On a Pair of Diophantine Equations

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Abstract. For relatively prime natural numbers a and b, we study the two equations $ax + by = (a-1)(b-1)/2$ and $ax + by + 1 = (a-1)(b-1)/2$, which arise from the study of cyclotomic polynomials. Previous work showed that exactly one equation has a nonnegative solution, and the solution is unique. Our first result gives criteria to determine which equation is used for a given pair (a, b). We then use the criteria to study the sequence of equations used by the pair $(a_n/\gcd(a_n, a_{n+1}), a_{n+1}/\gcd(a_n, a_{n+1}))$ from several special sequences $(a_n)_{n\ge 1}$. Finally, fixing $k \in \mathbb{N}$, we investigate the periodicity of the sequence of equations used by the pair $(k/\gcd(k, n), n/\gcd(k, n))$ as n increases.

Keywords: Diophantine equation, sequence, periodic

1 Introduction

Given $a, b \in \mathbb{N}$, the following two Diophatine equations arise in the study of cyclotomic polynomials $\Phi_{pq}(x)$ [7] for primes $p < q$ [2]:

$$ax + by = \frac{(a-1)(b-1)}{2}, \quad (1)$$

$$ax + by + 1 = \frac{(a-1)(b-1)}{2}. \quad (2)$$

Beiter [2] showed that if a, b are primes, then exactly one of the equations has a nonnegative integral solution (x, y). Chu [3, Theorem 1.1] extended the result to any pair of relatively prime numbers (a, b); furthermore, the solution is unique. For a fixed pair of relatively prime numbers $(a, b) \in \mathbb{N}^2$, we say that (a, b) uses Equation (1) if (1) has a nonnegative solution; otherwise, we say that (a, b) uses Equation (2). As a corollary of [3, Theorem 1.1], Chu considered which equation is used by the pairs (F_n, F_{n+1}) and (F_n, F_{n+2}), where $(F_n)_{\geq 1}$ is the Fibonacci sequence[8], and established new identities involving Fibonacci numbers. In particular, [3, Theorems 1.4 and 1.6] stated that the pair (F_n, F_{n+1}) uses (1) and (2) alternatively in groups of three; the same conclusion holds for (F_n, F_{n+2}). Following the work, Chen et al. studied which equation is used by (F_n^2, F_{n+1}^2) and (F_n^3, F_{n+1}^3) and discovered the following identities: for $n \geq 2$,

$$1 + \frac{F_n^2 - 3}{2} F_n^2 + \frac{F_n^2 - F_{n-1}^2 - 1}{2} F_{n+1}^2 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}, \text{ if } n \text{ is odd},$$

$$1 + \frac{F_n^2 + 1}{2} F_n^2 + \frac{F_n^2 - F_{n-1}^2 - 1}{2} F_{n+1}^2 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}, \text{ if } n \text{ is even},$$

$$\left(F_n^2 - \frac{F_{n-1}^2 + 1}{2}\right) F_n^2 + \frac{F_{n-1}^2 - 1}{2} F_{n+1}^2 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2},$$

and

$$\left(\sum_{k=1}^{2n-1} (-1)^{k-1} F_k^3\right) F_{2n-1}^3 + \left(\sum_{k=2}^{2n-2} F_k^3\right) F_{2n}^3 = \frac{(F_{2n-1}^3 - 1)(F_{2n}^3 - 1)}{2},$$

$$1 + \left(\sum_{k=1}^{2n} (-1)^k F_k^3 - 1\right) F_{2n}^3 + \left(\sum_{k=2}^{2n-1} F_k^3\right) F_{2n+1}^3 = \frac{(F_{2n}^3 - 1)(F_{2n+1}^3 - 1)}{2}.$$

Continuing these work, the present paper studies the sequence of equations used by consecutive terms of some special sequences $(a_n)_{n\geq 1}$. Moreover, we avoid requiring consecutive terms of $(a_n)_{n\geq 1}$ to be relatively prime (as in the case of $(F_n)_{n\geq 1}$) by considering $(a_n/d, a_{n+1}/d)$ for $d = \gcd(a_n, a_{n+1})$ instead. This was recently done by Davala [6] for (B_n, B_{n+2}), where $(B_n)_{\geq 1}$ is the balancing

[7] The equations are used in calculating the midterm coefficient of $\Phi_{pq}(x)$ (see [2, pp. 770]).

[8] Note that $\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = 1$.

sequence[9]. Here $\gcd(B_n, B_{n+2}) = 6$. The main goal of the present paper is to treat the above-mentioned problem systematically. We first show a method to tell which equation is used by two arbitrary positive integers a and b, then apply the method to prove the pattern of equations used by various sequences.

For convenience, we introduce the following notation. Let $\Gamma(a,b)$ be the equation that the pair $(a/d, b/d)$ uses, where $d = \gcd(a,b)$. In particular, $\Gamma(a,b) = 1$ if $(a/d, b/d)$ uses (1); otherwise, $\Gamma(a,b) = 2$. For $b/d > 1$, define $\Theta(a,b)$ to be the unique multiplicative inverse of a/d modulo b/d such that $0 < \Theta(a,b) < b/d$. We are ready to state the first result that relates Γ to Θ. Note that unlike Γ, $\Theta(a,b)$ is not necessarily equal to $\Theta(b,a)$.

Theorem 1. *Let $a, b \in \mathbb{N}$. If a divides b or b divides a, then $\Gamma(a,b) = 1$. Otherwise,*

1. *if $a/\gcd(a,b)$ is odd, then $\Gamma(a,b) = 1$ if and only if $\Theta(b,a)$ is odd;*
2. *if $a/\gcd(a,b)$ is even, then $\Gamma(a,b) = 1$ if and only if $\Theta(a,b)$ is odd.*

Remark 1. Given $a, b \in \mathbb{N}$ with $b/\gcd(a,b) > 1$, we can find $\Theta(a,b)$ using Euler's Theorem. In particular, Euler's Theorem implies that

$$(a/d)^{\phi(b/d)-1} \times (a/d) \equiv 1 \mod b/d,$$

where $d = \gcd(a,b)$ and ϕ is the Euler totient function. Write $(a/d)^{\phi(b/d)-1} = (b/d)\ell + r$ for some $\ell \geq 0$ and $0 < r < b/d$. Then $\Theta(a,b) = r$.

For example, let $a = 15$ and $b = 85$. We have $d = \gcd(15, 85) = 5$, $\phi(b/d) = 16$, and $(a/d)^{\phi(b/d)-1} = 3^{15}$. Since $3^{15} = 17 \times 844053 + 6$, we obtain $\Theta(a,b) = 6$.

Given a sequence $(a_n)_n \subset \mathbb{N}$, let $\Delta((a_n)_n) = (\Gamma(a_n, a_{n+1}))_n$, i.e., Δ gives the sequence of equations used by consecutive terms of $(a_n)_n$. Inspired by [3, Theorem 1.4] that for $\Delta((F_n)_n)$, each equation appears in groups of three alternatively, we construct a sequence whose Δ has each equation appear in groups of k alternatively ($k \in \mathbb{N}$).

Theorem 2. *Fix $k \in \mathbb{N}$ and let $a_n = (\lceil 2^{n+k-1}/(2^k+1) \rceil)_{n\geq 1}$. Then $\Delta((a_n)_n)$ is*

$$\underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \underbrace{1, \ldots, 1}_{k}, \underbrace{2, \ldots, 2}_{k}, \ldots .$$

Remark 2. The case $k = 1$ gives us the sequence $(\lceil 2^n/3 \rceil)_{n\geq 1}$, which is A005578 in the Online Encyclopedia of Integer Sequences (OEIS) [7]. The case $k = 2$ gives $(\lceil 2^{n+1}/5 \rceil)_{n\geq 1}$, which appears to be unavailable in the encyclopedia at the time of the current writing.

[9] Balancing numbers were introduced by Behera and Panda [1] to be solutions of the Diophantine equation $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$ for some natural number r.

Next, for a fixed $k \in \mathbb{N}$, we investigate $\Delta((n^k)_{n\geq 1})$. When $k = 1$, $\Delta(\mathbb{N})$ has the simple form $1, 2, 1, 2, 1, 2, \ldots$. Interestingly, for $k > 1$, $\Delta((n^k)_{\geq 1})$ appears not to have a nice pattern among the first few terms. Let us take a look at $\Delta((n^4)_{n\geq 1})$, for example

$$1, 2, 1, 1, 1, 2, 2, 1, 2, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots ;$$

however, we have the following result about the pattern of $\Delta((n^k)_{n\geq 1})$ in the long run.

