Nature By Numbers

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Abstract

Mathematics often evokes images of numbers, equations, and abstract theorems—far removed from the world of art and beauty. However, for some, like the second named author, the boundaries between these disciplines blur, allowing them to see a harmonious interplay between these two worlds. This paper is an expanded version of a note of his, where he reflects on his journey from studying Fine Arts at the University of Barcelona to noticing how the mathematical structures manifest in nature, with additional material by the other two authors to expand on the mathematics discussed by giving a very brief description of occurrences and some references to the extensive literature.

1 Introduction

One of the blessings of the digital age is that publications no longer need to be static characters on a page. We expand on the adage that a picture is worth a thousand words by sharing reflections from the second named author on a beautiful animation he created on the Fibonacci numbers, *Nature by Numbers* (available at https://etereaestudios.com/works/nature-by-numbers/; we urge the reader to watch this short video, which has over 6 million views on YouTube, and then read on), using his comments as a springboard for the other two authors to discuss the resulting mathematics. His story not only demonstrates the connections between art, mathematics, and nature—which can be appreciated by audiences of all ages—but also serves as a prelude to the deeper exploration of the mathematical concepts behind his work. Through the author's experiences, we gain a unique perspective on how the patterns found in the natural world can be modeled, animated, and appreciated.

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2 Note on the Video

When I[1] studied Fine Arts at the University of Barcelona, between 1985 and 1990, we naturally focused on the importance of the golden ratio in the history of art, architecture, painting, drawing, and sculpture. However, it was a few years later when I was greatly surprised to find out that this type of proportion could also be found in nature. Later on, I was introduced to the world of 3D modeling and animation. In 2009, I began to cherish the idea of creating a personal 3D animation project where these kinds of ideas could be embodied. That's how *Nature by Numbers* was born. Starting with an image of a spiral formed from the Fibonacci Sequence was the initial concept; I felt compelled to model and animate the growth of one of those beautiful nautilus shells.

Curiously, after modeling the shell, I realized that its shape did not actually correspond to a golden spiral (although plenty of online sources assume it does). But it was too late to back out, so I decided to take this as a form of artistic license. The arrangement of seeds in the disk of a sunflower was the second idea I developed. I have always marveled at the geometric perfection of these seeds when they are so neatly arranged in those large floral formations. The third key idea for the animation, the use of Voronoi tessellations to illustrate the geometry of a dragonfly's wings, came to me in the most unexpected way. I was reading an article on one of my favorite blogs (not about mathematics, but about Japanese culture) where the author explained what a Voronoi tessellation is and how these formations are used in diverse applications, such as distributing pizzerias or antennas in a large city like Tokyo.

When I saw the tessellation, it suddenly became clear to me: I had encountered this kind of pattern in many other places! In fact, I remembered there was a Photoshop filter with that same name (Voronoi). Please bear in mind that my only mathematical training is from general high school content.

It became clear to me: I would use Voronoi to explain the cells in a dragonfly's wing. In the initial planning phases, while searching for many inspiring images, I had countless ideas: bee honeycombs, the Romanesco cauliflower, butterfly tongues, ferns, the famous Giant's Causeway, and so on. However, from previous animations, I learned it's best to focus on a few ideas and try to execute them well. In my native language, we have a saying: "He who covers too much does not grasp enough." So, the themes for *Nature by Numbers* were set. The project took nine months, between 2009 and 2010—not exclusively, as I also had some commercial work, but it occupied much of my time.

As a small anecdote, I remember explaining the project to a friend while relaxing at the pool. My friend looked at me with a face of bemused incomprehension ("What strange things you do, Cristóbal," I could almost hear him thinking). During those nine months, I had no idea how the animation would be received. A part of me thought I'd post it online, and no one would be interested.

[1] This section contains reflections from the second named author on creating the video *Nature by Numbers*.

However, once I posted it in March 2010, the response was overwhelming. Without a doubt, it is my most recognized work. Unexpectedly, reviews appeared worldwide, from the *Huffington Post* to *The Guardian*, and on all kinds of blogs and specialized publications. Its release coincided with the rise of the first social networks, Twitter and Facebook, which helped it spread widely. Despite being a personal project (done for the "pure love of art"), it ended up being my most economically profitable work. Many companies approached me for rights to use the images, and I received several commercial commissions thanks to the recognition this project brought me. One of these came from the Belgian musician Wim Mertens, the author of the soundtrack.

