Sum of Consecutive Terms of Pell and Related Sequences

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Abstract

We study new identities related to the sums of adjacent terms in the Pell sequence, defined by $P_n := 2P_{n-1} + P_{n-2}$ for $n \geq 2$ and $P_0 = 0, P_1 = 1$, and generalize these identities for many similar sequences. We prove that the sum of $N > 1$ consecutive Pell numbers is a fixed integer multiple of another Pell number if and only if $4 \mid N$. We consider the generalized Pell (k, i)-numbers defined by $p(n) := 2p(n-1) + p(n-k-1)$ for $n \geq k+1$, with $p(0) = p(1) = \cdots = p(i) = 0$ and $p(i+1) = \cdots = p(k) = 1$ for $0 \leq i \leq k-1$, and prove that the sum of $N = 2k+2$ consecutive terms is a fixed integer multiple of another term in the sequence. We also prove that for the generalized Pell $(k, k-1)$-numbers such a relation does not exist when N and k are odd. We give analogous results for the Fibonacci and other related second-order recursive sequences.

1 Introduction

We first review some standard notation, and then describe our results. The *Fibonacci numbers* are defined by

$$F(n) \ := \ \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ F(n-1) + F(n-2) & n \geq 2, \end{cases} \tag{1.1}$$

and have a closed form given by *Binet's formula*:

$$F(n) \ = \ \frac{\varphi^n - \psi^n}{\sqrt{5}}, \tag{1.2}$$

where

$$\varphi \ = \ \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \psi \ = \ \frac{1-\sqrt{5}}{2} \ = \ -\frac{1}{\varphi}.$$

They are the simplest depth two constant-coefficient recurrence to study (the simpler depth one recurrences are just pure geometric sequences). They satisfy numerous interesting identities and arise in various areas; see for example [Ko]. We consider the Fibonacci numbers and other recursively defined sequences of numbers. In particular, we are interested in *Pell numbers* and their generalizations; we state these sequences and then our results.

Definition 1.1. Classical Pell numbers *are defined by the following recurrence and initial conditions:*

$$P(n) \ := \ \begin{cases} 0 & n=0 \\ 1 & n=1 \\ 2P(n-1)+P(n-2) & n \geq 2. \end{cases} \tag{1.3}$$

Definition 1.2. *The* Pell-Lucas sequence *or the* Companion Pell sequence *is defined by*

$$Q(n) \ := \ \begin{cases} 2 & n=0 \\ 2 & n=1 \\ 2Q(n-1)+Q(n-2) & n \geq 2. \end{cases} \tag{1.4}$$

Definition 1.3. *The* Lucas sequence *is defined by*

$$L(n) \ := \ \begin{cases} 2 & n=0 \\ 1 & n=1 \\ L(n-1)+L(n-2) & n \geq 2. \end{cases} \tag{1.5}$$

Definition 1.4. *If terms of a recursively defined infinite sequence can be expressed in a closed form similar to that of* (1.2)*, we call the closed form* a generalized Binet formula *(see [BBILMT, Le]).*

Example. Let

$$a \ := \ 1+\sqrt{2} \qquad \text{and} \qquad b \ := \ 1-\sqrt{2} \ = \ -\frac{1}{a}. \tag{1.6}$$

Then the n^{th} Pell Number is given by the generalized Binet formula:

$$P(n) \ = \ \frac{a^n - b^n}{2\sqrt{2}}. \tag{1.7}$$

Definition 1.5. *Throughout the paper, we let* $C : \mathbb{N} \to \mathbb{N}$ *denote a natural valued function on natural numbers.*

1.1 Motivation and Results

We analyze the relationships between sums of consecutive numbers in recurrence sequences. The first theorem below is a generalization of an observation made by the third named author to the fifth named author (for a problem for the Pi Mu Epsilon Journal) for sums of eight consecutive terms.

Theorem 1.6. *For any $N \in \mathbb{N}$, the sum of $4N$ consecutive Pell numbers is equal to a constant (depending on N) multiplied by the $(2N+1)^{\text{st}}$ term of the consecutive terms. In particular, we have*

$$\sum_{i=0}^{4N-1} P(n+i) \;=\; \frac{(a^{2N}-b^{2N})}{\sqrt{2}} P(n+2N), \tag{1.8}$$

where

$$a \;=\; 1+\sqrt{2} \qquad \text{and} \qquad b \;=\; 1-\sqrt{2} \;=\; -\frac{1}{a},$$

and therefore $(a^{2N}-b^{2N})/\sqrt{2}$ is an integer.

Proof. The n^{th} Pell Number is given by (1.7):

$$P(n) \;=\; \frac{a^n-b^n}{2\sqrt{2}}.$$

Therefore, we have

$$\begin{aligned}
\sum_{i=0}^{4N-1} P(n+i) &= \sum_{i=0}^{4N-1} \frac{a^{n+i}-b^{n+i}}{2\sqrt{2}} \\
&= \frac{a^n}{2\sqrt{2}} \sum_{i=0}^{4N-1} a^i - \frac{b^n}{2\sqrt{2}} \sum_{i=0}^{4N-1} b^i \\
&= \frac{a^n}{4}(a^{4N}-1) - \frac{b^n}{4}(1-b^{4N}) \\
&= \frac{a^{n+2N}}{4}(a^{2N}-b^{2N}) - \frac{b^{n+2N}}{4}(a^{2N}-b^{2N}) \\
&= \frac{(a^{2N}-b^{2N})}{4}(a^{n+2N}-b^{n+2N}) \\
&= \frac{(a^{2N}-b^{2N})}{\sqrt{2}} P(n+2N).
\end{aligned} \tag{1.9}$$

□

The above-mentioned excursion motivates the question: *For which numbers $n \in \mathbb{N}$ does the sum of n consecutive Pell numbers equal a fixed integer multiple of another Pell number?*

We answer this question for the Pell, Fibonacci and other related sequences. In particular, for the Pell sequence we observe that multiples of 4 (see Theorem 1.6) and the trivial case of $N = 1$ are the only values of N that work. We then extend our methods to generalized Pell numbers and present a conjecture regarding when the sum of consecutive generalized Pell numbers equals a fixed integer multiple of another generalized Pell number. Additionally, we describe several interesting properties of Pell numbers using tilings of an $n \times 1$ board with polyominoes.

2 Identities and Preliminary Results

The following lemmas describe standard identities relating Pell, Lucas and Fibonacci sequences, and are used extensively in the rest of the paper. For completeness, we provide the proofs.

Lemma 2.1. *For any non-negative integer k the Pell numbers satisfy*

$$P(n+k)+(-1)^k P(n-k) \;=\; Q(k)P(n), \tag{2.1}$$

where $Q(k)$ is the k^{th} term of the Pell-Lucas sequence given in Definition 1.2.

Proof. We proceed by induction on k, noting that the two base cases are $k=0$ and $k=1$. When $k=0$, we have

$$P(n+0)+(-1)^0 P(n-0) \;=\; 2P(n)=Q(0)P(n). \tag{2.2}$$

When $k=1$, we have

$$P(n+1)+(-1)^1 P(n-1) \;=\; [2P(n)+P(n-1)]-P(n-1) \;=\; 2P(n) \;=\; Q(1)P(n). \tag{2.3}$$

Now, we assume that

$$P(n+k-1)+(-1)^{k-1}P(n-k+1) \;=\; Q(k-1)P(n) \tag{2.4}$$

and

$$P(n+k-2)+(-1)^{k-2}P(n-k+2) \;=\; Q(k-2)P(n). \tag{2.5}$$

Using the recurrence relation (1.3), we have

$$P(n+k) \;=\; 2P(n+k-1)+P(n+k-2).$$

Rearranging (2.5), we get

$$P(n-k+2) \;=\; 2P(n-k+1)+P(n-k) \implies P(n-k) \;=\; P(n-k+2)-2P(n-k+1).$$

Thus,

$$\begin{aligned} &P(n+k)+(-1)^k P(n-k) \\ &= \; 2P(n+k-1)+P(n+k-2)+(-1)^k(P(n-k+2)-2P(n-k+1)). \end{aligned} \tag{2.6}$$

