

AN ORTHOGONAL TEST OF THE L -FUNCTIONS RATIOS CONJECTURE

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ABSTRACT. We test the predictions of (a weakened version of) the L -functions Ratios Conjecture for the family of cuspidal newforms of weight k and level N , with either k fixed and $N \rightarrow \infty$ through the primes or $N = 1$ and $k \rightarrow \infty$. We study the main and lower order terms in the 1-level density. We provide evidence for the Ratios Conjecture by computing and confirming its predictions up to a power savings in the family's cardinality, at least for test functions whose Fourier transforms are supported in $(-2, 2)$. We do this both for the weighted and unweighted 1-level density (where in the weighted case we use the Petersson weights), thus showing that either formulation may be used. These two 1-level densities differ by a term of size $1/\log(k^2N)$. Finally, we show that there is another way of extending the sums arising in the Ratios Conjecture, leading to a different answer (although the answer is such a lower order term that it is hopeless to observe which is correct).

1. INTRODUCTION

Zeros of L -functions are some of the most important objects in modern number theory. Numerous problems are connected to them, and frequently the more detailed information we have about zeros, the more we can say about difficult problems. We remark on just a few of these applications. We then discuss a new procedure to predict these properties, and discuss our tests of its predictions.

The Generalized Riemann Hypothesis (GRH) asserts that all non-trivial zeros of an L -function have real part $1/2$. Just knowing that there are no zeros on the line $\Re(s) = 1$ for $\zeta(s)$ suffices to prove the Prime Number Theorem. Similarly the non-vanishing of Dirichlet L -functions at $s = 1$ imply the infinitude of primes in arithmetic progression (see for example [Da]).

Assuming GRH, all the non-trivial zeros lie on the line $\Re(s) = 1/2$. We can thus ask more refined questions about their spacing. The Grand Simplicity Hypothesis asserts that the imaginary parts of zeros of Dirichlet L -functions are linearly independent over \mathbb{Q} ; this is one of the key inputs in Rubinstein and Sarnak's [RubSa] analysis of Chebyshev's bias, the observed preponderance of primes in some arithmetic progressions over others.

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Finally, Conrey and Iwaniec [CI] show that if a positive percentage of the spacings between normalized zeros of certain L -functions is less than half the average spacing, then the class number of $\mathbb{Q}(\sqrt{-q})$ satisfies $h(q) \gg \sqrt{q}(\log q)^{-A}$ for some $A > 0$.

Since the 1970s, random matrix theory has provided powerful models to predict the behavior of zeros of L -functions. The scaling limits of zeros of individual or of a family of L -functions are well-modeled by the scaling limits of eigenvalues of matrices of classical compact groups (see for example [CFKRS, Hej, KaSa1, KaSa2, KeSn1, KeSn2, KeSn3, Mon, Od1, Od2]). In particular, these models immediately imply that a positive percentage of zeros are less than half the average spacing apart.

While the corresponding classical compact group is naturally connected to the monodromy group in the function field case, the connection is far more mysterious for number fields. Further, these models often add the number theoretic pieces in an ad-hoc manner, and thus there is a real need to develop methods which naturally incorporate the arithmetic.¹

In this work we concentrate on one such approach, the L -functions Ratios Conjecture of Conrey, Farmer and Zirnbauer [CFZ1, CFZ2], which provides a recipe for predicting many properties of L -functions to a phenomenal degree, ranging from n -level correlations and densities to moments and mollifiers (see [CS] for numerous applications). In our analysis below we actually use a weaker version of the Ratios Conjecture recipe than is stated in [CFZ1], as we do not need one of their assumptions in our analysis; we comment on this in greater detail in Remark 1.8.

In [Mil4] we showed that the Ratios Conjecture successfully predicts all lower order terms up to size $O(N^{-1/2+\epsilon})$ in the 1-level density for certain families of quadratic Dirichlet characters, at least provided the Fourier transform of the test function is supported in $(-1/3, 1/3)$. In this paper we apply the Ratios Conjecture to families of cuspidal newforms. We chose these families as the 1-level density can be determined for test functions whose Fourier transform is supported in $(-2, 2)$.² To prove results for support exceeding $(-1, 1)$ requires us to take into account non-diagonal terms, specifically sums of Bessel functions and Kloosterman sums. Thus our hope is that this will be a very good test of the Ratios Conjecture.

1.1. Notation. We first set some notation. Let $f \in S_k(N)$, the space of cusp forms of weight k and level N , let $\mathcal{B}_k(N)$ be an orthogonal basis of $S_k(N)$, and let $H_k^*(N)$ be the subset of newforms. To each f we associate an L -function:

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}. \quad (1.1)$$

The completed L -function is

$$\Lambda(s, f) = \left(\frac{\sqrt{N}}{2\pi} \right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(s, f), \quad (1.2)$$

¹See [DM2] for some recent results on determining the symmetry group of convolutions of families, and [GHK] for an alternate approach which is a hybrid of the Euler product and the Hadamard expansion, which has the advantage of the arithmetic arising naturally.

²If we assume Hypothesis S from [ILS], we can extend the number theory calculations up to $(-22/9, 22/9)$; see (4.35).

and satisfies the functional equation $\Lambda(s, f) = \epsilon_f \Lambda(1 - s, f)$ with $\epsilon_f = \pm 1$. Thus $H_k^*(N)$ splits into two disjoint subsets, $H_k^+(N) = \{f \in H_k^*(N) : \epsilon_f = +1\}$ and $H_k^-(N) = \{f \in H_k^*(N) : \epsilon_f = -1\}$. We often assume the Generalized Riemann Hypothesis (GRH), namely that all non-trivial zeros of $L(s, f)$ have real part $1/2$.

From Equation 2.73 of [ILS] we have for $N > 1$ that

$$|H_k^\pm(N)| = \frac{k-1}{24}N + O\left((kN)^{\frac{5}{6}}\right). \quad (1.3)$$

If $N = 1$ then $H_k^+(1) = H_k^*(1)$ if $k \equiv 0 \pmod{4}$ and $H_k^-(1) = H_k^*(1)$ if $k \equiv 2 \pmod{4}$, where $|H_k^*(1)| = \frac{k-1}{12} + O(k^{2/3})$.

We let $D_{1, H_k^*(N); R}(\phi)$ denote the weighted 1-level density for the family $H_k^*(N)$:

$$D_{1, H_k^*(N); R}(\phi) = \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{\substack{\gamma_f \\ L(1/2 + i\gamma_f, f) = 0}} \phi\left(\gamma_f \frac{\log R}{2\pi}\right). \quad (1.4)$$

We discuss the weights $\omega_f^*(N)$ in greater detail in §1.2, and $R = k^2 N$ is (essentially³) the analytic conductor, which is constant throughout the family.⁴ Katz and Sarnak [KaSa1, KaSa2] conjectured that as the conductors tend to infinity, the 1-level density agrees with the scaling limit of a classical compact group. There are now many cases where, for suitably restricted test functions, we can show agreement between the main terms and the conjectures; see, for example [DM1, FI, Gao, Gü, HR, HM, ILS, KaSa2, Mil1, OS, RR1, Ro, Rub, Yo2]. Now that the main terms have been successfully matched in numerous cases, it is natural to try to analyze the lower order terms. Here we break universality. While the arithmetic of the family does not enter into the main terms, it does surface in the lower order term (see for example [FI, Mil2, Mil3, Mil4, RR2, Yo1]).

The Ratios Conjecture is a recipe for predicting the main and lower order terms (often up to square-root in the family's cardinality) for ratios of L -functions. Consider a family \mathcal{F} of L -functions with some weights ω_f . We shall be particularly interested in both

$$R_{\mathcal{F}}(\alpha, \gamma) = \sum_{f \in \mathcal{F}} \omega_f \frac{L(\frac{1}{2} + \alpha, f)}{L(\frac{1}{2} + \gamma, f)} \quad (1.5)$$

and $\partial R_{\mathcal{F}}(\alpha, \gamma) / \partial \alpha \Big|_{\alpha=\gamma=s}$. We are interested in the derivative as a contour integral of it yields the 1-level density.

1.2. Weights. To simplify some of the arguments, we content ourselves with investigating two cases: k is fixed and $N \rightarrow \infty$ through the primes,⁵ and $N = 1$ and $k \rightarrow \infty$.

³Sometimes the analytic conductor is defined slightly differently, incorporating some of the constants in the Gamma factors. In investigations of the 1-level density for these families, however, these factors are independent of k and N , and for our purposes all that matters is the k and N dependence in the conductor. Thus there is no loss in setting $R = k^2 N$.

⁴It greatly simplifies our analysis to have a family where the analytic conductors are constant. This allows us to pass the summation over the family past the test function to the Fourier transforms. Non-constant families can often be handled, at a cost of additional work and sieving (see for example [Mil1]).

⁵With additional work, the arguments should generalize to N square-free, though with worse error terms.

Throughout our analysis we shall need to investigate sums such as

$$\sum_{f \in H_k^*(N)} \lambda_f(m) \lambda_f(n). \quad (1.6)$$

It is technically easier to consider weighted sums

$$\sum_{f \in H_k^*(N)} \omega_f(N) \lambda_f(m) \lambda_f(n), \quad (1.7)$$

where the $\omega_f(N)$ are the harmonic (or Petersson) weights, though for completeness we study the unweighted sums as well. These are defined by

$$\omega_f^*(N) = \frac{\Gamma(k-1)}{(4\pi)^{k-1} (f, f)_N}, \quad (1.8)$$

where

$$(f, f)_N = \int_{\Gamma_0(N) \backslash \mathbb{H}} f(z) \bar{f}(z) y^{k-2} dx dy. \quad (1.9)$$

These weights are almost constant. We have the bounds (see [HL, Iw])

$$N^{-1-\epsilon} \ll_k \omega_f^*(N) \ll_k N^{-1+\epsilon}, \quad (1.10)$$

if we allow ineffective constants we can replace N^ϵ with $\log N$ for N large.

The main tool for evaluating these (weighted) sums is the Petersson formula; we state several useful variants in Appendix A.

If $N > 1$ we should use modified weights $\omega_f(N)/\omega(N)$, where

$$\omega(N) = \sum_{f \in H_k^*(N)} \omega_f(N). \quad (1.11)$$

The reason is that our family does not include the oldforms. One advantage of restricting to N prime is that the only oldforms in $S_k(N)$ are forms of level 1. We know there are only $O(k)$ such forms. As each $\omega_f(N) \ll N^{-1+\epsilon}$,

$$\sum_{f \in H_k^*(N)} \omega_f(N) = \sum_{f \in S_k(N)} \omega_f(N) + O\left(\frac{k}{N^{1-\epsilon}}\right) = 1 + O\left(\frac{k}{N^{1-\epsilon}}\right). \quad (1.12)$$

Thus for k fixed and $N \rightarrow \infty$, the difference between using $\omega_f(N)$ and $\omega_f(N)/\omega(N)$ is $O(N^{-1+\epsilon})$. We set

$$\omega_f^*(N) = \begin{cases} \omega_f(1) & \text{if } N = 1 \\ \omega_f(N)/\omega(N) & \text{if } N > 1; \end{cases} \quad (1.13)$$

note

$$\sum_{f \in H_k^*(N)} \omega_f^*(N) = 1 = (1 + O(N^{-1+\epsilon})) \sum_{f \in B_k(N)} \omega_f(N). \quad (1.14)$$

Remark 1.1. For some problems, such as bounding the order of vanishing at the central point for families of cuspidal newforms [HM, ILS], it is desirable to study the unweighted family. We shall see below that there is a difference of size $1/\log R$ between the weighted and unweighted 1-level densities. The predictions from the Ratios Conjecture (for weighted and unweighted families) agrees with the corresponding result from number theory in both cases.

1.3. Main results.

Theorem 1.2. *Assume GRH for $\zeta(s)$ and all $L(s, f)$ with $f \in H_k^*(N)$. The (weakened version of the) Ratios Conjecture predicts*

$$\begin{aligned} D_{1, H_k^*(N); R}(\phi) &= 2 \sum_p \widehat{\phi} \left(\frac{2 \log p}{\log R} \right) \frac{\log p}{p \log R} + M(\phi) \\ &+ \frac{1}{\log R} \int_{-\infty}^{\infty} \left(2 \log \frac{\sqrt{N}}{\pi} + \psi \left(\frac{1}{4} + \frac{k \pm 1}{4} + \frac{2\pi it}{\log R} \right) \right) \phi(t) dt \\ &+ O((kN)^{-1/2+\epsilon}), \end{aligned} \quad (1.15)$$

where⁶

$$\begin{aligned} M(\phi) &= \frac{2i^k \mu(N)}{N \log R} \int_{-\infty}^{\infty} X_L \left(\frac{1}{2} + \frac{2\pi it}{\log R} \right) \frac{\zeta(2)}{\zeta(2 + \frac{8\pi it}{\log R})} \zeta \left(1 + \frac{4\pi it}{\log R} \right) \\ &\cdot \prod_p \left(1 - \frac{p^{4\pi it / \log R} - 1}{p(p^{1+4\pi it / \log R} + 1)} \right) e^{-2\pi it \frac{\log N}{\log R}} \phi(t) dt \end{aligned} \quad (1.16)$$

and $X_L(s)$ is as in (2.9). If $N > 1$ then $M(\phi) \ll N^{-1}$. Let $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$. If $N = 1$ then $M(\phi) \ll 2009^k k^{-\frac{1-4\sigma}{3}k}$, which decays more rapidly than $k^{-\delta}$ for any $\delta > 0$ provided $\sigma < 1/4$.

Remark 1.3. Our estimate for $M(\phi)$ is significantly worse when $N = 1$; see Remark 3.5 for an explanation and a connection to other problems. Interestingly, if we change the order of some of the steps in the Ratios Conjecture's recipe, then $M(\phi)$ changes by a factor of $e^{-\gamma}$. See Appendix C for complete details, as well as Remark 2.10.

The 1-level density computation has some differences depending on whether or not $N \rightarrow \infty$ through the primes or $N = 1$ and $k \rightarrow \infty$. We therefore separate our results into two cases; see Remark 1.8 for an explanation of these differences. Further, we can often obtain results for smaller support without assuming GRH for Dirichlet L -functions, and thus we isolate these as well.

Theorem 1.4. *Let $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$ and let $N \rightarrow \infty$ through the primes.*

- (Density Theorem Limited) *If $\sigma < 3/2$ then the weighted 1-level density for the family $H_k^*(N)$ agrees with the prediction from the Ratios Conjecture up to errors of size $O(N^{\sigma - \frac{3}{2} + \epsilon} + N^{\frac{\sigma}{2} - 1 + \epsilon} + N^{\frac{\sigma}{4} - 1 + \epsilon'})$.*
- (Density Theorem Extended) *Assuming GRH for $\zeta(s)$, all Dirichlet L -functions and $L(s, f)$, the weighted 1-level density agrees with the prediction from the Ratios Conjecture up to errors of size $O(N^{\frac{\sigma}{2} - 1 + \epsilon} + N^{\frac{\sigma}{4} - 1 + \epsilon'})$.*

Remark 1.5. Theorem 1.4 implies we have agreement up to a power savings in N for $\sigma < 3/2$, and up to square-root cancellation for $\sigma < 1$. Assuming GRH, we can extend agreement up to $\sigma < 2$, again saving a power in N .

Theorem 1.6. *Let $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$ $N = 1$ and $k, K \rightarrow \infty$.*

⁶If $N = 1$, note the factor $\frac{\mu(N)}{N} \exp(-2\pi it \log N / \log R)$ below equals 1.

- (Density Theorem Limited) *The weighted 1-level density for the family $H_k^*(1)$ agrees with the prediction from the Ratios Conjecture up to errors of size $O(k^{-(5-3\sigma)/6+\epsilon})$ for $\sigma < 1/4$. If we knew $M(\phi) \ll k^{-(5-3\sigma)/6+\epsilon}$ for $\sigma < 1$, then we would have agreement up to $\sigma < 1$.*
- (Density Theorem Extended) *Let h be a Schwartz function compactly supported on $(0, \infty)$. Consider a weighted average (over k) of the weighted 1-level density*

$$A^*(K; \phi) = \frac{1}{A^*(K)} \sum_{k \equiv 0 \pmod{2}} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k^*(1)} D_{1, H_k^*(1); k^2}(\phi), \quad (1.17)$$

where

$$A^*(K) = \sum_{k \equiv 0 \pmod{2}} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) |H_k^*(1)| = \widehat{h}(0)K + O(K^{2/3}). \quad (1.18)$$

Assuming GRH for $\zeta(s)$, all Dirichlet L -functions and $L(s, f)$, the 1-level density agrees with the prediction from the Ratios Conjecture up to errors of size $O(K^{-(5-\sigma)/6+\epsilon} + K^{\sigma-2+\epsilon})$ for $\sigma < 1/4$. If we knew $M(\phi) \ll K^{-(5-\sigma)/6+\epsilon} + K^{\sigma-2+\epsilon}$ for $\sigma < 2$, then we would have agreement up to $\sigma < 2$.