One aspect of this work's impact that has struck me is the varied profile of its admirers.

- On one hand, people with a technical or scientific background, especially math teachers, who use it in their classes to inspire students.
- On the other, and quite unexpectedly, people with more spiritual or religious views who see in these images a validation of their beliefs, such as intelligent design. This isn't something I personally believe in, but people see what they want to see. This is evident in the more than 4000 comments on the YouTube video.

Ten years later, in 2019, I presented another animation, *Infinite Patterns* (available at https://etereaestudios.com/works/infinitepatterns/), which I see as a kind of continuation or second part. I am very proud of it; technically, it is far superior. Yet, its impact cannot compare with *Nature by Numbers*. I doubt I'll ever create something with that level of influence again—something that, nearly 15 years later, inspires people like you to reach out to me for a few lines about that work. Thank you for your interest. It makes me happy to know that it can inspire young people to appreciate the beauty behind mathematics.

Cristóbal Vila

While *Nature by Numbers* captures the beauty of Fibonacci sequences and Voronoi tessellations, their significance extends far beyond artistic inspiration. These mathematical concepts reveal patterns that can be found everywhere: from the growth of plants to the structure of the universe. In the next section, we'll take a deeper look into the mathematics behind Fibonacci sequences and Voronoi tessellations, exploring their surprising appearances in fields as diverse as finance, physics, and biology.

3 The Fibonacci Sequence and The Golden Ratio

The Fibonacci sequence, where each number is the sum of the two preceding it, has fascinated mathematicians, artists, and scientists for centuries. From this sequence we get an important quantity, the golden ratio (approximately 1.618), often denoted by the Greek letter ϕ (phi). One way the golden ratio appears is as the limit of the ratio between consecutive numbers in the Fibonacci sequence. When the larger number is divided by the smaller, the result converges to ϕ. As you progress further along the Fibonacci sequence, this ratio becomes increasingly close to ϕ, though it will never exactly equal it. Though this concept might seem somewhat random, the golden ratio is not just an abstract mathematical idea; it has profound implications in various fields, from financial markets to natural sciences.

Before reading further, try to prove the following facts about ϕ and its generalizations. Let F_n denote the n^{th} Fibonacci number, with initial conditions $F_0 = 0$ and $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, and set $r_n = F_n/F_{n-1}$.

- Prove if $\lim_{n\to\infty} r_n$ exists then it equals the golden mean, $(1+\sqrt{5})/2 \approx 1.61803$.
- Numerically compute the first 10 or so values of r_n. What do you observe about how this sequence converges to the golden mean? Can you prove your observation is correct?
- More generally, consider the sequence with $s_0 = 0, s_1 = 1$ but now $s_{n+2} = as_{n+1} + bs_n$. Let $\rho_{a,b;n} = s_n/s_{n-1}$; we include a and b in the subscripts for ρ to highlight how it may depend on the constants in the recurrence relation. Does the limit as $n \to \infty$ of $\rho_{a,b;n}$ exist, and if so what is it as a function of a and b? Again you're strongly encouraged to gather some data to make a conjecture. Choose nice values of a and b and see if you can sniff out the pattern; for example, try $a = 1$ and $b = 1 + m/4$ for some integer m (good choices include $m \in \{1, 2, 4, 11\}$).
- Instead of changing the coefficients of the recurrence we could change the initial conditions. Let $\mathcal{F}_{a,b;0} = a$, $\mathcal{F}_{a,b;1} = b$ and $\mathcal{F}_{a,b;n+2} = \mathcal{F}_{a,b;n+1} + \mathcal{F}_{a,b;n}$, and set $\mathfrak{r}_{a,b;n} = \mathcal{F}_{a,b;n}/\mathcal{F}_{a,b,n-1}$. Does $\lim_{n\to\infty} \mathfrak{r}_{a,b,n}$ exist, and if yes how does it depend on a and b?
- Try to find other generalizations of these questions to explore.

