VARIATION IN THE NUMBER OF POINTS ON ELLIPTIC CURVES AND APPLICATIONS TO EXCESS RANK

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ABSTRACT. Michel proved that for a one-parameter family of elliptic curves over $\mathbb{Q}(T)$ with non-constant $j(T)$ that the second moment of the number of solutions modulo p is $p^2 + O(p^{3/2})$. We show this bound is sharp by studying $y^2 = x^3 + Tx^2 + 1$. Lower order terms for such moments in a family are related to lower order terms in the n-level densities of Katz and Sarnak, which describe the behavior of the zeros near the central point of the associated L-functions. We conclude by investigating similar families and show how the lower order terms in the second moment may affect the expected bounds for the average rank of families in numerical investigations.

1. INTRODUCTION

Let $\mathcal{E}$ be a one-parameter family of elliptic curves (equivalently, an elliptic surface) over $\mathbb{Q}(T)$:

$$y^2 + a_1(T)xy + a_3(T)y \ = \ x^3 + a_2(T)x^2 + a_4(T)x + a_6(T), \quad a_i(T) \in \mathbb{Z}[T]. \quad (1.1)$$

For each integer t we have an elliptic curve E_t over $\mathbb{Q}$, with $N_t(p)$ the number of solutions modulo p. Set $a_t(p) = p - N_t(p)$. If $a_1(T) = a_3(T) = 0$ we have

$$a_t(p) \ = \ -\sum_{x \bmod p} \left(\frac{x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)}{p} \right). \quad (1.2)$$

We are interested in evaluating the second moment for the family:

$$A_{2,\mathcal{E}}(p) \ := \ \sum_{t \bmod p} a_t^2(p). \quad (1.3)$$

For one-parameter families of elliptic curves with $j(T)$ non-constant, Michel [Mic] proves $A_{2,\mathcal{E}}(p) = p^2 + O(p^{3/2})$ by using the Lefschetz-Groethendieck trace formula. We show his result is sharp by constructing a family where the second moment has a term of size $p^{3/2}$.

Moments of the number of solutions modulo p provide enormous amounts of information about the family. Rosen and Silverman [RoSi] prove a conjecture of Nagao (unconditionally for rational elliptic surfaces, conditional on Tate's conjecture in general) that the first moment is related to the rank of the family over $\mathbb{Q}(T)$, denoted rank $\mathcal{E}(\mathbb{Q}(T))$:

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p \le X} \frac{\log p}{p} \sum_{t \bmod p} a_t(p) \ = \ -\text{rank } \mathcal{E}(\mathbb{Q}(T)). \quad (1.4)$$

In [Mil1, ALM] it is shown how to construct families with moderate rank by choosing the $a_i(T)$ so that the first moment is computable and large.

Another application is in the connections between number theory and random matrix theory [KaSa1, KaSa2]. In showing the behavior of the low lying zeros (zeros near the central point) of L-functions of a family of elliptic curves agrees

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with that of eigenvalues near 1 of orthogonal groups, the only needed inputs are the first and second moment of the number of solutions modulo p; to first order all families of elliptic curves with rank r over $\mathbb{Q}(T)$ agree with the same random matrix ensemble (see [Mil2], though a similar result with a global rather than local rescaling of the zeros is implicit in [Sil2]). An analysis of the lower order terms in the first and second moments leads to breaking this universality; i.e., seeing lower-order family dependent behavior in the low lying zeros. See [Mil1, Yo1] for more details on family dependent behavior. One application of these correction terms is a refinement on predicting the number of curves in a family with rank above the family rank. While these corrections vanish in the limit of large conductor, they lead to slight modifications of excess rank bounds for conductors in the range accessible by computers.

In §2 we determine the second moment of $a_t(p)$ for a specific family, showing Michel's result is sharp. In §3 we analyze how the lower order terms in Michel's theorem are related to bounds for the average rank of the family.

2. The Second Moment for $y^2 = x^3 + Tx^2 + 1$

We may expand the Legendre symbol $\left(\frac{x}{p}\right)$ in (1.2) by

$$\left(\frac{x}{p}\right) \ = \ \frac{1}{G_p}\sum_{c=1}^{p-1}\left(\frac{c}{p}\right)\mathbf{e}_p\left(cx\right), \quad \mathbf{e}_p\left(x\right) \ = \ e^{2\pi \mathrm{i} x/p}. \tag{2.5}$$

Here $G_p = \sum_{a \bmod p}\left(\frac{a}{p}\right)\mathbf{e}_p\left(a\right)$ is the Gauss sum, which equals $\sqrt{p}$ for $p \equiv 1 \bmod 4$ and $\mathrm{i}\sqrt{p}$ for $p \equiv 3 \bmod 4$. See, for example, [BEW]. Using Gauss sums to evaluate Legendre sums is a common technique; we sketch an alternate approach which avoids Gauss sums in Remark 2.2.

For the family $\mathcal{E} : y^2 = x^3 + Tx^2 + 1$, $j(T) = -\frac{256T^6}{4T^3+27}$ and thus by Michel's Theorem $A_{2,\mathcal{E}}(p) = p^2 + O(p^{3/2})$. We determine an exact formula for $A_{2,\mathcal{E}}(p)$:

Theorem 2.1. *For the one-parameter family $\mathcal{E} : y^2 = x^3 + Tx^2 + 1$ over $\mathbb{Q}(T)$, for $p > 2$ the second moment of $a_t(p)$ is*

$$A_{2,\mathcal{E}}(p) \ = \ \sum_{t \bmod p} a_t(p)^2 \ = \ p^2 - n_{3,2,p}p - 1 + p\sum_{x \bmod p}\left(\frac{4x^3+1}{p}\right), \tag{2.6}$$

where $n_{3,2,p}$ denotes the number of cube roots of 2 modulo p. For any $[a,b] \subset [-2,2]$ there are infinitely many primes $p \equiv 1 \bmod 3$ such that

$$A_{2,\mathcal{E}}(p) - \left(p^2 - n_{3,2,p}p - 1\right) \ \in \ [a\cdot p^{3/2}, b \cdot p^{3/2}]. \tag{2.7}$$

