OPTIMAL POINT SETS DETERMINING FEW DISTINCT TRIANGLES

Alyssa Epstein [1]
Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267
ale2@williams.edu

Adam Lott
Department of Mathematics, University of Rochester, Rochester, NY 14627
Department of Mathematics, UCLA, Los Angeles, CA 90095
alott@u.rochester.edu
adamlott99@math.ucla.edu

Steven J. Miller
Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267
sjm1@williams.edu, Steven.Miller.MC.96@aya.yale.edu

Eyvindur A. Palsson
Department of Mathematics, Virginia Tech, Blacksburg, VA 24061
palsson@vt.edu

Received: , Revised: , Accepted: , Published:

Abstract

We generalize work of Erdős and Fishburn to study the structure of finite point sets that determine few distinct triangles. Specifically, we ask for a given t, what is the maximum number of points that can be placed in the plane to determine exactly t distinct triangles? Denoting this quantity by $F(t)$, we show that $F(1) = 4$, $F(2) = 5$, and we completely characterize the optimal configurations for $t = 1, 2$. We also discuss the general structure of optimal configurations and conjecture that regular polygons are always optimal. This differs from the structure of optimal configurations for distances, where it is conjectured that optimal configurations always exist in the triangular lattice. We also prove that the number of distinct triangles determined by a regular n-gon is asymptotic to $n^2/12$; so if the conjecture about regular n-gons being optimal is true, we identify the constant for the lower bound of distinct triangles determined by any point configuration.

[1]This work was supported by NSF Grants DMS1265673, DMS1561945, and DMS1347804, Simons Foundation Grant #360560, Williams College, and the Clare Boothe Luce program. We also thank Paul Baird-Smith and Xiaoyu Xu for helpful conversations.

1. Introduction

Finite point configurations are a central object of study in discrete geometry. Perhaps the most well-known problem is the Erdős distinct distances conjecture, which states that any set of n points in the plane determines at least $\Omega(n/\sqrt{\log n})$ distinct distances between points. This problem, first proposed by Erdős in 1946 [2], was essentially resolved by Guth and Katz who proved that n points determined at least $\Omega(n/\log n)$ distinct distances [5]. Higher dimensional analogs still remain open. A closely related question is: given a fixed positive integer k, what is the maximum number of points that can be placed in the plane to determine exactly k distances? Furthermore, can the optimal configurations be completely characterized? Erdős and Fishburn [3] introduced this question in 1996 and characterized the optimal configurations for $1 \leq k \leq 4$. Shinohara [8] and Wei [10] have characterized the optimal configurations for $k = 5$ and $k = 6$, respectively. Erdős also conjectured that an optimal configuration always exists in the triangular lattice given k large enough (see Figure 1) and this conjecture remains open.

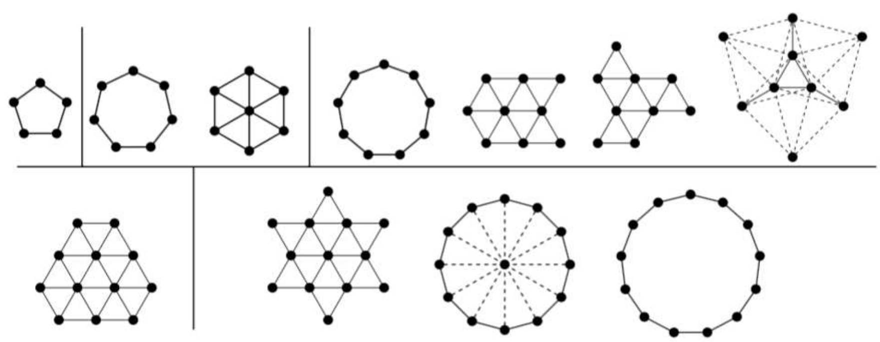


Figure 1: Maximal configurations determining exactly k distances, for $2 \leq k \leq 6$ [1]. For each $k > 2$, there is an example from the triangular lattice; it is conjectured that this is always the case for k large enough.

As a distance is just a pair of points, distances can be phrased as the set of 2-point configurations determined by a set. Analogously, we can study the set of 3-point configurations (i.e., triangles) determined by a set. The analogue of the Erdős distinct distance problem would ask for the minimum number of distinct triangles determined by n points in the plane. It follows directly from Guth and Katz's result on the number of distinct distances that a set of n points in the plane determines at least $\Omega(n^2/\log(n))$ distinct triangles, but Misha Rudnev [7] adapted their argument and improved this bound to $\Omega(n^2)$. It is also known that this bound is best possible up to the implicit constant. We study the following analogue of Erdős and Fishburn's question: given a fixed t, what is the maximum number of points that can be placed in the plane to determine exactly t distinct triangles? Our main result is the following.