- (Hypothesis S and Density Theorem Extended) *Assume Hypothesis S from [ILS] (i.e., (4.35)) with $A = 0$ and $\alpha = 1/2$. Then as $K \rightarrow \infty$ the weighted average (over k) of the weighted 1-level density agrees with the prediction from the Ratios Conjecture for $\sigma < 1/4$. If we knew $M(\phi) \ll K^{-2(2.5-\sigma)} + K^{-(5-\sigma)/6+\epsilon} + K^{-\frac{11}{2}(1-\frac{9}{22}\sigma)}$ for $\sigma < 22/9$, then we would have agreement up to $\sigma < 22/9$.*

Remark 1.7. Theorem 1.6 implies we have agreement up to a power savings in K for $\sigma < 1/4$; in fact, we agree *beyond* square-root cancelation in this range. Assuming GRH, by averaging over k we can extend our calculations up to $\sigma < 2$ (or, if we assume Hypothesis S, up to $\sigma < 22/9$). If we knew $M(\phi)$ were small, we would again save a power in N (with agreement up to square-root cancelation for $\sigma < 3/2$ if we assume GRH for Dirichlet L -functions, or up to $\sigma < 22/9$ if we assume Hypothesis S).

Remark 1.8. We briefly comment on the differences in the calculations for $N \rightarrow \infty$ through the primes and $N = 1$ and $k \rightarrow \infty$. In the first family, the sign of the functional equation is basically 1 half the time and -1 the other half, while in the second family it is either always 1 (if $k \equiv 0 \pmod{4}$) or -1 (if $k \equiv 2 \pmod{4}$). It is natural to argue in constructing the Ratios Conjecture recipe similarly as in other conjectures, such as in the Moments Conjecture (see [CFKRS]). In both any product of the signs of the functional equations is replaced with the average of the signs of the functional equations in the family.⁷ When $N \rightarrow \infty$ we thus do not expect any term, as the average of the functional equations is zero; however, for $N = 1$ the average is non-zero (either 1 or -1), and there will be a predicted term (which may or may not be small, but which must be analyzed). This is one reason why our results are significantly weaker when $N = 1$.⁸ If we were to follow the Ratios Conjecture recipe fully, we would not have the $M(\phi)$

⁷See for instance the comments after equation (4.1.5) and the analysis after equation (4.5.4) in [CFKRS] or the comments after (5.5) in [CFZ1] (note that in an earlier draft of [CFZ1] there was no mention of replacing products of epsilon factors with their averages, though this is a natural thing to do).

⁸Another is that the analytic conductor is $k^2 N$. If $N = 1$ then the conductor is essentially the square of the family's cardinality, while if $N \rightarrow \infty$ the conductor is of the same size as the family's cardinality.

term when $N \rightarrow \infty$. We choose to include $M(\phi)$ as our analysis that it is $O(1/N)$ provides support for replacing the epsilon factors in the Ratios Conjecture recipe with their average over the family (in this case, zero).

Theorem 1.9. *Assume GRH for $\zeta(s)$, all Dirichlet L -functions and all $L(s, f)$. The unweighted 1-level density for $H_k^*(N)$ agrees with the predictions of the Ratios Conjecture for the unweighted family, up to a power savings in the family's cardinality, as $N \rightarrow \infty$ through the primes; this answer differs from the weighted 1-level density by an additional term of size $1/\log R$. The Ratios Conjecture applied to the unweighted family predicts*

$$\begin{aligned} D_{1, H_k^*(N); R}^{\text{unwt}}(\phi) &= \frac{1}{\log R} \int_{-\infty}^{\infty} \left(2 \log \frac{\sqrt{N}}{\pi} + \psi \left(\frac{1}{4} + \frac{k \pm 1}{4} + \frac{2\pi it}{\log R} \right) \right) \phi(t) dt \\ &\quad + 2 \sum_{\substack{\nu \equiv 0 \pmod{2} \\ \nu \geq 2}} \sum_{p \neq N} \frac{p-1}{p^\nu} \widehat{\phi} \left(\nu \frac{\log p}{\log R} \right) \frac{\log p}{\log R} \\ &\quad + O((kN)^{-1/2+\epsilon}), \end{aligned} \tag{1.19}$$

which agrees with number theory up to errors of size $O(N^{-(2-\sigma)/6+\epsilon})$.

Remark 1.10. Theorem 1.9 and our other results imply that the predictions from the L -functions Ratios Conjecture agree with number theory for both the weighted and unweighted families. Thus, when investigating cuspidal newforms, we may study either family.

The paper is organized as follows. In §2 we describe the Ratios Conjecture's recipe, and determine its prediction for the 1-level density for our families. In §3 we analyze these predictions and prove Theorem 1.2. In §4 we prove Theorems 1.4 and 1.6, which show the 1-level densities agree (up to a power savings in the cardinality of the families, at least for suitably restricted test functions) with what can be proved. Finally, in §5 we analyze the unweighted 1-level density, and prove Theorem 1.9.

2. RATIOS CONJECTURE

The Ratios Conjecture is a recipe to predict the main and lower order terms for a variety of problems. We analyze its predictions for the 1-level density for families of cuspidal newforms. We first briefly describe its recipe for predicting quantities related to

$$R_{\mathcal{F}}(\alpha, \gamma) = \sum_{f \in \mathcal{F}} \omega_f \frac{L(\frac{1}{2} + \alpha, f)}{L(\frac{1}{2} + \gamma, f)}. \tag{2.1}$$

In the description below, we actually use a slightly weakened version of the Ratios Conjecture; the difference is in step (3).

- (1) Use the approximate functional equation to expand the numerator into two sums plus a remainder. The first sum is over m up to x and the second over n up to y , where xy is of the same size as the analytic conductor (typically one takes $x = y$). We ignore the remainder term.

- (2) Expand the denominator by using the generalized Mobius function:

$$\frac{1}{L(s, f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $h = p$, $\chi_0(p)$ if $h = p^2$ (with χ_0 the trivial character modulo N) and 0 otherwise.

- (3) Execute the sum over \mathcal{F} , keeping only main (diagonal) terms. In [CFZ1] there is an additional step before executing the sums over m and n , as they replace any product over epsilon factors (arising from the signs of the functional equations) with the average value of the sign of the functional equation in the family. For families of constant sign (such as $N = 1$), there is no difference; however, for families where the sign varies (such as $N \rightarrow \infty$) there is a significant difference, as they would not have our term $M(\phi)$. We are thus using a weaker recipe of the Ratios Conjecture; we choose to include the analysis of the contribution from the signs of the functional equation as this provides support for the claim that we may ignore any term multiplied by signs of functional equations which average to zero.⁹ See Remark 1.8 for additional details.
- (4) Extend the m and n sums to infinity (i.e., complete the products).
- (5) Differentiate with respect to the parameters, and note that the size of the error term does not significantly change upon differentiating.
- (6) A contour integral involving $\left. \frac{\partial}{\partial \alpha} R_{\mathcal{F}}(\alpha, \gamma) \right|_{\alpha=\gamma=s}$ yields the 1-level density.

We now describe these steps in greater detail and deduce the Ratios Conjecture's prediction for the 1-level density.

Remark 2.1. It is almost miraculous how well the Ratios Conjecture works, given that several of the steps involve throwing away significant error terms. The miracle is that all these errors seem to cancel, and the resulting expression is correct to a remarkable order. See Remark 2.11 for more details.

Remark 2.2. Differentiating is essentially harmless because we have analytic functions. If the error were $N^{-1/2} \cos(N^2 \alpha)$ and α was forced to be real, then differentiating increases the error from size $N^{-1/2}$ to $N^{3/2}$! For us, α will be complex. By Cauchy's integral theorem, if f is analytic at z_0 then

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz, \quad (2.2)$$

where C is a circle of very small radius about z_0 . The sum of the ratios is analytic, and we shall see later that the main term is analytic. Thus their difference, the error term, is also analytic. Applying Cauchy's argument with a circle of very small radius, say $\log^{-2009} R$, we see the effect of differentiating is only to increase the error by some powers of $\log R$. We thank David Farmer for pointing this out to us.

⁹Also, of course, $M(\phi)$ is present when $N = 1$.

2.1. Approximate Functional Equation. We state the approximate functional equation in greater generality than we need, though not the greatest generality possible; see Section 1 of [CFKRS] for more details. Let

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad (2.3)$$

be a nice L -function with real coefficients ($a_n \in \mathbb{R}$),

$$\gamma_L(s) = P(s)Q^s \prod_{j=1}^w \Gamma(w_j s + \mu_j) \quad (2.4)$$

with $Q, w_j > 0$, $\mu_j \geq 0$ and $P(s)$ a real polynomial whose zeros in $\Re(s) > 0$ are at the poles of $L(s)$ (so if $L(s)$ has no poles then $P(s)$ is constant). Let

$$\xi_L(s) = \gamma_L(s)L(s) = \epsilon \bar{\xi}_L(1-s) \quad (2.5)$$

be the completed L -function, with $|\epsilon| = 1$ the sign of the functional equation and $\bar{\xi}_L(s) = \xi_L(\bar{s})$. Our assumptions imply that $\bar{\xi}_L(s) = \xi_L(s)$. Set

$$X_L(s) = \frac{\gamma_L(1-s)}{\gamma_L(s)} = \frac{P(1-s)Q^{1-s} \prod_{j=1}^w \Gamma(w_j(1-s) + \mu_j)}{P(s)Q^s \prod_{j=1}^w \Gamma(w_j s + \mu_j)}. \quad (2.6)$$

Then

Lemma 2.3 (The Approximate Functional Equation). *Notation and assumptions as above,*

$$L(s) = \sum_{m \leq x} \frac{a_m}{m^s} + \epsilon X_L(s) \sum_{n \leq y} \frac{a_n}{n^{1-s}} + \text{remainder}, \quad (2.7)$$

where xy is of the same size as the analytic conductor.

Remark 2.4. The Ratios Conjecture's recipe for generating predictions ignores the remainder term in the approximate functional equation. Thus we too shall ignore these errors in our arguments below, and treat the approximate functional equation as exact.

For us, $L(s)$ will be a weight k cuspidal newform of level N , which we shall denote by $L(s, f)$. In this case, we have (see [ILS] for instance) that

$$\begin{aligned} \gamma_L(s) &= \left(\frac{2^k}{8\pi}\right)^{1/2} \left(\frac{\sqrt{N}}{\pi}\right)^s \Gamma\left(\frac{s}{2} + \frac{k-1}{4}\right) \Gamma\left(\frac{s}{2} + \frac{k+1}{4}\right) \\ &= \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right); \end{aligned} \quad (2.8)$$

note that $\gamma_L(s)$ depends only on the weight k and the level N of the cuspidal newform f . This yields the following expressions for $X_L(s)$:

$$\begin{aligned} X_L(s) &= \left(\frac{\sqrt{N}}{\pi}\right)^{1-2s} \frac{\Gamma\left(\frac{1-s}{2} + \frac{k-1}{4}\right) \Gamma\left(\frac{1-s}{2} + \frac{k+1}{4}\right)}{\Gamma\left(\frac{s}{2} + \frac{k-1}{4}\right) \Gamma\left(\frac{s}{2} + \frac{k+1}{4}\right)} \\ &= \left(\frac{\sqrt{N}}{2\pi}\right)^{1-2s} \frac{\Gamma\left(\frac{1-s}{2} + \frac{k-1}{2}\right)}{\Gamma\left(\frac{s}{2} + \frac{k-1}{2}\right)}. \end{aligned} \quad (2.9)$$

Finally, the analytic conductor of a cuspidal newform of weight k and level N is (up to a constant) $k^2 N$. Thus we will typically take $x = y \sim \sqrt{k^2 N}$ in the approximate functional equation.

2.2. Ratios Conjecture. Let χ_0 denote the principal character with conductor N . For f a weight k cuspidal newform of level N we have

$$\begin{aligned} L(s, f) &= \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}} \right)^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \\ \frac{1}{L(s, f)} &= \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}} \right) = \sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^s}, \end{aligned} \quad (2.10)$$

where $\mu_f(n)$ is the multiplicative function such that $\mu_f(1) = 1$, $\mu_f(p) = -\lambda_f(p)$, $\mu_f(p^2) = \chi_0(p)$, and $\mu_f(p^k) = 0$ for $k \geq 3$.

Let \mathcal{F} be a family of weight k cuspidal newforms of level N . The Ratios Conjecture for the family gives an expansion for

$$R_{\mathcal{F}}(\alpha, \gamma) = \sum_{f \in \mathcal{F}} \omega_f \frac{L(\frac{1}{2} + \alpha, f)}{L(\frac{1}{2} + \gamma, f)}, \quad (2.11)$$

where α and γ satisfy

- (1) $\Re(\alpha) \in (-1/4, 1/4)$;
- (2) $\Re(\gamma) \in (1/\log |\mathcal{F}|, 1/4)$;
- (3) $\Im(\alpha), \Im(\gamma) \ll_{\epsilon} |\mathcal{F}|^{1-\epsilon}$ for all $\epsilon > 0$.

We have introduced weights ω_f , as often in practice the weighted sum is significantly easier to control. For example, we may take ω_f to be the Petersson weights, which facilitates applying the Petersson formula (see Appendix A for statements). As remarked in §1.2, it is convenient to choose $\omega_f = \omega_f^*(N)$ (see (1.13)).

We shall concentrate on the diagonal terms in the Petersson formula. Thus if our family is $H_k^*(N)$ then by the Petersson formula we have for n_1 and n_2 relatively prime to N ,

$$\sum_{f \in H_k^*(N)} \omega_f^*(N) \lambda_f(n_1) \lambda_f(n_2) = \delta_{n_1, n_2} + \text{small}. \quad (2.12)$$

The weights are normalized to sum to 1. If $N > 1$ our sums do not include the oldforms; however, the oldforms do not contribute to the main term of the Petersson formula in this case.

In general, we must be careful by what we mean by ‘small’ when we apply the Petersson formula. The error term is a Bessel-Kloosterman sum, and is typically small only if n_1 and n_2 are not too large with respect to k and N (and are relatively prime to N). It is *very* important that our sums are restricted. It is only after we compute the main term that the heuristics of the Ratios Conjecture tells us to extend the sums to infinity. Depending on how (and when!) we extend our sums to infinity can lead to different answers.¹⁰

¹⁰These differences, however, involve terms of size $1/N$, which is unimaginable beyond anything we can hope to prove (except possibly if $N = 1$; we hope to return to this in a future paper). Interestingly,

Unless our family is all of $H_k^*(N)$ and $N = 1$, however, the sign of the functional equation is not constant.¹¹ For square-free N we have

$$\epsilon_f = i^k \mu(N) \lambda_f(N) \sqrt{N}; \quad (2.13)$$

thus the sign of the functional equation only weakly depends on the specific form f . Further $\lambda_f(q)^2 = 1/q$ if $q|N$. Note $\mu(1) = 1$ and $\mu(N) = -1$ if N is prime, and since k is even we have $i^k = \pm 1$. Thus there is at most one ‘bad’ prime, namely N .

Remark 2.5. We consider just the case $\mathcal{F} = H_k^*(N)$ with N either 1 or prime here; more involved arguments should be able to handle the case of N square-free (at a cost of worse error terms), and we will investigate the sub-families $H_k^\pm(N)$ in a future paper [HuMi].

Lemma 2.6. *The Ratios Conjecture predicts that*

$$\begin{aligned} R_{H_k^*(N)}(\alpha, \gamma) &= \prod_p \left(1 - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \\ &\quad - \frac{i^k \mu(N) X_L \left(\frac{1}{2} + \alpha \right)}{N^{1+\gamma} \zeta(1-\alpha+\gamma)} \prod_p \left(1 + \frac{p^{1-\alpha+\gamma}}{p^{1+2\gamma}(p^{1-\alpha+\gamma}-1)} \right) \\ &\quad + O \left(|H_k^*(N)|^{-1/2+\epsilon} \right), \end{aligned} \quad (2.14)$$

where the N -factors are present only if N is prime.

Proof. From the Approximate Functional Equation (Lemma 2.3) and (2.13) we have

$$L \left(\frac{1}{2} + \alpha, f \right) = \sum_{m \leq x} \frac{\lambda_f(m)}{m^{\frac{1}{2}+\alpha}} + i^k \mu(N) \lambda_f(N) \sqrt{N} X_L \left(\frac{1}{2} + \alpha \right) \sum_{n \leq y} \frac{\lambda_f(n)}{n^{\frac{1}{2}-\alpha}}, \quad (2.15)$$

where $x = y \sim \sqrt{k^2 N}$. From (2.10) we have

$$\frac{1}{L(1/2 + \gamma, f)} = \sum_{h=1}^{\infty} \frac{\mu_f(h)}{h^{\frac{1}{2}+\gamma}}. \quad (2.16)$$

Therefore

$$\begin{aligned} R_{H_k^*(N)}(\alpha, \gamma) &= \\ &\sum_{f \in H_k^*(N)} \omega_f^*(N) \left[\sum_{\substack{m \leq x \\ h}} \frac{\mu_f(h) \lambda_f(m)}{h^{\frac{1}{2}+\gamma} m^{\frac{1}{2}+\alpha}} + i^k \mu(N) X_L \left(\frac{1}{2} + \alpha \right) \sqrt{N} \sum_{\substack{n \leq y \\ h}} \frac{\mu_f(h) \lambda_f(N) \lambda_f(n)}{h^{\frac{1}{2}+\gamma} m^{\frac{1}{2}-\alpha}} \right]. \end{aligned} \quad (2.17)$$

If $N > 1$ then the presence of the $\lambda_f(N)$ factor requires us to handle the two sums in slightly different manners. We first analyze the sum without the $\lambda_f(N)$ factor. By the Petersson formula, we have $\sum_{f \in H_k^*(N)} \omega_f^*(N) \lambda_f(n_1) \lambda_f(n_2) = \delta_{n_1, n_2} + \text{small}$ if at least one of n_1 and n_2 is relatively prime to N . There are two cases: either $N = 1$ and

however, the difference between these two terms is related to sieving actual versus random primes. See Appendix C.