Proof. Combining (1.2) and (2.5) yields

$$\begin{aligned} A_{2,\mathcal{E}}(p) \ &= \ \sum_{t \bmod p}\ \sum_{x \bmod p}\ \sum_{y \bmod p}\left(\frac{x^3+1+x^2t}{p}\right)\left(\frac{y^3+1+y^2t}{p}\right) \\ &= \ \sum_{x,y \bmod p}\ \sum_{c,d=1}^{p-1}\frac{1}{p}\left(\frac{cd}{p}\right)\mathbf{e}_p\left(c(x^3+1)-d(y^3+1)\right)\sum_{t \bmod p}\mathbf{e}_p\left((cx^2-dy^2)t\right); \end{aligned} \tag{2.8}$$

above we used the complex conjugate of (2.5) in expanding $\left(\frac{y^3+1+y^2t}{p}\right)$. The two Gauss sum expansions give $\frac{1}{G_p\overline{G_p}} = \frac{1}{p}$. It will be convenient to set $g(x,y) = (x-y)(x^2y^2-(x+y))$.

Note c and d are invertible modulo p in (2.8). If the numerator in the t-exponential is non-zero, the t-sum vanishes. Thus it suffices to study (2.8) for $cx^2 \equiv dy^2 \bmod p$. If exactly one of x and y vanishes, then $cx^2 \not\equiv dy^2 \bmod p$. Hence x and y are both zero or non-zero. If both are zero the t-sum gives p, the c-sum gives G_p, the d-sum gives $\overline{G}_p$, for a total contribution of p. If x and y are non-zero then we must have $d \equiv cx^2y^{-2} \bmod p$. The t-sum gives p. Thus (2.8) is

$$\begin{aligned} A_{2,\mathcal{E}}(p) &= p + \sum_{x,y=1}^{p-1} \sum_{c=1}^{p-1} \frac{1}{p} \left(\frac{x^2y^2}{p}\right) \mathbf{e}_p\left(cy^{-2}(x^3y^2 + y^2 - x^2y^3 - x^2)\right) p \\ &= p + \sum_{x,y=1}^{p-1} \sum_{c=1}^{p-1} \mathbf{e}_p\left(cy^{-2}(x-y)(x^2y^2 - (x+y))\right) \\ &= p + \sum_{x,y=1}^{p-1} \sum_{c=0}^{p-1} \mathbf{e}_p\left(cy^{-2}g(x,y)\right) - \sum_{x,y=1}^{p-1} 1 \\ &= p + \sum_{x,y=1}^{p-1} \sum_{c=0}^{p-1} \mathbf{e}_p\left(cy^{-2}g(x,y)\right) - (p-1)^2 \\ &= p \cdot \#\{x, y \not\equiv 0 \bmod p : g(x,y) \equiv 0 \bmod p\} + p - (p-1)^2, \quad (2.9) \end{aligned}$$

where the last equality follows from the fact that if $g(x,y) \equiv 0 \bmod p$ then the c-sum is p and otherwise it is 0. We are left with counting how often $g(x,y) \equiv 0 \bmod p$ for x, y non-zero.

Whenever $x = y$ then $g(x,y) \equiv 0 \bmod p$; therefore there are $p-1$ solutions from $x = y$. Consider now $x^2y^2 \equiv x + y \bmod p$, which we may rewrite as a quadratic in y: $x^2y^2 - y - x \equiv 0 \bmod p$. By the Quadratic Formula modulo p (recall p is odd), if the discriminant $4x^3 + 1$ is a non-zero square modulo p there are two distinct roots, if it is not a square modulo p there are no roots, and if the discriminant vanishes there is one root. Equivalently, if $\left(\frac{4x^3+1}{p}\right) = 1$ (respectively, -1 or 0) there are two (respectively, none or one) solutions to $x^2y^2 \equiv x + y \bmod p$.

Recall neither x nor y is allowed to be zero. If $y = 0$ then $x^2y^2 \equiv x + y \bmod p$ reduces to $x = 0$. Hence our solutions have $x, y \not\equiv 0 \bmod p$, and for a non-zero x the number of non-zero y with $x^2y^2 \equiv x + y \bmod p$ is $1 + \left(\frac{4x^3+1}{p}\right)$. Hence the number of non-zero pairs with $x^2y^2 \equiv x + y \bmod p$ is

$$\sum_{x=1}^{p-1} \left(1 + \left(\frac{4x^3+1}{p}\right)\right) = p - 1 + \sum_{x \bmod p} \left(\frac{4x^3+1}{p}\right) - 1. \quad (2.10)$$

We must be careful about double counting solutions. If both $x - y \equiv 0 \bmod p$ and $x^2y^2 \equiv x + y \bmod p$, then we find $x^4 \equiv 2x \bmod p$. As $x \not\equiv 0 \bmod p$, we obtain $x^3 \equiv 2 \bmod p$. We have double counted all pairs (x,x) with x a cube root of 2 modulo p. Let $n_{3,2,p}$ denote the number of cube roots of 2 modulo p; $|n_{3,2,p}| \le 3$. We have shown

$$\begin{aligned} &\#\{x, y \in \{1, \dots, p-1\} : g(x,y) \equiv 0 \bmod p\} \\ &\quad = (p-1) + \left(p - 1 + \sum_{x \bmod p} \left(\frac{4x^3+1}{p}\right) - 1\right) - n_{3,2,p} \\ &\quad = 2p - 3 - n_{3,2,p} + \sum_{x \bmod p} \left(\frac{4x^3+1}{p}\right). \quad (2.11) \end{aligned}$$

Thus (2.9) becomes

$$\begin{aligned} A_{2,\mathcal{E}}(p) &= p\left(2p - 3 - n_{3,2,p} + \sum_{x \bmod p} \left(\frac{4x^3+1}{p}\right)\right) + p - (p-1)^2 \\ &= p^2 - n_{3,2,p}p - 1 + p \sum_{x \bmod p} \left(\frac{4x^3+1}{p}\right). \quad (2.12) \end{aligned}$$