Theorem 3. *Fix $k \in \mathbb{N}$. Then $\Delta((n^k)_n)$ is eventually $1, 2, 1, 2, 1, 2, \ldots$. In particular, define*

$$g(x) \; = \; \left(\sum_{i=1}^{k} x^{i-1}\right)^k \quad \bmod x^k .$$

When k is odd, let M_k be the smallest positive integer such that

$$0 \; < \; n^k - g(n) \; < \; n^k \text{ and } 0 \; < \; g(-n) \; < \; n^k, \text{ for all } n \geq M_k ;$$

when k is even, let M_k be the smallest positive integer such that

$$0 \; < \; n^k + g(-n) \; < \; n^k \text{ and } 0 \; < \; g(n) \; < \; n^k, \text{ for all } n \geq M_k .$$

Then the sequence $(\Gamma(n^k, (n+1)^k))_{n\geq 1}$ starts to be $1, 2, 1, 2, 1, 2, \ldots$ at $n \leq M_k + 1$.

The next natural sequences to consider are arithmetic progressions of the form $(a + (n-1)r)_{n\geq 1}$ for fixed $a, r \in \mathbb{N}$. We do not consider geometric progressions with an integral multiplier because every given term in the sequence divides the next term; hence, the sequence Δ is constantly 1 according to Theorem 1. We instead investigate $\Delta((a_n)_{n\geq 1})$, where $(a_n)_n$ is a shifted geometric sequence, i.e., $a_n = ar^{n-1} + 1$ for some $a, r \in \mathbb{N}$.

Theorem 4. *Let $(a_n)_{n\geq 1}$ be an arithmetic progression. Then $\Delta((a_n)_n)$ is either $1, 2, 1, 2, \ldots$ or $2, 1, 2, 1, \ldots$.*

Theorem 5. *Let $a, r \in \mathbb{N}$ with $r \geq 2$. For each $n \geq 1$, define the sequence $(a_n)_n = (ar^{n-1} + 1)_n$. Set $d = \gcd(a + 1, r - 1)$.*

1. *Suppose that d is odd. If $a + 1$ does not divide $r - 1$, then $\Delta((a_n)_{n\geq 1})$ is constant. If $a + 1$ divides $r - 1$, then $\Delta((a_n)_{n\geq 1})$ has the first term be 1 and the later terms be 2.*
2. *If d is even, then $\Delta((a_n)_{n\geq 2})$ alternates between 1 and 2.*

What we have done so far is to determine $\Delta((a_n)_n)$, for a fixed sequence $(a_n)_n$. Note that $\Delta((a_n)_n)$ is the sequence $(\Gamma(a_n, a_{n+1}))_{n\geq 1}$, so both inputs of the function Γ change as we move along the sequence. Our final result instead considers the case when one of the parameters is fixed. Specifically, we study properties of the sequence $(\Gamma(k, n))_{n\geq 1}$ when $k \in \mathbb{N}$ is given. Given a sequence $(a_n)_n$, let T be the smallest positive integer (if any) such that $a_n = a_{n+T}$ for all $n \geq 1$; the number T is called the period of $(a_n)_n$.

Theorem 6. *Fix $k \in \mathbb{N}$. For $n \geq 1$, let $a_n = \Gamma(k, n)$. The following hold.*

1. *If k is odd, $(a_n)_n$ has period k; furthermore, in each period, the number of 1's is one more than the number of 2's.*
2. *If k is even, $(a_n)_n$ has period $2k$; furthermore, in each period, the number of 1's is two more than the number of 2's.*

Our paper is structured as follows. In Sect. 2, we prove Theorem 1, which gives a way to compute $\Gamma(a, b)$ using $\Theta(a, b)$, and provide some preliminary results for our later proofs. In Sect. 3, we use the framework developed in Sect. 2 to investigate $\Delta((a_n)_n)$ for some special sequences $(a_n)_n$. In Sect. 4, we prove Theorem 6, which describes the period of the sequence $(\Gamma(k, n)_n)_{n\geq 1}$ for any fixed $k \in \mathbb{N}$. In Sect. 5, we study sequences satisfying certain linear recurrence relations of order two, thus extending [3, Theorem 1.4].

2 On $\Gamma(a, b)$ and Preliminary Results

The main goal of this section is to prove Theorem 1, which tells us which equation is used by two arbitrary numbers a, b, given the parity of $a/\gcd(a, b)$ and of Θ. The key idea is to use the modulo argument to express Θ in terms of the solution of (1) or (2). We then present other useful results to be used in later sections.

Proof (Proof of Theorem 1). We can rewrite (1) and (2) as

$$a(2x + 1) + b(2y + 1) \;=\; ab + \ell\,, \tag{3}$$

where $\ell = 1$ and -1, respectively.

If a divides b, then $\Gamma(a, b) = \Gamma(1, b/a)$. Note that $(0, 0)$ is the solution of

$$1 \times (2x + 1) + \frac{b}{a} \times (2y + 1) \;=\; 1 \times \frac{b}{a} + 1\,.$$

Hence, $\Gamma(a, b) = 1$. Similarly, if b divides a, $\Gamma(a, b) = 1$.

Next, we prove Item (1). Let $d = \gcd(a, b)$. Suppose that $d \notin \{a, b\}$ and a/d is odd. Assuming $\Gamma(a, b) = 1$, we need to show $\Theta(b, a)$ is odd. By (3), there exist nonnegative integers x and y such that

$$\frac{a}{d}(2x + 1) + \frac{b}{d}(2y + 1) \;=\; \frac{ab}{d^2} + 1\,. \tag{4}$$

Taking modulo a/d gives

$$\frac{b}{d}(2y + 1) \;\equiv\; 1 \mod \frac{a}{d}\,.$$

Hence,

$$\Theta(b, a)\frac{b}{d}(2y + 1) \;\equiv\; \Theta(b, a) \mod \frac{a}{d}\,.$$

By definition, $\Theta(b,a)b/d \equiv 1 \mod a/d$; therefore,

$$\Theta(b,a) \equiv 2y+1 \mod \frac{a}{d}. \tag{5}$$

We claim that $\Theta(b,a) = 2y+1$. To do so, it suffices to verify that $0 < 2y+1 < a/d$. Multiplying both sides of (4) by d/b, we have

$$\frac{a}{b}(2x+1) + 2y + 1 = \frac{a}{d} + \frac{d}{b}. \tag{6}$$

Since $d < a$, (6) implies that

$$\frac{d}{b}(2x+1) + 2y + 1 < \frac{a}{d} + \frac{d}{b},$$

which gives

$$0 < 2y+1 < \frac{a}{d},$$

as desired.

Conversely, assuming that $\Gamma(a,b) = 2$, we prove that $\Theta(b,a)$ is even. By (3), there exist nonnegative integers x and y such that

$$\frac{a}{d}(2x+1) + \frac{b}{d}(2y+1) = \frac{ab}{d^2} - 1. \tag{7}$$

Taking modulo a/d then multiplying both sides by $\Theta(b,a)$ gives

$$\Theta(b,a)\frac{b}{d}(2y+1) \equiv -\Theta(b,a) \mod \frac{a}{d}.$$

Hence,

$$2y+1 \equiv \frac{a}{d} - \Theta(b,a) \mod \frac{a}{d}. \tag{8}$$

It is easily seen from (7) that $0 < 2y+1 < a/d$. It follows from the definition of Θ that $0 < a/d - \Theta(b,a) < a/d$. Therefore, (8) implies that $2y+1 = a/d - \Theta(b,a)$. Since a/d is odd, $\Theta(b,a)$ must be even. This completes our proof of Item (1).

It remains to prove Item (2), but (2) follows directly from Item (1). Indeed, suppose that a/d is even. Since $\gcd(a/d, b/d) = 1$, we know that b/d is odd. By Item (1), $\Gamma(a,b) = 1$ if and only if $\Theta(a,b)$ is odd.

The next two results are used in the proof of Theorem 2.

