new ideas. Throughout this note we mark possible generalizations inspired by these investigations for the interested reader to pursue.

2 Mersenne numbers

Given a natural number $n > 1$, we denote the n^{th} *Mersenne number* by

$$M_n \;:=\; 2^n - 1.$$

It's well known that if n is a composite number then M_n is also composite. The proof is easy. If $n = a \cdot b$ with a, b natural numbers greater than 1, then

$$M_n \;=\; (2^a)^b - 1 \;=\; (2^a - 1)\left(1 + 2^a + 2^{2a} + \cdots + 2^{(b-1)a}\right)$$

is a factorization with each term exceeding 1. For example, if $n = 15 = a \cdot b$, with $a = 3$ and $b = 5$, then

$$\begin{aligned} M_{15} &= 2^{15} - 1 \;=\; \left(2^3\right)^5 - 1 \;=\; 32767 \\ &= \left(2^3 - 1\right)\left(1 + 2^{1\cdot 3} + 2^{2\cdot 3} + 2^{3\cdot 3} + 2^{(5-1)\cdot 3}\right) \;=\; 7 \cdot 4681. \end{aligned}$$

An equivalent formulation is that if M_n is prime then n is prime. For example, since $M_2 = 2^2 - 1 = 3$ is prime then $n = 2$ is prime. The question that quickly comes to mind is whether the other direction is true: if n is prime, then M_n is prime? To see that the answer is no observe that $n = 11$ is prime although $M_{11} = 2047$ is composite, indeed $M_{11} = 23 \cdot 89$. The numbers M_n that are prime are called *Mersenne primes*, after the 17th-century French monk Marin Mersenne (see Grosslight's work [Gro] on his life and deeds), a philosopher and mathematician who, in his *Cogitata Physico-Mathematica* [Mer] provided a list of primes of the form M_n and conjectured that the only prime numbers of this form with $n \leq 257$ are M_2, M_3, M_5, M_7, M_{13}, M_{17}, M_{19}, M_{31}, M_{67}, M_{127}, and M_{257}. Drake's paper [Dra] discusses why Mersenne made this conjecture. His list has, however, some errors: M_{67} and M_{257} are not primes, while M_{61}, M_{89}, and M_{107} are.[2] It's still an open question whether or not there are infinitely many such primes.

Mersenne primes, by their shape, have typically generated the largest prime numbers [GIMPS]. To date the largest known prime number is $M_{136279841}$, which has more than 41 million digits. This prime was found using the "Great Internet Mersenne Prime Search" (GIMPS) project with a probable prime (PRP) test and subsequently demonstrated its primality using the Lucas–Lehmer test [Leh1, Leh2, Luc], and is the 52nd known Mersenne prime number.

The Lucas-Lehmer test tells us when a Mersenne number M_n is prime, and works as follows.

Define the sequence $(s_i)_{i=0,1,\ldots}$ by

$$s_i \;=\; \begin{cases} 4 & \text{if } i = 0, \\ s_{i-1}^2 - 2 & \text{otherwise.} \end{cases}$$

Then, one has that M_n is prime if, and only if, M_n is a divisor of s_{n-2}. In terms of modular arithmetic:

$$M_n \text{ is prime} \quad \Leftrightarrow \quad s_{n-2} \equiv 0 \pmod{M_n}.$$

[2] It's worth noting that $M_{67} = 147\ 573\ 952\ 589\ 676\ 412\ 927$ or about 10^{20}, which factors as 193707721 times 761838257287; while trivial to do on today's computers this would be quite a challenge almost half a millenium ago!