¹¹See Remark 1.8 for comments on how the non-constancy of the signs of the functional equation affects the Ratios Conjecture’s prediction.

$k \rightarrow \infty$ or k is fixed and $N \rightarrow \infty$ through the primes. As $x \sim \sqrt{k^2 N}$, if $N > 1$ then N does not divide m for sufficiently large N . Thus we may assume $(n_2, N) = 1$. Using the multiplicativity of the Fourier coefficients, from the Petersson formula (Lemma A.3) we see that if $p|n_1$ then there is negligible contribution unless $p^\ell || n_1$ and $p^\ell || n_2$. From the definition of the multiplicative function $\mu_f(h)$, we see immediately that h must be cube-free (if not, $\mu_f(h) = 0$). Thus we may write $h = p_1 \cdots p_r \cdot q_1^2 \cdots q_\ell^2$ where p_1, \dots, q_ℓ are distinct primes, and $\mu_f(h) = (-1)^r \lambda_f(p_1 \cdots p_r) \chi_0(q_1 \cdots q_\ell)$. We immediately see that unless m is square-free and equal to $p_1 \cdots p_r$ and the q_i are relatively prime to N then the main term from $\mu_f(h) \lambda_f(m)$ is zero. Further, the p_i must also be prime to N , as $p_i \leq m \leq x \sim \sqrt{k^2 N}$. Thus the only contribution from the m and h -sum is

$$\prod_{p \leq x} \left(1 - \frac{\lambda_f(p)^2}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \cdot \prod_{\substack{p > x \\ p \neq N}} \left(1 + \frac{1}{p^{1+2\gamma}} \right). \quad (2.18)$$

To see this, use multiplicativity to replace the sum in (2.17) with a product over primes, dropping all terms which will give a negligible contribution after applying the Petersson formula. For each prime $p \leq x$ we either have 1, $\mu_f(p) \lambda_f(p)$ or $\mu_f(p^2) \lambda_f(1)$. The product over $p > x$ arises from the fact that, for such large primes, we must either have 1 or $\mu_f(p^2) \lambda_f(1)$ (as the m -sum is only up to primes at most x , and the prime $p = N$ can be ignored because $\chi_0(N) = 0$). Thus when we use the Petersson formula we always have two Fourier coefficients relatively prime to the level N . Summing over $f \in H_k^*(N)$ allows us to replace $\lambda_f(p)^2$ with 1 + small (and, as always, we ignore all ‘small’ terms), so the first half of $R_{H_k^*(N)}(\alpha, \gamma)$ is

$$\prod_{p \leq x} \left(1 - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \cdot \prod_{\substack{p > x \\ p \neq N}} \left(1 + \frac{1}{p^{1+2\gamma}} \right); \quad (2.19)$$

see Remark 2.8 for an explanation as to why we have chosen to write this factor in this way. As is customary in applications of the Ratios Conjecture, we complete the m -sum by extending it to infinity. This is equivalent to sending x to infinity. Thus the first term of $R_{H_k^*(N)}(\alpha, \gamma)$ is

$$\prod_p \left(1 - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right). \quad (2.20)$$

We now study the $\lambda_f(N) \mu_f(h) \lambda_f(n)$ terms in (2.17), noting that N does not divide n (since $n \leq y \sim \sqrt{k^2 N}$). There is thus negligible contribution unless $N || h$. For $p = N$ the factor is now

$$\frac{\mu_f(N)}{N^{\frac{1}{2}+\gamma}} = -\frac{\lambda_f(N)}{N^{\frac{1}{2}+\gamma}} \quad (2.21)$$

(again, this factor is not present if $N = 1$). Remember we have a truncated sum, with $n \leq y$. Thus for $p \leq y$ the factors are the same as before (except we replace α with $-\alpha$), arising from factors of 1, $\mu_f(p) \lambda_f(p)$ or $\mu_f(p^2) \lambda_f(1)$. However, for $y < p \neq N$ the factor is $1 + p^{-1-2\gamma}$ (arising from 1 or $\mu_f(p^2)$ – there is no $\mu_f(p) \lambda_f(p)$ term as

$p > y$). Thus our factors are

$$\begin{aligned} & i^k \mu(N) \lambda_f(N) \sqrt{N} X_L \left(\frac{1}{2} + \alpha \right) \\ & \cdot \prod_{p \leq y} \left(1 - \frac{\lambda_f(p)^2}{p^{1-\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \cdot \prod_{\substack{p > y \\ p \neq N}} \left(1 + \frac{1}{p^{1+2\gamma}} \right) \cdot \frac{-\lambda_f(N)}{N^{\frac{1}{2}+\gamma}}, \end{aligned} \quad (2.22)$$

where as before the N -factor is present only if N is prime. If $N > 1$ we replace $\lambda_f(N)^2$ with $1/N$, so when we apply the Petersson formula all Fourier coefficients will be relatively prime to the level N . If $N = 1$ we do not have this second factor of $\lambda_f(N)$; however, as $\lambda_f(1) = 1$ the resulting expression is the same.

Summing over $f \in H_k^*(N)$ allows us to replace the $\lambda_f(p)^2$ factors above with $1 +$ small. Thus the product becomes

$$\begin{aligned} & -\frac{i^k \mu(N)}{N^{1+\gamma}} X_L \left(\frac{1}{2} + \alpha \right) \prod_{p \leq y} \left(1 - \frac{1}{p^{1-\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \cdot \prod_{p > y} \left(1 + \frac{1}{p^{1+2\gamma}} \right) \\ = & -\frac{i^k \mu(N)}{N^{1+\gamma}} X_L \left(\frac{1}{2} + \alpha \right) \\ & \cdot \prod_{p \leq y} \left(1 - \frac{1}{p^{1-\alpha+\gamma}} \right) \cdot \left(1 + \frac{p^{1-\alpha+\gamma}}{p^{1+2\gamma}(p^{1-\alpha+\gamma} - 1)} \right) \cdot \prod_{p > y} \left(1 + \frac{1}{p^{1+2\gamma}} \right). \end{aligned} \quad (2.23)$$

As before, we complete the n -sum by sending y to infinity. We have deliberately pulled out the p -factor of $1/\zeta(1 - \alpha + \gamma)$ to improve the convergence of the remaining piece. We thus find this factor is

$$-\frac{i^k \mu(N)}{N^{1+\gamma}} X_L \left(\frac{1}{2} + \alpha \right) \cdot \frac{1}{\zeta(1 - \alpha + \gamma)} \prod_p \left(1 + \frac{p^{1-\alpha+\gamma}}{p^{1+2\gamma}(p^{1-\alpha+\gamma} - 1)} \right). \quad (2.24)$$

Substituting the above completes the proof. \square

Remark 2.7. In the Ratios Conjecture, the size of the error term is added in a somewhat ad-hoc manner. The predicted size of the error term is amazing, as it implies the lower order terms depending on the arithmetic of the family are calculated basically up to square-root cancelation in the family's cardinality. As the recipe involves throwing away numerous remainders and arguing their aggregate does not matter, it is not possible to rigorously derive the size of the error term (unless, of course, we make significant progress towards proving the Ratios Conjecture!), and the standard assumptions in practice are that it is typically smaller than the main term by approximately the square-root of the family's cardinality. See Remark 2.11 for additional comments on the discarded error terms.

Remark 2.8. We briefly discuss how we chose to write some of the factors above; we are thankful to the referee for pointing this out, and paraphrase their comments below. The usual convention in the Ratios Conjecture is to pull out factors of zeta, which leaves a product over primes that is convergent when the shift parameters α and γ are very small. When this is done then every orthogonal family ratios conjecture looks the same, except for the convergent prime product, which is specific to the family; the polar

structure of the terms is determined by the factors of zeta. For us, we could have pulled out a factor of $\zeta(1+2\gamma)/\zeta(1+\alpha+\gamma)$ to leave something convergent for small α and γ . One expects this to be the same for every orthogonal family (for example, we can see the same factor in equation (2.55) of [CS]). The structure of the zetas pulled out to ensure convergence is an identifying feature of the orthogonal symmetry as they mirror the structure of the random matrix result (compare the structure of the ratios of “z”s, the quantities called “y”, in section 4 of [CFZ1] for different symmetry types). The number theory results exactly mirror these RMT results. It is not always convenient to write the number theory results in this way when you want to calculate with them, but there is a universal structure there. This is similar to some of the computations in [HM], where an alternate expression for the n-level density (different than the determinantal one from Katz-Sarnak [KaSa1, KaSa2]) was derived to facilitate comparisons between number theory and RMT).

Lemma 2.9. *Let $R'_{H_k^*(N)}(r, r) = \frac{\partial}{\partial \alpha} R_{H_k^*(N)}(\alpha, \gamma) \Big|_{\alpha=\gamma=r}$. Then for $\Re r > 0$ the Ratios Conjecture predicts*

$$\begin{aligned} R'_{H_k^*(N)}(r, r) &= \sum_p \frac{\log p}{p^{1+2r}} + \frac{i^k \mu(N)}{N^{1+r}} X_L \left(\frac{1}{2} + r \right) \prod_p \left(1 + \frac{1}{(p-1)p^{2r}} \right) \\ &\quad + O \left(|H_k^*(N)|^{-1/2+\epsilon} \right), \end{aligned} \quad (2.25)$$

where, as always, the N -factors are present only if $N > 1$.

Proof. We must differentiate the two terms in Lemma 2.6, and investigate the limit as $y \rightarrow \infty$; see Lemma 2.2 for an explanation as to why the size of the error term is unaffected. The first term is easily handled. Using $d \log f(\alpha)/d\alpha = f'(\alpha)/f(\alpha)$, we see that

$$\begin{aligned} \frac{\partial}{\partial \alpha} \left[\prod_p \left(1 - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \right] \Big|_{\alpha=\gamma=r} &= \prod_p \left(1 - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \Big|_{\alpha=\gamma=r} \\ &\quad \cdot \frac{\partial}{\partial \alpha} \log \left[\prod_p \left(1 - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \right] \Big|_{\alpha=\gamma=r} \\ &= \sum_p \frac{\log p}{p^{1+2r}}, \end{aligned} \quad (2.26)$$

where we need $\Re(r) > 0$ to ensure that the sum converges.

We now handle the second term in Lemma 2.6. We must differentiate, with respect to α ,

$$-\frac{i^k \mu(N) X_L \left(\frac{1}{2} + \alpha \right)}{N^{1+\gamma} \zeta(1-\alpha+\gamma)} \prod_p \left(1 + \frac{p^{1-\alpha+\gamma}}{p^{1+2\gamma}(p^{1-\alpha+\gamma}-1)} \right). \quad (2.27)$$

We use the following observation (see page 7 of [CS]): if $f(z, w)$ is analytic at $(z, w) = (r, r)$, then

$$\frac{\partial}{\partial \alpha} \frac{f(\alpha, \gamma)}{\zeta(1-\alpha+\gamma)} \Big|_{\alpha=\gamma=r} = -f(r, r). \quad (2.28)$$

Thus the derivative of (2.27) with respect to α , evaluated at $\alpha = \gamma = r$, is

$$\frac{i^k \mu(N)}{N^{1+r}} X_L \left(\frac{1}{2} + r \right) \prod_p \left(1 + \frac{1}{(p-1)p^{2r}} \right). \quad (2.29)$$

□

Remark 2.10. If we don't extend the sums to infinity before differentiating, we get from Mertens' theorem (see Appendix C) a factor of $e^{-\gamma}$ in the second sum, where γ here is Euler's constant. This is very interesting, as $e^{-\gamma}$ is related to sieving primes. The sieving constant of $e^{-\gamma}$ in Mertens theorem is not 1, though for a generic sequence of random primes (also called Hawkins primes) it is. While it is fascinating that there are two procedures which lead to different answers, this term is of size $1/N$, well beyond any plausible hope of testing.¹² See [BK, Ha, HW, Gr, NW, Wu] for some additional comments on $e^{-\gamma}$.

Remark 2.11. We briefly comment on the size of the errors made at various steps in the Ratios Conjecture. For example, consider the first piece of $R'_{H_k^*(N)}(r, r)$, namely $\sum_p \frac{\log p}{p^{1+2r}}$. This piece arose from a product originally over $p \leq \sqrt{R}$ which we extended to be over all p ; thus the error between what we should have had and what we wrote is $\sum_{p \geq \sqrt{R}} \frac{\log p}{p^{1+2r}}$. We typically evaluate this when $r = \epsilon + it$, and thus we have introduced an error of size $O(R^{-\epsilon'})$. Thus while this is smaller than any power of $1/\log R$, it is significantly more than $R^{-1/2+\epsilon}$. Thus this sizable error *must* be canceled by other errors if the Ratios Conjecture is to yield the correct prediction.

3. WEIGHTED 1-LEVEL DENSITY FROM THE RATIOS CONJECTURE

3.1. Main Expansion. We now compute the 1-level density for the family $H_k^*(N)$, with either $N = 1$ and $k \rightarrow \infty$ or k a fixed even integer and N tending to infinity through the primes. We follow closely the arguments in [CS, Mil4].

Lemma 3.1. *Assume GRH for $\zeta(s)$ and all $L(s, f)$ with $f \in H_k^*(N)$, and let ϕ be an even Schwartz function whose Fourier transform has compact support. Denote the weighted 1-level density for the family $H_k^*(N)$ by*

$$D_{1, H_k^*(N); R}(\phi) = \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{\substack{\gamma_f \\ L(1/2+i\gamma_f, f)=0}} \phi \left(\gamma_f \frac{\log R}{2\pi} \right). \quad (3.1)$$

¹²This term is related to $M(\phi)$. If $N = 1$ we can only show $M(\phi)$ is small for $\sigma < 1/4$, though based on number theory computations we expect it to be small for $\sigma < 2$ or even $22/9$.