Rearranging the right-hand side of (2.6) yields

$$2(P(n+k-1)+(-1)^{k-1}P(n-k+1))+(P(n+k-2)+(-1)^{k-2}P(n-k+2)). \tag{2.7}$$

We apply the inductive hypotheses (2.4) and (2.5) along with Definition 1.2 to this expression to conclude

$$\begin{aligned} 2Q(k-1)P(n)+Q(k-2)P(n) &= (2Q(k-1)+Q(k-2))P(n) \\ &= Q(k)P(n), \end{aligned}$$

which completes the proof. □

Lemma 2.2. *For the Fibonacci and Lucas numbers we have*

$$\sum_{i=0}^{4N-1} F(n+i) \;=\; F(2N)L(n+2N+1). \tag{2.8}$$

Proof. We use induction and the following well-known properties of the Fibonacci and Lucas Numbers [Ko, §5.3, §5.8]:

$$(i) \quad F(n-1)F(n+1) - F(n)^2 \;=\; (-1)^n. \tag{2.9}$$

$$(ii) \quad F(n+k) \;=\; F(n)F(k-1) + F(n+1)F(k). \tag{2.10}$$

$$(iii) \quad F(n-1) + F(n+1) \;=\; L(n). \tag{2.11}$$

$$(iv) \quad \sum_{i=0}^{n-1} F(i) + 1 \;=\; F(n+1). \tag{2.12}$$

$$(v) \quad \sum_{i=0}^{k} F(n+i) = F(n+k+2) - F(n+1). \tag{2.13}$$

We now prove Lemma 2.2. First, begin by noting that for $n = 0$, we have

$$\begin{aligned}
\sum_{i=0}^{4N-1} F(i) &= F(4N+1) - 1 \qquad \text{(Using (2.12))} \\
&= F(2N)F(2N+2) + F(2N-1)F(2N+1) - 1 \qquad \text{(Using (2.10))} \\
&= F(2N)(F(2N+2) + F(2N)) \qquad \text{(Using (2.9)} \\
&= F(2N)L(2N+1). \qquad \text{(Using (2.11))}
\end{aligned}$$

Now, by our induction hypothesis, $\sum_{i=0}^{4N-1} F(m+i) \;=\; F(2N)L(m+2N+1)$ holds for all $m < n+1$. We now expand $\sum_{i=0}^{4N-1} F(n+1+i)$ using the following manipulations:

$$\begin{aligned}
\sum_{i=0}^{4N-1} F(n+1+i) &= F(n+4N+2) - F(n+2) \text{ (Using (2.13))} \\
&= F(n+4N+1) + F(n+4N) - (F(n+1) + F(n)) \\
&\quad \text{(Using (1.1))} \\
&= F(n+4N+1) - F(n+1) + (F(n+4N) - F(N)) \qquad \text{(Rearranging terms)} \\
&= \sum_{i=0}^{4N-1} F(n+i) + \sum_{i=0}^{4N-1} F(n-1+i) \\
&\quad \text{(Using (2.13))} \\
&= F(2N)(L(n+2N+1) + L(n+2N) \\
&\quad \text{(Induction hypothesis)} \\
&= F(2N)L(n+2N+2) \qquad \text{(Using Definition 1.3)},
\end{aligned}$$

which yield the desired result.

□

Lemma 2.3. *For the Fibonacci numbers, we have for all positive integers n*

$$\varphi^n \;=\; F(n)\varphi + F(n-1) \quad \textit{where} \quad \varphi \;=\; \frac{1+\sqrt{5}}{2}.$$

Proof. We proceed by induction on n. When $n = 1$, the statement of the lemma is $\varphi = \varphi$, which is trivially true. Similarly, $n = 2$ is true because $\varphi^2 = \varphi + 1$ is true as $\frac{1+\sqrt{5}}{2}$ is a root of the characteristic polynomial. Thus we may assume the statement holds for all natural numbers less than $k \geq 3$. Then

$$\begin{aligned}
\varphi^k &= \varphi^{k-1} + \varphi^{k-2} \\
&= (F(k-1)\varphi + F(k-2)) + (F(k-2)\varphi + F(k-2)) \\
&= (F(k-1) + F(k-2))\varphi + (F(k-2) + F(k-3)) \\
&= F(k)\varphi + F(k-1),
\end{aligned}$$

which completes the proof. □

3 Some General Results

One of our main goals is to determine not only when the sum of consecutive terms in a recurrence is a fixed multiple of a term of the recurrence, but further to determine which term. In this section we shall consider the following sequence.

Definition 3.1. *Let r be a non-negative integer. Consider a sequence $\{f(n)\}$ of non-negative integers recursively defined by*

$$f(n) \;:=\; rf(n-1) + f(n-2)$$

with initial conditions so that it is not identically zero (we call this a non-degenerate sequence).

If we set

$$\alpha \;:=\; \frac{r+\sqrt{r^2+4}}{2} \qquad \text{and} \qquad \beta \;:=\; \frac{r-\sqrt{r^2+4}}{2}$$

then the generalized Binet formula (see [BBILMT, Le])) yields

$$f(n) \;=\; a\alpha^n + b\beta^n.$$

Theorem 3.2. *Fix any integer $N > 0$. If there is an integer $C(N)$ such that for every sufficiently large n there exists an integer index $j(n;N)$ such that the following equation holds*

$$\sum_{i=0}^{N-1} f(n+i) \;=\; C(N) \cdot f(j(n;N)),$$

then there is an integer $k(N)$ such that

$$j(n;N) \;=\; n + k(N) \qquad \textit{and} \qquad k(N) \in \left[\frac{N}{2}, N\right].$$

Proof. Define

$$b \;:=\; \frac{\alpha^N - 1}{C(N)(\alpha - 1)} \qquad \text{and} \qquad k(N) \;:=\; \log_\alpha b,$$

with α, β and f as above.
Note that $|\alpha| \;>\; |\beta|$ and $|\beta| \;<\; 1$. Then by the generalized Binet's formula

$$f(n) \;=\; a\alpha^n + b\beta^n$$

and

$$\lim_{n\to\infty} |f(n) - a\alpha^n| \;=\; 0.$$

This implies that for any $\varepsilon > 0$ there exists a natural number M such that for all $n > M$,

$$|f(n) - a\alpha^n| \;<\; \frac{C(N)\cdot\varepsilon}{2N}. \tag{3.1}$$

We choose M sufficiently large such that

$$|f(j(n;N)) - a\alpha^{j(n;N)}| \;<\; \frac{\varepsilon}{2} \tag{3.2}$$

for all $n > M$.

Then

$$\begin{aligned}
&\left| a\alpha^{j(n;N)} - \frac{1}{C(N)} \sum_{i=0}^{N-1} a\alpha^{n+i} \right| \\
&< \left| a\alpha^{j(n;N)} - f(j(n;N)) \right| + \frac{1}{C(N)} \left| C(N)\cdot f(j(n;N)) - \sum_{i=0}^{N-1} a\alpha^{n+i} \right| \\
&< \frac{\varepsilon}{2} + \frac{1}{C(N)} \left| \sum_{i=0}^{N-1} f(n+i) - \sum_{i=0}^{N-1} a\alpha^{n+i} \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon.
\end{aligned} \tag{3.3}$$

We now have

$$\frac{1}{C(N)} \sum_{i=0}^{N-1} \alpha^{n+i} \;=\; \alpha^{n+\log_\alpha b}.$$

If $k(N) \notin \mathbb{N}$ then consider $m = \min\{n + k(N), j(n; N)\}$. The conditions on f imply that it is an increasing sequence, therefore $j(n; N) \to \infty$ as $n \to \infty$. Hence $m \to \infty$ as $n \to \infty$. We also note that

$$\begin{aligned} \left| a\alpha^{j(n;N)} - \frac{1}{C(N)} \sum_{i=0}^{N-1} a\alpha^{n+i} \right| &= |a|\,\alpha^m \left| \alpha^{|j(n;N)-n-k(N)|} - 1 \right| \\ &\geq |a|\,\alpha^m \left(\alpha^{|j(n;N)-n-k(N)|} - 1 \right), \end{aligned} \tag{3.4}$$

and since $j(n; N) \in \mathbb{N}$ we have $j(n; N) - n - k(N) \notin \mathbb{Z}$. Similarly, since $\alpha > 1$ we have $\alpha^{|j(n;N)-n-k(N)|} - 1 > 0$. Lastly, $m \to \infty$ as $n \to \infty$ and thus for large enough n, the left hand side of (3.4) tends to infinity:

$$\lim_{n\to\infty} \left| a\alpha^{j(n;N)} - \frac{1}{C(N)} \sum_{i=0}^{N-1} a\alpha^{n+i} \right| \to \infty,$$

which contradicts (3.3), implying that $k(N) \in \mathbb{N}$, and $j(n; N) = n + k(N)$.