Theorem 1.1. *Let $F(t)$ denote the maximum number of points that can be placed in the plane to determine exactly t distinct triangles. Then*

1. *$F(1) = 4$ and the only configuration that achieves this is a rectangle, and*
2. *$F(2) = 5$ and the only configurations that achieve this are a square with its center and a regular pentagon.*

We also make two conjectures: first, that $F(3) = 6$, with a regular hexagon being a representative optimal configuration, and second, that a regular polygon always minimizes the number of distinct triangles in an n-point set. If true, this second conjecture determines the true leading constant for Guth and Katz's asymptotic of at least $\Omega(n^2)$ distinct triangles for a set of n points: $1/12$.

We prove Theorem 1.1 by classifying all potential arrangements of 4-point sets in the plane and sorting them by the minimum number of distinct triangles they create. To show part 1, we look at the 4-point sets that do not trivially determine more than one triangle. Through elementary geometry, we eliminate all non-trivial cases that have at least two distinct triangles except the rectangle. This immediately implies that $F(1) = 4$, and the rectangle uniquely satisfies this equation. Proving part 2, we take the 4-point sets that determine fewer than three distinct triangles, and we examine all possible ways to add a fifth point to the set. After removing all cases where the fifth point causes at least three distinct triangles, the only remaining configurations are the square with a point at its center and the regular pentagon. Thus, $F(2) = 5$.

2. Conjectures

In this section, we present some conjectures and investigate their consequences.

Conjecture 2.1. *Any set of seven points in the plane determines at least four distinct triangles; thus $F(3) = 6$.*

In Figure 2 we see that the vertices of a regular hexagon determine exactly three distinct triangles, so we know $F(3) \geq 6$.

Another interesting question to ask concerns the general structure of the optimal configurations. For example, are regular polygons always optimal? What about regular polygons with their centers? As we discussed in the introduction, Erdős and Fishburn conjectured in [3] that optimal configurations for distinct distances always exist in the triangular lattice. For triangles, we make an analogous but qualitatively different conjecture.

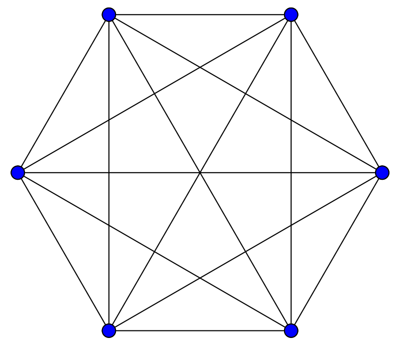


Figure 2: A regular hexagon determines three distinct triangles.

Conjecture 2.2. *The regular n-gon minimizes (not necessarily uniquely) the number of distinct triangles determined by an n-point set.*[2]

If true, Conjecture 2.2 establishes the following best-possible result on the number of distinct triangles. We offer a proof of this claim in Section 6.

Theorem 2.3. *Unconditionally, the vertices of a regular n-gon determine $[n^2/12]$ distinct triangles, where $[y]$ denotes the nearest integer to y. Assuming Conjecture 2.2, this implies that $[n^2/12]$ is the minimum number of distinct triangles that can be determined by a set of n points in the plane.*

Remark 2.4. *It is known from the work of Rudnev, expanding on a result of Guth and Katz, that a set of n points in the plane determines at least $\Omega(n^2)$ distinct triangles, and that this bound is best possible. If true, Conjecture 2.2 establishes the true leading constant, namely $1/12$.*

3. Definitions and setup

We make precise the notion of distinct triangles.

Definition 3.1. *Given a finite point set $P \subset \mathbb{R}^2$, we say two triples $(a, b, c), (a', b', c') \in P^3$ are equivalent if there is an isometry mapping one to the other, and we denote this as $(a, b, c) \sim (a', b', c')$.*

Definition 3.2. *Given a finite point set $P \subset \mathbb{R}^2$, we denote by P^3_{nc} the set of noncollinear triples $(a, b, c) \in P^3$.*

[2]In [1] it was stated as an open problem, due to Brass, whether the regular n-gon minimizes the number of distinct triangles determined by an n-point set. Given the evidence in this paper, we go further and conjecture that it is true.

Definition 3.3. *Given a finite point set $P \subset \mathbb{R}^2$, we define the set of distinct triangles determined by P as*

$$T(P) := P^3_{nc}/\sim . \tag{3.1}$$

We prove Theorem 1.1 by enumerating cases and disposing of them one by one via elementary geometry. We then conclude with a conjecture analogous to that of Erdős concerning the structure of optimal configurations in general.

In the proof of Theorem 1.1, we also use the following lemma, which we prove in Section 7.

