Anil,

The informal write-up below is not optimized for elegance but simply to meet your request that I produce the proof that Stan told me (and many others, in my presence). He said he discovered it in the early 1950s (but I leave all this for you to remember and to judge).

Obviously, I did not produce anything that could be checked by a machine (I assume, of course, that you are not a machine). I have "lifted" a couple nice drawings from the site www.divisbyzero.com and the page http://divisbyzero.com/2009/10/06/tennenbaums-proof-of-the-irrationality-of-the-square-root-of-2/ .

It would be fabulous if you lent support here to the history of science. The current issues are miniscule, obviously—certainly compared pressing realities-- but nobody can predict what lies ahead.

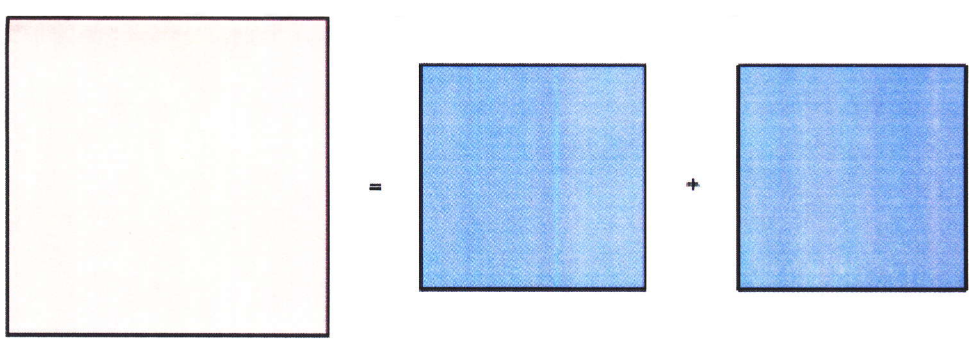
Peter

Theorem (Pythagoreans circa 500 BC): The square root of two is irrational.

Geometric Proof (Stanley Tennenbaum circa 1950 – 1954, as presented by his younger son, Peter Tennenbaum, December 12, 2009):

This is a proof by contradiction. Two slight variants exist; both are given.

Assume that a and b are positive integers such that $(a/b) = \sqrt{2}$. Then $a^2 = 2b^2$. Geometrically, this means there exists a square with sides of length a (depicted in pink below) whose AREA equals the sum of two smaller squares (depicted in blue) that have sides of length b.



Assume that the a, above, is the smallest integer that solves the equation.

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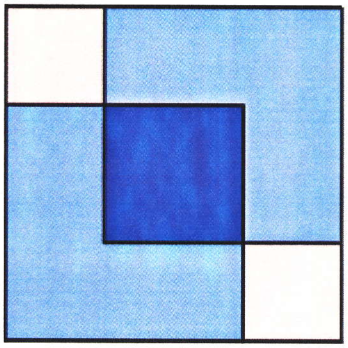
Here is the key idea of Stanley Tennenbaum's from which everything else flows naturally:

Place one of the smaller squares in the upper right hand corner of larger square and the other one in the lower left hand corner. Pictorially this looks as follows:



Note that the two smaller squares overlap in the dark blue area. As an aside, it is trivial to prove that they MUST overlap. Otherwise this proof via pictures might be fraudulent and, if so, should be added to the long list of examples used to demonstrate how pictorial methods alone can lead to erroneous mathematical conclusions.

The area of the dark blue overlap must equal the sum of the non-covered pink areas in the upper left and lower right because the original assumption implies that the area of the two blue squares equals the area of the larger pink square. A moment's reflection reveals that the dark blue overlap is indeed a square with integral sides and the upper and lower pink areas are two identical squares having integral sides. But the dark blue square is smaller than the original pink square of side a. This contradicts the assumption above, that a is the smallest integral solution that solves the equation. Therefore, no such rational number exits and the square root of two is irrational.

To obtain a slight variant of the proof, drop the assumption that the original a is the smallest integral solution to the original equation. Clearly, the basic method can be repeated ad infinitum. For example, simply place the two pink squares, above, into the larger dark blue square. As before, this generates a square that is the sum of two identical smaller squares. Repetition yields a (never ending) strictly decreasing sequence of squares, a method akin to Fermat's method of infinite decent. Regardless of where one begins, this cannot continue forever while still satisfying the assumption that the a in the original equation is positive. Q.E.D.

Dec 12, 2009.

This proof was shown to me by Stanley Tennenbaum in about 1952 in his apartment in Hyde Park. He showed it to many mathematicians at that time, and noone could find an earlier exposition by any one else. He taught it to numerous children as well, using cardboard + wooden cut outs.

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