We now prove that

$$k(N) \in \left[\frac{N}{2}, N \right].$$

We begin by noting that

$$\frac{\alpha^N - 1}{C(N)(\alpha - 1)} = \alpha^{k(N)} \implies C(N) = \sum_{i=0}^{N-1} \alpha^{i-k(N)}. \tag{3.5}$$

Let $k(N) < \frac{N}{2}$, then

$$C(N) = 1 + \sum_{i=1}^{k(N)} \left(\alpha^i + \frac{1}{\alpha^i} \right) + \sum_{i=2k(N)+1}^{N-1} \alpha^{i-k(N)}.$$

Now, note that the coefficient of the irrational part of $\sum\limits_{i=2k(N)+1}^{N-1} \alpha^{i-k(N)}$ is a positive integer. We now have

$$\alpha^i + \frac{1}{\alpha^i} = \alpha^i + (-\beta)^i. \tag{3.6}$$

Applying the binomial theorem to the above-mentioned equation gives the coefficient of the irrational part in $\alpha^i + (-\beta)^i$ for $i > 1$ to be

$$\sum_{j=1}^{\lfloor \frac{i-1}{2} \rfloor} (r^2 + 4)(r^{i-2j+1} + (-r)^{i-2j+1}) \geq 0,$$

which implies that $C(N)$ is irrational, resulting in a contradiction. Therefore $k(N) \geq N/2$.

Now, by induction we get the following inequality:

$$\sum_{i=0}^{n+N-1} f(i) \;<\; f(n+N+1),$$

which proves $k(N) \leq N$.
This implies that

$$j(n;N) \in \left[n + \frac{N}{2}, n+N\right].$$

$\square$

Theorem 3.3. *Given a non-degenerate sequence of non-negative integers recursively defined by*

$$f(n) \;:=\; rf(n-1) + f(n-2),$$

where $r \in \mathbb{N}$*, if*

$$\sum_{i=0}^{3} f(n+i) \;=\; C(N) \cdot f(j(n;N))$$

then $r = 2$*.*

Proof. From the proof of Theorem 3.2 we have $C(N) = \sum_{i=0}^{3} \alpha^{i-k(N)}$ for $C(N)$ a positive integer, where $2 \leq k(N) \leq 4$. Therefore, the only possible values for $k(N)$ are $2, 3, 4$. We will now do casework based on the value of $k(N)$.

Case 1: $k(N) = 2$.

$$\begin{aligned} C(N) &= \frac{1}{\alpha^2} + \frac{1}{\alpha} + 1 + \alpha \\ &= 1 + \frac{2r^2 + 4 - 2r\sqrt{r^2+4}}{4} + \frac{\sqrt{r^2+4} - r}{2} + \frac{r + \sqrt{r^2+4}}{2} \\ &= 1 + \frac{r^2+2}{2} + \left(\frac{2-r}{2}\right)\sqrt{r^2+4}. \end{aligned}$$

Since r is an integer and there is no Pythagorean triple with 2 as one of the terms, therefore $\sqrt{r^2+4}$ is irrational. Thus for $C(N)$ to be an integer, we must have $\frac{2-r}{2} = 0$, therefore $r = 2$.

Case 2: $k(N) = 3$.

$$\begin{aligned} C(N) &= \frac{1}{\alpha^3} + \frac{1}{\alpha^2} + \frac{1}{\alpha} + 1 \\ &= \frac{(r^2+4)\sqrt{r^2+4} - 3r(r^2+4) + 3r^2\sqrt{r^2+4} - r^3}{8} \\ &+ \frac{2r^2+4-2r\sqrt{r^2+4}}{4} + \frac{\sqrt{r^2+4}-r}{2} \\ &= \frac{-4r^3+4r^2-16r+8}{8} + \left(\frac{4r^2+4-4r+4}{8}\right)\sqrt{r^2+4} \\ &= \frac{-r^3+r^2-4r+2}{2} + \left(\frac{r^2-r+2}{2}\right)\sqrt{r^2+4}. \end{aligned}$$

Since $C(N)$ is an integer, we must have $r^2 - r + 2 = 0$. But since this equation has no integer roots, no such r exists.

Case 3: $k(N) = 4$.

$$\begin{aligned} \text{Then } C(N) &= \frac{1}{\alpha^4} + \frac{1}{\alpha^3} + \frac{1}{\alpha^2} + \frac{1}{\alpha} \\ &= \frac{r^4 - r^3 + 5r^2 - 3r = 4}{2} + \left(\frac{-r^3+r^2-3r+1}{2}\right)\sqrt{r^2+4}. \end{aligned} \tag{3.7}$$

Since $C(N)$ is an integer, we must have $-r^3 + r^2 - 3r + 1 = 0$. But this has no integer roots, so no such r exists. □

Clearly, only looking at rational multiples of terms in the sequence is sufficient, because the desired multiple can be written as a ratio of integers. The following theorem proves why only looking at *integer* multiples of terms in the sequence is sufficient.

Theorem 3.4. *Define $\{f(n)\}$ by the recurrence relation*

$$f(n) \;:=\; rf(n-1) + f(n-2)$$

where $r \in \mathbb{N}$, and choose initial conditions so that f is not identically zero. Then if the sum of $N > 1$ consecutive terms of $\{f(n)\}$ is a fixed rational constant times another term in the sequence, then the rational constant is an integer.

Proof. Let $d = \gcd(f(0), f(1))$. We notice that $d \mid f(n)$ for all $n \in \mathbb{N}$ and therefore consider the equivalent sequence $h(n) := f(n)/d$ instead. Then

$$\gcd(h(n), h(n+1)) \;=\; 1$$

for all $n \geq 1$ since

$$\begin{aligned}
\gcd(h(n), h(n+1)) &= \gcd(h(n), rh(n) + h(n-1)) \\
&= \gcd(h(n), h(n-1)) \\
&\;\;\vdots \\
&= \gcd(h(0), h(1)).
\end{aligned}$$

Now suppose

$$\sum_{i=0}^{N-1} f(n+i) = \frac{a}{b} f(j(n;N)), \tag{3.8}$$

where $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$. Dividing both sides by d, we get

$$\sum_{i=0}^{N-1} h(n+i) = \frac{a}{b} h(j(n;N)). \tag{3.9}$$

From Theorem 3.2 we know that there exists $M \in \mathbb{N}$ such that $j(n;N) = n + k(N)$ for $n > M$. Applying (3.8) tells us that $b \mid h(n+k(N))$ for all $n > M$. However, if $b > 1$ we reach the contradiction that $\gcd(f(n), f(n+1)) \neq 1$ for all $n \in \mathbb{N}$. Therefore $b = 1$, which completes our proof. $\square$

4 Pell numbers

In Theorem 1.6, we proved that the sum of $4N$ consecutive Pell numbers is a constant integer multiple of the $(2N+1)^{\text{st}}$ term. We generalize to other related partial sums.

4.1 Sum of $4N+2$ Consecutive Terms

Theorem 4.1. *Let $P(n)$ denote the n^{th} Pell number. Fix any integer $N > 0$. There is no integer $C(N)$ such that for every n there exists an integer index $j(n;N)$ such that the following equation holds:*

$$\sum_{i=0}^{2N} P(n+i) = C(N) P(j(n;N)).$$

Proof. We note that

$$\sum_{k=0}^{4N+1} P(n+k) = \sum_{k=0}^{n+4N+1} P(k) - \sum_{k=0}^{n-1} P(k). \tag{4.1}$$

The following result

$$\sum_{k=0}^{n} P(k) \;=\; \frac{1}{2}\Big[P(n+1)+P(n)-1\Big], \tag{4.2}$$

from [Br, §2, equation (2)] implies that

$$\sum_{k=0}^{4N+1} P(n+k) \;=\; \frac{1}{2}\Big[P(n+4N+2)+P(n+4N+1)-P(n-1)-P(n)\Big]. \tag{4.3}$$

We also note that

$$\begin{aligned} P(n+4N+2) &= P(n+2N+1+(2N+1)), && (4.4)\\ P(n+4N+1) &= P(n+2N+(2N+1)), && (4.5)\\ -P(n-1) &= P(n+2N-(2N+1)), && (4.6)\\ -P(n) &= P(n+2N+1-(2N+1)). && (4.7) \end{aligned}$$

Applying Lemma 2.1 on expressions (4.4) and (4.7) in equation (4.2), we get

$$P(n+4N+2)+(-1)^{2N+1}P(n) \;=\; Q(2N+1)P(n+2N+1). \tag{4.8}$$

Furthermore, applying Lemma 2.1 on expressions (4.5) and (4.6) in equation (4.2), we get

$$P(n+4N+1)+(-1)^{2N+1}P(n-1) \;=\; Q(2N+1)P(n+2N), \tag{4.9}$$

where $\{Q(n)\}$ is the companion Pell sequence given in Definition 1.2. Therefore,

$$\begin{aligned} \sum_{k=0}^{4N+1} P(n+k) &= \frac{1}{2}\Big[Q(2N+1)P(n+2N+1)+Q(2N+1)P(n+2N)\Big]\\ &= Q(2N+1)\left(\frac{P(n+2N+1)+P(n+2N)}{2}\right). && (4.10) \end{aligned}$$

Since $Q(n)$ is even for all n (see Definition 1.2), we have $\frac{Q_{2N+1}}{2} \in \mathbb{N}$.