Lemma 3.4. *For a set of four noncollinear points in the plane, exactly one of the following holds.*

1. *The four points are not in convex position.*
2. *The four points are in convex position.*
   (a) *Three of the points are collinear.*
   (b) *The determined quadrilateral has four distinct side lengths.*
   (c) *The determined quadrilateral has exactly one pair of congruent sides.*
      i. *The congruent sides are adjacent.*
      ii. *The congruent sides are opposite.*
   (d) *The determined quadrilateral has two distinct pairs of congruent sides.*
      i. *The congruent sides are adjacent to each other (a kite).*
      ii. *The congruent sides are opposite each other (a parallelogram).*
   (e) *Three sides are congruent and the fourth is distinct.*
   (f) *All four sides are congruent (a rhombus).*

Cases 2b, 2(c)i, 2(c)ii, and 2(d)i determine at least three distinct triangles. Cases 1, 2a, and 2e determine at least two distinct triangles.

4. Classifying optimal 1-triangle sets

In this section, we prove part (1) of Theorem 1.1. We show that the only four-point configuration that determines exactly one triangle is a rectangle. This proves that $F(1) = 4$ because there is no five-point configuration such that every four-point subconfiguration is a rectangle.

By Lemma 3.4, we only need to consider the cases 2(d)ii and 2f because all of the other cases trivially lead to at least two triangles. We consider first the case 2(d)ii, when there are two pairs of congruent sides opposite each other.

Proof of case 2(d)ii: two pairs of opposite congruent sides. Since two pairs of opposite sides are congruent, the quadrilateral must be a parallelogram (Figure 3). We claim $\triangle ABC$ and $\triangle BCD$ are congruent if and only if $ABCD$ is a rectangle. They share side BC and $AB = CD$, so $\triangle ABC \cong \triangle BCD$ if and only if $BD = AC$, which happens if and only if $ABCD$ is a rectangle. □

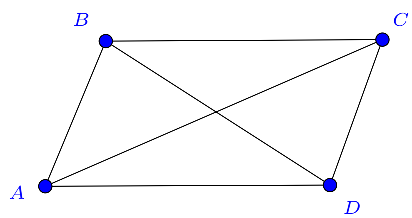


Figure 3: A quadrilateral with two pairs of opposite congruent sides. If $ABCD$ is a rectangle, then it determines only one triangle, but if $ABCD$ is not a rectangle, then $\triangle ABC$ and $\triangle BCD$ are distinct.

Proof of case 2f: four congruent sides. Any quadrilateral with four sides congruent is a rhombus, and a rhombus is a parallelogram. So, by the argument in case 2(d)ii, a rhombus determines two distinct triangles if and only if it is not a square. Thus, we have shown that the only four-point configuration that determines one triangle is a rectangle. This completes the proof of part (1) of Theorem 1.1. □

5. Classifying optimal 2-triangle sets

In this section, we prove part (2) of Theorem 1.1. As in the proof of part (1), we show that the only possible configurations determining exactly two triangles are the square with its center and the regular pentagon. We consider the possible four-point configurations enumerated in Lemma 3.4, and we show that the addition of a fifth point to any of them (unless it creates one of the two claimed configurations) necessarily determines a third triangle. Moreover, adding a sixth point to either of the demonstrated optimal configurations also must determine a third triangle. By Lemma 3.4, the only cases we need to consider are 1, 2a, 2(d)ii, 2e, and 2f because the other four point configurations already contain more than two distinct triangles.

Proof of case 1: not in convex position. Using the notation of Figure 4, if $\triangle ABC$ is not equilateral, or if $\triangle ABC$ is equilateral but D is not the center of $\triangle ABC$, then there are already three distinct triangles, so no more work is needed.

If $\triangle ABC$ is equilateral and D is its center, we show that the addition of a fifth point anywhere necessarily determines a new triangle. When we add a fifth point E, it will necessarily determine a triangle with AB (Figure 4). If $\triangle EAB$ is not congruent to $\triangle ABC$ or $\triangle ABD$, we're done, so assume it's congruent to one of those. Either way, $\triangle ECB$ will be distinct from the other two, so we have three distinct triangles, so this case is done. $\square$

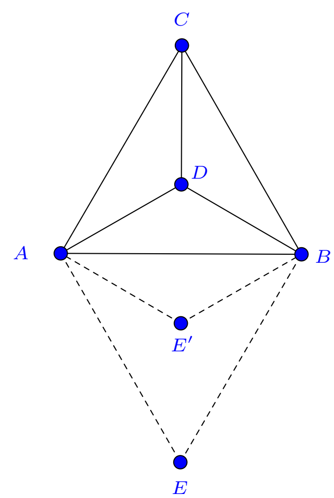


Figure 4: Possibilities for adding a fifth point to a non-convex set.