To complete the analysis, we need to determine the size of $\sum_{x \bmod p} \left(\frac{4x^3+1}{p}\right)$. Note this is the number of solutions modulo p to the elliptic curve $y^2 = 4x^3 + 1$, and this curve is equivalent to $E: y^2 = x^3 + 16$. This curve has analytic rank 0, as can be seen from $L(E,1) \sim .5968$. It has complex multiplication, and for $p \equiv 2 \bmod 3$, $a_E(p) = 0$. Write $a_E(p) = 2\sqrt{p}\cos\theta_{E,p}$. As E has complex multiplication, for $p \equiv 1 \bmod 3$ the distribution of the angles $\theta_{E,p}$ is known; all we need is that for any $[\Theta, \Theta'] \subset [0, \pi]$, a positive percent of the time $\theta_{E,p} \in [\Theta, \Theta']$. This implies that a typical $a_E(p)$ is of size $\sqrt{p}$ if $p \equiv 1 \bmod 3$, and hence for any $[a, b] \subset [-2, 2]$ we can find infinitely many primes p with

$$A_{2,\mathcal{E}}(p) - \left(p^2 - n_{3,2,p}p - 1\right) \ \in \ [a \cdot p^{3/2}, b \cdot p^{3/2}]. \tag{2.13}$$

$\square$

Note that if $p \equiv 2 \bmod 3$, as $x \mapsto x^3 \bmod p$ is an automorphism then $n_{3,2,p} = 1$ and $a_E(p) = 0$. Thus, at least half the time, $A_{2,\mathcal{E}}(p) = p^2 - p - 1$.

Remark 2.2. A few words should be said about how we cooked up this family. If instead of $y^2 = x^3 + Tx^2 + 1$ we had $y^2 = x^3 + Tx + 1$, we would have found the condition $d \equiv cxy^{-1} \bmod p$. As we have $\left(\frac{cd}{p}\right)$ this would lead to $\left(\frac{c^2}{p}\right)\left(\frac{xy}{p}\right)$ times a similar c-exponential. It would not suffice to determine how often a similar $g(x, y)$ vanished; we would need to know the value of $\left(\frac{xy}{p}\right)$. Our analysis was greatly aided by the presence of $\left(\frac{x^2y^2}{p}\right)$. We also want to change the order of summation and do the t-sum first, which basically forces our family to be at most quadratic in t, and such that $g(x, y)$ factors easily. Instead of expanding by using Gauss sums, we could write the product of Legendre symbols (2.8) as the Legendre symbol of $h(x, y, t)$, where h is quadratic in t with leading term $x^2y^2t^2$:

$$\begin{aligned}\left(\frac{x^3 + 1 + x^2t}{p}\right)&\left(\frac{y^3 + 1 + y^2t}{p}\right) \\ &= \left(\frac{x^2y^2 \cdot t^2 + \left(y^2(x^3 + 1) + x^2(y^3 + 1)\right) \cdot t - (x^3 + 1)(y^3 + 1)}{p}\right).\end{aligned} \tag{2.14}$$

We execute the t-sum first. Quadratic Legendre sums are easily determined; what matters is the discriminant modulo p. After some algebra we find the discriminant is $g(x, y)^2$ (with $g(x, y)$ as before), and then the argument proceeds identically. See [Mil1, ALM] for more on determining tractable families where the summation can be done in closed form. These families will be quadratic in t, although not necessarily in Weierstrass form.

3. Other Families and Applications to Excess Rank

We give some additional examples of families where the first and second moments can be determined exactly; see [Mil1] for the calculations (though we provide calculations of a representative set of these families in Appendices A through C).

Recall $n_{3,2,p}$ denotes the number of cube roots of 2 modulo p, and set $c_0(p) = \left(\frac{-3}{p}\right) + \left(\frac{3}{p}\right)$, $c_1(p) = \left[\sum_{x \bmod p} \left(\frac{x^3-x}{p}\right)\right]^2$ and $c_{3/2}(p) = p\sum_{x \bmod p} \left(\frac{4x^3+1}{p}\right)$.

Family	$A_{1,\mathcal{E}}(p)$	$A_{2,\mathcal{E}}(p)$
$y^2 = x^3 + Sx + T$	0	$p^3 - p^2$
$y^2 = x^3 + 2^4(-3)^3(9T+1)^2$	0	$\begin{cases} 2p^2-2p & p\equiv 2 \bmod 3 \\ 0 & p\equiv 1 \bmod 3 \end{cases}$
$y^2 = x^3 \pm 4(4T+2)x$	0	$\begin{cases} 2p^2-2p & p\equiv 1 \bmod 3 \\ 0 & p\equiv 3 \bmod 3 \end{cases}$
$y^2 = x^3 + (T+1)x^2 + Tx$	0	$p^2 - 2p - 1$
$y^2 = x^3 + x^2 + 2T + 1$	0	$p^2 - 2p - \left(\frac{-3}{p}\right)$
$y^2 = x^3 + Tx^2 + 1$	$-p$	$p^2 - n_{3,2,p}p - 1 + c_{3/2}(p)$
$y^2 = x^3 - T^2x + T^2$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 - T^2x + T^4$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$

The first family is the family of all elliptic curves; it is a two parameter family and we expect the main term of its second moment to be p^3. Note that except for our family $y^2 = x^3 + Tx^2 + 1$, all the families $\mathcal{E}$ have $A_{2,\mathcal{E}}(p) = p^2 - h(p)p + O(1)$, where $h(p)$ is non-negative. Further, many of the families have $h(p) = m_{\mathcal{E}} > 0$. Note $c_1(p)$ is the square of the coefficients from an elliptic curve with complex multiplication. It is non-negative and of size p for $p \not\equiv 3 \bmod 4$, and zero for $p \equiv 1 \bmod 4$ (send $x \mapsto -x \bmod p$ and note $\left(\frac{-1}{p}\right) = -1$). It is somewhat remarkable that all these families have a correction to the main term in Michel's theorem in the same direction, and we analyze the consequence this has on the average rank. For our family which has a $p^{3/2}$ term, note that on average this term is zero and the p term is negative.