We now suppose the sum of $4k+2$ Pell numbers is equal to a constant multiple of another Pell number. Then for some $t_1 \geq t_2 \in \mathbb{N}$, we have the following equations:

$$\begin{aligned} rP(t_1) &= Q(2N+1)\cdot\frac{P(n+2N+2)+P(n+2N+1)}{2} && (4.11)\\ rP(t_2) &= Q(2N+1)\cdot\frac{P(n+2N+1)+P(n+2N)}{2}. && (4.12) \end{aligned}$$

Dividing the two, we get

$$\frac{P(n+2N+1)+P(n+2N+2)}{P(n+2N)+P(n+2N+1)} \;=\; \frac{P(t_1)}{P(t_2)}. \tag{4.13}$$

Now, let

$$T_m \;=\; P(m) + P(m-1) \text{ for } m \in \mathbb{N},$$

then T_m satisfies

$$\begin{aligned}
2T_{m-1} + T_{m-2} &= 2[P(m-1) + P(m-2)] + [P(m-2) + P(m-3)] \\
&= 2P(m-1) + 3P(m-2) + P(m+3) \\
&= [2P(m-1) + P(m-2)] + [2P(m-2) + P(m+3)] \\
&= P(m) + P(m-1) \\
&= T_m,
\end{aligned}$$

which gives us the following recurrence:

$$T_m \;=\; 2T_{m-1} + T_{m-2}. \tag{4.14}$$

By applying induction on (4.14), we deduce

$$2 \;<\; \frac{T_{m+1}}{T_m} \;\leq\; 3, \tag{4.15}$$

which implies that $t_1 > t_2$ as otherwise we would have $T_{m+1}/T_m < 1$.
We now notice that if $t_1 \geq t_2 + 2$ then

$$\begin{aligned}
\frac{P(t_2+2)}{P(t_2)} &= \left(\frac{P(t_2+2)}{P(t_2+1)}\right)\left(\frac{P(t_2+1)}{P(t_2)}\right) \;>\; 4 \\
&\implies \frac{P(n+2N+1) + P(n+2N+2)}{P(n+2N) + P(n+2N+1)} \;>\; 4 \\
&\implies P(n+2N+2) \;>\; 4P(n+2N) + 3P(n+2N+1) \\
&\implies 2P(n+2N+1) + P(n+2N) > 4P(n+2N) + 3P(n+2N+1)
\end{aligned}$$

which leads to a contradiction. Therefore, we must have $t_1 = t_2 + 1$. We now note that

$$\begin{aligned}
P(n+2N+2) &< P(n+2N+1) + P(n+2N+2) \;=\; \frac{2rP(t_1)}{Q(2N+1)} \\
&< 2P(n+2N+2) + P(n+2N+1) \;=\; P(n+2N+3) \\
&\implies \frac{2r}{Q(2N+1)} \;<\; \frac{P(n+2N+3)}{P(t_1)} \\
&\implies \frac{2r}{Q(2N+1)} \;\neq\; 1.
\end{aligned} \tag{4.16}$$

Additionally, since $T_{m+1} \;=\; 2T_m + T_{m-1}$, we have

$$\begin{aligned}
\gcd(T_{m+1}, T_m) &= \gcd(2T_m + T_{m-1}, T_m) \\
&= \gcd(T_m, T_{m-1}).
\end{aligned} \tag{4.17}$$

Now, note that $\gcd(T_1, T_2) = 1$. Therefore, by induction we can conclude that $\gcd(T_{m+1}, T_m) = 1$. Utilizing the same argument, we deduce that $\gcd(P(m+1), P(m)) = 1$, but this contradicts the following statements:

$$rP(t_2 + 1) = \frac{Q(2N+1)}{2} T_{n+2N+2}, \tag{4.18}$$

$$rP(t_2) = \frac{Q(2N+1)}{2} T_{n+2N+1}, \tag{4.19}$$

$$\text{and} \quad \frac{2r}{Q(2N+1)} > 1, \tag{4.20}$$

which are obtained by trivial substitution. □

4.2 Sums of Odd Numbers of Consecutive Terms

Theorem 4.2. *Let $P(n)$ denote the n^{th} Pell number. Fix any integer $N > 0$. There is no integer $C(N)$ such that for every n there exists an integer index $j(n; N)$ such that the following equation holds:*

$$\sum_{i=0}^{2N} P(n+i) = C(N) P(j(n; N)).$$

Proof. Suppose that we can write the sum of any N consecutive Pell numbers as $C(N)$ times a Pell number for some positive integer $C(N)$ where N is odd. Consider the Pell sequence modulo $C(N)$. By using the Pigeonhole Principle and the fact that two consecutive Pell numbers uniquely determine the terms before and after them we see that $\{P_{n,C(N)}\}_{n\geq 0} := \{P(n) \bmod C(N)\}_{n\geq 0}$ is periodic. The period is called the *Pisano Period* and is denoted by $\pi(C(N))$.

Notice that $\pi(1) = 1$ and $\pi(2) = 2$. Now consider $C(N) > 2$. Since $\{P_{n,C(N)}\}_{n\geq 0}$ is not a constant sequence, therefore $\pi(C(N)) \geq 2$, which implies that

$$P_{c+\pi(C(N)),C(N)} = P_{c,C(N)}.$$

Now since

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{\pi(C(N))+1} = \begin{pmatrix} P(\pi(C(N)) + 2) & P(\pi(C(N)) + 1) \\ P(\pi(C(N)) + 1) & P(\pi(C(N))) \end{pmatrix}, \tag{4.21}$$

this implies that, with I_2 the 2×2 identity matrix,

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{\pi(C(N))} = I_2 \text{ in } GL_2\left(\mathbb{Z}/\, C(N)\, \mathbb{Z}\right). \tag{4.22}$$

Taking the determinant, we get $(-1)^{\pi(C(N))} = 1$, which implies that $\pi(C(N))$ is even for all $C(N) > 2$.

Now let N be an odd number, to emphasize this we change notation and write it as $2N+1$. Suppose the sum of any $2N+1$ consecutive Pell numbers is $C(N)$ times another Pell Number. Then

$$\sum_{i=0}^{2N+1-1} P(n+i) \equiv 0 \pmod{C(N)}, \text{ for } n \geq 0.$$

Replacing n by $n+1$ we get

$$P(n+2N+1) \equiv P(n) \pmod{C(N)} \text{ for all } n \geq 0,$$

which implies that $\pi(C(N)) \mid N$. However, since $\pi(C(N))$ is even when $C(N) \geq 2$, this implies that $C(N) = 1$. Thus the sum of any $2N+1$ consecutive Pell numbers must be equal to a Pell number. In other words,

$$\sum_{i=0}^{2N+1-1} P(n+i) = P(j(n;N))$$

for some integer $j(n;N)$ and $n \geq 0$. Notice that when $N = 0$, we obtain $j(n;N) = n$. Now suppose $2N+1$ is an odd integer greater than 1. Then, we have

$$\sum_{i=0}^{2N+1-1} P(n+i) > P(n+2N+1-1).$$

However,

$$\sum_{i=0}^{2N+1-1} P(n+i) < \sum_{i=0}^{n+2N+1-1} P(i) < P(n+2N+1)$$

where the last inequality can be proven by using induction based on the value of $n+2N+1$. We conclude that

$$P(n+2N+1-1) < \sum_{i=0}^{2N+1-1} P(n+i) < P(n+2N+1),$$

and hence the sum of $2N+1$ consecutive Pell numbers is a fixed integer multiple of a Pell Number if and only if $N = 0$. □

We will use a proof of a similar flavor in 6.5.

5 Fibonacci Numbers

We now prove similar results for the Fibonacci numbers. In particular, we show that the sum of N consecutive Fibonacci numbers is equal to a fixed constant multiple of a *Fibonacci number* if and only if $N \equiv 2 \pmod 4$, $N = 3$, or $N = 1$.