Corollary 1. *For $a \geq 1$, $\Gamma(a,2a) = 1$; for $a \geq 2$, $\Gamma(a,2a-1) = 2$.*

Proof. By Theorem 1, $\Gamma(a,2a) = 1$. To see that $\Gamma(a,2a-1) = 2$, we first compute $\Theta(2a-1,a) = a-1$ and $\Theta(a,2a-1) = 2$. If a is odd, then $\Theta(2a-1,a)$ is even; Theorem 1 states that $\Gamma(a,2a-1) = 2$. If a is even, then $\Gamma(a,2a-1) = 2$ because $\Theta(a,2a-1)$ is even.

Lemma 1. *Let $k, n \in \mathbb{N}$ and $m \in \{1, \ldots, 2k\}$ such that $n \equiv m \mod 2k$. Then*

$$2^{n+k-1} \equiv \begin{cases} -2^{m-1} \mod 2^k + 1\,, & \text{if } 1 \le m \le k\,, \\ 2^{m-k-1} \mod 2^k + 1\,, & \text{if } k+1 \le m \le 2k\,. \end{cases}$$

Proof. Write $n = 2kj + m$ for some $j \ge 0$.

If $1 \le m \le k$, then

$$2^{n+k-1} - 2^{m+k-1} = 2^{2kj+m+k-1} - 2^{m+k-1} = 2^{m+k-1}(2^{2kj} - 1)\,,$$

which is divisible by $2^k + 1$. Hence, $2^{n+k-1} \equiv 2^{m+k-1} \equiv -2^{m-1} \mod 2^k + 1$.

If $k + 1 \le m \le 2k$, then

$$2^{n+k-1} - 2^{m-k-1} = 2^{2kj+2k+m-k-1} - 2^{m-k-1} = 2^{m-k-1}(2^{2k(j+1)} - 1)\,,$$

which is divisible by $2^k + 1$. Hence, $2^{n+k-1} \equiv 2^{m-k-1} \mod 2^k + 1$.

Our next two lemmas show that for arithmetic progressions and shifted geometric progressions $(a_n)_n$, $\gcd(a_n, a_{n+1})$ is a constant.

Lemma 2. *Fix $a, r \in \mathbb{N}$. Let $a_n = a + (n-1)r$. Then* $\gcd(a_n, a_{n+1}) = \gcd(a, r)$.

Proof. Let $n \in \mathbb{N}$. We have that $\gcd(a_n, a_{n+1})$ divides $2a_{n+1} - a_n = a_{n+2}$. Hence, $\gcd(a_n, a_{n+1})$ divides $\gcd(a_{n+1}, a_{n+2})$. Conversely, $\gcd(a_{n+1}, a_{n+2})$ divides $2a_{n+1} - a_{n+2} = a_n$. Hence, $\gcd(a_{n+1}, a_{n+2})$ divides $\gcd(a_n, a_{n+1})$. Consequently,

$$\gcd(a_n, a_{n+1}) = \gcd(a_{n+1}, a_{n+2})\,.$$

Therefore, for all n, $\gcd(a_n, a_{n+1}) = \gcd(a_1, a_2) = \gcd(a, a + r) = \gcd(a, r)$.

Lemma 3. *Fix $a, r \in \mathbb{N}$ with $r \ge 2$. Let $a_n = ar^{n-1} + 1$. Then* $\gcd(a_n, a_{n+1}) = \gcd(a + 1, r - 1)$ *for all $n \in \mathbb{N}$.*

Proof. Let $k \ge 0$. We shall show that

$$\gcd(ar^k + 1, r - 1) = \gcd(a + 1, r - 1)\,. \tag{9}$$

The equality clearly holds when $k = 0$. Suppose that it holds for some $k = \ell \ge 0$. We have

$$\begin{aligned} \gcd(ar^{\ell+1} + 1, r - 1) &= \gcd(r(ar^\ell + 1), r - 1) \\ &= \gcd(ar^\ell + 1, r - 1) = \gcd(a + 1, r - 1)\,. \end{aligned}$$

By mathematical induction, we are done.

For $n \in \mathbb{N}$, by (9), we have

$$\begin{aligned} \gcd(a_n, a_{n+1}) &= \gcd(ar^{n-1} + 1, ar^n + 1) = \gcd(ar^{n-1} + 1, ar^{n-1}(r - 1)) \\ &= \gcd(ar^{n-1} + 1, r - 1) = \gcd(a + 1, r - 1)\,. \end{aligned}$$

3 The Sequence $\Delta((a_n)_n)$ for Some Special $(a_n)_n$

In this section, we look at various sequences $(a_n)_n$ and determine their $\Delta((a_n)_n)$. Results from Sect. 2 will be used in due course.

3.1 Sequences Whose Δ Has a Periodic Form

Proof (Proof of Theorem 2). Fix $k \in \mathbb{N}$. Let $a_n = \lceil 2^{n+k-1}/2^{k+1} \rceil$. Thanks to Corollary 1, it suffices to prove that $a_1 = 1$ and for $n \geq 1$,

$$a_{n+1} = \begin{cases} 2a_n\,, & \text{if } n \equiv 1\,, \ldots, k \mod 2k\,, \\ 2a_n - 1\,, & \text{if } n \equiv k+1, \ldots, 2k \mod 2k\,. \end{cases}$$

Fix $n \in \mathbb{N}$ and pick $m \in \{1, \ldots, 2k\}$ such that $n \equiv m \mod 2k$.

Case 1: $1 \leq m \leq k$. By Lemma 1, $2^{n+k-1} \equiv 2^k + 1 - 2^{m-1} \mod 2^k + 1$, so

$$\begin{aligned} 2^{n+k-1} &= (2^k+1)\left\lfloor \frac{2^{n+k-1}}{2^k+1} \right\rfloor + 2^k + 1 - 2^{m-1} \\ &= (2^k+1)\left(\left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil - 1\right) + 2^k + 1 - 2^{m-1}\,. \end{aligned}$$

Multiplying both sides by 2 gives

$$2^{n+k} = 2(2^k+1)\left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil - 2^m\,.$$

Hence,

$$\frac{2^{n+k}}{2^k+1} = 2\left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil - \frac{2^m}{2^k+1}\,.$$

Therefore,

$$\left\lceil \frac{2^{n+k}}{2^k+1} \right\rceil = 2\left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil\,,$$

i.e., $a_{n+1} = 2a_n$.

Case 2: $k+1 \leq m \leq 2k$. By Lemma 1, $2^{n+k-1} \equiv 2^{m-k-1} \mod 2^k + 1$, so

$$\begin{aligned} 2^{n+k-1} &= (2^k+1)\left\lfloor \frac{2^{n+k-1}}{2^k+1} \right\rfloor + 2^{m-k-1} \\ &= (2^k+1)\left(\left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil - 1\right) + 2^{m-k-1}. \end{aligned}$$

Multiplying both sides by 2 gives

$$2^{n+k} = (2^k+1)\left(2\left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil - 1\right) - (2^k + 1 - 2^{m-k}).$$

Hence,

$$\left\lceil \frac{2^{n+k}}{2^k+1} \right\rceil \;=\; 2\left\lceil \frac{2^{n+k-1}}{2^k+1} \right\rceil - 1\,,$$

i.e., $a_{n+1} = 2a_n - 1$. This completes our proof.