Consider a one-parameter family of elliptic curves $\mathcal{E}$ of rank r over $\mathbb{Q}(T)$. With our normalizations, under GRH the non-trivial zeros of E_t are $1 + i\gamma_t$, $\gamma_t \in \mathbb{R}$. We typically study $t \in [N, 2N]$ with $N \to \infty$. Let C_t be the conductor of the elliptic curve E_t, and let $\log R = \frac{1}{N}\sum_{t=N}^{2N} \log C_t$ be the average log-conductor. For many families there is an integer a such that $\log C_t \sim \log N^a$ for most curves; this is true for the families listed above (see [Mil1] for the calculations). Assuming the Birch and Swinnerton-Dyer conjecture, by Silverman's specialization theorem eventually all curves E_t have rank at least r, and under natural standard conjectures (see [He]) a typical family will have equidistribution of signs of the functional equations. What is typically seen in studying the ranks of curves in a family is that roughly 30% have rank r and 20% rank $r+2$, while about 48% have rank $r+1$ and 2% rank $r+3$. Random matrix theory predicts that in the limit 50% should be rank r and 50% rank $r+1$ for an average rank of $r + \frac{1}{2}$, markedly different from the observed (approximately) $r + \frac{1}{2} + .40$. See [Fe1, Fe2, Wa] for numerical investigations and [Br, H-B, FP, Mic, Sil2, Yo2] for theoretical bounds of the average rank.

The excess rank question is whether this disagreement persists or is a result of small data. We often expect the rate of convergence for problems such as this to be like the logarithm of the conductors. As the conductors are often at most 10^{12}, it is reasonable to believe the data is misleading (especially as random matrix theory predicts sub-families of higher rank of size $N^{3/4}$, and for small N such families are a noticeable percentage; see for example [CKRS, DFK, Go, GM, Mai, Ono, RoSi, ST, Yu] for discussions of random matrix predictions and results from number theory).

For an even Schwartz test function ϕ with $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$, the 1-level density (which is basically just the sum of the explicit formula for each curve) is defined by

$$\begin{aligned} \frac{1}{N}\sum_{t=N}^{2N}\sum_{\gamma_t} \phi\left(\gamma_t \frac{\log R}{2\pi}\right) &= \widehat{\phi}(0) + \phi(0) - \frac{2}{N}\sum_{t=N}^{2N}\sum_p \frac{\log p}{\log R}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log R}\right) a_t(p) \\ &- \frac{2}{N}\sum_{t=N}^{2N}\sum_p \frac{\log p}{\log R}\frac{1}{p^2}\widehat{\phi}\left(\frac{2\log p}{\log R}\right) a_t(p)^2 + O\left(\frac{\log\log R}{\log R}\right). \end{aligned} \tag{3.15}$$

If ϕ is non-negative, we obtain a bound for the average rank in the family by restricting the sum to be only over zeros at the central point. The error $O\left(\frac{\log\log R}{\log R}\right)$ comes from trivial estimation and ignores probable cancellation, and we expect $O\left(\frac{1}{\log R}\right)$ or smaller to be the correct magnitude. For most families $\log R \sim \log N^a$ for some integer a.

The main term of the first and second moments of the $a_t(p)$ give $r\phi(0)$ and $-\frac{1}{2}\phi(0)$, respectively, in (3.15). Assume the second moment of $a_t(p)^2$ is $p^2 - m_{\mathcal{E}}p + O(1)$, $m_{\mathcal{E}} > 0$. We have already handled the contribution from p^2, and $-m_{\mathcal{E}}p$ contributes

$$\begin{aligned} S_2 &\sim \frac{-2}{N}\sum_p \frac{\log p}{\log R}\widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{1}{p^2}\frac{N}{p}(-m_{\mathcal{E}}p) \\ &= \frac{2m_{\mathcal{E}}}{\log R}\sum_p \widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p^2}. \end{aligned} \tag{3.16}$$

Thus there is a contribution of size $\frac{1}{\log R}$. A good choice of test functions (see Appendix A of [ILS]) is the Fourier pair

$$\phi(x) = \frac{\sin^2(2\pi\frac{\sigma}{2}x)}{(2\pi x)^2}, \qquad \widehat{\phi}(u) = \begin{cases} \frac{\sigma-|u|}{4} & \text{if } |u| \le \sigma \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

Note $\phi(0) = \frac{\sigma^2}{4}$, $\widehat{\phi}(0) = \frac{\sigma}{4} = \frac{\phi(0)}{\sigma}$, and evaluating the prime sum in (3.16) gives

$$S_2 \sim \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}\,\phi(0). \tag{3.18}$$

Let r_t denote the number of zeros of E_t at the central point (i.e., the analytic rank). Then up to our $O\left(\frac{\log\log R}{\log R}\right)$ errors (which we think should be smaller), we have

$$\begin{aligned} \frac{1}{N}\sum_{t=N}^{2N} r_t\phi(0) &\le \frac{\phi(0)}{\sigma} + \left(r+\frac{1}{2}\right)\phi(0) + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}\phi(0) \\ \text{Ave Rank}_{[N,2N]}(\mathcal{E}) &\le \frac{1}{\sigma} + r + \frac{1}{2} + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}. \end{aligned} \tag{3.19}$$

Remark 3.1. The Density Conjecture states that the 1-level density (in the limit) should hold for all σ. In that case, the lower order terms from the second moment will *not* contribute to the bound for the average rank, as their contribution vanishes as $\sigma \to \infty$. Of course, the agreement with random matrix theory is a statement about the limit as $N \to \infty$; the correct finite-conductor model is still unknown

Let us examine the boost the $-m_{\mathcal{E}}p$ term from the second moment gives to the upper bound for the average rank. As remarked, if our 1-level density were true for all σ then there would be no contribution from the correction term to the second sum, nor would the $\frac{1}{\sigma}$ term contribute, and we would obtain the average rank is bounded by $r + \frac{1}{2}$.

Let us assume we know the 1-level density up to $\sigma = 1$. (This is well beyond the range of current technology; the best result to date is for the family of all elliptic curves, where Young [Yo2] proves we may take any $\sigma < \frac{7}{9}$). Assume $m_{\mathcal{E}} = 1$. The $\frac{1}{\sigma}$ term would contribute 1, the lower correction would contribute .03 for conductors of size 10^{12}, and thus the average rank is bounded by $1 + r + \frac{1}{2} + .03 = r + \frac{1}{2} + 1.03$. This is significantly higher than Fermigier's observed $r + \frac{1}{2} + .40$.