5.1 Sum of $4N+2$ Consecutive Terms

Theorem 5.1. *Let $F(n)$ denote the n^{th} Fibonacci number, and $L(n)$ denote the n^{th} Lucas number. Fix any $N > 0$. The following equation*

$$\sum_{i=0}^{4N+1} F(n+i) \;=\; L(2N+1)F(n+2N+2),$$

holds for all n.

Proof. The Fibonacci numbers are the solutions to

$$F(n) \;=\; F(n-1) + F(n-2)$$

with $F(0) = 0$ and $F(1) = 1$, while the Lucas Sequence are the solutions to the same recurrence

$$L(n) \;=\; L(n-1) + L(n-2),$$

but with initial conditions $L(0) = 2$ and $L(1) = 1$. It is well known (see [Ko, Theorem 5.1]) that

$$\sum_{i=0}^{n} F(i) \;=\; F(n+2) - 1.$$

A straightforward induction yields $F(n+k)+(-1)^k F(n-k) \;=\; L(k)F(n)$, and therefore

$$\begin{aligned} \sum_{i=0}^{4N+1} F(n+i) \;&=\; F(n+4N+3) - F(n+1) \\ &=\; L(2N+1)F(n+2N+2), \end{aligned} \tag{5.1}$$

which completes the proof. $\square$

5.2 Sum of $4N$ Consecutive Terms

Theorem 5.2. *Let $F(n)$ denote the n^{th} Fibonacci number. Fix any integer $N > 0$. There is no integer $C(N)$ such that for every n there exists an integer index $j(n;N)$ such that the following equation holds:*

$$\sum_{i=0}^{4N-1} F(n+i) \;=\; C(N)F(j(n;N)).$$

Proof. From Lemma 2.2 we have

$$\sum_{i=0}^{4N-1} F(n+i) \;=\; F(2N)L(n+2N+1) \;=\; F(2N)(F(n+2N)+F(n+2N+2)).$$

Now setting $T_m = F(m)+F(m+2)$ and repeating the proof of the $4N+2$ case for Pell numbers gives us the desired result. $\square$

5.3 Sums of Odd Numbers of Consecutive Terms

We note that any Fibonacci number is one times itself and the sum of any three consecutive Fibonacci numbers is two times the third term. We prove that these are the only solutions for odd cases with the following theorem.

Theorem 5.3. *Let $F(n)$ denote the n^{th} Fibonacci number. Fix any integer $N \geq 2$. There is no integer $C(N)$ such that for every n there exists an integer index $j(n;N)$ such that the following equation holds:*

$$\sum_{i=0}^{2N} F(n+i) = C(N) \cdot F(j(n;N)).$$

Proof. The proof of Theorem 3.2, specifically, 3.5 tells us that if the sum of N consecutive Fibonacci numbers is $C(N)$-times another Fibonacci number, then

$$b \;=\; \frac{(\varphi^N - 1)}{C(N)(\varphi - 1)} \;=\; \frac{\sum_{i=0}^{N-1} \varphi^i}{C(N)} \;=\; \varphi^\gamma \quad \text{for some } \gamma \in \mathbb{N}.$$

Using an argument of a similar flavor to Section 4.2, we deduce that $C(N)$ must either be 1 or 2.

Now, Lemma 2.3 tells us that

$$b \;=\; \frac{\varphi \sum_{i=1}^{N-1} F(i) + \sum_{i=1}^{N-2} F(i) + 1}{C(N)} \tag{5.2}$$

$$\;=\; \frac{(F(N+1)-1)\varphi + F(N)}{C(N)}, \tag{5.3}$$

where $C(N)$ is either 1 or 2. Since for $n \geq 3$, $b \geq \alpha$, we let $b = \alpha^m$ where $m \geq 1$. Thus, we get

$$\frac{(F(N+1)-1)\varphi + F(N)}{C(N)} \;=\; F(m)\varphi + F(m-1),$$

which implies that

$$F(m-1) \;=\; \frac{F(N)}{C(N)} \quad \text{and} \quad F(m) \;=\; \frac{F(N+1)-1}{C(N)}.$$

We now consider the two cases, $C(N) = 1$ and $C(N) = 2$.

Case 1: $C(N) = 1$**.**

If $C(N) = 1$, we see that if $m \neq 3$ then $m = N + 1$ leads to a contradiction, therefore $m = 3$ and thus $N = 1$.

Case 2: $C(N) = 2$**.**

If $C(N) = 2$ then Carmichael's Theorem [Ca] tells us that for $n > 13$, $F(n)$ has a prime factor not present in the previous Fibonacci numbers. Therefore, we only need to check the cases where $n \leq 13$. Checking for the smaller cases we realize that $N = 3$ is the only case where $F(N)/2$ is another Fibonacci number. $\square$

6 Generalized Pell and Fibonacci Numbers

We adapt our previous results to a generalization of the Pell numbers that satisfies a $(k+1)^{\text{st}}$ order recursion, where $k \in \mathbb{N}$. We also conjecture that for $k > 1$, the sum of N consecutive *generalized Pell numbers* is a fixed integer multiple of another term of the sequence if and only if $N = 2k + 2$. Finally, we prove similar properties for a generalization of the Fibonacci numbers.

6.1 Definition

In [Ki] the authors consider the following generalization of the Pell numbers (we slightly modify their notation as we start our indexing at $n = 0$).

Definition 6.1. Generalized Pell (k, i)-numbers *are the solutions to the following recursion with given initial conditions:*

$$\begin{aligned} P_k^i(n) &= 2P_k^i(n-1) + P_k^i(n-k-1) \\ \textit{with} \quad P_k^i(0) &= P_k^i(1) = \cdots = P_k^i(i) = 0 \\ \textit{and} \quad P_k^i(i+1) &= P_k^i(i+2) = \cdots = P_k^i(k) = 1 \\ \textit{where} \quad 0 &\leq i \leq k-1. \qquad (k \in \mathbb{N}). \end{aligned} \tag{6.1}$$

6.2 Sum of $2k + 2$ Consecutive Terms

Applying the following formula from [Ki, §4, Theorem 19], we get

$$\sum_{i=0}^{n} P_k^{k-1}(i) = \frac{1}{2}\left(-1 + \sum_{i=0}^{k} P_k^{k-1}(n-i+1)\right) \tag{6.2}$$

where $n \geq k - 1$. We prove a result similar to Theorem 1.6 for the generalized Pell sequence.

Theorem 6.2. *For $n \geq k$ we have*

$$\sum_{i=0}^{2k+1} P_k^{k-1}(n+i) = 4P_k^{k-1}(n+2k). \tag{6.3}$$

Proof. Let

$$S_n \;=\; \sum_{i=0}^{n} P_k^{k-1}(i). \tag{6.4}$$

Note that the first $k-1$ terms in this sum are 0.

We proceed by induction on n, starting at $n = k$ for the base case. Shifting indices on (6.4) for $n = k$ gives

$$\sum_{i=0}^{2k+1} P_k^{k-1}(k+i) \;=\; \sum_{i=k}^{3k+1} P_k^{k-1}(i) \;=\; S_{3k+1}.$$

Noting that the first $k-1$ terms of S_n are all zero, we find

$$\begin{aligned} S_{3k+1} &= S_{3k} + 2P_k^{k-1}(3k) + P_k^{k-1}(2k) \\ &= \underbrace{S_{2k-1} + 2P_k^{k-1}(2k) + \sum_{i=2k+1}^{3k-1} P_k^{k-1}(i)}_{(6.5)} + 3P_k^{k-1}(3k). \end{aligned}$$

Now, we consider

$$S \;:=\; S_{2k-1} + 2P_k^{k-1}(2k) + \sum_{i=2k+1}^{3k-1} P_k^{k-1}(i). \tag{6.5}$$

Since the first $k-1$ terms of the sum in (6.4) are zero,

$$\begin{aligned} S_{2k-1} &= \sum_{i=0}^{2k-1} P_k^{k-1}(i) \\ &= \sum_{i=k}^{2k-1} P_k^{k-1}(i) \\ &= P_k^{k-1}(k) + \sum_{i=k+1}^{2k-1} P_k^{k-1}(i). \end{aligned}$$