Proof of case 2a: three collinear points. With the notation of Figure 5, if D does not lie on the perpendicular bisector of AB, then $\triangle ACD$, $\triangle BCD$, and $\triangle ABD$ are all distinct, so no more work is needed. Also note that if a fifth point E is added to the interior of $\triangle ABD$, it creates a non-convex four-point subconfiguration, so the previous case applies to show that there are at least 3 distinct triangles. Thus we assume the fifth point E is added outside $\triangle ABD$.

If D lies on the perpendicular bisector of AB but $DC \neq AB$, the addition of a fifth point E will create a triangle with AC. Triangle $\triangle EAC$ can't be congruent to $\triangle ABD$ because AC is shorter than any side of $\triangle ABD$, so to avoid a third triangle we must have $\triangle EAC \cong \triangle ACD$. There are three choices for E that satisfy this (Figure 5), but either way, $\triangle EAC$, $\triangle EAB$, and $\triangle EDB$ are all distinct.

If D lies on the perpendicular bisector of AB and $DC = AB$, then the same argument from above still applies; however, in this case, choosing E to form the square $ADBE$ leaves us with only two triangles, but the other two choices for E give us three (see Figure 5), so this case is done. $\square$

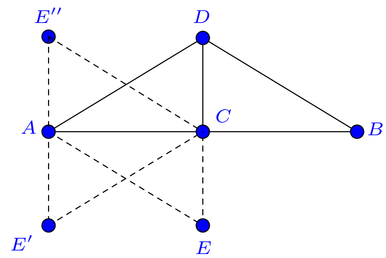


Figure 5: Addition of a fifth point when three points are collinear. If $DC \neq AC$, then any choice of E forces a third triangle. If, on the other hand, $DC = AC$, then choosing E creates a square with its center but E' and E'' still generate a third triangles.

Proof of case 2(d)ii: two pairs of opposite congruent sides. This case has two subcases.

Subcase A: non-rectangle: Using the notation of Figure 6, if we add a fifth point E on line AB, then we have five points with three collinear, so we have 3 distinct triangles by case 5. So assume E does not lie on line AB. Then $\triangle EAB$ will be created. If $\triangle EAB$ is distinct from both $\triangle ABC$ and $\triangle ABD$, then we also have three distinct triangles, so assume otherwise. The only ways this can happen are enumerated in Figure 6. In Figure 6a, point E creates three collinear points (EAD), point E' creates a non-convex subconfiguration ($ACBE'$), and point E'' creates three collinear points (CDE''). Thus in any case there will be three distinct triangles. In Figure 6b, point E' creates three collinear points (CBE') and point E'' also creates three collinear points ($DE''C$). Point E creates a kite $ADBE$ if $AD \neq DB$, and if $AD = DB$, then CBE must be collinear, so in this case also, we have three distinct triangles no matter what.

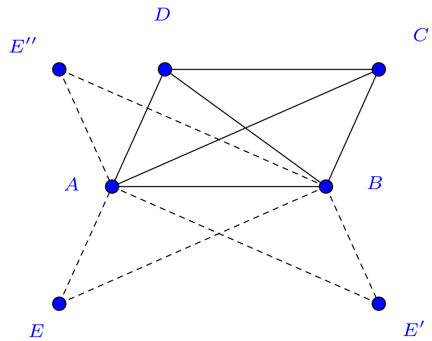
Subcase B: non-square rectangle: If the fifth point is added inside the rectangle, then we get either a non-convex configuration or a configuration with three collinear points (Figure 7a). So assume that the fifth point is added outside the rectangle. Using the notation of Figure 7b, to add a fifth point E without creating three distinct triangles there are three potential possibilities.

1. $\triangle EAB \cong \triangle ABC$. In this case, we get three collinear points, so we have three triangles.

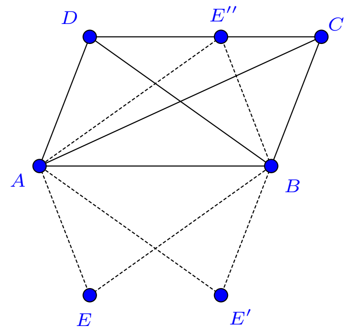
2. $\triangle E'AD \cong \triangle ABC$. Here, DCE are collinear, so we have three triangles.

3. $\triangle E''DC \cong \triangle E''CB \ncong \triangle ABC$. In this case, $E''DAB$ will form a kite, so we have three triangles.