If we were able to prove our 1-level density for $\sigma = 2$, then the $\frac{1}{\sigma}$ term would contribute $\frac{1}{2}$, and the lower order correction would contribute .02 for conductors of size 10^{12}. Thus the average rank would be bounded by $\frac{1}{2} + r + \frac{1}{2} + .02 = r + \frac{1}{2} + .52$.

While the main error contribution is from $\frac{1}{\sigma}$, there is still a noticeable effect from the lower order terms in $A_{2,\mathcal{E}}(p)$. Moreover, we are now in the ballpark of Fermigier's bound; of course, we were already there without the potential correction term.

It seems hopeless to think about obtaining a 1-level density for any family of elliptic curves with support $\sigma = 2$ or more. Iwawniec, Luo and Sarnak [ILS] obtain such large support for families of weight k cuspidal newforms of square-free level N, but only because of great averaging formulas (the Bessel-Kloosterman expansion in the Petersson formula) available for the family; the corresponding averaging formulas for elliptic curves are much weaker. We use the periodicity of $a_t(p)$ as a function of $t \bmod p$ to analyze complete sums of the moments for each prime; however, the error from the incomplete sum is bounded by Hasse and contributes a large error (this problem is avoided in [ILS] because the Petersson formula gives us sums over a basis of newforms, and there is no incomplete piece to be approximated). The random matrix models for the behavior of the zeros near the central point have been shown to hold as the conductors tend to infinity; in the small conductor ranges investigated, it is not surprising that there is disagreement. While it would be desirable to find a good model for small conductors (similar to Keating and Snaith's [KeSn1, KeSn2] modeling zeros of $\zeta(s)$ at height T by $N \times N$ matrices with $N = \frac{\log T}{2\pi}$), we can identify potential family dependent lower order terms in the 1-level density arising from lower order terms in the second moment. For finite conductors these do lead to slightly larger predicted upper bounds for the average rank in a family.

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The following appendices contain calculations for the second moment of a representative set of the other families mentioned in §3, and for completeness proofs of standard quadratic Legendre sums. For general one-parameter families of elliptic curves, there are not closed form expressions for the moments. Our hope is that these families may be useful for other investigations.

Appendix A. The Family $y^2 = x^3 + T^2$

Theorem A.1. *For the family $\mathcal{E} : y^2 = x^3 + T^2$,*

$$A_{2,\mathcal{E}}(p) \ = \ \begin{cases} 2p^2 - 2p & \text{if } p \equiv 1 \bmod 3 \\ 0 & \text{if } p \equiv 2 \bmod 3. \end{cases} \tag{A.20}$$

Proof. If $p \equiv 2 \bmod 3$ then $a_t^2(p) = 0$ as $x \mapsto x^3 \bmod p$ is an automorphism. Assume $p \equiv 1 \bmod 3$.

$$\begin{aligned} A_{2,\mathcal{E}}(p) \quad &= \quad \sum_{t \bmod p} \sum_{x \bmod p} \sum_{y \bmod p} \left(\frac{x^3+t^2}{p}\right)\left(\frac{y^3+t^2}{p}\right) \\ &= \quad \sum_{t=1}^{p-1} \sum_{x \bmod p} \sum_{y \bmod p} \left(\frac{x^3+t^2}{p}\right)\left(\frac{y^3+t^2}{p}\right) \\ &= \quad \sum_{t=1}^{p-1} \sum_{x \bmod p} \sum_{y \bmod p} \left(\frac{t^4}{p}\right)\left(\frac{tx^3+1}{p}\right)\left(\frac{ty^3+1}{p}\right) \\ &= \quad \sum_{x \bmod p} \sum_{y \bmod p} \sum_{t \bmod p} \left(\frac{tx^3+1}{p}\right)\left(\frac{ty^3+1}{p}\right) - p^2. \end{aligned} \tag{A.21}$$

We use inclusion / exclusion to reduce to $xy \neq 0$. If $x = 0$, the t-sum vanishes unless $y = 0$, in which case we get p. Similarly if $y = 0$, the t and x-sums give p. We subtract the doubly counted contribution from $x = y = 0$, which gives p. Thus

$$A_{2,\mathcal{E}}(p) \quad = \quad \sum_{x=1}^{p-1}\sum_{y=1}^{p-1}\sum_{t(p)} \left(\frac{tx^3+1}{p}\right)\left(\frac{ty^3+1}{p}\right) + 2p - p - p^2. \tag{A.22}$$

By Lemma D.1, the t-sum is $(p-1)\left(\frac{x^3y^3}{p}\right)$ if $p|(x^3-y^3)^2$ and $-\left(\frac{x^3y^3}{p}\right)$ otherwise. As $p = 6m+1$, let g be a generator of the multiplicative group $\mathbb{Z}/p\mathbb{Z}$. Solving $g^{3a} \equiv g^{3b} \bmod p$ yields $b = a$, $a+2m$, or $a+4m$. Thus, $x^3 \equiv y^3 \bmod p$ three times, and in each instance y equals x times a square (1, g^{2m}, g^{4m}).

$$\begin{aligned} A_{2,\mathcal{E}}(p) \quad &= \quad \sum_{x=1}^{p-1} \sum_{\substack{y=1 \\ y^3 \equiv x^3}}^{p-1} p - \sum_{x=1}^{p-1}\sum_{y=1}^{p-1} \left(\frac{x^3y^3}{p}\right) + p - p^2 \\ &= \quad (p-1)3p + p - p^2 \\ &= \quad 2p^2 - 2p. \end{aligned} \tag{A.23}$$

□

Appendix B. The Family $y^2 = x^3 + x^2 + T$

Theorem B.1. *For the family $\mathcal{E} : y^2 = x^3 + x^2 + T$ we have*

$$A_{2,\mathcal{E}}(p) \;=\; p^2 - 2p - p\left(\frac{-3}{p}\right). \tag{B.24}$$

Proof.

$$\begin{aligned} A_{2,\mathcal{E}}(p) &= \sum_{t=0}^{p-1}\sum_{x=0}^{p-1}\sum_{y=0}^{p-1} \left(\frac{t+(x^3+x^2)}{p}\right)\left(\frac{t+(y^3+y^2)}{p}\right) \\ &= \sum_{t=0}^{p-1}\sum_{x=0}^{p-1}\sum_{y=0}^{p-1} \left(\frac{t^2 + \big((x^3+x^2)+(y^3+y^2)\big)\,t + (x^3+x^2)(y^3+y^2)}{p}\right). \end{aligned} \tag{B.25}$$