Then, applying recursion (6.1) to (6.5), we find

$$\begin{aligned}
S &= \sum_{i=k+1}^{2k-1} P_k^{k-1}(i) + P_k^{k-1}(2k+1) + \sum_{i=2k+1}^{3k-1} P_k^{k-1}(i) \\
&= \sum_{i=k+1}^{2k-1} P_k^{k-1}(i) + 2P_k^{k-1}(2k+1) + \sum_{i=2k+2}^{3k-1} P_k^{k-1}(i) \\
&= \sum_{i=k+2}^{2k-1} P_k^{k-1}(i) + 2P_k^{k-1}(2k+1) + P_k^{k-1}(k+1) + \sum_{i=2k+2}^{3k-1} P_k^{k-1}(i) \\
&\vdots \\
&= 2P_k^{k-1}(3k-1) + P_k^{k-1}(2k-1) \;=\; P_k^{k-1}(3k),
\end{aligned} \tag{6.6}$$

where the final reduction of S results from alternatively removing terms indexed by the lower bounds of each of the summations and then applying recursion (6.1). Thus we have $S_{3k+1} = S + 3P_k^{k-1}(3k) = 4P_k^{k-1}(3k)$, proving the base case.
Now, by the induction hypothesis we have

$$\sum_{i=0}^{2k+1} P_k^{k-1}(n-1+i) \;=\; 4P_k^{k-1}(n+2k-1), \tag{6.7}$$

and by (6.2) we have

$$\begin{aligned}
\sum_{i=0}^{2k+1} P_k^{k-1}(n-1+i) &= \sum_{i=0}^{n+2k} P_k^{k-1}(i) - \sum_{i=0}^{n-2} P_k^{k-1}(i) \\
&= \frac{1}{2}\left(\sum_{i=0}^{k} P_k^{k-1}(n+2k+1-i) - \sum_{i=0}^{k} P_k^{k-1}(n-1-i)\right).
\end{aligned} \tag{6.8}$$

Combining (6.7) and (6.8) we get

$$\sum_{i=0}^{k} P_k^{k-1}(n+2k+1-i) - \sum_{i=0}^{k} P_k^{k-1}(n-1-i) \;=\; 8P_k^{k-1}(n+2k-1). \tag{6.9}$$

Furthermore, we have

$$\begin{aligned}
\sum_{i=0}^{2k+1} P_k^{k-1}(n+i) &= \frac{1}{2}\left(\sum_{i=0}^{k} P_k^{k-1}(n+2k+2-i) - \sum_{i=0}^{k} P_k^{k-1}(n-i)\right) \\
&= \frac{1}{2}\big(P_k^{k-1}(n+2k+2) - P_k^{k-1}(n+k+1) + 8P_k^{k-1}(n+2k-1) \\
&\quad + P_k^{k-1}(n-k-1) - P_k^{k-1}(n)\big) \\
&= \frac{1}{2}\left(2P_k^{k-1}(n+2k+1) + 8P_k^{k-1}(n+2k-1) - 2P_k^{k-1}(n-1)\right) \\
&= \frac{1}{2}\big(2P_k^{k-1}(n+2k+1) - 2P_k^{k-1}(n+k) + 2P_k^{k-1}(n+k) \\
&\quad - 2P_k^{k-1}(n-1) + 8P_k^{k-1}(n+2k-1)\big) \\
&= \frac{1}{2}\left(4P_k^{k-1}(n+2k) + 4P_k^{k-1}(n+k-1) + 8P_k^{k-1}(n+2k-1)\right) \\
&= 4P_k^{k-1}(n+2k). \qquad (6.10)
\end{aligned}$$

We obtain the last equation by using

$$P_k^{k-1}(n+k-1) + 2P_k^{k-1}(n+2k-1) = P_k^{k-1}(n+2k). \qquad (6.11)$$

Now, consider

$$\sum_{i=0}^{2k+1} P_k^{k-1}(n+i) = \frac{1}{2}\left(\underbrace{\sum_{i=0}^{k} P_k^{k-1}(n+2k+2-i)}_{s_1\ (6.13)} - \underbrace{\sum_{i=0}^{k} P_k^{k-1}(n-i)}_{s_2\ (6.14)}\right). \qquad (6.12)$$

We have the following explicit formulas for s_1 and s_2:

$$s_1 = P_k^{k-1}(n+2k+2) - P_k^{k-1}(n+k+1) + \sum_{i=0}^{k} P_k^{k-1}(n+2k+1-i), \qquad (6.13)$$

$$s_2 = P_k^{k-1}(n) - P_k^{k-1}(n-k-1) + \sum_{i=0}^{k} P_k^{k-1}(n-i-1). \qquad (6.14)$$

We now note that the RHS of 6.13 and 6.14 are particularly amenable to manipulation, and therefore turn our attention towards $\frac{1}{2}(s_1 - s_2)$.

Thus, we have

$$\begin{aligned}
\frac{1}{2}(s_1 - s_2) &= \frac{1}{2}\big(P_k^{k-1}(n+2k+2) - P_k^{k-1}(n+k+1) + 8P_k^{k-1}(n+2k-1) \\
&+ \ P_k^{k-1}(n-k-1) - P_k^{k-1}(n)\big) \\
&= \frac{1}{2}\left(2P_k^{k-1}(n+2k+1) + 8P_k^{k-1}(n+2k-1) - 2P_k^{k-1}(n-1)\right) \\
&= \frac{1}{2}\big(2P_k^{k-1}(n+2k+1) - 2P_k^{k-1}(n+k) + 2P_k^{k-1}(n+k) \\
&- 2P_k^{k-1}(n-1) + 8P_k^{k-1}(n+2k-1)\big) \\
&= \frac{1}{2}\left(4P_k^{k-1}(n+2k) + 4P_k^{k-1}(n+k-1) + 8P_k^{k-1}(n+2k-1)\right) \\
&= 4P_k^{k-1}(n+2k). \qquad (6.15)
\end{aligned}$$

□

Setting $k = 1$ in Theorem 6.3, we obtain the following corollary.

Corollary 6.3. *The sum of any four consecutive Pell numbers is a four times the third Pell number:*

$$\sum_{i=0}^{3} P(n+i) = 4P(n+2). \qquad (6.16)$$

The partial sum formula of Pell numbers along with the identity

$$P(n+k) + (-1)^k P(n-k) = Q(k)P(n), \qquad k \in \mathbb{N} \cup \{0\} \qquad (6.17)$$

proven in Lemma 2.1, can be used to give an alternate proof of Theorem 1.6. Although we have a similar partial sum formula for P_k^{k-1}, there is no obvious way to extend this partial sum to a general property of adding consecutive generalized Pell numbers to get a multiple of another generalized Pell number for arbitrary $k > 1$. For $k > 1$, we haven't been able to find $N \neq 2k+2$ such that the sum of N consecutive generalized Pell-(k, i) numbers is an integer multiple of another generalized Pell-(k, i), suggesting the following conjecture.

Conjecture 1. *Fix any integer $N > 0$. There exists an integer $C(N)$ such that for every n there exists an integer index $j(n; N)$ such that the following equation holds*

$$\sum_{i=0}^{N} P_k^i(n+i) = C(N) \cdot P_k^i\left(j(n;N)\right)$$

if and only if $N = 2k + 2$.

Theorem 6.4. *For $0 \le i \le k-1$ we have*

$$\sum_{j=0}^{2k+1} P_k^i(n+j) = 4P_k^i(n+2k). \qquad (6.18)$$

Proof. In [Ki, §2, Corollary 2] they prove for $n > k$ that

$$P_k^{k-1-j}(n) \;=\; P_k^{k-1}(n) + \sum_{i=0}^{j-1} P_k^{k-1}(n-k+i). \tag{6.19}$$

This along with Theorem 6.3 gives the result. $\square$

6.3 Sum of Odd Number of Consecutive Terms

The same argument given for the classical Pell numbers (4.2) can be generalized for the sequence $\{P_k^{k-1}(n)\}_{n\geq 0}$ where k is an odd natural number. This is because from [Ki, §2,Theorem 2] we have the following equality:

$$\begin{pmatrix} 2 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^n =$$

$$\begin{pmatrix} P_k^{k-1}(n+p+1) & P_k^{k-1}(n+1) & \cdots & P_k^{k-1}(n+p-1) & P_k^{k-1}(n+p) \\ P_k^{k-1}(n+p) & P_k^{k-1}(n) & \cdots & P_k^{k-1}(n+p-2) & P_k^{k-1}(n+p-1) \\ P_k^{k-1}(n+p-1) & P_k^{k-1}(n-1) & \cdots & P_k^{k-1}(n+p-3) & P_k^{k-1}(n+p-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_k^{k-1}(n+1) & P_k^{k-1}(n-p+1) & \cdots & P_k^{k-1}(n-1) & P_k^{k-1}(n) \end{pmatrix}.$$

Since

$$\det \begin{pmatrix} 2 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \;=\; (-1)^k \;=\; -1,$$

we obtain the following theorem.