3.1 Trading

The golden ratio, often applied in trading and technical analysis, is used to predict market movements due to its psychological influence on herd behavior. Traders frequently rely on Fibonacci-based strategies, which revolve around key percentages derived from the golden ratio, such as 38.2%, 50%, and 61.8%. This reliance can create a self-fulfilling prophecy, making these strategies more effective as more traders use them. Fibonacci techniques are applied in various ways, including retracements, arcs, fans, and time zones, to identify potential support and resistance levels on charts, guiding traders in making informed decisions; see [Ku22].

3.2 Music

The Fibonacci sequence and the golden ratio also play a significant role in Western music, impacting everything from the structure of scales to the composition of classical pieces. For example, an octave on a piano contains 13 notes, with 8 white and 5 black keys, reflecting Fibonacci numbers. In music theory, the 5^{th} and 8^{th} notes in a scale create harmony, with the ratio 13/8 closely approximating the golden ratio (1.625 versus approximately 1.618, a difference of less than half a percent). Composers like Mozart have used this ratio in their works, arranging sonatas so that the division of bars aligns with the golden ratio, as seen in his Piano Sonata No. 1 in C Major (the one Sophia plays during a conservatory competition in Amor Towles's *A Gentleman in Moscow*). The golden ratio's influence can be seen in modern music as well, with artists like Lady Gaga incorporating it into their compositions, such as the key change in her song "Perfect Illusion" at the golden ratio point of the track; see [Pi22].

3.3 Biology

The fascinating connection between the Fibonacci numbers and various phenomena has also been observed in the natural world. As Cristóbal Vila demonstrated in his project *Nature by Numbers*, the arrangement of leaves in plants (phyllotaxis) is often linked to the golden angle derived from Fibonacci numbers, optimizing their exposure to sunlight and rain (although this view is debated). Research has also shown that Fibonacci patterns can be replicated in both biological systems, like the arrangement of cactus spines, and artificial systems, such as stress patterns in microstructures. Additionally, Fibonacci sequences have been observed in genetic codes and even in a novel self-replicating system designed without the need for proteins; see [Pl18].

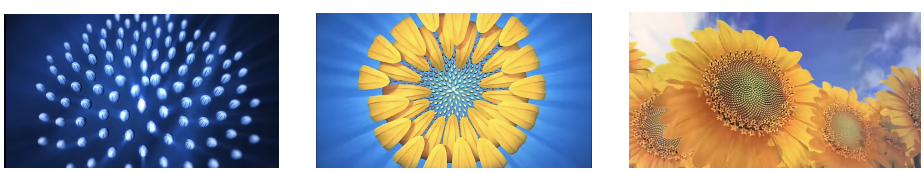

Figure 1: Visualization of the connection between the arrangement of seeds and petals and the golden angle, taken from the video *Nature by Numbers*.

However, even though the golden ratio is frequently observed in biological phenomena, new research using chemically realistic self-replicating systems indicates other algebraic numbers[2] – such as $2^{(1/3)} \approx 1.25992\ldots$ and $1.22074\ldots$,

[2] A number is algebraic if it's a root of a polynomial of finite degree and integer coefficients, otherwise the number is transcendental. Note the golden mean ϕ solves $x^2 - x - 1 = 0$ and is thus algebraic. Not all infinities are equal; though both sets are infinite, there are "more" transcendentals than algebraic numbers. While almost every number is transcendental, proving specific numbers are transcendental is hard (such as e, π and Liouville's number $\sum_{n=0}^{\infty} 10^{-n!}$); see [MTB06]

which is also known as the third lower golden ratio – are more common in these systems. These "universal constants" are derived from specific equations and often reflect idealized scenarios with unlimited resources, which is unlikely in the real world. Consequently, the golden ratio may not be as universally significant in self-replicating systems as previously thought, yet it occurs quite frequently; see [Li18].

3.4 Physics

The golden ratio has been discovered in atomic physics related to the Bohr radius and the bond-valence parameters of hydrogen bonds in specific borates. Such conjurations reveal a strict relation between the golden ratio and atomic structures, consequently affecting how atoms bond and place themselves in these materials; see [Pl18].