Let $\delta(x,y) = (x^3+x^2)-(y^3+y^2)$; note $\delta(x,y)^2$ is the discriminant of the quadratic regarded as a function of t. The t-sum is $p-1$ if $p|\delta(x,y)$ and -1 otherwise. Note

$$\delta(x,y) \;=\; (x-y)(y^2+(x+1)y+(x^2+x)). \tag{B.26}$$

The first factor is congruent to zero when $x = y$; for fixed x, the discriminant of the second factor is $(x+1)^2 - 4(x^2+x) = 1-2x-3x^2$. Thus the number of solutions of the second factor, for fixed x, is $1 + \left(\frac{1-2x-3x^2}{p}\right)$. As the discriminant of $1-2x-3x^2$ is 16, summing over x for $p > 2$ yields $p - \left(\frac{-3}{p}\right)$ by Lemma D.2.

We must be careful about double counting. If both factors are congruent to zero, then $3x^2 + 2x \equiv 0$, or $x \equiv 0, -2 \cdot 3^{-1}$. Hence we always double count two solutions.

$$\begin{aligned} A_{2,\mathcal{E}}(p) &= \left[p + p - \left(\frac{-3}{p}\right) - 2\right]p - \sum_{x=0}^{p-1}\sum_{y=0}^{p-1} 1 \\ &= p^2 - 2p - p\left(\frac{-3}{p}\right). \end{aligned} \tag{B.27}$$

□

Appendix C. $y^2 = x^3 - T^2x + T^2$

Consider the family $\mathcal{E} : y^2 = x^3 - T^2x + T^2$. We calculate the first moment of $a_t(p)$, which shows the family is of rank 2 over $\mathbb{Q}(T)$, and then determine the second moment.

Theorem C.1. *For $\mathcal{E} : y^2 = x^3 - T^2x + T^2$, $A_{1,\mathcal{E}}(p) = -2p$. Thus by Rosen and Silverman the family has rank 2 over $\mathbb{Q}(T)$.*

Proof.

$$\begin{aligned}
-A_{1,\mathcal{E}}(p) &= -\sum_{t(p)} a_t(p) \;=\; \sum_{t(p)}\sum_{x(p)} \left(\frac{x^3-t^2x+t^2}{p}\right) \\
&= \sum_{t=1}^{p-1}\sum_{x(p)} \left(\frac{x^3-t^2x+t^2}{p}\right) \;=\; \sum_{t=1}^{p-1}\sum_{x(p)} \left(\frac{t^3x^3-t^3x+t^2}{p}\right) \\
&= \sum_{t=1}^{p-1}\sum_{x(p)} \left(\frac{t^2}{p}\right)\left(\frac{t(x^3-x)+1}{p}\right) \\
&= \sum_{t(p)}\sum_{x(p)} \left(\frac{t(x^3-x)+1}{p}\right) - \sum_{x(p)}\left(\frac{1}{p}\right) \\
&= \sum_{t(p)}\sum_{x=0,\pm1} \left(\frac{t(x^3-x)+1}{p}\right) + \sum_{t(p)}\sum_{\substack{x(p)\\ x\neq 0,\pm1}} \left(\frac{t(x^3-x)+1}{p}\right) - p \\
&= \sum_{t(p)}\sum_{x=0,\pm1} \left(\frac{1}{p}\right) + \sum_{\substack{x(p)\\ x\neq 0,\pm1}}\sum_{t(p)} \left(\frac{t+1}{p}\right) - p \\
&= 3p+0-p \;=\; 2p. \qquad \text{(C.28)}
\end{aligned}$$

□

Theorem C.2. *For* $\mathcal{E} : y^2 = x^3 - T^2x + T^2$,

$$A_{2,\mathcal{E}}(p) = p^2 - p - \left[\sum_{x(p)}\left(\frac{(x^3-x)}{p}\right)\right]^2 - \left(\frac{-3}{p}\right) - \left(\frac{3}{p}\right) = p^2 + O(p). \qquad \text{(C.29)}$$

Proof.

$$\begin{aligned}
A_{2,\mathcal{E}}(p) &= \sum_{t(p)} a_t^2(p) \\
&= \sum_{t(p)}\sum_{x,y(p)} \left(\frac{x^3-t^2x+t^2}{p}\right)\left(\frac{y^3-t^2y+t^2}{p}\right) \\
&= \sum_{t=1}^{p-1}\sum_{x,y(p)} \left(\frac{x^3-t^2x+t^2}{p}\right)\left(\frac{y^3-t^2y+t^2}{p}\right) \\
&= \sum_{t=1}^{p-1}\sum_{x,y(p)} \left(\frac{t^3x^3-t^3x+t^2}{p}\right)\left(\frac{t^3y^3-t^3y+t^2}{p}\right) \\
&= \sum_{t=1}^{p-1}\sum_{x,y(p)} \left(\frac{t^4}{p}\right)\left(\frac{t(x^3-x)+1}{p}\right)\left(\frac{t(y^3-y)+1}{p}\right) \\
&= \sum_{t=0}^{p-1}\sum_{x,y(p)} \left(\frac{t(x^3-x)+1}{p}\right)\left(\frac{t(y^3-y)+1}{p}\right) - \sum_{x,y(p)}\left(\frac{1}{p}\right) \\
&= \sum_{x,y(p)}\sum_{t(p)} \left(\frac{t(x^3-x)+1}{p}\right)\left(\frac{t(y^3-y)+1}{p}\right) - p^2. \qquad \text{(C.30)}
\end{aligned}$$

In Lemma D.2 we showed that, if a and b are not both zero,

$$\sum_{t=0}^{p-1}\left(\frac{at^2+bt+c}{p}\right) \;=\; \begin{cases} (p-1)\left(\frac{a}{p}\right) & \text{if } p|b^2-4ac \\ -\left(\frac{a}{p}\right) & \text{otherwise.}\end{cases} \qquad \text{(C.31)}$$