Assuming the Ratios Conjecture, we have

$$\begin{aligned}
D_{1, H_k^*(N); R}(\phi) &= 2 \sum_p \widehat{\phi} \left(\frac{2 \log p}{\log R} \right) \frac{\log p}{p \log R} \\
&+ \frac{2i^k \mu(N)}{N \log R} \int_{-\infty}^{\infty} X_L \left(\frac{1}{2} + \frac{2\pi it}{\log R} \right) \prod_{p \neq N} \left(1 + \frac{1}{(p-1)p^{4\pi it / \log R}} \right) e^{-2\pi it \frac{\log N}{\log R}} \phi(t) dt \\
&+ \frac{1}{\log R} \int_{-\infty}^{\infty} \left(2 \log \frac{\sqrt{N}}{\pi} + \psi \left(\frac{1}{4} + \frac{k \pm 1}{4} + \frac{2\pi it}{\log R} \right) \right) \phi(t) dt \\
&+ O((kN)^{-1/2+\epsilon}).
\end{aligned} \tag{3.2}$$

Proof. We first compute the unscaled, weighted 1-level density $S_{1; H_k^*(N)}(g)$ for the family $H_k^*(N)$ with g an even Schwartz function,

$$S_{1; H_k^*(N)}(g) = \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{\substack{\gamma_f \\ L(1/2+i\gamma_f, f)=0}} g(\gamma_f). \tag{3.3}$$

Let $c \in \left(\frac{1}{2} + \frac{1}{\log k^2 N}, \frac{3}{4} \right)$; thus

$$\begin{aligned}
S_{1; H_k^*(N)}(g) &= \sum_{f \in H_k^*(N)} \frac{1}{2\pi i} \left(\int_{(c)} - \int_{(1-c)} \right) \omega_f^*(N) \frac{L'(s, f)}{L(s, f)} g \left(-i \left(s - \frac{1}{2} \right) \right) ds \\
&= S_{1, c; H_k^*(N)}(g) + S_{1, 1-c; H_k^*(N)}(g).
\end{aligned} \tag{3.4}$$

We argue as in §3 of [CS]. We first analyze the integral on the line $\Re(s) = c$. By GRH and the rapid decay of g , for large t the integrand is small. We use the Ratios Conjecture (Lemma 2.9 with $r = c - \frac{1}{2} + it$) to replace the $\sum_f \omega_f^*(N) L'(s, f)/L(s, f)$ term when t is small. We may then extend the integral to all of t because of the rapid decay of g . As the integrand is regular at $r = 0$ we can move the path of integration to $c = 1/2$. The contribution from the error term in the Ratios Conjecture is negligible, due to g being a Schwartz function. Thus the integral on the c -line is

$$\begin{aligned}
S_{1, c; H_k^*(N)}(g) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} g \left(t - i \left(c - \frac{1}{2} \right) \right) \sum_{f \in H_k^*(N)} \omega_f^*(N) \frac{L'(\frac{1}{2} + (c - \frac{1}{2} + it), f)}{L(\frac{1}{2} + (c - \frac{1}{2} + it), f)} i dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) \left[\sum_p \frac{\log p}{p^{1+2it}} \right. \\
&\quad \left. + \frac{i^k \mu(N)}{N^{1+it}} X_L \left(\frac{1}{2} + it \right) \prod_p \left(1 + \frac{1}{(p-1)p^{2it}} \right) \right] dt. \\
&+ O((kN)^{-1/2+\epsilon}).
\end{aligned} \tag{3.5}$$

As

$$\int_{-\infty}^{\infty} g(t) p^{-2it} dt = \int_{-\infty}^{\infty} g(t) e^{-2\pi i \left(\frac{2 \log p}{2\pi} \right) t} dt = \widehat{g} \left(\frac{2 \log p}{2\pi} \right) \tag{3.6}$$

we have

$$\begin{aligned} S_{1,c;H_k^*(N)}(g) &= \frac{1}{2\pi} \sum_p \widehat{g} \left(\frac{2 \log p}{2\pi} \right) \frac{\log p}{p} \\ &+ \frac{i^k \mu(N)}{2\pi N} \int_{-\infty}^{\infty} X_L \left(\frac{1}{2} + it \right) \prod_p \left(1 + \frac{1}{(p-1)p^{2it}} \right) N^{-it} g(t) dt \\ &+ O((kN)^{-1/2+\epsilon}). \end{aligned} \quad (3.7)$$

We now study $S_{1,1-c;H_k^*(N)}(g)$:

$$S_{1,1-c;H_k^*(N)}(g) = \sum_{f \in H_k^*(N)} \frac{-\omega_f^*(N)}{2\pi i} \int_{\infty}^{-\infty} \frac{L'(1-(c+it), f)}{L(1-(c+it), f)} g \left(-i \left(\frac{1}{2} - c \right) - t \right) (-idt). \quad (3.8)$$

We use the functional equation

$$L(s, f) = \epsilon_f X_L(s) L(1-s, f) \quad (3.9)$$

to find that

$$\frac{L'(1-(c+it), f)}{L(1-(c+it), f)} = -\frac{L'(c+it, f)}{L(c+it, f)} + \frac{X'_L(c+it)}{X_L(c+it)}. \quad (3.10)$$

This yields

$$\begin{aligned} S_{1,1-c;H_k^*(N)}(g) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{f \in H_k^*(N)} \omega_f^*(N) \frac{L'(c+it, f)}{L(c+it, f)} g \left(-i \left(\frac{1}{2} - c \right) - t \right) dt \\ &- \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{X'_L(c+it)}{X_L(c+it)} g \left(-i \left(\frac{1}{2} - c \right) - t \right) dt. \end{aligned} \quad (3.11)$$

The first term yields the same contribution as $S_{1,c;H_k^*(N)}(g)$; this follows by sending c to $1/2$ and noting g is an even function. Thus

$$\begin{aligned} S_{1;H_k^*(N)}(g) &= \frac{2}{2\pi} \sum_p \widehat{g} \left(\frac{2 \log p}{2\pi} \right) \frac{\log p}{p} \\ &+ \frac{2i^k \mu(N)}{2\pi N} \int_{-\infty}^{\infty} X_L \left(\frac{1}{2} + it \right) \prod_p \left(1 + \frac{1}{(p-1)p^{2it}} \right) N^{-it} g(t) dt \\ &- \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{X'_L(1/2+it)}{X_L(1/2+it)} g(t) dt + O((kN)^{-1/2+\epsilon}). \end{aligned} \quad (3.12)$$

In investigating zeros near the central point, it is convenient to renormalize them by the logarithm of the analytic conductor. Let $g(t) = \phi \left(\frac{t \log R}{2\pi} \right)$. A straightforward computation shows that $\widehat{g}(\xi) = \frac{2\pi}{\log R} \widehat{\phi}(2\pi\xi/\log R)$. The (scaled) weighted 1-level density for the family $H_k^*(N)$ is

$$D_{1,H_k^*(N);R}(\phi) = \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{\substack{\gamma_f \\ L(1/2+i\gamma_f, f)=0}} \phi \left(\gamma_f \frac{\log R}{2\pi} \right) = S_{1;H_k^*(N)}(g) \quad (3.13)$$

(where $g(t) = \phi\left(\frac{t \log R}{2\pi}\right)$ as before). Thus

$$\begin{aligned} D_{1, H_k^*(N); R}(\phi) &= 2 \sum_p \widehat{\phi}\left(\frac{2 \log p}{\log R}\right) \frac{\log p}{p \log R} \\ &+ \frac{2i^k \mu(N)}{2\pi N} \int_{-\infty}^{\infty} X_L\left(\frac{1}{2} + it\right) \prod_{p \neq N} \left(1 + \frac{1}{(p-1)p^{2it}}\right) N^{-it} \phi\left(\frac{t \log R}{2\pi}\right) dt \\ &- \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{X'_L(1/2 + it)}{X_L(1/2 + it)} \phi\left(\frac{t \log R}{2\pi}\right) dt + O((kN)^{-1/2+\epsilon}). \end{aligned} \quad (3.14)$$

Changing variables yields

$$\begin{aligned} D_{1, H_k^*(N); R}(\phi) &= 2 \sum_p \widehat{\phi}\left(\frac{2 \log p}{\log R}\right) \frac{\log p}{p \log R} \\ &+ \frac{2i^k \mu(N)}{N \log R} \int_{-\infty}^{\infty} X_L\left(\frac{1}{2} + \frac{2\pi it}{\log R}\right) \prod_{p \neq N} \left(1 + \frac{1}{(p-1)p^{4\pi it / \log R}}\right) e^{-2\pi it \frac{\log N}{\log R}} \phi(t) dt \\ &- \frac{1}{\log R} \int_{-\infty}^{\infty} \frac{X'_L\left(\frac{1}{2} + \frac{2\pi it}{\log R}\right)}{X_L\left(\frac{1}{2} + \frac{2\pi it}{\log R}\right)} \phi(t) dt + O((kN)^{-1/2+\epsilon}). \end{aligned} \quad (3.15)$$

Set $\psi(z) = \Gamma'(z)/\Gamma(z)$. As the derivative of $\log X_L(s)$ is $X'_L(s)/X_L(s)$, we find

$$-\frac{X'_L\left(\frac{1}{2} + \frac{2\pi it}{\log R}\right)}{X_L\left(\frac{1}{2} + \frac{2\pi it}{\log R}\right)} = 2 \log \frac{\sqrt{N}}{\pi} + \frac{1}{2} \psi\left(\frac{1}{4} + \frac{k \pm 1}{4} \pm \frac{2\pi it}{\log R}\right) \quad (3.16)$$

(note there are four ψ -terms). As ϕ is an even function, the $+t$ and $-t$ terms yield the same integral, completing the proof. \square

The first sum and the last integral in Lemma 3.1 will match up perfectly with terms from the number theory calculation. In §3.2 we finish the proof of Theorem 1.2 by analyzing the middle term.

3.2. Proof of Theorem 1.2.

Proof of Theorem 1.2. Most of the analysis for the first part of the theorem has been done in §3.1; in particular, the expansion in (3.15). The proof is completed by Lemmas 3.2 and 3.4 below, which derive a simpler expression for the middle piece and then show it yields a negligible contribution. \square

Lemma 3.2. *Let $\Re(u) = 0$. Then*

$$\prod_p \left(1 + \frac{1}{(p-1)p^u}\right) = \frac{\zeta(2)}{\zeta(2+2u)} \cdot \zeta(1+u) \cdot \prod_p \left(1 - \frac{p^u - 1}{p(p^{1+u} + 1)}\right); \quad (3.17)$$

note the product over primes converges rapidly for $\Re(u) = 0$, as each term in the product is like $1 + O(1/p^2)$.

Proof. We have

$$\begin{aligned}
\prod_p \left(1 + \frac{1}{(p-1)p^u}\right) &= \prod_p \left(1 + \frac{1}{p^{1+u}}\right) \cdot \left(1 + \frac{1}{(p-1)(p^{1+u} + 1)}\right) \\
&= \prod_p \frac{\left(1 + \frac{1}{p^{1+u}}\right) \cdot \left(1 - \frac{1}{p^{1+u}}\right)}{1 - \frac{1}{p^{1+u}}} \cdot \left(1 + \frac{1}{(p-1)(p^{1+u} + 1)}\right) \\
&= \frac{\zeta(1+u)}{\zeta(2+2u)} \cdot \prod_p \left(1 + \frac{1}{(p-1)(p^{1+u} + 1)}\right). \quad (3.18)
\end{aligned}$$

We can rewrite this a little further, using

$$\begin{aligned}
\prod_p \left(1 + \frac{1}{(p-1)(p^{1+u} + 1)}\right) &= \prod_p \frac{p^2}{p^2 - 1} \left(1 - \frac{p^u - 1}{p(p^{1+u} + 1)}\right) \\
&= \prod_p \frac{1}{1 - \frac{1}{p^2}} \cdot \left(1 - \frac{p^u - 1}{p(p^{1+u} + 1)}\right) \\
&= \zeta(2) \prod_p \left(1 - \frac{p^u - 1}{p(p^{1+u} + 1)}\right). \quad (3.19)
\end{aligned}$$

Substituting this into (3.18) completes the proof. \square

Remark 3.3. When arguing along the lines of the Ratios Conjecture, it often greatly simplifies the calculations to rewrite the prime products in a more rapidly convergent manner by factoring out zeta or L -functions. In Lemma B.1 we use the above expansion to show that the X_L term in the 1-level density is negligible. When $N = 1$ this is the hardest part of the proof, and follows by shifting contours.

Lemma 3.4. *Let*

$$\begin{aligned}
M(\phi) &= \frac{2i^k \mu(N)}{N \log R} \int_{-\infty}^{\infty} X_L \left(\frac{1}{2} + \frac{2\pi it}{\log R}\right) \frac{\zeta(2)}{\zeta\left(2 + \frac{8\pi it}{\log R}\right)} \zeta\left(1 + \frac{4\pi it}{\log R}\right) \\
&\quad \cdot \prod_p \left(1 - \frac{p^{4\pi it / \log R} - 1}{p(p^{1+4\pi it / \log R} + 1)}\right) e^{-2\pi it \frac{\log N}{\log R}} \phi(t) dt. \quad (3.20)
\end{aligned}$$

If $N > 1$ we have $M(\phi) = O(1/N)$. Assume $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$. If $N = 1$ then $M(\phi) = O\left(2009^k \cdot k^{-\frac{1-4\sigma}{3}k}\right)$, which tends to zero more rapidly than $k^{-\delta}$ for any $\delta > 0$ for $\sigma < 1/4$.

Proof. We use the lemmas from Appendix B to bound the relevant quantities. As ϕ is an even function, there is no contribution from the pole of the Riemann zeta function.

Assume first $N > 1$. If $u \geq 0$ then

$$\left| \zeta\left(2 + 2u + \frac{8\pi it}{\log R}\right) \right| \geq 1 - \sum_{n=2}^{\infty} \frac{1}{n^2} = 1 - \left(\frac{\pi^2}{6} - 1\right) > 0. \quad (3.21)$$

As remarked above, there is no contribution from the pole of the Riemann zeta function (since ϕ is even). We may thus subtract off the pole without changing the value of the

integral, and note that

$$\left| \zeta \left(1 + \frac{4\pi it}{\log R} \right) - \frac{\log R}{4\pi it} \right| \ll (t^2 + 1) \log R \quad (3.22)$$

(we could of course do far better, but a very weak bound suffices for large t due to the rapid decay of ϕ). Thus the product of the zeta terms is $O((t^2 + 1) \log N)$. The product over primes is bounded by

$$\prod_p \left(1 - \frac{2}{p(p-1)} \right), \quad (3.23)$$

which is $O(1)$. Finally, the X_L -term is $O(1)$ by Lemma B.1. Thus

$$M(\phi) \ll \frac{1}{N \log R} \int_{-\infty}^{\infty} (t^2 + 1) \log R \cdot \phi(t) dt \ll \frac{1}{N} \quad (3.24)$$

(as ϕ is a Schwartz function).

Assume now that $N = 1$. We follow the method used in [Mil4], and replace t with $t - iw \frac{\log R}{4\pi}$ (where initially $w = 0$), shift contours and exploit the decay in w . By analyzing X_L and the zeta factors, we see we may shift the contour to $w = 2k - 1 - \epsilon$ without passing through any zeros or poles. We shift to $w = \frac{2k-1}{3}$ as this will simplify some of the computations. We have

$$\begin{aligned} M(\phi) &= -\frac{2i^k}{\log R} \int_{-\infty}^{\infty} X_L \left(\frac{1+w}{2} + \frac{2\pi it}{\log R} \right) \frac{\zeta(2)}{\zeta(2+2w + \frac{8\pi it}{\log R})} \zeta \left(1 + w + \frac{4\pi it}{\log R} \right) \\ &\quad \cdot A \left(w + \frac{2\pi it}{\log R} \right) \cdot \phi \left(t - iw \frac{\log R}{2\pi} \right) dt \end{aligned} \quad (3.25)$$

where

$$A(x + iy) = \prod_p \left(1 - \frac{p^{x+iy} - 1}{p(p^{1+x+iy} + 1)} \right). \quad (3.26)$$

As ϕ is even, there is no contribution from the pole of the zeta function. In the arguments below, we could be more explicit and subtract off this pole. The shifted term will have a factor of size $O((w^2 + (t/\log R)^2)^{-1}) = O(1)$, which will not change any of the arguments.

From Lemma B.4 we have $A \left(w + \frac{2\pi it}{\log R} \right) = O(1)$. For any $w > 0$, by (3.21) the ratio of the zeta factors $\zeta(2)/\zeta(2+2w + \frac{8\pi it}{\log R})$ is $O(1)$. From Lemma B.1 we know that for $w = \frac{2k-1}{3}$ the X_L -term is $O(2009^k \cdot k^{-k/3})$, and from Lemma B.3 we have

$$\phi \left(t - iw \frac{\log R}{2\pi} \right) \ll \exp(\sigma w \log R) \cdot \left(t^2 + \frac{\log^2 R}{16\pi^2} \right)^n \ll \frac{R^{\sigma w}}{(t^2 + 1)^n}. \quad (3.27)$$

Thus

$$M(\phi) \ll (2009^k \cdot k^{-k/3}) \cdot \frac{R^{\sigma w}}{\log R} \int_{-\infty}^{\infty} \frac{dt}{(t^2 + 1)^n} \ll \frac{2009^k R^{\sigma w}}{k^{k/3} \log R}. \quad (3.28)$$

For cuspidal newforms of level 1 and weight k , one takes (see (1.14) and (4.29) of [ILS]) $R \sim k^2$. As $w = \frac{2k-1}{3}$, the above decays more rapidly than

$$2009^k \cdot k^{\frac{4k\sigma}{3} - \frac{k}{3}} = 2009^k \cdot k^{-\frac{1-4\sigma}{3}k}, \quad (3.29)$$

thus as long as $\sigma < 1/4$, this term decays faster than $k^{-\delta}$ for any $\delta > 0$. \square

Remark 3.5. Note the results in Lemma 3.4 are significantly worse for $N = 1$ than for $N \rightarrow \infty$.¹³ This is due to the rapid growth of $\widehat{\phi}(x + iy)$ in y , and leads to a significantly reduced support. This is very similar to the difficulties encountered in studying families of quadratic characters [Mil4], where we again had to perform a contour shift, which restricted our results to $\sigma < 1$ (with square-root agreement for $\sigma < 1/3$). Our result is weaker than the corresponding result in [Mil4] (we have $\sigma < 1/4$ instead of $\sigma < 1$) because here the conductor is k^2 (whereas in [Mil4] the conductor is d) and k appears in the Gamma factors.

Remark 3.6. Another approach to analyzing $M(\phi)$ when $N = 1$ is to shift the contour very far to the right, picking up contributions from the poles of the Gamma function in the numerator of X_L . Unfortunately the resulting expressions can only be shown to be small for $\sigma < 1/4$. The poles arise when $w = k - \frac{1}{2} + 2\ell$ for $\ell \in \{0, 1, 2, \dots\}$, and yield contributions of

$$\frac{-2i^k \zeta(2)\zeta(1+k-\frac{1}{2}+2\ell) A(k-\frac{1}{2}+2\ell)}{\log R \zeta(2+2k-1+4\ell)\Gamma(k-\frac{1}{2}+\ell)} \cdot \left(\frac{2\pi}{\sqrt{N}}\right)^{k-\frac{1}{2}+2\ell} \cdot \phi\left(\frac{-i(k-\frac{1}{2}+2\ell)\log R}{2\pi}\right). \quad (3.30)$$

The ϕ term is at most $R^{\sigma(k-\frac{1}{2}+2\ell)}$, while the main term of $\Gamma(k-\frac{1}{2}+\ell)$ is of size $k^\ell k^{k-\frac{1}{2}+\ell}$. The problem is the resulting sum over ℓ is only small if $\sigma < 1/4$, though based on our number theory computations we expect it to be small for $\sigma < 1$ or even $\sigma < 2$. The difficulty is that we are ignoring all oscillation when we shift contours.