Another interesting piece of research on cobalt niobate, a material known to be magnetic and have a chain of linked atoms, showed an exciting relationship between quantum systems and the golden ratio. If this material is put in a magnetic field, it attains a 'quantum critical' state similar to that of a quantum fractal pattern. An earlier study found that the resonant frequencies of the material are in the golden ratio, which indicates a deep and hidden symmetry of the quantum system; see [He]. This finding confirms the ability of quantum mechanics to expose fundamental and elegant relations which, in their turn, could indeed influence modern technological achievements.

The Fibonacci sequence also arises in the pattern of period-doubling leading into chaos, known as Feigenbaum scaling. This pattern is seen in many systems, like turbulent flows, chemical reactions, and biological processes, showing a connection between the golden ratio and the start of chaotic and complex behaviors; see [Pl18].

3.5 Astrophysics

Recent research has shown intriguing connections between the golden ratio and astrophysical phenomena. In the study of pulsating stars, scientists using data from the Kepler space telescope found that some stars exhibit brightness fluctuations at frequencies that are close to the golden ratio. These pulsations, driven by a nonlinear dynamical system with irrational frequency ratios, often result in a strange but nonchaotic attractor, revealing the golden ratio's influence in the behavior of stellar systems; see [Pl18].

In black hole physics, the golden ratio also plays a significant role. Black holes, particularly spinning Kerr black holes, are known for their complex thermodynamic properties. Research has shown that the ratio of a black hole's angular momentum squared to its mass raised to the fourth power is the inverse of the golden ratio when considering quantum effects, such as Hawking radiation. This relationship highlights the fundamental and unexpected presence of the golden ratio in the structure and behavior of black holes, connecting it to both classical and quantum realms of physics; see [Pl18].

Furthermore, we see the golden ratio and its associated Fibonacci sequence playing a crucial role in the stability and organization of celestial mechanics. Recent studies suggest that gravitational resonance, a fundamental force responsible for the cohesion of planetary systems, is closely linked to harmonic configurations that mirror the Fibonacci series. Remarkably, these correlations are more pronounced when the Fibonacci sequence is expressed in a period of 24 hours rather than the traditional 365-day year. This finding applies to various celestial cycles, including rotation, precession, and orbital patterns, indicating that the long-term stability of the solar system may be inherently connected to the Fibonacci sequence. These findings deepen our theoretical understanding of gravitational resonance and suggest that the golden ratio and Fibonacci numbers are integral to the underlying order of the cosmos; see [Sa19].

4 Voronoi Tessellations

Voronoi tessellation is a mathematical method used to divide a plane into regions based on the distance to a specific set of points. Assume we have a set of points scattered across a surface; the Voronoi tessellation partitions the surface into regions where each region contains all the points closer to one particular point than to any other. This creates a grid of polygons, each surrounding one of the initial points, such that any location within a polygon is nearer to its associated point than to any other.

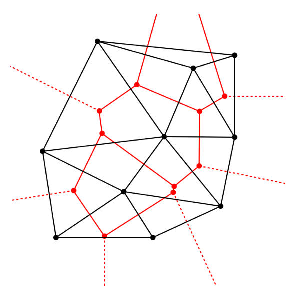

Figure 2: Voronoi tessellation and its Delaunay triangulation— image from [Wi11].

Now that we have a clear definition of what the Voronoi diagram is, we can discuss how to compute it. Many algorithms can accomplish this task, but the most accessible approach involves computing the Delaunay triangulation of our set of points first. The Delaunay triangulation is a collection of triangles where each triangle's vertices are the original set of points, with the key condition that no point lies within the circumcircle (the circle going through all the points on the polygon) of any triangle.[3] The circumcircles' origins are crucial for constructing the Voronoi diagram. By comparing the Voronoi diagram

[3] Prove or disprove the following geometric claim: given any triangle, there is a unique circumcircle going through its three vertices. Do you think there is always a circumcircle for

and its Delaunay triangulation, you can observe that each circumcircle in the triangulation corresponds to a vertex in the Voronoi diagram. Thus, with a Delaunay triangulation at hand, constructing the Voronoi diagram is straightforward: just draw edges between neighboring triangles' circumcenters; we give an example in Figure 3. The Bowyer-Watson algorithm is one effective method for computing the Delaunay triangulation, where points are iteratively added to an initial triangulation, and triangles are adjusted as necessary to maintain the Delaunay property; see [Hr24].