Therefore, to check if M_n is prime, we can do the following. Consider the residual sequence

$$r_i = \begin{cases} 4 & \text{if } i = 0, \\ \operatorname{mod}\left(r_{i-1}^2 - 2, M_n\right) & \text{otherwise,} \end{cases}$$

where $\operatorname{mod}\left(r_{i-1}^2 - 2, M_n\right)$ is the remainder of dividing $r_{i-1}^2 - 2$ by M_n. Then

$$M_n \text{ is prime} \quad \Leftrightarrow \quad r_{n-2} = 0.$$

For example, to check if $M_5 = 31$ is a Mesenne prime or a composite number, we compute the terms of r_i up to $i = 5 - 2 = 3$:

$$\begin{aligned} r_0 &= 4, \\ r_1 &= \operatorname{mod}\left(4^2 - 2, 31\right) = 14, \\ r_2 &= \operatorname{mod}\left(14^2 - 2, 31\right) = 8, \\ r_3 &= \operatorname{mod}\left(8^2 - 2, 31\right) = 0. \end{aligned}$$

Since $r_3 = 0$, then M_5 is a Mersenne prime.

Let's see now if $M_{11} = 2047$ is a Mersenne prime or a composite number. We need to find r_9:

$$\begin{aligned} r_0 &= 4, & r_5 &= \operatorname{mod}\left(701^2 - 2, 2047\right) = 119, \\ r_1 &= \operatorname{mod}\left(4^2 - 2, 2047\right) = 14, & r_6 &= \operatorname{mod}\left(119^2 - 2, 2047\right) = 1877, \\ r_2 &= \operatorname{mod}\left(14^2 - 2, 2047\right) = 194, & r_7 &= \operatorname{mod}\left(1877^2 - 2, 2047\right) = 240, \\ r_3 &= \operatorname{mod}\left(194^2 - 2, 2047\right) = 788, & r_8 &= \operatorname{mod}\left(240^2 - 2, 2047\right) = 282, \\ r_4 &= \operatorname{mod}\left(788^2 - 2, 2047\right) = 701, & r_9 &= \operatorname{mod}\left(282^2 - 2, 2047\right) = 1736, \end{aligned}$$

therefore, since $r_9 \neq 0$, then M_{11} is a composite number.

As discussed in [GIMPS], GIMPS is a collaborative project of volunteers who search for Mersenne prime numbers. It was founded in 1996 by George Woltman and has discovered the last 18 Mersenne primes (see the last three rows of the list (2.1)). One can find a good overview of the history of GIMPS at
https://en.wikipedia.org/wiki/Great_Internet_Mersenne_Prime_Search
.

Below is the complete list of the indices n corresponding to the known Mersenne primes:

$$\begin{gathered} 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, \\ 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, \\ 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, \\ 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, \\ 25964951, 30402457, 32582657, 37156667, 42643801, 43112609, \\ 57885161, 74207281, 77232917, 82589933, 136279841. \end{gathered} \tag{2.1}$$

See https://t5k.org/mersenne/ for information about the discoverer of each of the above primes.

Lenstra, Pomerance and Wagstaff [Cal, Pom] conjectured that there are infinitely many Mersenne primes, and that as $x \to \infty$ the number of such primes at most x is approximately $e^{\gamma} \log_2 (\log_2 x)$, where $\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right) \approx 0.5772$ is the Euler–Mascheroni constant. For more on Mersenne numbers, see, for instance, [Bur, Chapter 11] or https://t5k.org/mersenne/ .

3 Perfect numbers

3.1 Definitions

Related to the Mersenne primes, though the connection should not be obvious, are the *perfect numbers*. These are numbers that equal the sum of their proper divisors. Remember that the *proper divisors* of n are all of its positive divisors except n itself. Thus the proper divisors of 6 are 1, 2, and 3, and those of 12 are 1, 2, 3, 4, and 6; hence 6 is a perfect number because

$$6 = 1 + 2 + 3,$$

while 12 is not as

$$12 \neq 1 + 2 + 3 + 4 + 6.$$

The next three perfect numbers after 6 are 28, 496 and 8128. Many centuries before Mersenne, Euclid, around 300 BC, had already discovered [Euc] that the first 4 perfect numbers were of the form $2^{n-1}(2^n - 1)$ with $n = 2$, 3, 5 and 7. The fact that for these cases $2^n - 1$ is prime led Euclid to prove that if $2^n - 1$ is prime then $2^{n-1}(2^n - 1)$ is a perfect number. Many centuries later, in the 18th century, Leonhard Euler proved that every *even* perfect number had to be of the form $2^{n-1}(2^n - 1)$ with $2^n - 1$ prime [Eul].