So we see both subcases yield at least three triangles, so the proof of case 2(d)ii is complete. $\square$

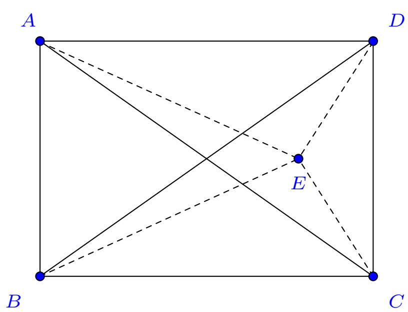


(a) Possibilities for E so that $\triangle EAB \cong \triangle ABC$. Any one of these choices creates a 4-point subconfiguration determining at least 3 distinct triangles.

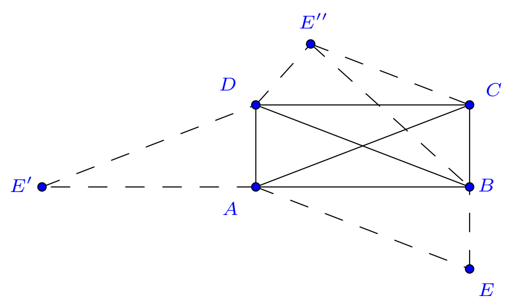


(b) Possibilities for E so that $\triangle EAB \cong \triangle ABD$. Here also, any choice creates a bad 4-point subconfiguration.

Figure 6: Possible additions of a fifth point when two pairs of opposite sides are congruent.



(a) Any way to place a fifth point inside a rectangle results in at least 3 distinct triangles.



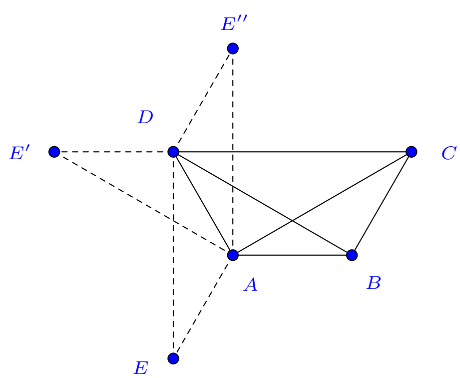
(b) Any way to place a fifth point outside a rectangle also results in at least 3 distinct triangles.

Figure 7: Any way to add a fifth point to a rectangle results in at least 3 distinct triangles.

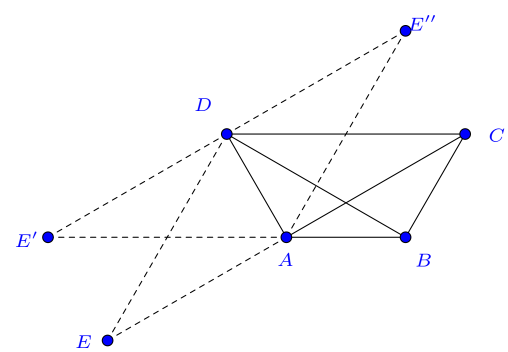
Proof of case 2e: three congruent sides. Using the notation of Figure 8, if the quadrilateral $ABCD$ is not a trapezoid, then in particular $AC \neq BD$. Then we claim $\triangle ABD$, $\triangle BDC$, and $\triangle ABC$ are all distinct. Triangle $\triangle ABC \ncong \triangle ABD$ because $AC \neq BD$. If $\triangle ABC \cong \triangle BDC$, then $AB = BD$ and $CD = AC$, but this is impossible because then there would be two isoceles triangles based on AD.

So we can assume $ABCD$ is a trapezoid. When we add a fifth point E, $\triangle EAD$ is created (Figure 8). As in case 2(d)ii, we must have $\triangle EAD \cong \triangle ABD$ or $\triangle EAD \cong \triangle ACD$. Suppose $\triangle EAD \cong \triangle ABD$ (Figure 8a). In the figure, point E creates a non-convex configuration $EABD$ and point E' creates three collinear points $E'DC$. For point E'', if $E''C$ is a new distance then we obviously have a new triangle. If $E''C = DC$, then $E''DC$ is a new triangle. If $E''C = AC$, then $E''DAC$ is a kite, so we have three triangles. If $E''C = BC$, then $ABCE''D$ is a regular pentagon, and this is one of our claimed optimal configurations.