3.2 The k^{th}-Power Sequence

Proof (Proof of Theorem 3). Let us first assume that k is odd. Choose $n \in \mathbb{N}_{\geq 2}$. Let $x = \Theta((n-1)^k, n^k)$ and $y = \Theta((n+1)^k, n^k)$. Since $(n-1)^k x \equiv 1 \mod n^k$, we have, in modulo n^k,

$$(-1)^k x \;\equiv\; (n^k-1)^k x \;=\; \left(\sum_{i=1}^{k} n^{i-1}\right)^k (n-1)^k x \;\equiv\; \left(\sum_{i=1}^{k} n^{i-1}\right)^k . \tag{10}$$

As k is odd, it follows that

$$x \;\equiv\; n^k - \left(\sum_{i=1}^{k} n^{i-1}\right)^k \mod n^k\,. \tag{11}$$

Since $(n+1)^k y \equiv 1 \mod n^k$, we have, in modulo n^k,

$$y \;\equiv\; (n^k+1)^k y \;=\; \left(\sum_{i=1}^{k} (-1)^{i-1} n^{i-1}\right)^k (n+1)^k y \;\equiv\; \left(\sum_{i=1}^{k} (-1)^{i-1} n^{i-1}\right)^k . \tag{12}$$

Define $u(x) = \left(\sum_{i=1}^{k} x^{i-1}\right)^k$ and $v(x) = \left(\sum_{i=1}^{k} (-1)^{i-1} x^{i-1}\right)^k$. Since $u(x)+v(x) = u(-x)+v(-x)$, $u(x)+v(x)$ is an even function; therefore, the coefficients of odd powers in $u(x)$ are equal to the negative of the corresponding coefficients of $v(x)$. On the other hand, $u(x)-v(x)$ is an odd function, so the coefficients of even powers in $u(x)$ are equal to the corresponding even powers in $v(x)$. Therefore, if we let $g(x)$ be the tail of $u(x)$ up to the power $k-1$, then $g(-x)$ is the tail of $v(x)$ up to the power $k-1$. From (11) and (12), we have

$$x \;\equiv\; n^k - g(n) \text{ and } y \;\equiv\; g(-n) \mod n^k\,. \tag{13}$$

Since $\deg(g(n)) = \deg(g(-n)) = k-1$, we can choose $N_1 \in \mathbb{N}$ such that $n^k > g(n)$, for all $n \geq N_1$. Observe that the coefficient of x^{k-1} is positive in both $g(x)$ and $g(-x)$; hence, we can choose $N_2 \in \mathbb{N}$ such that $g(n), g(-n) > 0$ whenever $n \geq N_2$. Set $N = \max\{N_1, N_2\}$ to have

$$0 \;<\; n^k - g(n) \;<\; n^k \text{ and } 0 \;<\; g(-n) \;<\; n^k\,. \tag{14}$$

From (13) and (14), we obtain

$$x \;=\; n^k - g(n) \text{ and } y \;=\; g(-n)\,, \quad \text{for all } n \geq N\,. \tag{15}$$

Therefore,

$$x + y \;=\; n^k - (g(n) - g(-n))\,, \qquad \text{for all } n \geq N\,. \tag{16}$$

As discussed above, all coefficients of $g(x) - g(-x)$ are even. Hence, the parity of $x + y$ is the same as the parity of n^k.

Taking an even $M \geq N - 1$, we show that

$$\Gamma(M^k, (M+1)^k) \;\neq\; \Gamma((M+1)^k, (M+2)^k)\,.$$

To do so, we invoke Theorem 1. By Theorem 1, $\Gamma(M^k, (M+1)^k) = 1$ if and only if $\Theta(M^k, (M+1)^k)$ is odd; furthermore, $\Gamma((M+1)^k, (M+2)^k) = 1$ if and only if $\Theta((M+2)^k, (M+1)^k)$ is odd. By (16), $\Theta(M^k, (M+1)^k) + \Theta((M+2)^k, (M+1)^k)$ has the same parity as $(M+1)^k$, which is odd. Hence, $\Theta(M^k, (M+1)^k)$ and $\Theta((M+2)^k, (M+1)^k)$ have different parities and so, $\Gamma(M^k, (M+1)^k) \neq \Gamma((M+1)^k, (M+2)^k)$.

To show that Γ alternates between 1 and 2, it remains to verify that $\Theta(M^k, (M+1)^k)$ and $\Theta((M+2)^k, (M+3)^k)$ have the same parity. According to (15),

$$\begin{aligned} \Theta(M^k, (M+1)^k) \;&=\; (M+1)^k - g(M+1)\,, \\ \Theta((M+2)^k, (M+3)^k) \;&=\; (M+3)^k - g(M+3)\,. \end{aligned}$$

Hence, $\Theta((M+2)^k, (M+3)^k) - \Theta(M^k, (M+1)^k) = (M+3)^k - (M+1)^k - (g(M+3) - g(M+1))$, which is clearly even. It follows that $\Theta((M+2)^k, (M+3)^k)$ and $\Theta(M^k, (M+1)^k)$ have the same parity. Invoking Theorem 1, we finish the proof when k is odd.

Proof (Proof of Theorem 3 for even k). Let n, x, y, and $g(x)$ be chosen as in the proof of Theorem 3 for odd k. From (10) and that k is even, we know that

$$x \;\equiv\; g(n) \mod n^k\,.$$

On the other hand, in modulo n^k,

$$y \;=\; (-1)^k y \;\equiv\; (n^k - 1)^k y \;=\; \left(\sum_{i=1}^{k} (-1)^i n^{i-1}\right)^k (n+1)^k y \;\equiv\; \left(\sum_{i=1}^{k} (-1)^i n^{i-1}\right)^k .$$

Since k is even,

$$y \;\equiv\; \left(\sum_{i=1}^{k} (-1)^{i-1} n^{i-1}\right)^k \mod n^k\,;\ \text{hence, } y \;\equiv\; n^k + g(-n) \mod n^k\,.$$

Choose $M \in \mathbb{N}$ such that $0 < n^k + g(-n) < n^k$ and $0 < g(n) < n^k$ whenever $n \geq M$. This can be done since $\deg(g(n)) = \deg(g(-n)) = k - 1$, and the coefficients of n^{k-1} in $g(n)$ and $g(-n)$ are positive and negative, respectively. Therefore, for $n \geq M$, $x = g(n)$ and $y = n^k + g(-n)$. That $x + y = n^k + (g(n) + g(-n))$ shows that $x + y$ has the same parity as n^k. Using the same argument as in the proof of Theorem 3 for odd k completes the proof.

3.3 Arithmetic Progression

Proof (Proof of Theorem 4). For $a_n = a + r(n-1)$, using Lemma 2, we let $\gcd(a_n, a_{n+1}) = d$ for all $n \in \mathbb{N}$. Fix $n \geq 2$ and consider the consecutive terms $a_n, a_n + r, a_n + 2r$ in the sequence. Note that $a_n \nmid a_{n+1}$ because $n \geq 2$. Our goal is to show that $\Gamma(a_n, a_{n+2}) \neq \Gamma(a_{n+1}, a_{n+2})$.

Case 1: a_{n+1}/d is odd. Let $x = \theta(a_n, a_{n+1})$ and $y = \theta(a_{n+2}, a_{n+1})$. Then

$$\begin{aligned} \frac{a_n}{d} x &\equiv 1 \mod \frac{a_n}{d} + \frac{r}{d}, \\ \left(\frac{a_n}{d} + \frac{2r}{d}\right) y &\equiv 1 \mod \frac{a_n}{d} + \frac{r}{d}. \end{aligned}$$

These imply that

$$\frac{r}{d}\left(\frac{a_n}{d} + \frac{r}{d} - x\right) \equiv 1 \mod \frac{a_n}{d} + \frac{r}{d}, \tag{17}$$

$$\frac{r}{d} y \equiv 1 \mod \frac{a_n}{d} + \frac{r}{d}. \tag{18}$$

Since $0 < x < a_n/d + r/d$, we get

$$0 < \frac{a_n}{d} + \frac{r}{d} - x < \frac{a_n}{d} + \frac{r}{d}. \tag{19}$$

From (17), (18), and (19), we deduce that

$$y = \frac{a_n}{d} + \frac{r}{d} - x.$$

Hence, $x + y = (a_n + r)/d$. That $\gcd(a_n/d, r/d) = \gcd(a_n/d, (a_n + r)/d) = 1$ implies that $(a_n + r)/d$ is odd. Consequently, x and y have different parities. Using Theorem 1, we conclude that $\Gamma(a_n, a_{n+1}) \neq \Gamma(a_{n+1}, a_{n+2})$.