In $A_{2,\mathcal{E}}(p)$ we have

$$\begin{aligned} a &= (x^3-x)(y^3-y) = y(x^2-1)x(y^2-1) \\ b &= (x^3-x)+(y^3-y) \\ c &= 1 \\ \delta(x,y) &= b^2-4ac = \Big((x^3-x)-(y^3-y)\Big)^2. \end{aligned} \tag{C.32}$$

We use inclusion / exclusion on x^3-x and y^3-y vanishing. Assume first that x^3-x equals zero (happens three ways: $x=0,\pm1$). Then we have $\sum_t \left(\frac{t(y^3-y)+1}{p}\right)$, which is $3p$ from our $A_{1,\mathcal{E}}(p)$ computation, giving $3\cdot 3p$. Similarly we get $3\cdot 3p$ if y^3-y is zero. We subtract the doubly counted $x^3-x\equiv y^3-y\equiv 0$ (nine ways), each of which gives $\sum_t \left(\frac{1}{p}\right)=p$. Hence the contribution from at least one of x^3-x and y^3-y vanishing is $9p$.

Assume $x,y\notin\{0,\pm1\}$. When is $\delta(x,y)=(x^3-x)-(y^3-y)\equiv 0(p)$?

$$\delta(x,y) = (x-y)\cdot(x^2+xy+y^2-1). \tag{C.33}$$

Therefore

$$A_{2,\mathcal{E}}(p) = \sum_{\substack{x,y\neq 0,\pm1\\ \delta(x,y)\equiv 0}} p\left(\frac{(x^3-x)(y^3-y)}{p}\right) - \sum_{x,y\neq 0,\pm1}\left(\frac{(x^3-x)(y^3-y)}{p}\right) + 9p - p^2. \tag{C.34}$$

Clearly, $\delta(x,y)\equiv 0(p)$ if $x=y$, which happens $p-3$ times. If $x=y$ then the second factor is $3x^2-1$, which is congruent to zero at most twice.

When is $\delta_2(x,y)=x^2+xy+y^2-1\equiv 0$? By the Quadratic Formula mod p,

$$y = \frac{-x\pm\sqrt{4-3x^2}}{2}, \tag{C.35}$$

which reduces to finding when $4-3x^2$ is a square mod p. We get two values of y if it is equivalent to a non-zero square, one value if it is equivalent to zero, and no values if it is not equivalent to a square. When solving $\delta_2(x,y)\equiv 0(p)$, we make sure such $y\notin\{0,\pm1\}$. If $y=0$, $x=\pm1$; $y=1$, $x=0$ or -1; $y=-1$, $x=0$ or 1. Therefore, we don't get an excluded y (and similarly if we reverse the rolls of y and x). Thus the number of solutions to $\delta_2(x,y)\equiv 0(p)$ is

$$\begin{aligned} \sum_{x=2}^{p-2}\left[1+\left(\frac{4-3x^2}{p}\right)\right] &= p-3+\sum_{x=2}^{p-2}\left(\frac{4-3x^2}{p}\right) \\ &= p-6+\sum_{x(p)}\left(\frac{4-3x^2}{p}\right). \end{aligned} \tag{C.36}$$

We again use Lemma D.2. The discriminant now is $0^2-4\cdot(-3)\cdot 4$. For $p\geq 5$, p does not divide the discriminant, hence this sum is $-\left(\frac{-3}{p}\right)$.

Thus, for $x\neq 0,\pm1$, the number of solutions with $x^2+xy+y^2\equiv 1$ is $p-6-\left(\frac{-3}{p}\right)$; the number with $x-y\equiv 0$ is $p-3$. At most two of the pairs (x,y) satisfying $x^2+xy+y^2-1\equiv 0(p)$ also satisfy $x=y$. These pairs satisfy $3x^2\equiv 1$, thus, if $\left(\frac{3}{p}\right)=1$ we have doubly counted two solutions; if it is -1, there was no double counting. Thus, the number of doubly counted pairs is $1+\left(\frac{3}{p}\right)$, and the total number of pairs is

$$2p-10-\left(\frac{-3}{p}\right)-\left(\frac{3}{p}\right). \tag{C.37}$$

When $x=y\neq 0,\pm1$, clearly $\left(\frac{(x^3-x)(y^3-y)}{p}\right)=1$. Hence these terms contribute 1.

Consider $x \neq y$ and $x^2 + xy + y^2 - 1 \equiv 0$. Thus $x, y \neq 0, \pm 1$. Then $y^2 - 1 \equiv -x(x+y)$ and $x^2 - 1 \equiv -y(x+y)$ and

$$\left(\frac{(x^3-x)(y^3-y)}{p}\right) = \left(\frac{x(x^2-1)y(y^2-1)}{p}\right) = \left(\frac{x^2y^2(x+y)^2}{p}\right). \tag{C.38}$$

As long as $x \neq -y$, this is 1. If $x = -y$ then we would have $x^2 - x^2 + x^2 - 1 \equiv 0$. This implies $x = \pm 1$, which cannot happen as $x, y \neq 0, \pm 1$. Therefore all pairs have their Legendre factor $+1$, and we need only count how many such pairs there are. We've previously shown this to be $p + O(1)$, therefore

$$\begin{aligned} A_{2,\mathcal{E}}(p) &= p\left[2p - 10 - \left(\frac{-3}{p}\right) - \left(\frac{3}{p}\right)\right] - \sum_{x,y \neq 0,\pm 1} \left(\frac{(x^3-x)(y^3-y)}{p}\right) + 9p - p^2 \\ &= p^2 - p - \left[\sum_{x(p)} \left(\frac{x^3-x}{p}\right)\right]^2 - \left(\frac{-3}{p}\right) - \left(\frac{3}{p}\right). \end{aligned} \tag{C.39}$$

As $x^3 - x$ is a non-singular elliptic curve, by Hasse its sum above is bounded by $4p$. It has complex multiplication and analytic rank 0. For $p \equiv 3 \bmod 4$ its $a_E(p) = 0$ (change variables $x \to -x$); for the remaining p, the angles of $\frac{a_E(p)}{2\sqrt{p}}$ are uniformly distributed. Hence $A_{2,\mathcal{E}}(p) = p^2 + O(p)$. □

Remark C.3. The reason this calculation succeeds is we have a very tractable expression for $x(x^2-1)y(y^2-1)$ when $x^2 + xy + y^2 - 1 \equiv 0 \bmod p$. It was non-trivial to find a family with high rank over $\mathbb{Q}(T)$ and $A_{2,\mathcal{E}}(p)$ computable.