Figure 3: Voronoi Diagram for the Lower 48 States and Major Leagxue Baseball, from Jason Davies: https://www.jasondavies.com/maps/voronoi/mlb/ (fortunately the school of the first and third named authors is not part of the Evil Empire).

4.1 Computer Vision

The practical benefits of Voronoi diagrams extend beyond theoretical interest, being very useful in fields like computer vision, where they enhance our ability to analyze and interpret shapes. Once we've computed the Voronoi diagram, we can create Voronoi skeletons, which are crucial for precise and resilient shape representation. This approach ensures that the resulting skeleton not only maintains strong connectivity but also remains stable even when the shape's boundaries are slightly altered. By capturing the core structure of the shape and systematically pruning redundant edges, the Voronoi skeletons provide a streamlined and accurate depiction. This method proves best in reli-

a polygon with four vertices? If not, what additional condition on the vertices / shape is needed?

ably representing both simple and complex shapes, including those with holes; see [Ma96].

In facial landmark detection, Delaunay triangulations and Voronoi diagrams play a crucial role in accurately mapping and analyzing facial features. The Delaunay triangulation helps by creating a network of triangles that connect a set of facial landmarks, ensuring that no landmark lies within the circumcircle of any triangle, which avoids overfitting and provides a clear, non-overlapping mesh of the facial surface. This triangulated network is essential for tasks such as facial alignment and feature extraction. On the other hand, the Voronoi diagram, derived from the Delaunay triangulation, segments the facial area into regions closest to each landmark. This segmentation helps in localizing specific facial regions and enhancing the precision of feature detection by defining boundaries around each landmark's influence. Together, these techniques help more effective facial recognition, expression analysis, and image processing by structuring and contextualizing facial data; see [NO15].

This connection between Voronoi diagrams and Delaunay triangulation is not a coincidence. Delaunay's Ph.D. advisor was Georgy Voronoy, the mathematician after whom Voronoi diagrams are named. An interesting bit of trivia is that Boris Nikolaevich Delaunay, a Russian mathematician, used two different spellings for his last name: "Delaunay" for French and German publications and "Delone" for others. Remarkably, Delaunay has two mathematical concepts named after him, one for each spelling: Delaunay triangulation and Delone sets; see [NO15].

One of the interesting implementations of Voronoi diagrams in computer vision is NPC (non-player character) pathfinding in game development. In modern game engines, popular navigation methods like grid-based pathfinding, waypoint navigation, and navigation meshes often fall short in terms of performance and path quality. Unreal Engine's reliance on navigation meshes, for instance, is limited by its focus on the shortest path and poor parameterization.

A Voronoi diagram-based pathfinding algorithm addresses these issues by offering enhanced flexibility and realism. It introduces parameters like penalties for path rotation, crouching, and poor visibility, allowing for more nuanced pathfinding. The algorithm also supports dynamic adjustments during gameplay, enabling AI to react to real-time environmental changes. The IVoronoiQuerier interface, developed by a Russian student, further allows for tailored bot behavior, making navigation decisions more contextually relevant. By combining the strengths of traditional methods, while eliminating their weaknesses, the Voronoi-based approach helps to significantly improve the effectiveness of AI pathfinding, making gaming more interesting and realistic; see [HS15].

4.2 Sports

Voronoi diagrams can also be used to analyze spatial dynamics in sports, particularly in football, where understanding space and positioning is extremely important for both defensive and offensive strategies. Traditionally, football and fútbol statistics have focused on individual actions, such as tackles or goals, without fully considering the broader context of player positioning. However, the use of Voronoi diagrams provides a new dimension to these analyses by visualizing the space each player controls relative to their opponents. This approach allows for a more nuanced evaluation of a player's effectiveness in controlling space, which is especially valuable when assessing defenders whose primary role is to prevent attackers from exploiting dangerous areas. For example, the analysis of Brazil's defensive breakdown during their infamous 7-1 loss to Germany in the 2014 World Cup revealed how poor positioning, visualized through Voronoi diagrams, contributed to the team's collapse; see [Ta19].