Case 2: a_{n+1}/d is even, then a_n/d and $a_n/d + 2r/d$ are both odd. Let $x = \theta(a_{n+1}, a_n)$ and $y = \theta(a_{n+1}, a_{n+2})$. Then

$$\begin{aligned} \left(\frac{a_n}{d} + \frac{r}{d}\right) x &\equiv 1 \mod \frac{a_n}{d}, \\ \left(\frac{a_n}{d} + \frac{r}{d}\right) y &\equiv 1 \mod \frac{a_n}{d} + \frac{2r}{d}. \end{aligned}$$

Equivalently, there exist positive integers $k_1, k_2 < (a_n + r)/d$ such that

$$\left(\frac{a_n}{d} + \frac{r}{d}\right) x = 1 + k_1 \frac{a_n}{d}, \tag{20}$$

$$\left(\frac{a_n}{d} + \frac{r}{d}\right) y = 1 + k_2 \left(\frac{a_n}{d} + \frac{2r}{d}\right). \tag{21}$$

Subtracting (20) and (21) side by side gives

$$\frac{a_n + r}{d}(x - y + 2k_2) = \frac{a_n}{d}(k_1 + k_2).$$

Since $\gcd((a_n+r)/d, a_n/d) = 1$, $(a_n+r)/d$ divides k_1+k_2. Observe that $0 < k_1+k_2 < 2(a_n+r)/d$ because $0 < k_1, k_2 < (a_n+r)/d$. Therefore,

$$k_1 + k_2 \;=\; \frac{a_n+r}{d} \text{ and } x - y + 2k_2 = \frac{a_n}{d}\,.$$

Since a_n/d is odd, x and y must have different parities. We again have $\Gamma(a_n, a_{n+1}) \neq \Gamma(a_{n+1}, a_{n+2})$.

We have shown that $\Gamma(a_n, a_{n+1}) \neq \Gamma(a_{n+1}, a_{n+2})$ for all $n \geq 2$. It remains to show that $\Gamma(a_1, a_2) \neq \Gamma(a_2, a_3)$. If $a_1 \nmid a_2$, the same reasoning as above gives $\Gamma(a_1, a_2) \neq \Gamma(a_2, a_3)$. If $a_1 \mid a_2$, $\Gamma(a_1, a_2) \;=\; 1$. Let $a_2 \;=\; pa_1 \;=\; pa$ for some $p \geq 2$. Then $a_3 \;=\; 2a_2 - a_1 \;=\; (2p-1)a$ and

$$\Gamma(a_2, a_3) \;=\; \Gamma(pa, (2p-1)a) \;=\; \Gamma(p, 2p-1)\,.$$

By Corollary 1, $\Gamma(a_2, a_3) \;=\; 2 \neq \Gamma(a_1, a_2)$.

3.4 Shifted Geometric Progression

Proof (Proof of Theorem 5 when d is odd). For $n \in \mathbb{N}$, let $b_n = ar^{n-1}$. Then we can write the three consecutive terms a_n, a_{n+1}, a_{n+2} as

$$b_n + 1, rb_n + 1, r^2 b_n + 1\,.$$

Since d is odd, $rb_n + 1$ is odd. Indeed, suppose otherwise that $rb_n + 1$ is even. Then both r and b_n are odd. Hence, $r^2 b_n + 1$ is even, thus $\gcd(a_{n+1}, a_{n+2}) = \gcd(rb_n+1, r^2 b_n+1)$ is even. This contradicts Lemma 3 that $\gcd(a_{n+1}, a_{n+2}) = d$, which is odd.

Observe that $rb_n + 1$ does not divide $r^2 b_n + 1$ because

$$(rb_n+1)(r-1) \;=\; r^2 b_n + r - ar^n - 1 \;<\; r^2 b_n + 1 \;<\; (rb_n+1)r\,.$$

Let $x = \Theta(b_n+1, rb_n+1)$ and $y = \Theta(r^2 b_n + 1, rb_n + 1)$. Then

$$\frac{(b_n+1)x}{d} \;\equiv\; 1 \mod \frac{rb_n+1}{d}\,, \tag{22}$$

$$\frac{(r^2 b_n+1)y}{d} \;\equiv\; 1 \mod \frac{rb_n+1}{d}\,. \tag{23}$$

Note that (23) is equivalent to

$$\frac{(1-r)y}{d} \;\equiv\; 1 \mod \frac{rb_n+1}{d}\,. \tag{24}$$

Multiplying (24) by b_n gives

$$\frac{(b_n+1)y}{d} \;\equiv\; b_n \mod \frac{rb_n+1}{d}\,,$$

which implies that

$$\frac{(b_n+1)((rb_n+1)/d+d-y)}{d} \;\equiv\; 1 \quad \mathrm{mod}\; \frac{rb_n+1}{d}\,. \tag{25}$$

Case 1: $a+1$ does not divide $r-1$. Then $d < a+1$, so $d \le b_1$. We have $b_n \ge d$ for all $n \in \mathbb{N}$. By (24), $(rb_n+1)/d$ divides $(r-1)y/d+1$; hence,

$$\frac{(r-1)y}{d} \;\ge\; \frac{rb_n+1}{d} - 1 \;>\; r-1\,,$$

where the second inequality is due to $b_n+1/r > d$. As a result, $y > d$. Combining with the definition of y, we obtain

$$0 \;<\; \frac{rb_n+1}{d} + d - y \;<\; \frac{rb_n+1}{d}\,. \tag{26}$$

It follows from (22), (25), and (26) that

$$x+y \;=\; \frac{rb_n+1}{d} + d\,.$$

Since both rb_n+1 and d are odd, x and y have the same parity.

We now apply Theorem 1 Item (1). We claim that b_n+1 does not divide rb_n+1. Indeed, when $n=1$, we have $b_1+1 = a+1$, while $rb_1+1 = ar+1$. If $a+1$ divides $ar+1$, then $a+1$ divides $a(r-1)$. This contradicts our assumption that $a+1$ does not divide $r-1$. For all $n \ge 2$,

$$(r-1)(b_n+1) \;=\; rb_n - ar^{n-1} + r - 1 \;<\; rb_n+1 \;<\; r(b_n+1)\,.$$

The condition $n \ge 2$ is used in claiming the first inequality. Hence, $\Gamma(rb_n+1, b_n+1)$ is determined by the parity of x. Similarly, $\Gamma(rb_n+1, r^2b_n+1)$ is determined by the parity of y. Since x and y have the same parity, $\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2})$.

Case 2: $a+1$ divides $r-1$. Then $d = a+1$. We have $b_n = ar^{n-1} \ge d$ for all $n \ge 2$. The same argument as in Case 1 shows that $\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2})$ for all $n \ge 2$. By Lemma 3, $\gcd(a+1, ar+1) = \gcd(a+1, r-1) = a+1$. Hence,

$$\Gamma(a_1, a_2) \;=\; \Gamma(a+1, ar+1) \;=\; \Gamma(1, (ar+1)/(a+1)) \;=\; 1\,.$$

It remains to verify that $\Gamma(a_2, a_3) = \Gamma(ar+1, ar^2+1) = 2$. This follows directly from the observation that the equation

$$\frac{ar+1}{a+1}x + \frac{ar^2+1}{a+1}y + 1 \;=\; \frac{1}{2}\left(\frac{ar+1}{a+1} - 1\right)\left(\frac{ar^2+1}{a+1} - 1\right)$$

has solution

$$(x,y) \;=\; \left(\frac{ar}{2} - 1, \frac{a}{2}\left(\frac{r-1}{a+1} - 1\right)\right)\,. \tag{27}$$

Proof (Proof of Theorem 5 when d is even). For $n \in \mathbb{N}_{\geq 2}$, let $b_n = ar^{n-1}$. Then we can write the three consecutive terms a_n, a_{n+1}, a_{n+2} as

$$b_n + 1, rb_n + 1, r^2 b_n + 1\,.$$

Our goal is to show that $\Gamma(a_n, a_{n+1}) \neq \Gamma(a_{n+1}, a_{n+2})$.

Case 1: $(rb_n + 1)/d$ is odd. Let $u = \Theta(b_n + 1, rb_n + 1)$ and $v = \Theta(r^2 b_n + 1, rb_n + 1)$. For $n \geq 2$, as in the proof for odd d,

$$u + v \;=\; \frac{rb_n + 1}{d} + d\,,$$

which implies that x and y have different parities. Hence, $\Gamma(a_n, a_{n+1}) \neq \Gamma(a_{n+1}, a_{n+2})$.