Theorem 6.5. *Fix any integer $N > 0$. There is no integer $C(N)$ such that for every n there exists an integer index $j(n;N)$ such that the following equation holds:*

$$\sum_{i=0}^{2N} P_k^{k-1}(n+i) \;=\; C(N) \cdot P_k^{k-1}\left(j(n;N)\right).$$

Proof. The proof follows from the discussion above and from the proof of the odd case for the classical Pell numbers. $\square$

We now prove the following stronger theorem for the generalized Pell sequence.

Theorem 6.6. *Fix any odd integer $N > 0$, and even integer $k \geq 0$. Suppose that there exists an integer $C(N)$ such that for every n there exists an integer index $j(n; N; k)$ such that the following equation holds:*

$$\sum_{i=0}^{N-1} P_k^{k-1}(n+i) \;=\; C(N) P_k^{k-1}(j(n;N;k)).$$

Then,

i) $2 \nmid C(N)$ *and*

ii) $N > 2k+2$.

Proof. i) By induction we know that $a_n := P_k^{k-1}(n) \pmod 2$ is of the form

$$a_n \;=\; \begin{cases} 1, & \text{if } (k+1) \mid (n+1) \\ 0, & \text{otherwise.} \end{cases}$$

Let $N = q(2k+2)+r$. Since N is odd, we have $1 \leq r \leq 2k+1$. Now take any n such that $n \geq k+1$ and $(k+1) \mid n$. We now prove that $2 \nmid C(N)$. Define

$$\begin{cases} S_{n,N} \;:=\; \sum_{i=-r+1}^{q(2k+2)} P_k^{k-1}(n+i) & \text{if } r \leq k+1, \\ \\ S_{n,N} \;:=\; \sum_{i=0}^{N-1} P_k^{k-1}(n+i) & \text{if } r > k+1. \end{cases}$$

Using the explicit form of a_n we conclude that $S_{n,N}$ is odd in both cases, and therefore $C(N)$ must be odd.

ii) By the same argument, we know that $\pi(p) \mid N$ where p is any prime dividing $C(N)$. Similarly, using the previous argument, we also know that $p > 3$. Now since

$$P_k^{k-1}(0) \;=\; P_k^{k-1}(1) \;=\; \cdots \;=\; P_k^{k-1}(k-1) \;=\; 0,$$

and $P_k^{k-1}(k) = 1$, we must have

$$P_k^{k-1}(\pi(p)) \;\equiv\; P_k^{k-1}(\pi(p)+1) \;\equiv\; \cdots \;\equiv\; P_k^{k-1}(k-1) \;\equiv\; 0 \pmod p.$$

However, we know that

$$P_k^{k-1}(k+i) \;=\; 2^i \text{ for } 1 \leq i \leq k,$$

and since $k \geq 2$, we have

$$\begin{aligned} P_k^{k-1}(2k+1) &= 2^{k+1}+1 \\ P_k^{k-1}(2k+2) &= 2^{k+2}+4 \\ P_k^{k-1}(2k+3) &= 2^{k+3}+12. \end{aligned}$$

Now as $p > 2$, we know that $\pi(p) > 2k$. We notice that if $\pi(p) = 2k+1$, then $p \mid 2^{k+1} + 1$ and $p \mid 2^{k+2} + 4$ which implies $p \mid 2$: a contradiction. Similarly, if $p = 2k+2$, then $p \mid 2^{k+2} + 4$ and $p \mid 2^{k+3} + 12$ which implies $p \mid 4$ which is also a contradiction. Therefore, $\pi(p) > 2k+2$ which implies $N > 2k+2$.

□

6.4 Tilings and Generalized Pell Sequence

In [BSP] the authors proved certain properties related to the Pell numbers using tilings of an $n \times 1$ board. We generalized some of the properties for $P_k^{k-1}(n)$. Let us first define a sequence $(p_{k,n})_{n \geq 0}$ such that

$$p_{k,n} \ := \ P_k^{k-1}(n+k). \tag{6.20}$$

It is not difficult to see that $p_{k,n}$ counts the number of tilings of an $n \times 1$ board using black 1×1 squares, white 1×1 squares and grey $(k+1) \times 1$ polyominoes.

Theorem 6.7. *We have*

$$p_{k,(k+1)n+r+1} \ = \ \begin{cases} 2\sum_{m=0}^{n} p_{k,m(k+1)+r}, & 0 \leq r \ < \ k \\ 2\sum_{m=0}^{n} p_{k,m(k+1)+r} + 1, & r \ = \ k. \end{cases} \tag{6.21}$$

Proof. Firstly, assume that $r < k$. Now, consider the tiling of a $[(k+1)n + r + 1] \times 1$ board, with the cells on the board numbered from left to right 1 to $(k+1)n+r+1$. Let t be the location of the last 1×1 cell in the tiling. Black or white squares cannot cover the cells to the right of t, so they must be covered by $(k+1) \times 1$ polyominoes. Therefore, t is of the form $(k+1)m + r + 1$. In this case, the number of tilings of the board is $2p_{k,mk}$ (accounting for the fact that cell t can be covered by either black or white 1×1 squares), proving the identity.

Now, let us assume that $r = k$. We can still cover the board with black and white squares as well as grey polyominoes as we discussed in the previous case, yielding $2\sum\limits_{m=0}^{n} p_{k,m(k+1)+r}$ tilings of the board. However, since the length of the board is now $(k+1)(n+1)$, it is possible the board can be covered without black and white squares altogether. We add this new case to the total number of tilings, proving the second identity. □

Note that (6.2) also follows from this result. An alternate proof using matrices is given in [Ki, §4, Theorem 19], which can be generalized further.

Definition 6.8. *Define the sequence* $\{f_k(n)\}$ *as follows:*

$$f_k(n) \;:=\; af_k(n-1) + bf_k(n-k-1) \qquad a, b \in \mathbb{N}$$

with $f_k(0) = f_k(1) = \cdots = f_k(k-1) = 0$ *and* $f_k(k) = 1$ *where* $k \in \mathbb{N}$.

Definition 6.9. *Define the sequence* $\{p_{k,n}\}$ *as follows:*

$$p_{k,n} \;:=\; f_k(n+k) \text{ for } n \in \mathbb{N} \cup \{0\}.$$

Theorem 6.10.

$$p_{k,(k+1)n+r+1} \;=\; \begin{cases} a \sum\limits_{m=0}^{n} b^{n-m} p_{k,m(k+1)+r}, & 0 \le r \;<\; k \\ \\ a \sum\limits_{m=0}^{n} b^{n-m} p_{k,m(k+1)+r} + 1, & r = k \end{cases} \tag{6.22}$$

Proof. Analogous to the proof of Theorem 6.7. $\square$

6.5 Generalized Fibonacci sequence

Define the order-k generalized Fibonacci sequence by

$$f_k(n) \;:=\; \sum_{i=1}^{k} f_k(n-i) \tag{6.23}$$

with $f_k(1) = f_k(2) = \cdots = f_k(k-1) = 0$ and $f_k(k) = 1$.

Its generating matrix (see [KiTa]) is given by

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{6.24}$$

A similar argument to the generalized Pell case in Section 6.3 tells that the Pisano Period for $f_k(n)$ is even modulo n whenever $n > 2$ and k is even, and yields the following Theorem.

Theorem 6.11. *Let* $F_k(n)$ *denote the* n^{th} *order-*k *Fibonacci number where* k *is even. Fix any* $N > 0$. *There is no integer* $C(N)$ *such that for every* n *there exists an integer index* $j(n;N)$ *such that the following equation holds:*

$$\sum_{i=0}^{2N} F_k(n+i) \;=\; C(N) \cdot F_k(\,j(n;N)\,).$$

Proof. Let the sum of any $2N+1$ consecutive terms in the kth Fibonacci sequence be $C(N)$ times another integer in the Fibonacci Sequence. An argument similar to Theorem 4.2 rules out the cases $C(N) > 2$. Therefore, we just need to take care of the cases when $C(N) = 1, 2$.

Case 1: $C(N) = 1$.

By induction on $r > 2k+1$, we have

$$\sum_{n=0}^{r} f_k(n) \; < \; f_k(r+2), \tag{6.25}$$

which implies

$$\sum_{i=0}^{2N} f_k(n+i) \; \leq \; \sum_{i=0}^{n+2N} f_k(i) \; < \; f_k(n+2N+2).$$

From the definition of the order-k generalized Fibonacci sequence, for $2N > k+1$ we have

$$f_k(n+2N+1) \; < \; \sum_{i=0}^{2N} f_k(n+i),$$

and thus $C(N) \neq 1$.

Case 2: $C(N) = 2$.