While even perfect numbers are now completely characterized, the situation is very different for possible odd perfect numbers. It's unknown if any exist, though if there is one we do know some properties it must have. The difficulty of achieving the conditions below is what leads many to conjecture that there are no odd perfect numbers.

An odd perfect number n must satisfy the following (see [Bur, page 234]):

- $n > 10^{300}$;
- n must be divisible at least by 9 distinct primes, the largest of which is greater than 108, and the next largest exceeds 104; and
- if 3 is not a divisor of n, then n must be divisible at least by 12 distinct primes.

Remark 3.1 *Note that* $2^{n-1}(2^n - 1)$ *is the sum of all natural numbers up to* $2^n - 1$*, so, roughly speaking, Euler proved that even perfect numbers, which by definition satisfy a summation property, are numbers that are obtained by some related summation. Should something similar happen for odd perfect numbers, if they exist? That is, if an odd perfect number exists, should it be obtained by some related summation? Observe that the necessary conditions given above are not of this type.*

3.2 Euclid-Euler Theorem

We extract the following from the results above.

Theorem 3.2 (Euclid-Euler theorem) *Let* $N > 1$ *be a natural number, then*

$$N \text{ is an even perfect number } \iff N = 2^{n-1}\left(2^n - 1\right) \text{ and } 2^n - 1 \text{ is prime.}$$

There are many proofs of the Euclid–Euler theorem, see for example Voight's work [Voi], where Euclid's implication is given in Theorem 7, while six proofs are given for Euler's assertion in Theorem 9. See also [Bur, Theorem 11.1]. Thus proving that there are infinitely many Mersenne primes is equivalent to proving that there are infinitely many even perfect numbers. For completeness we give a proof of this theorem at the end of the paper in Appendix A.

As a consequence of Theorem 3.2 and the list of primes n given in (2.1), we directly obtain the known 52 perfect numbers of the form $2^{n-1}\left(2^n - 1\right)$.

Remark 3.3 *The list suggests every even perfect number ends in a 6 or an 8. We leave a proof of this to the reader, and encourage them to explore the alternations between 6's and 8's. The latter is of course quite difficult as we have an incomplete list; note for investigations we only need to compute the numbers modulo 10.*

The sum of the first n integers, denoted T_n and called the n^{th} *Triangular number* as we can view the increasing summands as rows of a triangle, is well-known[3]

$$T_n := 1 + 2 + \cdots + n = \frac{n\,(n+1)}{2}.$$

Thus

$$T_{M_n} = \frac{M_n(M_n + 1)}{2} = \frac{(2^n - 1)2^n}{2} = 2^{n-1}\left(2^n - 1\right).$$

[3]The simplest proof is to write the sum in reverse order and add, obtaining $2T_n = (1 + n) + (2 + n - 1) + \cdots + (n + 1)$.

3.3 Summation Results

Let's start by visualizing what was discussed in Remark 3.1 about the summation relations for even perfect numbers. Thus T_{M_n} being perfect means that the sum of the proper divisors of T_{M_n} equals the sum of all the natural numbers up to M_n. For example, since $M_3 = 7$ is a Mersenne prime, then $T_{M_3} = T_7 = 28$ is perfect and, therefore,

$$\underbrace{1+2+3+4+5+6+7}_{\text{Sum of all natural numbers up to 7}} \quad = \quad T_7 \quad = \quad \underbrace{1+2+4+7+14}_{\text{Sum of proper divisors of } T_7} \quad .$$

Following up on what was said in the Remark 3.1, as our numbers T_n are defined by sums of integers, it is not surprising that they have certain summation properties. We record a few below that are necessary, but not sufficient, conditions for T_{M_n} to be a perfect number.