4. WEIGHTED 1-LEVEL DENSITY FROM NUMBER THEORY

We now determine the main and lower order terms in the 1-level density for the family $H_k^*(N)$ for as large support as possible for the Fourier transform of the test function. In [ILS] the main term is determined for $\text{supp}(\widehat{\phi}) \subset (-2, 2)$; however, as they are only concerned with the main term they are a little crude in bounding the error terms. We perform a more careful analysis below.

In Section 4 of [ILS] the explicit formula is used to compute the 1-level density for the family $H_k^*(N)$. In their paper $Q = \sqrt{N}/\pi$. Noting that $\widehat{\phi}(0) = \int_{-\infty}^{\infty} \phi(t) dt$, we may

¹³In fact, for $N \rightarrow \infty$ the Ratios Conjecture states we should not include $M(\phi)$. We only have it as we are using a weaker version of their conjecture; see Remark 1.8 for additional details.

rewrite the weighted sum over $f \in H_k^*(N)$ of their equation (4.11) as

$$D_{1, H_k^*(N); R}(\phi) = \frac{1}{\log R} \int_{-\infty}^{\infty} \left(2 \log \frac{\sqrt{N}}{\pi} + \psi \left(\frac{1}{4} + \frac{k \pm 1}{4} + \frac{2\pi it}{\log R} \right) \right) \phi(t) dt \\ - 2 \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_p \sum_{\nu=1}^{\infty} \frac{\alpha_f^\nu(p) + \beta_f^\nu(p)}{p^{\nu/2}} \widehat{\phi} \left(\frac{\nu \log p}{\log R} \right) \frac{\log p}{\log R}, \quad (4.1)$$

where

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s} \right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s} \right)^{-1}. \quad (4.2)$$

Note the first term agrees exactly with the last term from the Ratios Conjecture (Theorem 1.2).

The following identities for the Fourier coefficients (for $p \nmid N$) are standard:

$$\begin{aligned} \lambda_f(p) &= \alpha_f(p) + \alpha_f(p)^{-1}, \quad |\alpha_f(p)| = 1, \quad \alpha_f(p)^{-1} = \beta_f(p) \\ \lambda_f(p^\nu) &= \alpha_f(p)^\nu + \alpha_f(p)^{\nu-2} + \cdots + \alpha_f(p)^{2-\nu} + \alpha_f(p)^{-\nu} \\ \alpha_f(p)^\nu + \alpha_f(p)^{-\nu} &= \lambda_f(p^\nu) - \lambda_f(p^{\nu-2}). \end{aligned} \quad (4.3)$$

Trivially bounding the contribution from $p = N$, we may thus rewrite $D_{1, H_k^*(N); R}(\phi)$ as

$$D_{1, H_k^*(N); R}(\phi) = \frac{1}{\log R} \int_{-\infty}^{\infty} \left(2 \log \frac{\sqrt{N}}{\pi} + \psi \left(\frac{1}{4} + \frac{k \pm 1}{4} + \frac{2\pi it}{\log R} \right) \right) \phi(t) dt \\ - 2 \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{p \neq N} \frac{\lambda_f(p)}{\sqrt{p}} \widehat{\phi} \left(\frac{\log p}{\log R} \right) \frac{\log p}{\log R} \\ - 2 \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{p \neq N} \frac{\lambda_f(p^2) - 1}{p} \widehat{\phi} \left(2 \frac{\log p}{\log R} \right) \frac{\log p}{\log R} \\ - 2 \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{p \neq N} \sum_{\nu=3}^{\infty} \frac{\lambda_f(p^\nu) - \lambda_f(p^{\nu-2})}{p^{\nu/2}} \widehat{\phi} \left(\frac{\nu \log p}{\log R} \right) \frac{\log p}{\log R} \\ + O \left(\frac{1}{\sqrt{N}} \right) \\ = \frac{1}{\log R} \int_{-\infty}^{\infty} \left(2 \log \frac{\sqrt{N}}{\pi} + \psi \left(\frac{1}{4} + \frac{k \pm 1}{4} + \frac{2\pi it}{\log R} \right) \right) \phi(t) dt \\ + 2 \sum_p \frac{1}{p} \widehat{\phi} \left(2 \frac{\log p}{\log R} \right) \frac{\log p}{\log R} \\ - S_1(\phi) - S_2(\phi) - S_3(\phi) + O \left(\frac{1}{\sqrt{N}} \right), \quad (4.4)$$

where

$$\begin{aligned}
S_1(\phi) &= 2 \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{p \neq N} \frac{\lambda_f(p)}{\sqrt{p}} \widehat{\phi} \left(\frac{\log p}{\log R} \right) \frac{\log p}{\log R} \\
S_2(\phi) &= 2 \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{p \neq N} \frac{\lambda_f(p^2)}{p} \widehat{\phi} \left(2 \frac{\log p}{\log R} \right) \frac{\log p}{\log R} \\
S_3(\phi) &= 2 \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{p \neq N} \sum_{\nu=3}^{\infty} \frac{\lambda_f(p^\nu) - \lambda_f(p^{\nu-2})}{p^{\nu/2}} \widehat{\phi} \left(\nu \frac{\log p}{\log R} \right) \frac{\log p}{\log R}. \quad (4.5)
\end{aligned}$$

The first and the second terms above perfectly match with terms from the Ratios Conjecture. We must therefore show the other three terms are negligible. We prove Theorems 1.4 and 1.6 in stages below; we first perform the analysis for limited support, and then extend the support by assuming various conjectures.

4.1. Density Theorem Limited. As the arguments are similar when $N \rightarrow \infty$ through the primes and when $N = 1$ and $k \rightarrow \infty$, we give complete details for $N \rightarrow \infty$ and sketch the arguments when $N = 1$.

Remark 4.1. It is important to note that we have included the harmonic (or Petersson) weights in our family to facilitate applications of the Petersson formula. When using results from [ILS], one must be careful as they have three related quantities involving averages of the Fourier coefficients over families. The first (converting to our notation) is their equation (2.7),

$$\Delta_k(m, n) = \sum_{f \in \mathcal{B}_k(N)} \omega_f(N) \lambda_f(m) \lambda_f(n); \quad (4.6)$$

the weights sum to 1, and thus in this expression we have effectively divided by the cardinality of the family. Note that we are summing over all cusp forms of weight k and level N , and not just the newforms. The second is their equation (2.54), where we sum over just the newforms:

$$\Delta_{k,N}^\sigma(m, n) = \zeta(2) \sum_{f \in H_k^\sigma(N)} \frac{\lambda_f(m) \lambda_f(n)}{L(1, \text{sym}^2 f)}, \quad \sigma \in \{*, +, -\}. \quad (4.7)$$

Finally, we have the unweighted, pure sums (their equation (2.59)):

$$\Delta_{k,N}^\sigma(n) = \sum_{f \in H_k^\sigma(N)} \lambda_f(n), \quad \sigma \in \{*, +, -\}. \quad (4.8)$$

Much effort was spent in [ILS] to remove the weights; thus when reading their paper we must look carefully to see which variant they are using.

Lemma 4.2. *Let $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$. Then $S_1(\phi) \ll N^{\sigma - \frac{3}{2} + \epsilon} + N^{\frac{\sigma}{2} - 1 + \epsilon}$ as $N \rightarrow \infty$ through the primes, and if $\sigma < 1$ then $S_1(\phi) \ll k^{2\sigma} 2^{-k}$ for $N = 1$ and $k \rightarrow \infty$.*

Proof. Assume $N > 1$ tends to infinity through the primes. We use the Petersson formula (Lemma A.4) to bound the weighted sum of $\lambda_f(p)$, and find

$$S_1(\phi) \ll 2 \sum_{\substack{p \neq N \\ p \leq R^\sigma}} \frac{\log R}{N\sqrt{p}} \left(\frac{\sqrt{p}}{\sqrt{N + \sqrt{p}}} + (pN)^\epsilon \right). \quad (4.9)$$

As $1/\sqrt{N + \sqrt{p}} \ll 1/\sqrt{N}$, we find

$$S_1(\phi) \ll \sum_{p \leq R^\sigma} \frac{\log N}{N\sqrt{N}} + N^{\epsilon' + \frac{\sigma}{2} - 1} \ll N^{\sigma - \frac{3}{2} + \epsilon} + N^{\frac{\sigma}{2} - 1 + \epsilon}. \quad (4.10)$$

If now $N = 1$ and $k \rightarrow \infty$, we use Lemma A.3 (which forces us to take $\sigma < 1$ as $R = k^2$) and find

$$S_1(\phi) \ll \frac{1}{2^k} \sum_{p \leq k^{2\sigma}} \frac{\log p}{\log R} \ll k^{2\sigma} 2^{-k}, \quad (4.11)$$

which is $O(k^{-1/2})$ for k large. \square

Remark 4.3. In the next lemma we shall see that $S_2(\phi)$ and $S_3(\phi)$ are extremely small if $\sigma < 2$. If $\sigma < 1$, then $S_1(\phi) \ll N^{-1/2}$ or $k^{-1/2}$, and we obtain square-root agreement of this term with the Ratios prediction (if $N = 1$ we must restrict to $\sigma < 1/4$ because of our estimate for $M(\phi)$). For $\sigma \geq 1$ we don't have such phenomenal agreement (we can take $\sigma < 3/2$ for $N \rightarrow \infty$, but if $k \rightarrow \infty$ the above arguments fail for $\sigma \geq 1$), but we do at least agree up to a power of N . We have not exploited any cancelation in the Bessel-Kloosterman terms (we shall do this in §4.2), contenting ourselves here to argue simply and crudely. The quality of our results is exactly the same as that in Theorem 5.1 of [ILS] (where they have not yet exploited properties of the Bessel-Kloosterman terms, which is required to increase the support).

Lemma 4.4. *Let $\text{supp}(\hat{\phi}) \subset (-\sigma, \sigma)$.*

- (1) *We have $S_2(\phi) \ll N^{\frac{\sigma}{4} - 1 + \epsilon''}$ as $N \rightarrow \infty$ through the primes, and $S_2(\phi) \ll k^{-(5-3\sigma)/6 + \epsilon}$ if $N = 1$ and $k \rightarrow \infty$.*
- (2) *We have $S_3(\phi) \ll N^{\frac{\sigma}{12} - 1 + \epsilon''}$ as $N \rightarrow \infty$ through the primes, and $S_3(\phi) \ll k^{-(5-\sigma)/6 + \epsilon}$ if $N = 1$ and $k \rightarrow \infty$.*

Proof. As the proofs are similar, we only prove the second statement. We first consider $N \rightarrow \infty$. We apply the Petersson Formula (Lemma A.4) to the sums of $\lambda_f(p^\nu)$ and $\lambda_f(p^{\nu-2})$. As the error from the $\lambda_f(p^{\nu-2})$ terms is dominated by the error from the $\lambda_f(p^\nu)$ terms, we only consider the former. As we evaluate $\hat{\phi}$ at $\nu \log p / \log R$ with $n \geq 3$, we may restrict the p -sums to $p \leq R^{\sigma/3}$ (where $R = k^2 N$). We find

$$\begin{aligned} S_3(\phi) &\ll \sum_{\substack{(p,N)=1 \\ p \leq R^{\sigma/3}}} \sum_{\nu=3}^{\log_p R} \frac{1}{p^{\nu/2}} \left(\frac{\log N}{N} \frac{p^{\nu/2}}{\sqrt{N + p^{\nu/2}}} + \frac{(p^\nu N)^\epsilon}{N} \right) \\ &\ll \frac{\log^2 N}{N} \sum_{p \leq R^{\sigma/3}} p^{-\frac{3}{4}} + N^{\epsilon' - 1} \\ &\ll N^{\frac{\sigma}{12} - 1 + \epsilon} + N^{\epsilon' - 1} \ll N^{\frac{\sigma}{12} - 1 + \epsilon''}. \end{aligned} \quad (4.12)$$

We now examine the case when $N = 1$ and $k \rightarrow \infty$. We use Lemma A.2. As $R = k^2$ and $\nu \geq 3$, the prime sum is restricted to $p \leq k^{2\sigma/3}$. We find

$$S_3(\phi) \ll \log^2 k \sum_{p \leq k^{2\sigma/3}} \frac{1}{k^{5/6}} \frac{1}{\sqrt{p^{3/2} + k}} \ll k^{-5/6+\epsilon} \sum_{p \leq k^{2\sigma/3}} p^{-3/4} \ll k^{-(5-\sigma)/6+\epsilon}. \quad (4.13)$$

□

Remark 4.5. Even for $\sigma < 6$ (which is *well* beyond current technology for analyzing $S_1(\phi)$!), $S_3(\phi)$ is $O(N^{-1/2})$; it is $O(k^{-1/2})$ for $\sigma < 2$, which *is* in the range of current technology. If $N > 1$ then $S_2(\phi) = O(N^{-1/2+\epsilon})$ for $\sigma < 2$; however, if $N = 1$ then we only have square-root cancelation up to $\sigma = 2/3$ (in fact, if $\sigma \geq 5/3$ then our argument is too crude to bound this term). Thus the difficulty in showing agreement between number theory and the Ratios Conjecture's predictions is entirely due to $S_1(\phi)$ on the number theory side and $M(\phi)$ on the Ratios side.

4.2. Density Theorem Extended. To improve our 1-level density results for $H_k^*(N)$, we need to improve our analysis of

$$S_1(\phi) = 2 \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{p \neq N} \frac{\lambda_f(p)}{\sqrt{p}} \widehat{\phi} \left(\frac{\log p}{\log R} \right) \frac{\log p}{\log R}. \quad (4.14)$$

We are able to show agreement with the Ratios Conjecture up to a power savings in N if $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$ with $\sigma < 2$ (with additional analysis of $S_2(\phi)$ we should be able to extend our results up to $\sigma < 2$ when $N = 1$). To do this we modify the arguments in [ILS]. There are two major differences. First, they were concerned only with the main term and N square-free, and thus some of their error terms can be significantly improved for N prime. Second, they studied the unweighted sum (i.e., they did not include the Petersson weights). Including the Petersson weights simplifies the computations, though they can be done with the unweighted sum as well (see §5).

Lemma 4.6. *Assume GRH for $\zeta(s)$, all Dirichlet L -functions and all $L(s, f)$ with $f \in S_k(N)$. If $N \rightarrow \infty$ through the primes then $S_1(\phi) \ll N^{\frac{\sigma}{2}-1+\epsilon}$.*

Proof. The most difficult part in the proofs in [ILS] were from handling the non-diagonal terms in the unweighted Petersson formula. We bypass some of these difficulties by using weighted sums. We have

$$S_1(\phi) = \sum_{p \neq N} \left(\sum_{f \in H_k^*(N)} \omega_f^*(N) \lambda_f(p) \right) \widehat{\phi} \left(\frac{\log p}{\log R} \right) \frac{2 \log p}{\sqrt{p} \log R}. \quad (4.15)$$

Let

$$Q_k^*(m; c) = 2\pi i^k \sum_{p \nmid N} S(m^2, p; c) J_{k-1} \left(\frac{4\pi m \sqrt{p}}{c} \right) \widehat{\phi} \left(\frac{\log p}{\log R} \right) \frac{2 \log p}{\sqrt{p} \log R}. \quad (4.16)$$

Applying the Petersson formula (Lemmas A.1 and A.4) to $S_1(\phi)$ yields

$$S_1(\phi) = \sum_{c \equiv 0 \pmod{N}} \frac{Q_k^*(1; c)}{c} + O \left(\frac{R^{\sigma/2}}{N^{1-\epsilon}} \right). \quad (4.17)$$

This is very similar to the sum $\mathcal{P}_k^*(\phi)$ in equation (5.15) of [ILS], with $X = Y = 1$, $L = 1$, $M = N$. The difference is that (5.15) has an extra factor of $(k-1)N/12$, which is basically the cardinality of $H_k^*(N)$. We can use the results from sections 5 through 7 of [ILS] to bound $Q_k^*(1; c)$. We have (see (7.1) of [ILS]) that

$$Q_k^*(m; c) \ll \tilde{\gamma}_k(z) m P^{1/2} (kN)^\epsilon (\log 2c)^{-2}, \quad (4.18)$$

where $R = k^2N$, $P = R^\sigma$, $z = 4\pi m \sqrt{P}/c$ and $\tilde{\gamma}(z) = 2^{-k}$ if $3z \leq k$ and $k^{-1/2}$ otherwise. Thus

$$S_1(\phi) \ll \sum_{c \equiv 0 \pmod N} \frac{(k^2N)^{\sigma/2} (kN)^\epsilon}{c(\log 2c)^2} + N^{\frac{\sigma}{2}-1+\epsilon} \ll N^{\frac{\sigma}{2}-1+\epsilon} \quad (4.19)$$

(write $c = c'N$), which is negligible so long as $\sigma < 2$. \square

Remark 4.7. We briefly comment on where we use GRH for Dirichlet L -functions. If χ is a character modulo c , then under GRH we have

$$\sum_{p \leq x} \chi(p) \log p = \delta_\chi x + O(x^{1/2} \log^2 cx), \quad (4.20)$$

where $\delta_\chi = 1$ if χ is the principal character and 0 otherwise. In Section 6 of [ILS] they expand the Kloosterman sum. Setting

$$G_\chi(n) = \sum_{a \pmod c} \chi(a) e^{2\pi i a n / c}, \quad (4.21)$$

we find

$$\begin{aligned} \sum_{\substack{p \leq c \\ p \neq c}} S(m, np; c) &= \frac{1}{\varphi(c)} \left(\sum_{\chi \pmod c} \chi(a) S(m, an; c) \right) \cdot \left(\sum_{p \leq x} \bar{\chi}(p) \log p \right) \\ &= \frac{1}{\varphi(c)} \sum_{\chi \pmod c} G_\chi(m) G_\chi(n) (\delta_\chi x + O(x^{1/2} \log^2 cx)). \end{aligned} \quad (4.22)$$

If we did not assume GRH, the error term above would have to be replaced with something significantly larger. This estimate is a key input in the bound for $Q_k^*(m; c)$.