Now suppose that $\triangle EAD \cong \triangle ACD$ (Figure 8b). Point E in the figure makes $EACD$ either a kite, a non-convex congfiguration, or a configuration with three collinear points, depending on the length of DC. In any case, we have at least three triangles. Point E' makes three collinear points $E'AB$. For point E'', if $E''C$ is a new distance, we have a new triangle. If $E''C = AD$, then $ADE''C$ is a non-rhombus parallelogram, so we have three triangles. If $E''C = AC$, then $DE''C$ is a new triangle. Finally, if $E''C = DC$, then $DE''C$ is also a new triangle. This shows that the only way to add a fifth point to a trapezoid configuration without generating a third triangle is to create a regular pentagon, which concludes the proof of case 2e. □



(a) Options for adding a fifth point E so that $\triangle EAD \cong \triangle ABD$. Adding E or E' will create a third triangle, and adding E'' will create a third triangle if and only if $DC \neq AC$. If $DC = AC$, E'' is the fifth vertex of a regular pentagon.



(b) Options for adding a fifth point E so that $\triangle EAD \cong \triangle ACD$. E and E' both generate a third triangle, and E'' generates a third triangle if $D \neq AC$. If $DC = AC$, then E'' and C are the same point.

Figure 8: Possible additions of a fifth point when three sides are congruent.

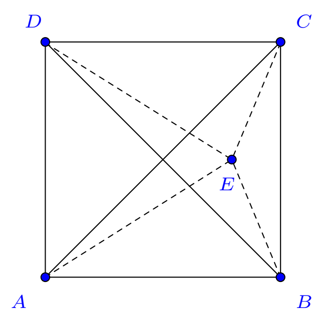
Proof of case 2f: four congruent sides. There are two subcases: the four points either form a non-square rhombus or a square.

If the four points form a non-square rhombus, then the argument presented in case 2(d)ii for a non-rectangle parallelogram also applies to show that the addition of a fifth point anywhere generates a third triangle (see Figure 6).

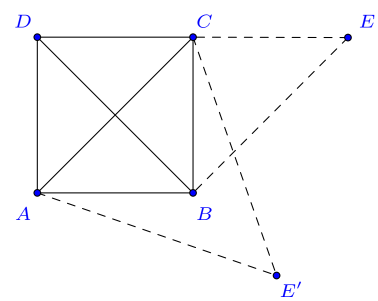
If the four points form a square, we must show that the addition of a fifth point anywhere but the center results in a configuration determining at least three triangles. If the fifth point is on the interior of the square but not in the center, then it creates a non-convex configuration (Figure 9a).

If the fifth point E is added outside the square, to avoid three distinct triangles, we must place it so that either $\triangle EBC \cong \triangle BCD$ or $\triangle EBC \cong \triangle EBA$ (see Figure 9b). If $\triangle EBC \cong \triangle BCD$, then ECD are collinear, so there are at least three triangles. If $\triangle EBC \cong \triangle EBA$, then we have a non-convex configuration, so there are at least three distinct triangles in this case also.

This shows that the addition of a fifth point to a square anywhere but the center generates at least three distinct triangles, and this completes the proof of case 2f. $\square$



(a) Addition of a fifth point inside the square but not at the center. $ABCE$ is a non-convex configuration, so we get three distinct triangles.



(b) Options for adding a fifth point to E to the outside of a square. Either option generates three distinct triangles.

Figure 9: Options for adding a fifth point to a square. Any choice except for the center of the square will result in a configuration with at least three distinct triangles.

6. Proof of Theorem 2.3

Proof. We show that the vertices of a regular n-gon determine $[n^2/12]$ distinct triangles. Conditional on Conjecture 2.2, this completes the proof. Label the vertices of a regular n-gon $\{P_0, \ldots, P_{n-1}\}$. By the symmetry of the configuration, every congruence class of a triangle has a member with P_0 as a vertex, so when counting triangles we can just count triangles incident on P_0. To form a triangle, we just have to pick two other vertices, P_a and P_b, and we can assume $a < b$. By symmetry, $\triangle P_0 P_a P_b$ will be distinct from $\triangle P_0 P_{a'} P_{b'}$ if and only if $\{a-0, b-a, n-b\}$ and $\{a'-0, b'-a', n-b'\}$ are not the same set (see Figure 10). Thus there is a bijection between distinct triangles determined by the regular n-gon and ways to write n as a sum of three positive integers. Using a result from the theory of integer partitions (see [6]), this quantity is equal to $[n^2/12]$, so this completes the proof.

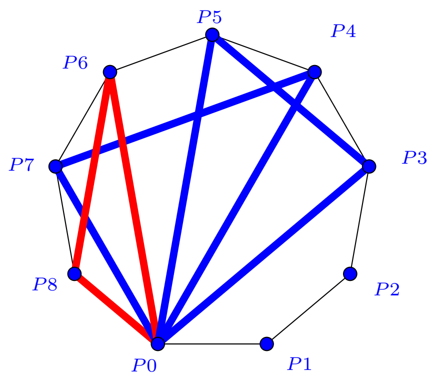


Figure 10: Illustrating the bijection described in the proof of Theorem 2.3 with $n = 9$. Note that triangles $\triangle P_0P_4P_7$ and $\triangle P_0P_3P_5$ represent the same partition of 9 $(\{4-0,\ 7-4,\ 9-7\} = \{3-0,\ 5-3,\ 9-5\} = \{4,3,2\})$. Thus they are congruent; however, $\triangle P_0P_6P_8$ represents a different partition $(\{6-0,\ 8-6,\ 9-8\} = \{6,2,1\})$, so it is a different triangle.