- $T_{M_n} \;=\; \sum_{k=0}^{M_{(n-1)/2}} (2k+1)^3.$

  For example

$$\begin{aligned} T_{M_5} &= 496 = \sum_{k=0}^{3} (2k+1)^3 = 1^3 + 3^3 + 5^3 + 7^3, \\ T_{M_{11}} &= 2096128 = \sum_{k=0}^{31} (2k+1)^3 \end{aligned} \tag{3.1}$$

  (recall $T_{M_{11}}$ is not perfect).

- $T_{M_n} \;=\; \sum_{k=n-1}^{2(n-1)} 2^k.$

  This implies that T_{M_n} in binary form has n ones followed by $n-1$ zeros. For example:

$$\begin{aligned} T_{M_5} = 496 &= \sum_{k=4}^{8} 2^k \\ &= 1\cdot 2^8 + 1\cdot 2^7 + 1\cdot 2^6 + 1\cdot 2^5 + 1\cdot 2^4 \\ &\qquad + 0\cdot 2^3 + 0\cdot 2^2 + 0\cdot 2^1 + 0\cdot 2^0 \\ &= 111110000_2 \\ T_{M_{11}} = 2096128 &= \sum_{k=10}^{20} 2^k \\ &= 111111111110000000000_2 \end{aligned} \tag{3.2}$$

  (recall again $T_{M_{11}}$ is not perfect).

Remark 3.4 *As we said before, the properties we just saw are necessary conditions for an even number to be perfect, but they are not sufficient. Note that if a sufficient condition of the previous type were found, we would have a constructive proof of the existence of infinitely many even perfect numbers, and therefore, of infinitely many Mersenne primes.*

Interestingly, another formulation of perfect numbers are those where the sum of the reciprocals of the proper divisors equals 1. To see this, note there is a 1-to-1 pairing of the divisors d and n/d. Thus summing $1/d$ over the proper divisors is the same as summing $1/(n/d) = d/n$, and clearly the second sums to 1 by the definition of being perfect (n is perfect if the sum of its proper divisors is n). For more on these and other curious properties of even perfect numbers, see https://en.wikipedia.org/wiki/Perfect_number and the references there, as well as [Bur, Section 11.2].

Remark 3.5 *This result suggests a family of generalizations of being perfect; what can we say about the numbers whose sum of the reciprocals of the proper divisors equals r for various rational r?*

4 Characterization of Mersenne primes

4.1 Preliminaries

Our goal is to extract a nice condition for when M_n is prime. Our starting point is the Euclid-Euler theorem, which provides the following characterization.

Proposition 4.1 (Characterization of Mersenne primes) *Let $n > 1$ be a natural number, then*

$$M_n \text{ is a Mersenne prime number} \iff T_{M_n} \text{ is a perfect number.} \tag{4.1}$$

Remark 4.2 *Proposition 4.1 differs slightly from the Euclid-Euler theorem we stated earlier, because it does not imply that if a perfect number is even then it has to be of the form T_{M_n} with M_n prime. This result can be stated independently and also proved independently, without resorting to the Euclid-Euler theorem, and is the motivation for writing this outreach note.*

We start by setting notation. We denote the *sum of the proper divisors (or aliquot sum) of* $m \in \mathbb{N}$ by $s(m)$. Below we give two proofs of Proposition 4.1. These are based on an explicit decomposition of the sum of proper divisors of T_{M_n} as

$$s\left(T_{M_n}\right) \;=\; T_{M_n} + e \tag{4.2}$$

where e will be described fully later, and is the quantity to determine. Our number M_n is prime and thus T_{M_n} is perfect if and only if $e = 0$.

The main tools of the proofs will be Lemma 4.3 and Corollary 4.6, where the proper divisors of M_n and T_{M_n} are computed. This determination allows us in the first proof to calculate $s\left(T_{M_n}\right)$ as a function of $s\left(M_n\right)$. In the second proof, the key will be to identify the proper divisors of T_{M_n} whose sum is exactly T_{M_n}, i.e., those divisors which ensure that T_{M_n} is perfect.