Lemma 4.8. *Assume GRH for $\zeta(s)$, all Dirichlet L -functions and all $L(s, f)$. Let $\text{supp}(\hat{\phi}) \subset (-\sigma, \sigma)$ with $\sigma < 2$, $N = 1$, and consider the 1-level density averaged over the weights (see Theorem 1.6 for an explicit statement). As $K \rightarrow \infty$ the 1-level density agrees with the prediction from the Ratios Conjecture up to errors of size $O(K^{-(5-\sigma)/6+\epsilon} + K^{\sigma-2+\epsilon})$.*

Proof. As the proof is similar to our previous results, we merely highlight the differences. Following [ILS] (Sections 8 and 9), we average over the weights as follows. Let h be a Schwartz function compactly supported on $(0, \infty)$. The weighted 1-level density is

$$\mathcal{A}^*(K; \phi) = \frac{1}{A^*(K)} \sum_{k \equiv 0 \pmod 2} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k^*(1)} D_{1, H_k^*(1); k^2}(\phi), \quad (4.23)$$

where

$$A^*(K) = \sum_{k \equiv 0 \pmod{2}} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) |H_k^*(1)| = \widehat{h}(0)K + O(K^{2/3}). \quad (4.24)$$

The only pieces whose errors cannot be trivially added arise from $S_1(\phi)$ and $S_2(\phi)$ for each k ; we now discuss how to handle these weighted averages.¹⁴ The main idea is to exploit the oscillation in the Bessel functions as k varies. The argument is easier than that in [ILS] due to the presence of the harmonic weights, though a similar result holds if we remove the weights (see §5).

We first handle the average of $S_1(\phi)$. Averaging over k allows us to exploit the oscillation in the Bessel functions; this is the reason we are able to double the support. The main input is their Corollary 8.2, which says

$$I(x) = \sum_{k \equiv 0 \pmod{2}} 2i^k h\left(\frac{k-1}{K}\right) J_{k-1}(x) \ll xK^{-4}, \quad (4.25)$$

where $x = 4\pi m\sqrt{p}/c$, $P = R^\sigma = K^{2\sigma}$, and for us $m = 1$ (as [ILS] remove the harmonic weights, they have a sum over $m \leq Y$). Corollary 8.2 requires $x \ll K^{2-\epsilon}$, i.e., $\sigma < 2 - \epsilon$. The analysis of the average of $S_1(\phi)$ is completed by feeding in the estimate from their equation (8.11), which yields a bound of $K^{\sigma+\epsilon-2}$ (remember we already executed the summation over k when we bounded $I(x)$). Thus the total error from the sum over k of the $S_1(\phi)$ terms is $O(K^{\sigma+\epsilon-2})$.

We now consider the average of $S_2(\phi)$. There are two major differences between this term and $S_1(\phi)$. The first is that the Kloosterman sums are $S(1, p^2; c)$ instead of $S(1, p; c)$. The second is that we have $\widehat{\phi}\left(\frac{2\log p}{\log R}\right) \frac{\log p}{p \log R}$ instead of $\widehat{\phi}\left(\frac{\log p}{\log R}\right) \frac{\log p}{\sqrt{p} \log R}$; this leads to a shorter prime sum of smaller terms. We can modify the arguments in Section 9 of [ILS] (remembering, as in Lemma 4.6, that our sum is simpler as $L = X = Y = m = M = 1$). Performing the averaging over k yields

$$\sum_c \frac{\mathcal{Q}^{(2)}(1; c)}{c}, \quad (4.26)$$

where

$$\mathcal{Q}^{(2)}(1; c) = 2\pi \sum_{p \neq N} S(1, p^2; c) I\left(\frac{4\pi p}{c}\right) \widehat{\phi}\left(\frac{2\log p}{\log R}\right) \frac{2\log p}{p \log R} \quad (4.27)$$

and $I(x)$ is the sum of Bessel functions (see their equation (8.7)). By their Corollary 8.2 we have

$$I(x) = -\frac{K}{\sqrt{x}} \operatorname{Im} \left\{ \bar{\zeta}_8 e^{ix\widehat{h}} \left(\frac{K^2}{2x} \right) \right\} + O\left(\frac{x}{K^4}\right). \quad (4.28)$$

¹⁴Actually, we need to be a little more careful. The problem is that the analytic conductors are no longer constant; if $\operatorname{supp}(h) \subset (a, b)$ then the conductors basically run from $(aK)^2$ to $(bK)^2$. Fortunately, an analysis of our previous arguments show that we do not need to localize the conductor exactly, but instead only up to a constant (see also equations (4.29) and (4.30) in [ILS], and the comments immediately after). Thus we may set $R = K^2$. The varying conductors here are significantly easier to handle than in other families, such as one-parameter families of elliptic curves [Mil1].

The error term yields to an insignificant contribution to $\sum_c \mathcal{Q}^{(2)}(1; c)/c$ (much less than in [ILS], due to the remarks above). Trivially estimating the Kloosterman sum by $c^{1/2+\epsilon}$ and recalling $R = K^2$ yields a contribution of

$$\sum_c \frac{1}{c} \sum_{p \leq R^{\sigma/2}} c^{\frac{1}{2}+\epsilon} \frac{p}{cK^4} \frac{1}{p} \ll K^{\sigma-4}, \quad (4.29)$$

which is negligible for $\sigma < 4$ (and smaller than $O(K^{-1/2})$ for $\sigma < 3.5$).

We now study the main term of $\mathcal{Q}^{(2)}(1; c)$. Following [ILS] it is

$$\mathcal{Q}^{(2)}(1; c) = -\frac{2K\sqrt{\pi c}}{\log R} \mathcal{T}(1; c), \quad (4.30)$$

where

$$\begin{aligned} \mathcal{T}(1; c) &= \sum_{p \neq N} S(1, p^2; c) \operatorname{Im} \left\{ \bar{\zeta}_8 \exp\left(\frac{4\pi i p}{c}\right) \hbar\left(\frac{cK^2}{8\pi p}\right) \right\} \hat{\phi}\left(\frac{2 \log p}{\log R}\right) \frac{\log p}{p^{3/2}} \\ \hbar(v) &= \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} e^{iuv} du. \end{aligned} \quad (4.31)$$

We do not need as delicate an analysis as in [ILS]. This is because of the extra \sqrt{p} in the denominator and the fact that the prime sums are up to $R^{\sigma/2}$ and not R^σ . We trivially estimate the Kloosterman sums and use the bound on \hbar from [ILS]: for any $A > 0$, $\hbar(v) \ll v^{-A}$. Taking $A = 1 + \delta$ yields

$$\mathcal{T}(1; c) \ll \sum_{p \leq R^{\sigma/2}} \frac{c^{\frac{1}{2}+\epsilon} p^{1+\delta} \log p}{c^{1+\delta} K^{2+2\delta} p^{3/2}} \ll \frac{K^{\frac{\sigma}{2}+\sigma\delta-2-2\delta}}{c^{\frac{1}{2}+\delta-\epsilon}}. \quad (4.32)$$

We substitute this into (4.26), and find a contribution bounded by

$$\sum_c \frac{K\sqrt{c}}{c} \frac{K^{\frac{\sigma}{2}+\sigma\delta-2-2\delta}}{c^{\frac{1}{2}+\delta-\epsilon}} \ll K^{-(\frac{1}{2}+\delta)(2-\sigma)}. \quad (4.33)$$

By taking δ sufficiently large, we can make this sum as small as we desire (and thus smaller than the contribution from the averaged $S_1(\phi)$). \square

Remark 4.9. There is a mistake right before equation (8.10) in [ILS]; it should read

$$4I'(x) = \frac{2}{K} h' \left(\frac{x+\eta}{K} \right) + O \left(\frac{x}{K^2} \right), \quad \eta \in (-1, 1); \quad (4.34)$$

fortunately all [ILS] use in their argument is that $I'(x) \ll K^{-1}$ when $x \ll K^{2-\epsilon}$, and that is true. Also, it is worth noting that our analysis of $\mathcal{Q}^{(2)}$ uses their results for the family $\{\operatorname{sym}^2 f : f \in H_k^*(N)\}$; our support is significantly larger because (1) this is now a $1/p$ term and not a $1/\sqrt{p}$; (2) we sum over $p \leq R^{\sigma/2}$ and not $p \leq R$.

4.3. Hypothesis S and further extensions. Iwaniec, Luo and Sarnak [ILS] show how a hypothesis on the size of some classical exponential sums over the primes can be used to increase the support to beyond $(-2, 2)$. They consider

Hypothesis S: For any $x \geq 1$, $c \geq 1$ and a with $(a, c) = 1$ we have

$$\sum_{\substack{p \leq x \\ p \equiv a \pmod{c}}} \exp\left(\frac{4\pi i \sqrt{p}}{c}\right) \ll_{\epsilon} c^A x^{\alpha+\epsilon}, \quad (4.35)$$

where α, A are constants with $A \geq 0$, $1/2 \leq \alpha \leq 3/4$ and ϵ is any positive number.

They present numerous arguments (see their Section 10 and their Appendix C) in support of the belief that Hypothesis S holds with $A = 0$ and $\alpha = 1/2$; however, any $\alpha < 3/4$ suffices to increase the support past $(-2, 2)$.¹⁵ We show how this hypothesis allows us to extend our computations. As [ILS] were only concerned with the main term, their error bounds are too crude; however, some additional book-keeping suffices to obtain all lower order terms up to a power savings in the family's cardinality.

To prove the third statement in Theorem 1.6 we need to study the weighted averages over k of $S_i(\phi)$ ($i \in \{1, 2, 3\}$). We note that they use the Petersson weights in their Section 10 (and thus we are using the same normalization for our sums). From Lemma 4.4, we see we may average $S_3(\phi)$ and obtain a contribution bounded by $O(K^{-(5-\sigma)/6+\epsilon})$. The analysis in Section 10 of [ILS] handles $S_1(\phi)$, and shows (under the assumption that Hypothesis S holds) that it is $O(K^{-2(2.5-\sigma)} + K^{-(2A+11/2+\epsilon)(1-\frac{2\alpha+A+5/4}{2A+11/2+\epsilon})})$. In particular, taking $A = 0$ and $\alpha = 1/2$ yields the weighted average of $S_1(\phi)$ is $O(K^{-2(2.5-\sigma)} + K^{-\frac{11}{2}(1-\frac{9}{22}\sigma)})$.

We are left with bounding the weighted average over k of $S_2(\phi)$, remembering $R = K^2$. In [ILS] it is shown to be $O\left(\frac{\log \log K}{\log K}\right)$, which does not suffice for our purposes. This term contributes

$$\frac{1}{B(K)} \sum_{p \leq R^{\sigma/2}} \mathcal{B}(p^2, 1) \widehat{\phi}\left(\frac{2 \log p}{\log R}\right) \frac{\log p}{p \log R}, \quad (4.36)$$

where $B(K) = \widehat{h}(0)K + O(1)$ (with $\widehat{h}(0) \neq 0$),

$$\mathcal{B}(p^2, 1) = -\frac{\sqrt{\pi}K}{\sqrt{p}} \operatorname{Im} \left\{ \bar{\zeta}_8 \sum_c \frac{S(1, p^2, c)}{c} e^{\frac{4\pi i p}{c}} \widehat{h}\left(\frac{cK^2}{8\pi p}\right) \right\} + O(pK^{-4}) \quad (4.37)$$

and $\widehat{h}(v) \ll v^{-\delta}$ for any $\delta > 0$. The $O(pK^{-4})$ term in $\mathcal{B}(p^2, 1)$ leads to a contribution of size $K^{-(5-\sigma)}$, which is dwarfed by the other error terms. We trivially bound the main term in $\mathcal{B}(p^2, 1)$ by using $S(1, p^2, c) \ll c^{\frac{1}{2}+\epsilon}$ and $\widehat{h}(cK^2/8\pi p) \ll p^{\delta}/(cK^2)^{\delta}$ for some $\delta > 1/2$ (we take $\delta > 1/2$ so that the resulting c -sum converges). This yields a contribution to the average of $S_2(\phi)$ of

$$\frac{1}{K} \sum_{p \leq R^{\sigma/2}} \frac{K}{\sqrt{p}} \sum_c \frac{c^{\frac{1}{2}+\epsilon}}{c} \frac{p^{\delta}}{c^{\delta} K^{2\delta}} \frac{1}{p} \ll K^{-2\delta} \sum_p p^{\delta-\frac{1}{2}-1} \ll K^{(\delta-\frac{1}{2})\sigma-2\delta}. \quad (4.38)$$

Taking δ just a little larger than $1/2$ shows that this error is also dwarfed by our existing errors (as well as being $O(N^{-1+\epsilon})$, which completes the proof.

¹⁵Vinogradov proved Hypothesis S with $\alpha = 7/8$; assuming the standard density hypothesis for Dirichlet L -functions allows one to take $\alpha = 3/4$.

5. CALCULATING THE UNWEIGHTED 1-LEVEL DENSITY

Much effort was spent removing the harmonic weights in [ILS]. Below we remove them for our family and calculate the lower order terms. We see some new, lower order terms which did not appear in either the expansion from the Ratios Conjecture or our number theory computations. This is not entirely surprising, as those computations were for weighted sums.

We prove Theorem 1.9. We first concentrate on the unweighted version of $S_1(\phi)$, which yields negligible contributions for $\sigma < 2$. We then analyze the unweighted versions of $S_2(\phi)$ and $S_3(\phi)$, and find new lower order terms. The other terms in (4.4) are unaffected by removing the weights. We conclude by determining the prediction from the Ratios Conjecture for the unweighted 1-level density, and show agreement with number theory.

5.1. Analyzing the unweighted $S_1(\phi)$. Below we modify the arguments in [ILS] to show that $S_{1,\text{unwt}}(\phi)$ has negligible contribution for $\sigma < 2$ when we do not include the harmonic weights.