However, we can also get this quantity explicitly, without using Honsberger's result. We denote the number of ways to write n as a sum of three positive integers as $p(n,3)$. Since the order of a partition doesn't matter, we view this quantity as the number of ways to pick two elements $k < l$ from $\{1,\ldots,n\}$ such that $k \geq l-k \geq n-l > 0$. Note that k can be any of the elements $\lceil n/3 \rceil, \ldots, n-2$. Once k is chosen, l can be any of the elements $k + \lceil (n-k)/2 \rceil, \ldots, \min(2k,\ n-1)$. Note $2k$ is the minimum when $k \leq \lfloor n/2 \rfloor$, and $n-1$ is the minimum otherwise. Thus the number of choices is given by

$$\begin{aligned} p(n,3) &= \sum_{k=\lceil n/3 \rceil}^{\lfloor n/2 \rfloor} \sum_{l=k+\lceil (n-k)/2 \rceil}^{2k} 1 + \sum_{k=\lfloor n/2 \rfloor + 1}^{n-2} \sum_{l=k+\lceil (n-k)/2 \rceil}^{n-1} 1 \\ &= \sum_{k=n/3}^{n/2} \sum_{l=k+(n-k)/2}^{2k} 1 + \sum_{k=(n+2)/2}^{n-2} \sum_{l=k+(n-k)/2}^{n-1} 1 \ + \ O(n) \end{aligned}$$

$$= \sum_{k=n/3}^{n/2} (3k/2 - n/2 + 1) + \sum_{k=(n+2)/2}^{n-2} (n/2 - k/2) \ + \ O(n)$$

$$= \frac{3}{4}\left(\frac{n^2}{4} - \frac{n^2}{9}\right) - \frac{n^2}{12} + \frac{n^2}{4} - \frac{1}{4}\left(n^2 - \frac{n^2}{4}\right) \ + \ O(n)$$

$$= \frac{n^2}{12} \ + \ O(n), \tag{6.1}$$

and this completes the proof. □

7. Proof of Lemma 3.4

Proof of case 1: not in convex position. In this case, the four points form a triangle with one point in the interior (Figure 11). Triangle $\triangle ABD$ is contained in $\triangle ABC$, so they must be distinct. □

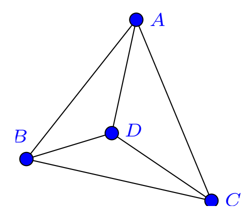


Figure 11: Four points not in convex position; $\triangle ABC$ and $\triangle ABD$ are distinct.

Proof of case 2a: three collinear points. Say point C lies on AB and D does not (Figure 12). Then $\triangle ACD$ is contained in $\triangle ABD$, so they are distinct. □

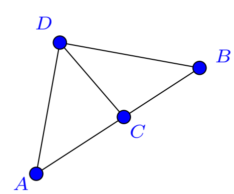


Figure 12: Four points containting three collinear points; $\triangle ACD$ and $\triangle ABD$ are distinct.

Proof of case 2b: no congruent sides. Say the four points form quadrilateral $ABCD$ (Figure 13). We have $\triangle ABD \ncong \triangle CBD$ because AB, AD, BC, and CD are all distinct. We claim $\triangle ABC$ is distinct from both of these. Triangle $\triangle ABC$ shares AB with $\triangle ABD$, and $BC \neq AD$, so if they are congruent then we must have $BC = BD$ and $AC = AD$. This is impossible because then $\triangle CBD$ and $\triangle CAD$ would both be isoceles triangles with CD as base, which is impossible unless one contains the other, which is not the case here. Thus $\triangle ABC \ncong ABD$. A similar argument shows that $\triangle ABC \ncong \triangle CBD$, so we have three distinct triangles. $\square$

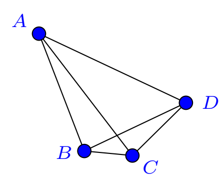


Figure 13: A quadrilateral with all distinct side lengths; $\triangle ABC$, $\triangle ABD$, and $\triangle CBD$ are all distinct.