Case 2: $(rb_n + 1)/d$ is even. It holds that $b_n + 1$ does not divide $rb_n + 1$. Indeed, for all $n \geq 2$,

$$(r-1)(b_n+1) \;=\; rb_n - ar^{n-1} + r - 1 \;<\; rb_n + 1 \;<\; r(b_n + 1)\,.$$

(The condition $n \geq 2$ is used in claiming the first inequality.) Let $x = \Theta(rb_n + 1, b_n + 1)$ and $y = \Theta(rb_n + 1, r^2 b_n + 1)$. Let $k_1, k_2 > 0$ such that

$$\frac{(rb_n+1)x}{d} \;=\; 1 + \frac{(b_n+1)k_1}{d}\,, \tag{28}$$

$$\frac{(rb_n+1)y}{d} \;=\; 1 + \frac{(r^2 b_n+1)k_2}{d}\,. \tag{29}$$

Since $0 < x < (b_n + 1)/d$ and $0 < y < (r^2 b_n + 1)/d$, we know that

$$0 \;<\; k_1, k_2 \;<\; \frac{rb_n + 1}{d}\,. \tag{30}$$

Claim. It holds that $(rb_n + 1)/d$ divides $k_1 + k_2 + d$.

Proof. It is easy to check that (29) is equivalent to

$$\frac{(rb_n+1)(yb_n - rk_2 b_n + k_2)}{d} \;=\; \frac{(b_n+1)(k_2+d)}{d} - 1\,. \tag{31}$$

Add (28) and (31) side by side to obtain

$$\frac{rb_n+1}{d}(x + yb_n - rk_2 b_n + k_2) \;=\; \frac{b_n+1}{d}(k_1 + k_2 + d)\,. \tag{32}$$

Since $\gcd((rb+1)/d, (b_n+1)/d) = 1$, we have the desired conclusion.

By the definition of x and (28), we have

$$\frac{k_1(b_n+1)}{d} \;<\; \frac{(rb_n+1)((b_n+1)/d - 1)}{d}\,,$$

which gives

$$k_1 \ < \ \frac{rb_n+1}{d} - \frac{rb_n+1}{b_n+1}\,. \tag{33}$$

Since $n \geq 2$,

$$d \ \leq \ r-1 \ < \ \frac{ar^n+1}{ar^{n-1}+1} \ = \ \frac{rb_n+1}{b_n+1}\,. \tag{34}$$

By (33) and (34),

$$k_1 \ < \ \frac{rb_n+1}{d} - d\,. \tag{35}$$

It then follows from (30) and (35) that

$$k_1+k_2+d \ < \ 2\frac{rb_n+1}{d}\,,$$

which, together with Claim 3.4, gives

$$k_1+k_2+d \ = \ \frac{rb_n+1}{d}\,.$$

Hence, by (32),

$$x+yb_n-rk_2b_n+k_2 \ = \ \frac{b_n+1}{d}\,. \tag{36}$$

We determine the parity of each integer in (36) as follows.

– Since $(rb_n+1)/d$ is even and $\gcd((rb_n+1)/d, (b_n+1)/d) = 1$, $(b_n+1)/d$ must be odd.
– Since d is even and $(b_n+1)/d$ is odd, b_n must be odd.
– The integer r is odd because $d = \gcd(a+1, r-1)$ is even.
– Finally, because the left side of Equation (29) is even, k_2 must odd.

Therefore, (36) guarantees that x and y have different parities. Applying Theorem 1, we conclude that $\Gamma(a_n, a_{n+1}) \neq \Gamma(a_{n+1}, a_{n+2})$. This completes our proof.

4 The Sequence $(\Gamma(k,n))_n$ for a Fixed k

We prove Theorem 6; the main ingredient is Theorem 1. We split the proof into two cases corresponding to the parity of k, where the case k is even is more technically involved. The two following lemmas are useful in proving the period. The proof of the following lemma is well-known and can be found, for example, at https://math.stackexchange.com/questions/3876246.

Lemma 4. *Suppose that $(a_n)_{n\geq 1}$ is periodic, and there exists $k \in \mathbb{N}$ with the property $a_n = a_{n+k}$ for all $n \geq 1$. If T is the period of the sequence $(a_n)_{n\geq 1}$, then T divides k.*

Lemma 5. *The sequence $\Delta(\mathbb{N})$ is $1, 2, 1, 2, 1, 2, \ldots$.*

Proof. The sequence $\Delta(\mathbb{N})$ starts with 1 because $\Gamma(1,2) = 1$. Apply Theorem 4 and we are done.

Proof (Proof of Theorem 6 when k is odd). The case $k = 1$ is trivial, so we assume that $k \geq 3$. Let $u, v \in \mathbb{N}$ such that $v = u + k$. We show that $\Gamma(k,u) = \Gamma(k,v)$.

Case 1: u divides k, giving $\Gamma(k,u) = 1$. Write $k = u\ell$ for some odd $\ell \in \mathbb{N}$. Then $v = u + k = u(\ell+1)$. We have $\Gamma(k,v) = \Gamma(u\ell, u(\ell+1)) = \Gamma(\ell, \ell+1)$. Since

$$\ell\frac{\ell-1}{2} + (\ell+1)\times 0 \;=\; \frac{(\ell-1)\ell}{2},$$

$\Gamma(\ell, \ell+1) = 1$; hence, $\Gamma(k,u) = \Gamma(k,v) = 1$.

Case 2: k divides u. Then k divides v, and $\Gamma(k,u) = \Gamma(k,v) = 1$.

Case 3: u does not divide k, and k does not divide u. Let $d = \gcd(k,u)$. Since k is odd, k/d is odd. By Theorem 1, $\Gamma(k,u)$ is determined by the parity of $x := \Theta(u,k)$. Similarly, $\Gamma(k,v)$ is determined by the parity of $y := \Theta(v,k)$. Hence, it suffices to show that $\Theta(u,k)$ and $\Theta(v,k)$ have the same parity. By definition,

$$\frac{ux}{d} \;\equiv\; 1 \mod \frac{k}{d}, \tag{37}$$

$$\frac{vy}{d} \;\equiv\; 1 \mod \frac{k}{d}. \tag{38}$$

Furthermore, that $v \equiv u \mod k$, and (38) implies that

$$\frac{uy}{d} \;\equiv\; 1 \mod \frac{k}{d}. \tag{39}$$

It follows from (37) and (39) that $x = y$. This completes our proof that $\Gamma(k,u) = \Gamma(k,v)$ whenever $u \equiv v \mod k$. However, this is not enough to conclude that the period of $(\Gamma(k,n))_{n\geq 1}$ is k; in the following, we show that this is indeed the case.

Consider the first k terms of $(\Gamma(k,n))_{n\geq 1}$. Pick $1 \leq s \leq (k-1)/2$ and let $r = \gcd(k,s) = \gcd(k-s,k)$.

Case I: if s divides k, then $\Gamma(k,s) = 1$. Write $k = s\ell$ for some odd $\ell \in \mathbb{N}_{\geq 3}$. By Lemma 5, we have $\Gamma(k, k-s) = \Gamma(\ell, \ell-1) = 2$. Hence, $\Gamma(k,s) \neq \Gamma(k, k-s)$.

Case II: suppose that s does not divide k. Observe that $k/2 < k - s < k$, so $k - s$ does not divide k. By Theorem 1, $\Gamma(k,s)$ and $\Gamma(k, k-s)$ are determined by the parity of $\Theta(s,k)$ and $\Theta(k-s,k)$, respectively. By definition,

$$\Theta(s,k)\frac{s}{r} \equiv 1 \mod \frac{k}{r}, \tag{40}$$

$$\Theta(k-s,k)\frac{k-s}{r} \equiv 1 \mod \frac{k}{r} \implies \left(\frac{k}{r} - \Theta(k-s,k)\right)\frac{s}{r} \equiv 1 \mod \frac{k}{r}. \tag{41}$$

It follows from (40) and (41) that

$$\Theta(s,k) + \Theta(k-s,k) = \frac{k}{r}.$$

Since k/r is odd, $\Theta(s,k) \not\equiv \Theta(k-s,k) \mod 2$, thus $\Gamma(k,s) \neq \Gamma(k,k-s)$.

We have shown that $\Gamma(k,s) \neq \Gamma(k,k-s)$ for all $1 \leq s \leq (k-1)/2$. Along with the fact that $\Gamma(k,k) = 1$, we know that within the first k terms of $(\Gamma(k,n))_{n\geq 1}$, the number of 1's is one more than the number of 2's.

We are ready to conclude the proof that $(\Gamma(k,n))_{n\geq 1}$ has period k. Let T be the period of $(\Gamma(k,n))_{n\geq 1}$. By Lemma 4, T divides k. Hence, within the first k terms, there are k/T copies of the period. Let p and q be the number of 1's and 2's within each period, respectively. Then $(p-q)(k/T) = 1$, which implies that $p - q = k/T = 1$. Hence, k is the period of $(\Gamma(k,n))_{n\geq 1}$.