Appendix D. Quadratics Sums of Legendre Symbols

For completeness we include proofs of standard sums of Legendre symbols.

Lemma D.1. *For $p > 2$*

$$S(n) = \sum_{x=0}^{p-1} \left(\frac{n_1+x}{p}\right)\left(\frac{n_2+x}{p}\right) = \begin{cases} p-1 & \text{if } p \mid n_1 - n_2 \\ -1 & \text{otherwise.} \end{cases} \tag{D.40}$$

Proof. Shifting x by $-n_2$, we need only prove the lemma when $n_2 = 0$. Assume $(n,p) = 1$ as otherwise the result is trivial. For $(a,p) = 1$ we have

$$\begin{aligned} S(n) &= \sum_{x=0}^{p-1} \left(\frac{n+x}{p}\right)\left(\frac{x}{p}\right) \\ &= \sum_{x=0}^{p-1} \left(\frac{n+a^{-1}x}{p}\right)\left(\frac{a^{-1}x}{p}\right) \\ &= \sum_{x=0}^{p-1} \left(\frac{an+x}{p}\right)\left(\frac{x}{p}\right) = S(an) \end{aligned} \tag{D.41}$$

Hence

$$\begin{aligned} S(n) &= \frac{1}{p-1}\sum_{a=1}^{p-1}\sum_{x=0}^{p-1} \left(\frac{an+x}{p}\right)\left(\frac{x}{p}\right) \\ &= \frac{1}{p-1}\sum_{a=0}^{p-1}\sum_{x=0}^{p-1} \left(\frac{an+x}{p}\right)\left(\frac{x}{p}\right) - \frac{1}{p-1}\sum_{x=0}^{p-1}\left(\frac{x}{p}\right)^2 \\ &= \frac{1}{p-1}\sum_{x=0}^{p-1}\left(\frac{x}{p}\right)\sum_{a=0}^{p-1}\left(\frac{an+x}{p}\right) - 1 \\ &= 0 - 1 = -1 \end{aligned} \tag{D.42}$$

Where do we use $p > 2$? We used $\sum_{a=0}^{p-1} \left(\frac{an+x}{p}\right) = 0$ for $(n,p) = 1$. This is true for all odd primes (as there are $\frac{p-1}{2}$ quadratic residues, $\frac{p-1}{2}$ non-residues, and 0); for $p = 2$, there is one quadratic residue, no non-residues, and 0. □

Lemma D.2 (Quadratic Legendre Sums)**.** *Assume a and b are not both zero mod p and $p > 2$. Then*

$$\sum_{t=0}^{p-1} \left(\frac{at^2+bt+c}{p}\right) \ = \ \begin{cases} (p-1)\left(\frac{a}{p}\right) & \text{if } p|b^2-4ac \\ -\left(\frac{a}{p}\right) & \text{otherwise.} \end{cases} \tag{D.43}$$

Proof. Assume $a \not\equiv 0(p)$ as otherwise the proof is trivial. Let $\delta = 4^{-1}(b^2-4ac)$. Then

$$\begin{aligned} \sum_{t=0}^{p-1} \left(\frac{at^2+bt+c}{p}\right) \ &= \ \sum_{t=0}^{p-1} \left(\frac{a^{-1}}{p}\right)\left(\frac{a^2t^2+bat+ac}{p}\right) \qquad \text{(D.44)} \\ &= \ \sum_{t=0}^{p-1} \left(\frac{a}{p}\right)\left(\frac{t^2+bt+ac}{p}\right) \\ &= \ \sum_{t=0}^{p-1} \left(\frac{a}{p}\right)\left(\frac{t^2+bt+4^{-1}b^2+ac-4^{-1}b^2}{p}\right) \\ &= \ \sum_{t=0}^{p-1} \left(\frac{a}{p}\right)\left(\frac{(t+2^{-1}b)^2-4^{-1}(b^2-4ac)}{p}\right) \\ &= \ \sum_{t=0}^{p-1} \left(\frac{a}{p}\right)\left(\frac{t^2-\delta}{p}\right) \\ &= \ \left(\frac{a}{p}\right)\sum_{t=0}^{p-1} \left(\frac{t^2-\delta}{p}\right) \end{aligned}$$

If $\delta \equiv 0 \bmod p$ we get $p-1$. If $\delta = \eta^2, \eta \neq 0$, then by the Lemma D.1

$$\sum_{t=0}^{p-1} \left(\frac{t^2-\delta}{p}\right) \ = \ \sum_{t=0}^{p-1} \left(\frac{t-\eta}{p}\right)\left(\frac{t+\eta}{p}\right) \ = \ -1. \tag{D.45}$$

We note that $\sum_{t=0}^{p-1} \left(\frac{t^2-\delta}{p}\right)$ is the same for all non-square δ's (let g be a generator of the multiplicative group, $\delta = g^{2k+1}$, change variables by $t \to g^k t$). Denote this sum by S, the set of non-zero squares by $\mathcal{R}$, and the non-squares by $\mathcal{N}$. Since $\sum_{\delta=0}^{p-1} \left(\frac{t^2-\delta}{p}\right) = 0$ we have

$$\begin{aligned} \sum_{\delta=0}^{p-1}\sum_{t=0}^{p-1} \left(\frac{t^2-\delta}{p}\right) \ &= \ \sum_{t=0}^{p-1} \left(\frac{t^2}{p}\right) + \sum_{\delta\in\mathcal{R}}\sum_{t=0}^{p-1} \left(\frac{t^2-\delta}{p}\right) + \sum_{\delta\in\mathcal{N}}\sum_{t=0}^{p-1} \left(\frac{t^2-\delta}{p}\right) \\ &= \ (p-1) + \frac{p-1}{2}(-1) + \frac{p-1}{2}S \ = \ 0 \qquad \text{(D.46)} \end{aligned}$$

Hence $S = -1$, proving the lemma. □

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