Lemma 5.1. *Assume GRH for $L(s, f)$. If $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$ with $\sigma < 2$, then $S_{1,\text{unwt}}(\phi) \ll N^{-(2-\sigma)/6+\epsilon}$ as $N \rightarrow \infty$ through the primes, where $S_{1,\text{unwt}}(\phi)$ is defined analogously as $S_1(\phi)$ except now we do not include the harmonic weights.*

Proof. We use the expansions in [ILS] for $\Delta_{k,N}^*(p)$, remembering to divide by $|H_k^*(N)|$. Let X and Y be two arbitrary parameters (depending on N) to be determined later. We let ϵ denote an arbitrarily small number (not necessarily the same value from line to line). We write

$$\Delta_{k,N}^*(p) = \Delta'_{k,N}(p) + \Delta_{k,N}^\infty(p), \quad (5.1)$$

where ((2.63) of [ILS])

$$\Delta'_{k,N}(p) = \frac{k-1}{12} \sum_{\substack{LM=N \\ L \leq X}} \frac{\mu(L)M}{\nu((n, L))} \sum_{\substack{(m,M)=1 \\ m \leq Y}} \frac{\Delta_{k,M}(m^2, n)}{m} \quad (5.2)$$

and $\Delta_{k,N}^\infty(p)$ is the complementary sum. Here

$$\nu(\ell) = [\Gamma_0(1) : \Gamma_0(\ell)] = \ell \prod_{p|\ell} \frac{p+1}{p}. \quad (5.3)$$

As N is prime, so long as $X < N$ then in $\Delta'_{k,N}(p)$ the only term is when $L = 1$ and $M = N$. Thus

$$\begin{aligned} S_{1,\text{unwt}}(\phi) &= \frac{1}{|H_k^*(N)|} \sum_{p \nmid N} \Delta'_{k,N}(p) \widehat{\phi} \left(\frac{\log p}{\log R} \right) \frac{2 \log p}{\sqrt{p} \log R} \\ &\quad + \frac{1}{|H_k^*(N)|} \sum_{p \nmid N} \Delta_{k,N}^\infty(p) \widehat{\phi} \left(\frac{\log p}{\log R} \right) \frac{2 \log p}{\sqrt{p} \log R} \\ &= S'_{1,\text{unwt}}(\phi) + S_{1,\text{unwt}}^\infty(\phi). \end{aligned} \quad (5.4)$$

We first show there is no contribution from the complementary sum. As we are going for a power savings in N and not just attempting to understand the main term, we choose

different values for X and Y then in [ILS], and argue slightly differently. Assuming the Riemann hypothesis for $L(s, f)$, if $\log Q \ll \log kN$ then (Lemma 2.12 of [ILS])

$$\sum_{\substack{(p,N)=1 \\ p \leq Q}} \Delta_{k,N}^\infty(p) \frac{\log p}{\sqrt{p}} \ll kN(pkNXY)^\epsilon (X^{-1} + Y^{-1/2}). \quad (5.5)$$

Using partial summation, the compact support of $\widehat{\phi}$ and $H_k^*(N) \ll kN$ shows that the complementary sum piece is bounded by

$$\begin{aligned} S_{1,\text{unwt}}^\infty(\phi) &= \frac{1}{kN \log R} \int^{R^\sigma} kN(pkNXY)^\epsilon (X^{-1} + Y^{-1/2}) \left| \widehat{\phi} \left(\frac{\log p}{\log R} \right) \right| \frac{dp}{p \log R} \\ &\ll N^\epsilon (X^{-1} + Y^{-1/2}). \end{aligned} \quad (5.6)$$

We now analyze the contribution from $\Delta'_{k,N}(p)$. The formulas from [ILS] simplify greatly as we only have one (L, M) pair, and as p is not a perfect square there are no main terms. We have

$$S'_{1,\text{unwt}}(\phi) = \frac{(k-1)N}{12|H_k^*(N)|} \sum_{\substack{(m,N)=1 \\ m \leq Y}} \frac{1}{m} \sum_{c \equiv 0 \pmod N} \frac{Q_k^*(m; c)}{c}, \quad (5.7)$$

where

$$Q_k^*(m; c) = 2\pi i^k \sum_{p \nmid N} \frac{S(m^2, p; c)}{c} J_{k-1} \left(\frac{4\pi m \sqrt{p}}{c} \right) \widehat{\phi} \left(\frac{\log p}{\log R} \right) \frac{2 \log p}{\sqrt{p} \log R}. \quad (5.8)$$

In (5.14) of [ILS] they set $X = Y = (kN)^\epsilon$; however, their estimates of $Q_k^*(m; c)$ are independent of X and Y , and we may thus use their results. We have (see (7.1) of [ILS]) that

$$Q_k^*(m; c) \ll \widetilde{\gamma}_k(z) m P^{1/2} (kN)^\epsilon (\log 2c)^{-2}, \quad (5.9)$$

where $R = k^2N$, $P = R^\sigma$, $z = 4\pi m \sqrt{P}/c$ and $\widetilde{\gamma}(z) = 2^{-k}$ if $3z \leq k$ and $k^{-1/2}$ otherwise. Thus

$$S'_{1,\text{unwt}}(\phi) \ll \sum_{\substack{(m,N)=1 \\ m \leq Y}} 1 \sum_{c \equiv 0 \pmod N} \frac{(k^2N)^{\sigma/2} (kN)^\epsilon}{c (\log 2c)^2} \ll N^{\frac{\sigma}{2}-1+\epsilon} Y. \quad (5.10)$$

Combining our estimates yields

$$S_{1,\text{unwt}}(\phi) \ll N^{\frac{\sigma}{2}-1+\epsilon} Y + N^\epsilon (X^{-1} + Y^{-1/2}). \quad (5.11)$$

We may take $X = N - 1$ (as N is prime). Equalizing the two errors involving Y , we find we should take $Y = N^{(2-\sigma)/3}$, which gives $S_{1,\text{unwt}}(\phi) \ll N^{(2-\sigma)/6}$. \square

Lemma 5.2. *Assume GRH for $L(s, f)$. If $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$ with $\sigma < 2$, then $S_{1,\text{unwt}}(\phi) \ll K^{-(2-\sigma)/6+\epsilon}$ as $N = 1$ and $K \rightarrow \infty$ (where we average over the weights).*

Proof. As the proof is similar to Lemma 5.1, we merely highlight the differences. Following [ILS] (Section 8), we average over the weights as follows. Let h be a Schwartz function compactly supported on $(0, \infty)$. We consider the weighted 1-level density

$$\mathcal{A}^*(K; \phi) = \frac{1}{A^*(K)} \sum_{k \equiv 0 \pmod 2} \frac{24}{k-1} h \left(\frac{k-1}{K} \right) \sum_{f \in H_k^*(1)} D_{1, H_k^*(1); k^2}(\phi), \quad (5.12)$$

where

$$A^*(K) = \sum_{k \equiv 0 \pmod{2}} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) |H_k^*(1)| = \widehat{h}(0)K + O(K^{2/3}). \quad (5.13)$$

The pieces whose errors cannot be trivially added arises from $S_{i,\text{unwt}}(\phi)$ ($i \in \{1, 2, 3\}$) for each k . We analyze the weighted average of $S_1(\phi)$ below, and then study the other two in §5.2. The main idea is to exploit the oscillation in the Bessel functions as k varies.

In their Lemma 2.12 we now take $X = 1$ and $Y = K^\delta$. The complementary sum gives an error bounded by $k^\epsilon Y^{-1/2}$. The averaging over k allows us to exploit the oscillation in the Bessel functions; this is the reason we are able to double the support. The main input is their Corollary 8.2, which says

$$I(x) = \sum_{k \equiv 0 \pmod{2}} 2i^k h\left(\frac{k-1}{K}\right) J_{k-1}(x) \ll xK^{-4}, \quad (5.14)$$

where $x = 4\pi m\sqrt{P}/c$ and $P = R^\sigma = K^{2\sigma}$. Corollary 8.2 requires $x \ll K^{2-\epsilon}$. In their arguments they take $Y = K^\epsilon$, and thus for them $m \leq K^\epsilon$ (recall $m \leq Y$). As we are interested in sharper error estimates, we must take Y a small power of K . This leads to a slight reduction in the support (our condition on x forces $\sigma < 2 - \delta$). The proof is completed by feeding in the estimate from their equation (8.11), which yields a bound of $K^{\sigma+\delta+\epsilon-2}$ for the term from the non-complementary piece (remember we already executed the summation over k when we bounded $I(x)$).

Thus the total error from the sum over k of the $S_{1,\text{unwt}}(\phi)$ terms is $O(K^\epsilon Y^{-1/2} + K^{\sigma+\delta+\epsilon-2})$. Equalizing the errors yields $\delta = (2 - \sigma)/3$, or the total error from the weighted $S_{1,\text{unwt}}(\phi)$ terms is $O(K^{-(2-\sigma)/6})$. \square

Remark 5.3. There is a mistake right before equation (8.10) in [ILS]; see Remark 4.9.

5.2. Analyzing the unweighted $S_2(\phi)$ and $S_3(\phi)$. We now modify our investigation of $S_2(\phi)$ and $S_3(\phi)$ and remove the weights. We set

$$\begin{aligned} S_{3,\text{unwt}}(\phi) &= \frac{2}{|H_k^*(N)|} \sum_{f \in H_k^*(N)} \sum_{p \neq N} \sum_{\nu=3}^{\infty} \frac{\lambda_f(p^\nu) - \lambda_f(p^{\nu-2})}{p^{\nu/2}} \widehat{\phi}\left(\nu \frac{\log p}{\log R}\right) \frac{\log p}{\log R} \\ S_{2,\text{unwt}}(\phi) &= \frac{2}{|H_k^*(N)|} \sum_{f \in H_k^*(N)} \sum_{p \neq N} \frac{\lambda_f(p^2)}{p} \widehat{\phi}\left(2 \frac{\log p}{\log R}\right) \frac{\log p}{\log R}. \end{aligned} \quad (5.15)$$

We argue as in the analysis of $S_{1,\text{unwt}}(\phi)$. As the two terms are handled analogously, we concentrate on $S_{3,\text{unwt}}(\phi)$. The analysis is significantly easier than the analysis of $S_{1,\text{unwt}}(\phi)$ due to the higher power of primes (both in dividing by larger quantities and restricting further the summation over primes). Let $v = \nu$ or $\nu - 2$. We must study the pure sums

$$\Delta_{k,N}^*(p^v) = \sum_{f \in H_k^*(N)} \lambda_f(p^v). \quad (5.16)$$

From Proposition 2.13 of [ILS] we have

$$\Delta_{k,N}^*(n) = \frac{(k-1)\varphi(N)}{12\sqrt{n}} \delta_{n,\square} + O\left(\frac{(kN)^{2/3}n^{1/6}}{\sqrt{(n,N)}}\right) \quad (5.17)$$

where the main term is present only if n is a square and $(n, N) = 1$. The contribution from the error term to $S_{3,\text{unwt}}(\phi)$ is bounded by

$$\sum_{\nu \leq \log_2 R} \sum_{p \leq R^{\sigma/3}} \frac{p^{3/6} (kN)^{2/3}}{p^{3/2} kN} \ll \frac{\log^2 R}{(kN)^{1/3}}. \quad (5.18)$$

Thus the error term yields a negligible contribution.

The main term from Proposition 2.13, however, is a different story. Whenever ν is even it *will* contribute, and yields

$$\sum_{\substack{\nu \equiv 0 \pmod{2} \\ \nu \geq 4}} \sum_{p \neq N} \frac{1-p}{p^\nu} \widehat{\phi}\left(\nu \frac{\log p}{\log R}\right) \frac{\log p}{\log R}. \quad (5.19)$$

The unweighted $S_2(\phi)$ term will also contribute, as it involves $\lambda_f(p^2)$. It gives another secondary term of size $1/\log R$, as well as an error of size $O(N^{-(6-\sigma)/6+\epsilon})$. Substituting everything into (4.4) yields

$$\begin{aligned} D_{1,H_k^*(N);R}(\phi) &= \frac{1}{\log R} \int_{-\infty}^{\infty} \left(2 \log \frac{\sqrt{N}}{\pi} + \psi\left(\frac{1}{4} + \frac{k \pm 1}{4} + \frac{2\pi it}{\log R}\right) \right) \phi(t) dt \\ &\quad + 2 \sum_{\substack{\nu \equiv 0 \pmod{2} \\ \nu \geq 2}} \sum_{p \neq N} \frac{p-1}{p^\nu} \widehat{\phi}\left(\nu \frac{\log p}{\log R}\right) \frac{\log p}{\log R} \\ &\quad + O\left(N^{-1/2} + N^{-(2-\sigma)/6+\epsilon}\right); \end{aligned} \quad (5.20)$$

the sum starts at $\nu = 2$ and not $\nu = 4$ as we have incorporated both $S_{2,\text{unwt}}(\phi)$ and the $\sum_p 1/p$ term in (4.4). This completes the analysis of the number theory terms in Theorem 1.9.

5.3. Unweighted Ratios Prediction. We sketch the derivation of the prediction for the unweighted 1-level density from the Ratios Conjecture, which completes the proof of Theorem 1.9. We concentrate on the case $N \rightarrow \infty$ through the primes. As the analysis is similar to the weighted case, we just highlight the new terms.

The Ratios Conjecture recipe states we should replace averages over the family by the main term, throwing away the ‘small’ error. The problem is that while $\sum_{f \in H_k^*(N)} \omega_f^*(N) \lambda_f(n)$ is small for $n \geq 2$, it is *not* small for n a perfect square if we drop the weights (see (5.17)).

We highlight the changes to Theorem 1.2 from studying the unweighted family $H_k^*(N)$. The first change is in Lemma 2.6. Originally we had the first term of $R_{H_k^*(N)}(\alpha, \gamma)$ as

$$\prod_p \left(1 - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right); \quad (5.21)$$

now, however, we shall see it is

$$\prod_p \left(1 - \frac{p+1}{p} \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \cdot \left(1 - \frac{1}{p^{2+2\alpha}} \right)^{-1}. \quad (5.22)$$

We do not worry about the changes to the second term, as it leads to a contribution of size $O(1/N)$.

The proof follows from mirroring the calculation in Lemma 2.6. We again split the sum into a product over primes. We constantly use (from (4.3)) $\lambda_f(p)\lambda_f(p^\nu) = \lambda_f(p^{\nu-1}) + \lambda_f(p^{\nu+1})$. We average over the family by using (5.17); which says there is no main term unless we are evaluating at a square. Thus below we drop all terms involving $\lambda_f(p)\lambda_f(p^{2k})$ or $\lambda_f(p^{2k+1})$, as these yield lower order terms. We also ignore the product over $p \geq x$, as those terms vanish when we complete the product by sending $x \rightarrow \infty$. Thus we have that

$$\sum_{\substack{m \leq x \\ h}} \frac{\mu_f(h)\lambda_f(m)}{h^{\frac{1}{2}+\gamma}m^{\frac{1}{2}+\alpha}} = \prod_{p \leq x} \left(1 - \frac{\lambda_f(p)}{p^{\frac{1}{2}+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \cdot \left(1 + \frac{\lambda_f(p)}{p^{\frac{1}{2}+\alpha}} + \frac{\lambda_f(p^2)}{p^{1+2\alpha}} + \dots \right) \quad (5.23)$$

contributes

$$\prod_{p \leq x} \left[\left(1 + \frac{1}{p^{1+2\gamma}} \right) \sum_{k=0}^{\infty} \frac{\lambda_f(p^{2k})}{(p^{1+2\alpha})^k} - \frac{1}{p^{1+\alpha+\gamma}} \sum_{k=0}^{\infty} \frac{\lambda_f(p^{2k}) + \lambda_f(p^{2k+2})}{(p^{1+2\alpha})^k} \right]. \quad (5.24)$$

We now average over the family and divide by the family's cardinality; this replaces $\lambda_f(p^{2\ell})$ with $1/p^\ell$ (remember we ignore all error terms). Using the geometric series formula and completing the product, after some simple algebra we find a contribution of

$$\prod_p \left(1 + \frac{1}{p^{1+2\gamma}} - \frac{p+1}{p} \frac{1}{p^{1+\alpha+\gamma}} \right) \left(1 + \frac{1}{p^{2+2\alpha}} \right)^{-1}. \quad (5.25)$$

For the Ratios Conjecture prediction, however, we need the derivative of this piece with respect to α when $\alpha = \gamma = r$. We must therefore modify Lemma 2.9 as well. This piece contributes to $R'_{H_k^*(N)}(r, r)$ a factor of

$$\sum_p \frac{(p-1) \log p}{p^{2+2r} - 1} = \sum_p (p-1) \log p \sum_{k=1}^{\infty} \frac{1}{p^{2k} p^{2rk}}; \quad (5.26)$$

this is very similar to what we previously had for $R'_{H_k^*(N)}(r, r)$, namely

$$\sum_p \frac{\log p}{p^{1+2r}}. \quad (5.27)$$

This change propagates to Lemma 3.1, where instead of

$$\int_{-\infty}^{\infty} g(t) \sum_p \frac{\log p}{p^{1+2it}} dt = \sum_p \hat{g} \left(\frac{2 \log p}{2\pi} \right) \quad (5.28)$$

we now have

$$\begin{aligned} \int_{-\infty}^{\infty} g(t) \sum_p \sum_{k=1}^{\infty} \frac{(p-1) \log p}{p^{2k} p^{2itk}} dt &= \sum_p \sum_{k=1}^{\infty} \frac{(p-1) \log p}{p^{2k}} \int_{-\infty}^{\infty} g(t) e^{-2\pi i \frac{2kt \log p}{2\pi}} dt \\ &= \sum_p \sum_{k=1}^{\infty} \frac{(p-1) \log p}{p^{2k}} \hat{g} \left(\frac{2k \log p}{2\pi} \right). \end{aligned} \quad (5.29)$$

Setting $g(t) = \phi \left(\frac{t \log R}{2\pi} \right)$ and collecting all the terms completes the proof of the Ratios Conjecture's prediction in Theorem 1.9.

APPENDIX A. PETERSSON FORMULA

Below we record several useful variants of the Petersson formula. We define

$$\Delta_{k,N}(m, n) = \sum_{f \in \mathcal{B}_k(N)} \omega_f(N) \lambda_f(m) \lambda_f(n). \quad (A.1)$$

We quote the following versions of the Petersson formula from [ILS] (to match notations, note that $\sqrt{\omega_f(N)} \lambda_f(n) = \psi_f(n)$).