Proof (Proof of Theorem 6 when k is even). Let $u, v \in \mathbb{N}$ such that $v = u + 2k$. We show that $\Gamma(k,u) = \Gamma(k,v)$.

Case 1: u divides k, giving $\Gamma(k,u) = 1$. Write $k = u\ell$ for some $\ell \in \mathbb{N}$. Then $v = u + 2k = u(2\ell+1)$. We have $\Gamma(k,v) = \Gamma(u\ell, u(2\ell+1)) = \Gamma(\ell, 2\ell+1)$. Since

$$\ell(\ell-1) + (2\ell+1)\times 0 = \frac{(\ell-1)((2\ell+1)-1)}{2},$$

$\Gamma(\ell, 2\ell+1) = 1$; hence, $\Gamma(k,u) = \Gamma(k,v) = 1$.

Case 2: k divides u. Then k divides v, and $\Gamma(k,u) = \Gamma(k,v) = 1$.

Case 3: u does not divide k, and k does not divide u. Let $d = \gcd(k,u) = \gcd(k,v)$. If k/d is odd, the exact same argument as in the proof of Theorem 6 when k is odd applies. Suppose that k/d is even. By Theorem 1, $\Gamma(k,u)$ is determined by the parity of $x := \Theta(k,u)$. Similarly, $\Gamma(k,v)$ is determined by the parity of $y := \Theta(k,v)$. Hence, it suffices to show that x and y have the same parity. By definition,

$$\frac{kx}{d} \equiv 1 \mod \frac{u}{d} \Rightarrow \frac{kx}{d} - 1 = \frac{u}{d}\ell_1, \quad \ell_1 > 0, \ \ell_1 \text{ is odd},$$

$$\frac{ky}{d} \equiv 1 \mod \frac{v}{d} \Rightarrow \frac{ky}{d} - 1 = \frac{u}{d}\ell_2 + 2\frac{k}{d}\ell_2, \quad \ell_2 > 0, \ \ell_2 \text{ is odd}.$$

Note that ℓ_1, $\ell_2 < k/d$ as $x < u/d$ and $y < v/d = (u+2k)/d$. Subtracting the two equations above gives

$$\frac{k}{d}(x-y) = \frac{u}{d}(\ell_1 - \ell_2) - 2\frac{k}{d}\ell_2\,, \tag{42}$$

which implies that $\ell_1 - \ell_2$ is divisible by k/d. However, $-k/d < \ell_1 - \ell_2 < k/d$; therefore, $\ell_1 - \ell_2 = 0$. Replacing $\ell_1 - \ell_2$ by 0 in (42), we obtain

$$x - y = -2\ell_2\,.$$

As a result, $x \equiv y \mod 2$.

Next, we prove that within the first $2k$ terms of $(\Gamma(k,n))_{n\geq 1}$, the number of 1's is two more than the number of 2's. Pick $1 \leq s \leq k-1$ and let $r = \gcd(k,s) = \gcd(k, 2k-s)$. We show that $\Gamma(k,s) \neq \Gamma(k,2k-s)$. Then we are done since $\Gamma(k,k) = \Gamma(k,2k) = 1$.

Case I: if s divides k, then $\Gamma(k,s) = 1$. Write $k = s\ell$ for some $\ell \in \mathbb{N}_{\geq 2}$. By Corollary 1, we have $\Gamma(k,2k-s) = \Gamma(\ell, 2\ell-1) = 2$. Hence, $\Gamma(k,s) \neq \Gamma(k,2k-s)$.

Case II: suppose that s does not divide k. Observe that $k+1 \leq 2k-s \leq 2k-1$, so k does not divide $2k-s$. If k/r is odd, the exact same argument as in the proof of Theorem 6 when k is odd applies. Suppose that k/r is even. By Theorem 1, $\Gamma(k,s)$ and $\Gamma(k,2k-s)$ are determined by the parity of $p := \Theta(k,s)$ and $q := \Theta(k,2k-s)$, respectively. By definition,

$$\frac{pk}{r} \equiv 1 \mod \frac{s}{r} \text{ and } \frac{qk}{r} \equiv 1 \mod \frac{2k-s}{r}\,.$$

Write

$$\begin{aligned} \frac{pk}{r} - 1 &= \frac{s}{r}\ell_1\,, \quad \ell_1 > 0\,, \ell_1 \text{ odd},\\ \frac{qk}{r} - 1 &= \frac{2k-s}{r}\ell_2\,, \quad \ell_2 > 0\,, \ell_2 \text{ odd}. \end{aligned}$$

Note that $\ell_1, \ell_2 < k/r$ because $p < s/r$ and $q < (2k-s)/r$. Hence,

$$\frac{k}{r}(p-q) = \frac{s}{r}(\ell_1 + \ell_2) - \frac{2k}{r}\ell_2\,, \tag{43}$$

which implies that $\ell_1 + \ell_2$ is divisible by k/r. However, $0 < \ell_1 + \ell_2 < 2k/r$; therefore, $\ell_1 + \ell_2 = k/r$. Replacing $\ell_1 + \ell_2$ by k/r in (43), we obtain

$$p - q = \frac{s}{r} - 2\ell_2\,.$$

As a result, $p \not\equiv q \mod 2$, so $\Gamma(k,s) \neq \Gamma(k,2k-s)$.

Finally, we prove that $(\Gamma(k,n))_{n\geq 1}$ has period $2k$. Let T be the period of $(\Gamma(k,n))_{n\geq 1}$. By Lemma 4, T divides $2k$. Hence, within the first $2k$ terms, there are $2k/T$ copies of the period. Let p and q be the number of 1's and 2's within each period, respectively. Then $(p-q)(2k/T) = 2$; equivalently, $(p-q)k/T = 1$. Since $p - q \leq 2$ by above, there are two cases: either $(p-q, T) = (1, k)$ or $(p-q, T) = (2, 2k)$. We show that the former cannot happen. Suppose, for a contradiction, that $T = k = 2j$ for some $j \in \mathbb{N}$. It follows that $\Gamma(2j, 2j+1) = \Gamma(k, k+1) = \Gamma(k, 1) = 1$, contradicting Lemma 5. This completes our proof.

5 Fibonacci-Type Recurrence

In this section, we find $\Delta((a_n)_n)$, where $(a_n)_n$ satisfies a certain linear recurrence of order two. This extends [3, Theorem 1.4].

Lemma 6. *Fix $a, b, k \in \mathbb{N}$ with $\gcd(a, b) = d$. Consider the sequence $(a_n)_n$ where $a_1 = a, a_2 = b$, and $a_n = ka_{n-1} + a_{n-2}$ for $n \geq 3$. Then $\gcd(a_n, a_{n+1}) = d$ for all $n \in \mathbb{N}$.*

Proof. We prove by induction. For $n = 1$, $\gcd(a_1, a_2) = \gcd(a, b) = d$. Suppose $\gcd(a_n, a_{n+1}) = d$ for some $n \geq 1$. We have

$$\gcd(a_{n+2}, a_{n+1}) \;=\; \gcd(ka_{n+1} + a_n, a_{n+1}) \;=\; \gcd(a_n, a_{n+1}) \;=\; d\,.$$

This completes our induction step.

Due to Lemma 6, if $(a_n)_n$ satisfies $a_n = ka_{n-1} + a_{n-2}$, then in finding $\Delta((a_n)_n)$, we can assume that $\gcd(a_1, a_2) = 1$ with any loss of generality.

Theorem 7. *Fix $a, b, k \in \mathbb{N}$ with $\gcd(a, b) = 1$. Let $a_1 = a, a_2 = b$, and $a_n = ka_{n-1} + a_{n-2}$ for $n \geq 3$. Let $(x_n)_n$ be the sequence $1, 1, 1, 2, 2, 2, 1, 1, 1, \ldots$ and $(y_n)_n$ be the sequence $2, 2, 2, 1, 1, 1, 2, 2, 2, \ldots$.*

1. *If k is even, $\Delta((a_n)_n)$ is constant.*
2. *If k is odd,*
   (a) and a, b are both odd, $\Delta((a_n)_{n\geq 3}) = (x_n)_n$ or $(y_n)_n$, or if
   (b) a is odd, and b is even, $\Delta((a_n)_{n\geq 2}) = (x_n)_n$ or $(y_n)_n$, or if
   (c) a is even, and b is odd, $\Delta((a_n)_n) = (x_n)_n$ or $(y_n)_n$.