4.2 Needed Results

The following lemma provides useful expressions for the positive divisors of a natural $m > 1$ and the sum of the positive divisors of m, denoted by $\sigma\left(m\right)$ (see, for example, [Bur, Theorem 6.1 and Theorem 6.2 (b)]). Note that $\sigma(m)$ includes all positive divisors and not just the proper ones; thus m is perfect if and only if $\sigma(m) = 2m$,

Lemma 4.3 (Positive divisors of a natural number) *Let $m > 1$ be a natural number with prime factorization $m := \prod_{i=1}^{r} p_i^{k_i}$. Then the positive divisors of m are*

$$\prod_{i=1}^{r} p_i^{a_i}, \text{ where } 0 \leq a_i \leq k_i, \tag{4.3}$$

and the sum of the positive divisors is given by

$$\sigma\left(m\right) \;=\; \prod_{i=1}^{r}\sum_{j=0}^{k_i} p_i^j \;=\; \prod_{i=1}^{r}\frac{p_i^{k_i+1}-1}{p_i-1}. \tag{4.4}$$

Remark 4.4 *The second equality in* (4.4) *follows immediately from the finite geometric series formula:*

$$\sum_{j=0}^{k} p^j \;=\; \frac{p^{k+1}-1}{p-1} \quad \text{for } p \;\neq\; 0.$$

Remark 4.5 (Proper divisors of a natural number) *Note that the proper divisors of m are $\prod_{i=1}^{r} p_i^{a_i}$, where $0 \leq a_i \leq k_i$, except for the case when $a_i = k_i$ for all $i = 1, \ldots, r$, since the corresponding divisor is $\prod_{i=1}^{r} p_i^{k_i} = m$.*

The following corollary specifies the proper divisors of T_{M_n} and the sum of the corresponding divisors.

Corollary 4.6 (Proper divisors of the triangular number T_{M_n}) *Let $n > 1$ be a natural number and let $M_n := \prod_{i=1}^{r} p_i^{k_i}$ be the prime factorization of M_n. Denote $p_0 := 2$ and $k_0 := n-1$. The proper divisors of T_{M_n} are $\prod_{i=0}^{r} p_i^{a_i}$, where*

$0 \le a_i \le k_i$, *except the case when* $a_i = k_i$ *for all* $i = 0, \ldots, r$, *since the corresponding divisor is* T_{M_n}. *Moreover, the sum of the proper divisors of* T_{M_n} *is:*

$$\begin{aligned}
s(T_{M_n}) &= \prod_{i=0}^{r} \frac{p_i^{k_i+1} - 1}{p_i - 1} - T_{M_n} && (4.5)\\
&= M_n \sigma(M_n) - T_{M_n} && (4.6)\\
&= T_{M_n} + M_n (s(M_n) - 1)\,. && (4.7)
\end{aligned}$$

Proof. For (4.5) and (4.6) just observe that $T_{M_n} = 2^{n-1} M_n$ with M_n odd and take into acount Lemma 4.3 and the subsequent remarks. For (4.7), take (4.4) for $m = M_n$ and follow from (4.6) as

$$\begin{aligned}
s(T_{M_n}) &= M_n \sigma(M_n) - T_{M_n} = M_n (s(M_n) - M_n) - T_{M_n} =\\
&= M_n (s(M_n) - 1 + 1 + M_n) - T_{M_n}\\
&= M_n (s(M_n) - 1) + M_n (1 + M_n) - T_{M_n}\\
&= M_n (s(M_n) - 1) + 2T_{M_n} - T_{M_n} = T_{M_n} + M_n (s(M_n) - 1)\,.
\end{aligned}$$

□

4.3 Example

To illustrate the arguments above, let's compute the proper divisors of M_6, T_{M_6}, and the corresponding sums.

As

$$M_6 = 2^6 - 1 = 63 = p_1^{k_1} p_2^{k_2}$$

where

$$p_1 = 3, \quad k_1 = 2, \quad p_2 = 7, \quad k_2 = 1,$$

the proper divisors of M_6 are of the form $3^{j_1} 7^{j_2}$, arising from the pairs

$$(j_1, j_2) \in \{0, 1, \underbrace{2}_{k_1}\} \times \{0, \underbrace{1}_{k_2}\},$$

except $M_6 = 3^2 7^1$, corresponding to the pair $(j_1, j_2) = (k_1, k_2) = (2, 1)$, which is not included.