Lemma A.1 ([ILS], Proposition 2.1). *We have*

$$\Delta_{k,N}(m, n) = \delta(m, n) + 2\pi i^k \sum_{c \equiv 0 \pmod N} \frac{S(m, n; c)}{c} J_{k-1} \left(\frac{4\pi \sqrt{mn}}{c} \right), \quad (A.2)$$

where $\delta(m, n)$ is the Kronecker symbol,

$$S(m, n; c) = \sum_{d \pmod c}^* \exp \left(2\pi i \frac{md + n\bar{d}}{c} \right) \quad (A.3)$$

is the classical Kloosterman sum ($\bar{d} \equiv 1 \pmod c$), and $J_{k-1}(x)$ is a Bessel function.

We expect the main term to arise only in the case when $m = n$ (though as shown in [HM, ILS], the non-diagonal terms require a sophisticated analysis for test functions with sufficiently large support). We have the following estimates.

Lemma A.2 ([ILS], Corollary 2.2). *We have*

$$\Delta_{k,N}(m, n) = \delta(m, n) + O \left(\frac{\tau(N)}{Nk^{5/6}} \frac{(m, n, N) \tau_3((m, n))}{\sqrt{(m, N) + (n, N)}} \left(\frac{mn}{\sqrt{mn} + kN} \right)^{1/2} \log 2mn \right), \quad (A.4)$$

where $\tau_3(\ell)$ denotes the corresponding divisor function (which is the sum of the cubes of the divisors of ℓ).

We can significantly decrease the error term if m and n are small relative to kN .

Lemma A.3 ([ILS], Corollary 2.3). *If $12\pi\sqrt{mn} \leq kN$ we have*

$$\Delta_{k,N}(m, n) = \delta(m, n) + O \left(\frac{\tau(N)}{2^k N^{3/2}} \frac{(m, n, N) \sqrt{mn}}{\sqrt{(m, N) + (n, N)}} \tau((m, n)) \right). \quad (A.5)$$

In this paper we consider two cases, $N = 1$ and $k \rightarrow \infty$ or k fixed and $N \rightarrow \infty$ through prime values. In the first case, there is no problem with using the above formulas; however, in the second case we must be careful. $\Delta_{k,N}(m, n)$ is defined as a sum over all cusp forms of weight k and level N ; in practice we often study the families $H_k^\sigma(N)$ of cuspidal newforms of weight k and level N (if $\sigma = +$ we mean the subset with even functional equation, if $\sigma = -$ we mean the subset with odd functional equation, and if $\sigma = *$ we mean all). Thus we should remove the contribution from the oldforms in our Petersson expansions. Fortunately this is quite easy if N is prime, as then the only oldforms are those of level 1 (following [ILS], with additional work we should be able to handle N square-free, though at the cost of worse error terms). We have (see (1.16) of [ILS])

$$|H_k^\pm(N)| \sim \frac{k-1}{24} \varphi(N), \quad (\text{A.6})$$

where $\varphi(N)$ is Euler's totient function (and thus equals $N-1$ for N prime). The number of cusp forms of weight k and level 1 is (see (1.15) of [ILS]) approximately $k/12$. As $\lambda_f(n) \ll \tau(n) \ll n^\epsilon$ and $\omega_f^*(N) \ll N^{-1+\epsilon}$, we immediately deduce

Lemma A.4. *Let $\mathcal{B}_k^{\text{new}}(N)$ be a basis for $H_k^*(N)$ and let $\omega_f^*(N)$ be as in (1.13). For N prime, we have*

$$\sum_{f \in \mathcal{B}_k^{\text{new}}(N)} \omega_f^*(N) \lambda_f(m) \lambda_f(n) = \Delta_{k,N}(m, n) + O\left(\frac{(mnN)^\epsilon k}{N}\right). \quad (\text{A.7})$$

Substituting yields

$$\begin{aligned} \sum_{f \in \mathcal{B}_k^{\text{new}}(N)} \omega_f^*(N) \lambda_f(m) \lambda_f(n) &= \delta(m, n) + O\left(\frac{(mnN)^\epsilon k}{N}\right) \\ &+ O\left(\frac{\tau(N)}{Nk^{5/6}} \frac{(m, n, N) \tau_3((m, n))}{\sqrt{(m, N) + (n, N)}} \left(\frac{mn}{\sqrt{mn} + kN}\right)^{1/2} \log 2mn\right), \end{aligned} \quad (\text{A.8})$$

while if $12\pi\sqrt{mn} \leq kN$ we have

$$\begin{aligned} \sum_{f \in \mathcal{B}_k^{\text{new}}(N)} \omega_f^*(N) \lambda_f(m) \lambda_f(n) &= \delta(m, n) \\ &+ O\left(\frac{\tau(N)}{2^k N^{3/2}} \frac{(m, n, N) \sqrt{mn}}{\sqrt{(m, N) + (n, N)}} \tau((m, n))\right) + O\left(\frac{(mnN)^\epsilon k}{N}\right). \end{aligned} \quad (\text{A.9})$$

Proof. The proof follows by using equations (1.13) and (1.14) in the Petersson lemmas. \square

APPENDIX B. USEFUL ESTIMATES

Lemma B.1. *Consider $\left|X_L\left(\frac{1+w}{2} + \frac{2\pi it}{\log R}\right)\right|$, with $X_L(s)$ given by (2.9). If $w = 0$ it is $O(1)$, while if $w = \frac{2k-1}{3}$ it is $O\left(\left(\frac{2009}{\sqrt{N}}\right)^{\frac{2k-1}{3}} \cdot k^{-k/3}\right)$.*

Proof. We have

$$X_L \left(\frac{1+w}{2} + \frac{2\pi it}{\log R} \right) = \left(\frac{\sqrt{N}}{2\pi} \right)^{-w - \frac{4\pi it}{\log R}} \frac{\Gamma \left(\frac{1-w}{4} + \frac{k-1}{2} - \frac{\pi it}{\log R} \right)}{\Gamma \left(\frac{1+w}{4} + \frac{k-1}{2} + \frac{\pi it}{\log R} \right)}. \quad (\text{B.1})$$

The claim follows from analyzing the ratio of the Gamma factors. As $|\Gamma(x + iy)| = |\Gamma(x - iy)|$ we may replace $-\pi it/\log R$ with $+\pi it/\log R$ in the Gamma function in the numerator. The proof is thus trivial for $w = 0$. If $w > 0$ then we use the identity

$$\frac{\Gamma(a + iy)\Gamma(b - a)}{\Gamma(b + iy)} = \int_0^1 t^{a+iy-1}(1-t)^{b-a-1} dt. \quad (\text{B.2})$$

Note that if $a = b$ then our bound is poor due to the presence of the $\Gamma(b - a)$ term; however, for us that would correspond to $w = 0$, and in that case the ratio of the Gamma factors is just 1.

We apply (B.2) with $a = \frac{1-w}{4} + \frac{k-1}{2}$ and $b - a = \frac{w}{2}$. We take $w = \frac{2k-1}{3}$ (chosen so that $b - a = a = \frac{2k-1}{6}$). We want $w < 2k - 1$ as in our applications we will be shifting contours, and we want to avoid the pole of the numerator. The ratio of the Gamma factors, when $w = \frac{2k-1}{3}$, is

$$\left| \frac{\Gamma \left(\frac{2k-1}{6} + \frac{\pi it}{\log R} \right)}{\Gamma \left(\frac{2k-1}{3} + \frac{\pi it}{\log R} \right)} \right| \leq \frac{1}{\Gamma \left(\frac{2k-1}{6} \right)} \int_0^1 (t \cdot (1-t))^{\frac{2k-1}{6}-1} dt \leq \frac{4(1/4)^{\frac{2k-1}{6}}}{\Gamma \left(\frac{2k-1}{6} \right)} \quad (\text{B.3})$$

(as the integrand is largest when $t = 1/2$). Thus for $w = \frac{2k-1}{3}$, applying Stirling's formula to $\Gamma \left(\frac{2k-1}{6} \right)$ we find

$$\left| X_L \left(\frac{k+1}{3} + \frac{2\pi it}{\log R} \right) \right| \leq \left(\frac{\pi}{\sqrt{N}} \right)^{\frac{2k-1}{3}} \frac{4}{\Gamma \left(\frac{2k-1}{6} \right)} \leq \left(\frac{2009}{\sqrt{N}} \right)^{\frac{2k-1}{3}} \cdot k^{-k/3}. \quad (\text{B.4})$$

□

Remark B.2. In the hope that this might be useful to other researchers on related problems, we sketch an alternative attack¹⁶ for estimating X_L in Lemma B.1. Unfortunately for our applications it also requires $\text{supp}(\hat{\phi}) \subset (-1/4, 1/4)$. In the proof above w was a function of k ; this was deadly as we had a factor of $R^{\sigma w} = k^{2\sigma w}$ from $\hat{\phi}$ from the contour shift (see (3.27)). This forced us to take $\sigma < 1/4$, as our denominator was (essentially) $k^{w/2}$. We sketch an alternative approach using Hölder's inequality; unfortunately this method also forces $\sigma < 1/4$ (and gives a worse error term).

We apply (B.2) with $a = \frac{1-w}{4} + \frac{k-1}{2}$ and $b - a = \frac{w}{2}$. We choose $w = 1/8$ for definiteness and ease of exposition; similar results hold for all w , always requiring $\sigma < 1/4$. For such w , we have $b - a - 1 < 0$; thus the factor of $(1-t)^{b-a-1}$ is very large for t near 1. We surmount this by using Hölder's inequality, which states that if $p, q \geq 1$ with $1/p + 1/q = 1$ then

$$\int_0^1 |f(t)g(t)| dt \leq \left(\int_0^1 |f(t)|^p dt \right)^{1/p} \cdot \left(\int_0^1 |g(t)|^q dt \right)^{1/q}. \quad (\text{B.5})$$

¹⁶We shall use Hölder's inequality. See [HM] for another application of Hölder's inequality to bounding error terms in n -level computations.

We let $f(t) = t^{a-1}$, $g(t) = (1-t)^{b-a-1}$, $p = \frac{16}{1-16\epsilon}$ and $q = \frac{16}{15+16\epsilon}$ in Hölder's inequality, yielding

$$\int_0^1 t^{a-1}(1-t)^{b-a-1} dt \leq \left(\int_0^1 t^{(a-1)p} \right)^{1/p} \cdot \left(\int_0^1 (1-t)^{(b-a-1)q} dt \right)^{1/q}. \quad (\text{B.6})$$

As $(b-a-1)q = -\frac{15}{15+16\epsilon} > -1$, the integral involving $1-t$ is just $O(1)$. The integral involving t is $((a-1)p+1)^{-1/p} \ll k^{-1/16+\epsilon}$. For $\sigma < 1/4$, the $k^{2\sigma w}$ term from (3.27) will be smaller than $k^{1/16}$.

Lemma B.3. *Let ϕ be an even Schwartz function such that $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$. Then*

$$\phi(t+iy) \ll_{n,\phi} e^{2\pi y \sigma} \cdot (t^2 + y^2)^{-n}. \quad (\text{B.7})$$

Proof. From the Fourier inversion formula, integrating by parts and the compact support of $\widehat{\phi}$, we have

$$\begin{aligned} \phi(t+iy) &= \int_{-\infty}^{\infty} \widehat{\phi}(\xi) e^{2\pi i(t+iy)\xi} d\xi \\ &= \int_{-\infty}^{\infty} \widehat{\phi}^{(2n)}(\xi) \cdot (2\pi i(t+iy))^{-2n} e^{2\pi i(t+iy)\xi} d\xi \\ &\ll e^{2\pi|y|\sigma} (t^2 + y^2)^{-n}. \end{aligned} \quad (\text{B.8})$$

□

Lemma B.4. *Let*

$$A(x+iy) = \prod_p \left(1 - \frac{p^{x+iy} - 1}{p(p^{1+x+iy} + 1)} \right). \quad (\text{B.9})$$

If $x > -1$ then $A(x+iy) = O(1)$.

Proof. We have

$$|A(x+iy)| \leq \prod_p \left(1 + \frac{2 \max(1, p^x)}{p^{2+x}} \right), \quad (\text{B.10})$$

and the product is $O(1)$ as long as $x > -1$. □

APPENDIX C. MERTENS' THEOREM AND HOW WE EXTEND THE SUMS.

We examine other ways of completing the product of the second factor in the definition of $R_{H_k^*(N)}(\alpha, \gamma)$, and the consequences of this alternate completion on $R'_{H_k^*(N)}(r, r)$. Recall this second factor contributes the product

$$\prod_{p \leq y} \left(1 - \frac{1}{p^{1-\alpha+\gamma}} \right) \cdot \left(1 + \frac{p^{1-\alpha+\gamma}}{p^{1+2\gamma}(p^{1-\alpha+\gamma} - 1)} \right) \cdot \prod_{p > y} \left(1 + \frac{1}{p^{1+2\gamma}} \right); \quad (\text{C.1})$$

we wrote it this way as we wanted to pull out factors of $1/\zeta(1-\alpha+\gamma)$ before sending $y \rightarrow \infty$. We now analyze this contribution in another manner. We do not pull out the factors of $1/\zeta(1-\alpha+\gamma)$, and we keep y fixed and finite. To find the derivative with

respect to α forces us to analyze the following (we ignore the product over $p > y$ for now as these terms have no α dependence):

$$\prod_{p \leq y} \left(1 - \frac{1}{p^{1-\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right); \quad (\text{C.2})$$

here the product over $p \leq y$ follows from brute force multiplication of the two terms in (C.1). Keeping y fixed, we now calculate the derivative of (C.2) with respect to α :

$$\begin{aligned} & \left. \frac{\partial}{\partial \alpha} \left[\prod_{p \leq y} \left(1 - \frac{1}{p^{1-\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \right] \right|_{\alpha=\gamma=r} \\ &= \prod_{p \leq y} \left(1 - \frac{1}{p^{1-\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \Big|_{\alpha=\gamma=r} \\ & \quad \cdot \left. \frac{\partial}{\partial \alpha} \log \left[\prod_{q \leq y} \left(1 - \frac{1}{q^{1-\alpha+\gamma}} + \frac{1}{q^{1+2\gamma}} \right) \right] \right|_{\alpha=\gamma=r} \\ &= \prod_{p \leq y} \left(1 - \frac{1}{p} + \frac{1}{p^{1+2r}} \right) \cdot \sum_{q \leq y} \left(1 - \frac{1}{q} + \frac{1}{q^{1+2r}} \right)^{-1} \cdot \frac{-\log q}{q}. \end{aligned} \quad (\text{C.3})$$

It is here that we must be careful in how we complete the sums (i.e., in how we let $y \rightarrow \infty$). For $\Re(r) > 0$ we write

$$\prod_{p \leq y} \left(1 - \frac{1}{p} + \frac{1}{p^{1+2r}} \right) = \prod_{p \leq y} \left(1 - \frac{1}{p} \right) \cdot \prod_{p \leq y} \left(1 + \frac{1}{(p-1)p^{2r}} \right); \quad (\text{C.4})$$

as $\Re(r) > 0$ the second factor is of size 1. By Mertens' Theorem (see [Da]) we have

$$\prod_{p \leq y} \left(1 - \frac{1}{p} \right) = \frac{e^{-\gamma}}{\log y} \left(1 + O\left(\frac{1}{\log y} \right) \right). \quad (\text{C.5})$$

Thus this product over primes tries to make our resulting term small; it is, however, balanced by the sum over q of $\log q/q$, as (see [Da])

$$\sum_{q \leq y} \frac{\log q}{q} \sim \log y + O(1). \quad (\text{C.6})$$

Completing the book-keeping, we find a very similar result for the second term in Lemma 2.9. Sending $y \rightarrow \infty$ gives us the second term in Lemma 2.9 but now multiplied by $e^{-\gamma}$.

This is a fascinating observation. It shows that there are at least *two* natural answers, and their main terms differ by $e^{-\gamma}$. Which is correct? It will almost surely be impossible to tell as $N \rightarrow \infty$, as this term contributes $O(1/N)$, and thus is *well* beyond current technology! We hope in a future paper to explore the case when $N = 1$ and $k \rightarrow \infty$ further.

Moreover, there is a lot of number theory and probability theory in $e^{-\gamma}$. Instead of the prime numbers, one could instead look at 'random' primes. There are many different models one can use to generate sequences of 'random' primes; often for each integer n one sets n to be prime with probability $p(n)$ (with $p(n)$ chosen so that the density

of our sequence mimics that of the primes). In these cases, the Riemann hypothesis is true with probability one; however, now the Riemann hypothesis is the statement that $\pi_{\text{random}}(x) = \text{Li}(x) + O(x^{1/2+\epsilon})$ (and not a statement about zeros of a corresponding function), where $\pi_{\text{random}}(x)$ denotes the number of ‘random’ primes less than x . In sieving heuristics, the number of primes at most x is about $2e^{-\gamma}x/\log x$, where $2e^{-\gamma} \approx 1.12292$. It is fascinating that the difference is equivalent to the differences in viewing the primes as random independent events versus including the congruence relations! See [BK, Ha, HW, Gr, NW, Wu] for additional remarks on $e^{-\gamma}$.

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