Proof. Fix $n \in \mathbb{N}$ and consider three consecutive terms a_n, a_{n+1}, a_{n+2}, which are $a_n, a_{n+1}, ka_{n+1} + a_n$. We proceed by case analysis.

Case 1: a_{n+1} is odd and $a_{n+1} \neq 1$. We show in this case that $\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2})$.

1. Case 1.1: a_n divides a_{n+1}. Write $a_{n+1} = a_n\ell$ for some odd $\ell \in \mathbb{N}$. Then $a_{n+2} = a_n(k\ell + 1)$, which gives $\Gamma(a_{n+1}, a_{n+2}) = \Gamma(\ell, k\ell + 1) = 1$ because

$$\ell \times \frac{(\ell-1)k}{2} + (k\ell+1) \times 0 \;=\; \frac{(\ell-1)k\ell}{2}\,.$$

   Therefore, $\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2})$.

2. Case 1.2: a_n does not divide a_{n+1}. We claim that a_{n+1} divides neither a_n nor a_{n+2}. Indeed, if $a_{n+1}|a_n$, then $\gcd(a_n, a_{n+1}) = a_{n+1} > 1$, contradicting $\gcd(a_n, a_{n+1}) = \gcd(a, b) = 1$ by Lemma 6. Similarly, a_{n+1} does not divide a_{n+2}.
   Let $c = \Theta(a_n, a_{n+1})$ and $d = \Theta(a_{n+2}, a_{n+1})$ to have

$$
\begin{aligned}
ca_n &\equiv 1 \mod a_{n+1}\,, && (44)\\
d(ka_{n+1} + a_n) &\equiv 1 \mod a_{n+1} \implies a_n d \equiv 1 \mod a_{n+1}\,. && (45)
\end{aligned}
$$

   Therefore, $c = d$, which, by Theorem 1, gives $\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2})$.

Case 2: a_{n+1} is odd and $a_{n+1} = 1$. Clearly, $\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2}) = 1$. From Case 1 and Case 2, we conclude

$$
\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2})\,, \quad \text{if } a_{n+1} \text{ is odd.} \tag{46}
$$

Case 3: a_{n+1} is even and $a_n \neq 1$.

1. Case 3.1: a_{n+1} divides a_n. Then $\gcd(a_n, a_{n+1}) = 1$ implies that $a_{n+1} = 1$, contradicting that a_{n+1} is even.
2. Case 3.2: a_{n+1} does not divide a_n. As in above cases, a_n does not divide a_{n+1}, and a_{n+1} does not divide a_{n+2} because otherwise, we violate the condition $\gcd(a_n, a_{n+1}) = \gcd(a_{n+1}, a_{n+2}) = 1$. Let $p = \Theta(a_{n+1}, a_n)$ and $q = \Theta(a_{n+1}, a_{n+2})$ to have

$$
\begin{aligned}
pa_{n+1} &\equiv 1 \mod a_n\,,\\
qa_{n+1} &\equiv 1 \mod ka_{n+1} + a_n\,.
\end{aligned}
$$

   There exist positive integers k_1, k_2 such that

$$
\begin{aligned}
pa_{n+1} &= 1 + k_1 a_n\,, && (47)\\
qa_{n+1} &= 1 + k_2(ka_{n+1} + a_n)\,. && (48)
\end{aligned}
$$

   By definition, $0 < p < a_n$ and $0 < q < ka_{n+1} + a_n$. Hence, (47) and (48) imply that $0 < k_1, k_2 < a_{n+1}$. Subtracting (48) from (47) side by side, we obtain

$$
a_{n+1}(p - q + kk_2) = a_n(k_1 - k_2)\,. \tag{49}
$$

   Since $\gcd(a_n, a_{n+1}) = 1$, it follows from (49) that a_{n+1} divides $k_1 - k_2$. However, that $0 < k_1, k_2 < a_{n+1}$ implies that $-a_{n+1} < k_1 - k_2 < a_{n+1}$. As a result, k_1 must be equal to k_2. Replacing k_1 by k_2 in (49) gives

$$
p - q + kk_2 = 0\,. \tag{50}
$$

   From (48) and the fact that a_{n+1} is even, we know that k_2 is odd. Therefore, from (50), we conclude that

– if k is even, p and q have the same parity, and
– if k is odd, p and q have different parities.

Therefore, when a_{n+1} is even and $a_n \neq 1$,

– if k is even, then $\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2})$, while
– if k is odd, then $\Gamma(a_n, a_{n+1}) \neq \Gamma(a_{n+1}, a_{n+2})$.

Case 4: a_{n+1} is even and $a_n = 1$. Write $a_{n+1} = 2r$ for some $r \in \mathbb{N}$ to have $a_{n+2} = 2kr + 1$. Clearly, $\Gamma(a_n, a_{n+1}) = 1$. We find $\Gamma(a_{n+1}, a_{n+2})$. Note that $\gcd(2r, 2kr + 1) = 1$ and $\Theta(2r, 2kr + 1) = 2kr + 1 - k$, which is odd if and only if k is even. Therefore, if k is even, $\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2}) = 1$, and if k is odd, then $\Gamma(a_n, a_{n+1}) = 1 \neq 2 = \Gamma(a_{n+1}, a_{n+2})$.

From Case 3 and Case 4, we conclude that

$$\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2}), \quad \text{if } a_{n+1} \text{ is even and } k \text{ is even.} \tag{51}$$

$$\Gamma(a_n, a_{n+1}) \neq \Gamma(a_{n+1}, a_{n+2}), \quad \text{if } a_{n+1} \text{ is even and } k \text{ is odd.} \tag{52}$$

From (46), (51), and (52), we have

– If k is even, then $\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2})$ for all $n \in \mathbb{N}$.
– If k is odd, then $\Gamma(a_n, a_{n+1}) = \Gamma(a_{n+1}, a_{n+2})$ if and only if a_{n+1} is odd.

Hence, we have proved Item (1) of Theorem 7. For Item (2), assume that k is odd.

If a and b are both odd, the parity of terms in $(a_n)_n$ is

$$\text{odd}\,, \text{odd}\,, \text{even}\,, \text{odd}\,, \text{odd}\,, \text{even}\,, \text{odd}\,, \text{odd}\,, \text{even}\,, \ldots.$$

It is easy to verify that (52) gives us Item (2) Part (a).

If a is odd and b is even, the parity of terms in $(a_n)_n$ is

$$\text{odd}\,, \text{even}\,, \text{odd}\,, \text{odd}\,, \text{even}\,, \text{odd}\,, \text{odd}\,, \text{even}\,, \text{odd}\,, \ldots.$$

In this case, (52) gives us Item (2) Part (b).

Similarly, we get Item (3) Part (c) when a is even and b is odd.

As discussed before, we can drop the condition $\gcd(a, b) = 1$ in Theorem 7.

Corollary 2. *Fix $a, b, k \in \mathbb{N}$. Let $a_1 = a, a_2 = b$, and $a_n = ka_{n-1} + a_{n-2}$ for $n \geq 3$. Let $(x_n)_n$ be the sequence $1, 1, 1, 2, 2, 2, 1, 1, 1, \ldots$ and $(y_n)_n$ be the sequence $2, 2, 2, 1, 1, 1, 2, 2, 2, \ldots$.*

1. *If k is even, $\Delta((a_n)_n)$ is constant.*
2. *If k is odd,*
   (a) *and a, b are both odd, $\Delta((a_n)_{n\geq 3}) = (x_n)_n$ or $(y_n)_n$.*
   (b) *a is odd, and b is even, $\Delta((a_n)_{n\geq 2}) = (x_n)_n$ or $(y_n)_n$.*
   (c) *a is even, and b is odd, $\Delta((a_n)_n) = (x_n)_n$ or $(y_n)_n$.*

Proof. The corollary follows immediately from how we define $\Delta((a_n)_n)$.

6 Problems for Future Investigation

It would be interesting to see $\Delta((a_n)_n)$ when $(a_n)_{n=1}^{\infty}$ satisfies more general recurrences to extend results in Sect. 5. For other problems in this topic, interested readers may refer to [5].

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