Proof of case 2(c)i: one pair of adjacent congruent sides. Let the points form quadrilateral $ABCD$ and suppose $AB = AD$ (Figure 14). Triangle $\triangle ABD \ncong \triangle BCD$ because $\triangle ABD$ is isoceles but $\triangle BCD$ is not. Also, by the same argument as in part 2b, we see that $\triangle ABC$ is distinct from both of these, so there are at least three distinct triangles. $\square$

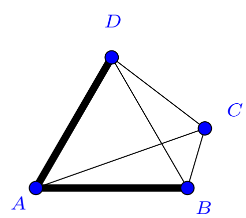


Figure 14: Quadrilateral with one pair of adjacent congruent sides (shown in bold); $\triangle ABD$, $\triangle BCD$, and $\triangle ABC$ are all distinct.

Proof of case 2(c)ii: one pair of opposite congruent sides. Suppose $AB = CD$ (Figure 15). Triangle $\triangle ABC \ncong \triangle DBC$ because they have two sides congruent to each other and the third is not. We now claim that $\triangle ACD$ is distinct from both of these. Triangle $\triangle ACD \ncong \triangle BCD$ by the same isoceles triangle argument from parts 2b and 2(c)i. If $\triangle ACD \cong \triangle ABC$, then BC must equal AD. But that would force AB to be parallel to CD, which would force $AC = BD$, a contradiction. Thus there are at least three distinct triangles. $\square$

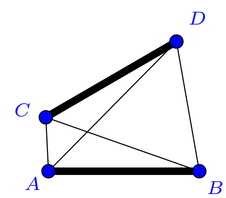


Figure 15: Quadrilateral with one pair of opposite congruent sides; $\triangle ACD$, $\triangle BCD$, and $\triangle ABC$ are all distinct.

Proof of case 2(d)i: two pairs of adjacent congruent sides. Say $AB = AD$ and $BC = CD$ and assume without loss of generality that $AC > BD$ (Figure 16). Triangle $\triangle ABD \ncong \triangle BCD$ because $AB \neq BC$. We claim that there is another triangle distinct from both of these. First note that it is impossible to have both $AC = CD = BC$ and $BD = AD = AB$. Because of this, the triangles $\triangle ABD$, $\triangle BCD$, and $\triangle ACD$ are necessarily distinct, so there are at least three distinct triangles.

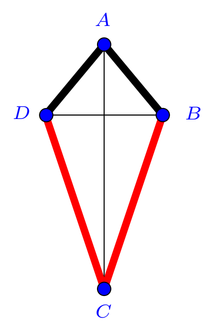


Figure 16: Quadrilateral with two pairs of adjacent congruent sides. Independently of the lengths of AC and BD, the triangles $\triangle ABD$, $\triangle BCD$, and $\triangle ACD$ are all distinct.

□

Proof of case 2e: three congruent sides. Say $AD = AB = BC$ (Figure 17). Triangle $\triangle ABC \ncong \triangle ADC$ because they have two sides congruent with each other and one side not congruent, thus there are at least two distinct triangles.

□

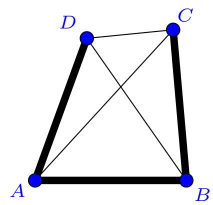


Figure 17: Quadrilateral with three congruent sides; $\triangle ABC$ and $\triangle ADC$ are distinct.

References

[1] P. Brass, W. Moser, and J. Pach, *Research Problems in Discrete Geometry*, Springer Verlag, 2005, 200.

[2] P. Erdős, On sets of distances of n points, *Amer. Math. Monthly* **53:5** (1946), 248-250.

[3] P. Erdős and P. Fishburn, Maximum planar sets that determine k distances, *Discrete Math.* **160:1-3** (1996), 115-125.

[4] A. Greenleaf and A. Iosevich, On three point configurations determined by subsets of the Euclidean plane, the associated bilinear operator and applications to discrete geometry, *Anal. PDE* **5:2** (2012), 397-409.

[5] L. Guth and N. H. Katz, On the Erdős distinct distances problem in the plane, *Ann. of Math.* **181:1** (2015), 155-190.

[6] R. Honsberger, *Mathematical Gems III*, Mathematical Association of America, Washington, D.C., 1985, 40-45.

[7] M. Rudnev, On the number of classes of triangles determined by N points in $\mathbb{R}^2$, arXiv: 1205.4865.

[8] M. Shinohara, Uniqueness of maximum planar five-distance sets, *Discrete Math.* **308:14** (2008), 3048-3055.

[9] L.A. Szekely, Crossing numbers and hard Erdős problems in discrete geometry, *Combin. Probab. Comput.* **11** (1993), 1-10.

[10] X. Wei, A proof of Erdős-Fishburn's conjecture for $g(6) = 13$, *Electron. J. Combin.* **19:4** (2012).