In Figure 1 the proper divisors of M_6 are in the row corresponding to $j_0 = 0$, except the last term that is $M_6 = 63$. The proper divisors of T_{M_6} are of the form $2^j 3^{j_1} 7^{j_2}$ for all the triples

$$(j_0, j_1, j_2) \in \left\{ 0, 1, 2, 3, 4, \underbrace{5}_{k_0 = n-1} \right\} \times \{0, 1, \underbrace{2}_{k_1}\} \times \{0, \underbrace{1}_{k_2}\},$$

except $T_{M_6} = 2^5 3^2 7^1$, corresponding to the triple $(j_0, j_1, j_2) = (k_0, k_1, k_2) = (5, 2, 1)$. In Figure 1, the proper divisors of T_{M_6} are all the values of the table except the last term of the row $j_0 = 5$, that is $T_{M_6} = 2016$.

j_0	2^{j_0}	$2^{j_0}7$	$2^{j_0}3$	$2^{j_0}21$	$2^{j_0}9$	$2^{j_0}63$
0	1	7	3	21	9	63
1	2	14	6	42	18	126
2	4	28	12	84	36	252
3	8	56	24	168	72	504
4	16	112	48	336	144	1008
5	32	224	96	672	288	2016

Figure 1: M_6, T_{M_6}, and their proper divisors.

The sum of the proper divisors of M_6 and T_{M_6} can be directly obtained by adding the corresponding values of Figure 1, and the formulas (4.4) for $m = M_6$ and (4.5) -(4.7) apply as follows:

$$\begin{aligned}
\sigma(M_6) &= \prod_{i=1}^{2} \frac{p_i^{k_i+1}-1}{p_i-1} = \frac{3^{2+1}-1}{3-1}\frac{7^{1+1}-1}{2-1} - 63 = 13 \cdot 8 = 104, \\
s(M_6) &= \sigma(M_6) - M_6 = 104 - 63 = 41, \\
s(T_{M_6}) &= M_6\sigma(M_6) - T_{M_6} = 63 \cdot 104 - 2016 = 4536.
\end{aligned}$$

Observe that T_{M_6} is obtained from Figure 1 by adding the proper divisors of the second and the last columns, corresponding to the extreme cases $(j_1, j_2) = (0,0)$ and $(j_1, j_2) = (2,1)$. The sum of the remaining values of the table corresponds to $M_6\,(s(M_6) - 1)$ as a consequence of (4.7):

$$M_6\,(s(M_6) - 1) = 63\,(41 - 1) = 2520.$$

The fact that the proper divisors of T_{M_n} whose sum is T_{M_n} itself are in the second and last columns of Figure 1 is in fact what happens in general, and leads us to another decomposition of T_{M_n}, given in Proposition 4.7.

We encourage the interested reader to reproduce the previous example for other values of n in order to see the simplicity of the calculation method.

4.4 Proofs

The main use of the decomposition (4.7) in Corollary 4.6 is to prove easily the characterization (4.1) stated in Proposition 4.1.

Proof 1 of Proposition 4.1. Observe that $s(M_n) = 1$ means that the unique proper divisor of M_n is 1, that is, M_n is prime. From equality (4.7) of Corollary 4.6 we have

$$s(T_{M_n}) = T_{M_n} + M_n\,(s(M_n) - 1)\,,$$

and thus

$$T_{M_n} \text{ is perfect } \Leftrightarrow\ s(T_{M_n})\ =\ T_{M_n}\ \Leftrightarrow\ s(M_n)\ =\ 1\ \Leftrightarrow\ M_n \text{ is prime},$$

which proves Proposition 4.1. □

Proposition 4.7 *The proper divisors of T_{M_n} in Corollary 4.6 corresponding to the extreme cases $a_i = 0$ and $a_i = k_i$ for all $i = 1, \dots, r$ are*

$$p_0^j \prod_{i=1}^{r} p_i^0\ =\ 2^j \prod_{i=1}^{r} p_i^0\ =\ 2^j \text{ for all } j = 0, \dots, n-1 (= k_0), \tag{4.8}$$

and

$$p_0^j \prod_{i=1}^{r} p_i^{k_i}\ =\ 2^j M_n \text{ for all } j = 0, \dots, n-2, \text{ (recall that } j = n-1 \text{ gives } T_{M_n}), \tag{4.9}$$

respectively, have the particular property that their sum is T_{M_n}.