In addition to enhancing our understanding of defensive strategies, Voronoi diagrams also have significant implications for attacking play. By mapping the areas controlled by each player, teams can evaluate how effectively they create and exploit space, which is key to breaking down organized defenses. For example, by stretching the pitch and moving the ball to areas with fewer defenders, teams can create opportunities for their most dangerous players. The randomness of player movement is far from chaotic; it's often a calculated response to the positioning of opponents, teammates, and the ball. Voronoi diagrams create a visual and analytical tool to capture these dynamics, offering insights into how teams and players navigate and manipulate space to gain a competitive advantage. Whether in football, basketball, or other sports with spatial elements, these diagrams are a crucial asset for coaches and analysts, who want to develop strategies that maximize spatial efficiency on the field; see [Ta19].

4.3 Nature

As it was shown in *Nature by Numbers*, Voronoi tessellation patterns are widespread in nature, appearing in structures like dragonfly wings, onion skin cells, jackfruit shells, and giraffe coats. Their prevalence can be attributed to their efficiency: Voronoi diagrams partition space completely, leaving no unused areas. This makes them ideal for optimizing space in systems like muscle fibers or beehives. These patterns also naturally emerge when growth occurs uniformly from distinct points. For example, the pattern of giraffe spots arises during embryonic development, where melanin-secreting cells are distributed randomly and radiate pigment outward over time; see [Bu23].

4.4 Urban Planning

Voronoi diagrams can be effectively applied to solve hybrid problems for placing facilities that need to adapt to changing needs by helping to determine optimal placement based on proximity and service areas. In a hybrid facility location problem, facilities not only need to be strategically placed to minimize distance to customers but also to dynamically respond to the distribution and demands of those customers. Attentive facilities, which can adapt or move based on real-time needs, require careful consideration of how they divide and cover a service area. By partitioning a plane into regions based on distance to predefined points (representing facilities), Voronoi diagrams can visualize the areas of influence for each facility. This helps in identifying optimal locations that minimize travel distances for clients while ensuring that the coverage area is well-balanced and responsive to changes in demand. Additionally, as demands shift, the Voronoi regions can be recalculated, enabling a continuous optimization process where facilities can adjust their locations or service areas in real-time to maintain efficient service delivery; see [Di13].

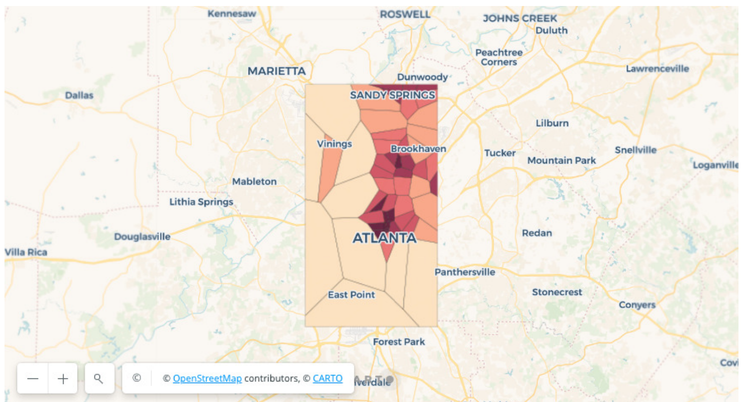

Figure 4: Coverage Areas of Starbucks Stores in Atlanta, image from [Ca24].

For example, we can analyze the coverage areas of Starbucks stores in Atlanta, with darker areas reflecting smaller areas of influence as shown in Figure 4. This simple approach helps us quickly spot areas where coverage could be improved and where new stores might be needed; see [Ca24].

5 Conclusion

Overall, the study of Fibonacci sequences, the golden ratio, and Voronoi tessellations uncovers the ways in which some mathematical concepts transcend their aesthetic value and leave their mark in a variety of fields: from finance to biology to physics, and beyond. From the rhythm of the stock market to the harmony of music, from growth patterns in nature to the very structure of the cosmos, we're given a window into the underpinning order of our world. The applications of Voronoi diagrams in areas like computer vision, sports analysis, and urban planning show that it has deep utility for answering complex real-world questions. As we look deeper beneath the surface of these and related ideas (we have only hit on a few of the many topics in the wonderful video), we discover more connections that show the universality and utility of these mathematical principles.

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Key Words

Fibonacci Sequence; Golden Ratio; Voronoi Tessellations.