Proof. Indeed,

$$\begin{aligned}\sum_{j=0}^{n-1} 2^j + \sum_{j=0}^{n-2} 2^j M_n \quad &= \quad 2^n - 1 + M_n\left(2^{n-1} - 1\right)\ =\ M_n + \frac{M_n}{2}\left(M_n - 1\right) \\ &= \quad M_n\left(1 + \frac{M_n - 1}{2}\right)\ =\ \frac{M_n\left(M_n + 1\right)}{2}\ =\ T_{M_n}.\end{aligned}$$

□

Corollary 4.8 *We have the following decomposition of the sum of proper divisors of T_{M_n}:*

$$s(T_{M_n})\ =\ T_{M_n} + \sum_{d \in \Omega} d, \tag{4.10}$$

where Ω is the set of proper divisors of T_{M_n} which are not in (4.8) and (4.9).

Proof. This is a direct consequence of Proposition 4.7. □

Example 4.9 (Example from Section4.3 revisited) *As we already commented in Section 4.3, observe that proper divisors of (4.8) and (4.9) correspond to those of the second and the last columns of Figure 1 , except the last element in the last column which is T_{M_n} itself.*

As a consequence of the previous corollary, we can provide another proof of the characterization (4.1) stated in Proposition 4.1.

Proof 2 of Proposition 4.1. Since $1 \notin \Omega$ because 1 is in (4.8), then, according to (4.10) one has

$$M_n \text{ is prime} \Longleftrightarrow \Omega \;=\; \varnothing \Longleftrightarrow \sum_{d\in\Omega} d \;=\; 0 \Longleftrightarrow s(T_{M_n}) \;=\; T_{M_n} \Longleftrightarrow T_{M_n} \text{ is perfect.}$$

□

Remark 4.10 *The extreme cases indicated in Proposition 4.7 are the key for Proof 2. Could this idea underlie perfect numbers in general?*

Remark 4.11 *Formulas (4.7) and (4.10) of Corollaries 4.6 and 4.8, respectively, are valid for any $n \in \mathbb{N}$, not necessarily for n prime.*

Corollary 4.12 *As an immediate consequence of expressions (4.7) or (4.10), one has that all numbers of the form T_{M_n} are either perfect numbers or abundant numbers. Remember that a number is called* abundant or excessive *if it is a positive integer for which the sum of its proper divisors is greater than the number. In such a case, from (4.7) and (4.10), one has the following formulas for the* abundancy*:*

$$s(T_{M_n}) - T_{M_n} \;=\; M_n\,(s(M_n) - 1) \;=\; \sum_{d\in\Omega} d\;.$$

Note that a general formula for calculating the abundance of an abundant number has to generalize the formulas in the previous corollary.

A A Standard Proof of the Euclid-Euler Theorem

For completeness we conclude with one of the standard proofs of the Euclid-Euler theorem. With the notation used above we may state the theorem as follows.

Theorem A.1 *Let $N > 1$ be a natural number, then*

$$N \text{ is an even perfect number} \quad\Longleftrightarrow\quad N \;=\; T_{M_n} \text{ and } M_n \text{ is prime.}$$

Proof. The Characterization (4.1) given in Proposition 4.1 gives us the implication from right to left, since if $N = T_{M_n}$ for some natural $n > 1$, and M_n is prime, then N is perfect and $N = 2^{n-1}M_n$ is even.

It only remains to show the implication from left to right. We rewrite the proof given in [Bur, Theorem 11.1] in our notation. Assume that N is an even perfect number. We can write

$$N \;=\; 2^{n-1}M \text{ with } n > 1 \text{ and } M \text{ odd.} \tag{A.1}$$

Lemma 4.3 then tells us that the sum of positive divisors of N is

$$\sigma(N) = \frac{2^{n-1+1}-1}{2-1}\sigma(M) = M_n\sigma(M). \tag{A.2}$$

Since N is perfect, then

$$\sigma(N) = s(N) + N = 2N. \tag{A.3}$$

From (A.1) and (A.3) we have

$$\sigma(N) = 2^n M. \tag{A.4}$$

Using (A.2) and (A.4) yields

$$M_n\sigma(M) = 2^n M. \tag{A.5}$$

Since M_n is odd and $\sigma(M)$ is a natural number, from (A.5) we find M_n is a divisor of M, and we can write

$$M = M_n K \text{ where } K \text{ is an integer.} \tag{A.6}$$

Replacing (A.6) in (A.5) and simplifying gives

$$\sigma(M) = 2^n K, \tag{A.7}$$

which entails that K is a divisor of $\sigma(M)$.

As $\sigma(M)$ is the sum of the positive divisors of M,

$$\sigma(M) \geq K + M, \tag{A.8}$$

and replacing now (A.6) in (A.8) and taking into account (A.7) we arrive at

$$\sigma(M) \geq K + M \geq K + M_n K = (M_n+1)K = 2^n K = \sigma(M), \tag{A.9}$$

which implies

$$\sigma(M) = K + M.$$

In other words, if the unique positive divisors of M are M and $K < M$, then necessarily $K = 1$ and M is prime, and from (A.6) we find

$$M_n = M \text{ prime.} \tag{A.10}$$

We only need to replace (A.10) in (A.1) to obtain

$$N = 2^{n-1}M_n = T_{M_n}, \text{ with } M_n \text{ prime,}$$

which is what we wanted to prove. □

As a final remark, it would be interesting to obtain a short proof, along the ones given in this paper, that any even perfect number must be of the form T_{M_n}. Different proofs may open up new approaches for exploration.

References

[Bur] D. M. Burton, *Elementary Number Theory*, Tata McGraw-Hill Publishing Company Limited, New York, 2006.

[Cal] C. K. Caldwell, *Heuristics: Deriving the Wagsta_Mersenne Conjecture*, The Prime Pages, retrieved on 2014-05-11.

[Dra] S. Drake, *The rule behind "Mersenne's numbers"*, Physis-Riv. Internaz. Storia Sci. **13** (1971), 421–424.

[Euc] Euclid, *The Thirteen Books of The Elements*, translated with introduction and commentary by Sir Thomas L. Heath, Vol. 2 (Books III–IX) (2nd ed.), Dover, 421–426, 1955.

[Eul] L. Euler, *De numeris amicabilibus*, Opera Omnia, **5** (1849), Series 1, 353–365.

[Gro] J. Grosslight, *Small Skills, Big Networks: Marin Mersenne as Mathematical Intelligencer*, Hist. Sci. **51** (2013), 337–374.

[GIMPS] Great Internet Mersenne Prime Search, http://www.mersenne.org, 1998.

[Leh1] D. H. Lehmer, *An Extended Theory of Lucas' Functions*, Annals of Mathematics **31** (1930), 419–448.

[Leh2] D. H. Lehmer, *On Lucas' test for the primality of Mersenne's numbers*, J. of the London Math. Soc. **10** (1935), 162–165.

[Luc] E. Lucas, *Théorie des fonctions numériques simplement périodiques*, American Journal of Mathematics **1** (1878), 289–321.

[Mer] M. Mersenne (1644), Cota*Physico Mathematica*, Parisiis, Praefatio Generalis No. **9.** (1935)

[Pom] C. Pomerance, *Recent developments in primality testing*, Math. Intelligencer, **3** (1980/1981), no. 3, 97-105.

[Voi] J. Voight, *Perfect numbers: an elementary introduction* (1998), https://math.dartmouth.edu/jvoight/notes/perfelem.pdf.