Define

$$\lambda_k \; := \; \lim_{n\to\infty} \frac{f_k(n+1)}{f_k(n)}.$$

We now note that [KuSi, §11, Theorem 9] states $\lambda_k + \lambda_k^{-k} = 2$, which implies that $\lambda_k < 2$ and hence for $n > 2k+1$ we have $f_k(n+1)/f_k(n) < 2$. Applying (6.25) thus implies

$$\sum_{n=0}^{r} f_k(n) \; < \; f_k(r+2) \; < \; 2f_k(r+1). \tag{6.26}$$

Now since, $2N > k+1$, from the definition of the order-k generalized Fibonacci sequence we have

$$f_k(n+2N) \; < \; \sum_{i=0}^{2N-1} f_k(n+i) \quad \Longrightarrow \quad 2f_k(n+2N) \; < \; \sum_{i=0}^{2N} f_k(n+i).$$

Lastly, we have

$$2f_k(n+2N) \; < \; \sum_{i=0}^{2N} f_k(n+i) \; < \; \sum_{i=0}^{n+2N} f_k(i) \; < \; 2f_k(n+2N+1), \tag{6.27}$$

which implies $C(N) \neq 2$, completing the proof. $\square$

7 Other Second-Order Recurrence Relations

Below we generalize some of the earlier results and proofs to other second-order recurrence relations. We prove novel results regarding the partial sums of consecutive terms of generalized Pell-like second-order recursive sequences.

7.1 Sum of $4N+2$ Consecutive Terms

We again consider sequences $f(n)$ satisfying the recurrence relation

$$f(n) \ := \ rf(n-1) + f(n-2),$$

but now we choose $f(0) = 0$ and $f(1) = 1$ and $r \geq 2$.

Theorem 7.1. *Let f be as above, and fix any integer $N > 0$. There is no integer $C(N)$ such that for every n there exists an integer index $j(n;N)$ such that the following equation holds:*

$$\sum_{i=0}^{4N+1} f(n+i) \ = \ C(N) f(\, j(n;N)\,).$$

Proof. Define the sequence $g(n)$ by $g(0) = 2$, $g(1) = r$ and

$$g(n) \ := \ rg(n-1) + g(n-2). \tag{7.1}$$

Using induction on k, we find

$$f(n+k) + (-1)^k f(n-k) \ = \ g(k) f(n). \tag{7.2}$$

Using (6.22) we get

$$\sum_{k=0}^{n} f(n) \ = \ \frac{1}{r}\Big(f(n) + f(n+1) - 1\Big). \tag{7.3}$$

Therefore, we have

$$\begin{aligned}
\sum_{k=0}^{4N+1} f(n+k) \ &= \ \sum_{k=0}^{n+4N+1} f(k) - \sum_{k=0}^{n-1} f(k) \\
&= \ \frac{1}{r}\Big[f(n+4N+1) + f(n+4N+2) - f(n-1) + f(n)\Big] \\
&= \ \frac{1}{r}\Big[g(2N+1) f(n+2N+1) + g(2N+1) f(n+2N)\Big] \\
&= \ g(2N+1) \frac{f(n+2N+1) + f(n+2N)}{r}.
\end{aligned}$$

Further by induction on N we know that $\dfrac{g(2N+1)}{r} \in \mathbb{N}$.

Now suppose the sum of $4N+2$ terms is a fixed multiple of another term. Then for some $t_1 \geq t_2 \in \mathbb{N}$, the following equations hold:

$$\begin{aligned} sf(t_1) &= g(2N+1)\frac{f(n+2N+2)+f(n+2N+1)}{r} \\ sf(t_2) &= g(2N+1)\frac{f(n+2N+1)+f(n+2N)}{r}. \end{aligned} \tag{7.4}$$

Dividing both sides yields

$$\frac{f(n+2N+1)+f(n+2N+2)}{f(n+2N)+f(n+2N+1)} = \frac{f(t_1)}{f(t_2)}. \tag{7.5}$$

Now for m a positive integer let

$$T_m := f(m)+f(m-1).$$

Then T_m satisfies the following recurrence:

$$\begin{aligned} rT_{m-1}+T_{m-2} &= rf(m-1)+rf(m-2)+f(m-2)+f(m-3) \\ &= f(m)+f(m-1) \\ &= T_m. \end{aligned}$$

Using this recursion and induction it follows that

$$r < \frac{T_{m+1}}{T_m} \leq \frac{3r}{2}, \tag{7.6}$$

which implies that $t_1 > t_2$. If $t_1 \geq t_2+2$ then

$$\begin{aligned} \frac{f(t_2+2)}{f(t_2)} &= \frac{rf(t_2+1)+f(t_2)}{f(t_2)} \\ &= \frac{r^2 f(t_2)+rf(t_2-1)}{f(t_2)}+1 \\ &> r^2+1, \end{aligned} \tag{7.7}$$

which leads to a contradiction as $r \geq 2 \implies r^2 > 3r/2$. Thus we must have $t_1 = t_2+1$.

Now we know that

$$\begin{aligned} f(n+2N+2) &< f_{n+2N+1}+f_{n+2N+2} \\ &= \frac{rsf_{t_1}}{g_{2N+1}} < f_{n+2N+3}, \end{aligned} \tag{7.8}$$

and therefore $c = rs/g_{2N+1}$ cannot possibly equal 1. Lastly, we note that

$$\begin{aligned} \gcd(T_{m+1},T_m) &= \gcd(rT_m+T_{m-1},T_m) \\ &= \gcd(T_m,T_{m-1}). \end{aligned} \tag{7.9}$$

Applying induction proves that this gcd is 1. The same argument shows that $\gcd(f_{m+1}, f_m) = 1$, but this contradicts the following statements:

$$sf(t_2+1) = \frac{g(2N+1)}{r} T_{n+2N+2}, \tag{7.10}$$

$$sf(t_2) = \frac{g(2N+1)}{r} T_{n+2N+1} \tag{7.11}$$

$$\text{and } c > 1, \tag{7.12}$$

which completes our proof. □

Note that in the above proof the result does not hold for $r = 1$, which is the Fibonacci sequence (see Theorem 5.2). Recall that the sum of $4N+2$ consecutive Fibonacci numbers is a fixed integer multiple of another Fibonacci number (see Theorem 5.1).

7.2 Sum of Odd Number of Consecutive Terms

Let a be any integer, $x, y \geq 0$ and y be odd. Define $\{f(n)\}$ to be the sequence following the recurrence relation

$$f(n) := 2af(n-1) + f(n-2)$$

with $f(0) = 2x$, $f(1) = y$ and $4axy + 4x^2 - y^2 = -1$.

Theorem 7.2. *Let f be as above and fix any $N > 0$. There is no integer $C(N)$ such that for every n there exists an integer index $j(n;N)$ such that the following equation holds:*

$$\sum_{i=0}^{N-1} f(n+i) = C(N)f(\,j(n;N)\,).$$

Proof. Assume that

$$\sum_{i=0}^{N-1} f(n+i) = C(N)f(j(n;N))$$

where $C(N) > 1$. We now prove the following lemma, which allows us to look at the distribution of residues modulo primes.

Lemma 7.3. *There exists no prime p such that $p \mid f(n)$ for all $n \geq 0$.*

Proof. We prove the lemma by contradiction, and divide the proof into two cases: $p = 2$ and $p > 2$.

Case I: $p = 2$**.**

Since $p = 2$, and $p \mid f(n)$ for all $n \geq 0$, this implies that $p \mid f(1) = y$, which is odd, which immediately leads to a contradiction. Hence, $p \neq 2$.

Case II: $p > 2$**.**

Since $p \mid 2x$ and $p \mid y$, this implies that $x, y \equiv 0 \pmod p$. Let $x = k_1 p$ and $y = k_2 p$ for some $k_1, k_2 \in \mathbb{N}^2$. We now employ the following manipulations:

$$\begin{aligned}
4axy + 4x^2 &= y^2 - 1 \\
\implies 4a(k_1p)(k_2p) + 4(k_1p)^2 &= (k_2p)^2 - 1 \\
\implies 4ak_1k_2p^2 + 4k_1^2p^2 &= k_2^2p^2 - 1 \\
\implies p^2(4ak_1k_2 + 4k_1^2) &= k_2^2p^2 - 1 \\
\implies 4ak_1k_2 + 4k_1^2 &= k_2^2 - \frac{1}{p^2}.
\end{aligned}$$

Note that the LHS is an integer, whereas the RHS is not. This results in a contradiction, which completes our proof. $\square$

Now, an argument of similar flavor to Theorem 4.2 tells us that the sequence is periodic modulo n for any natural number n with a period $\pi(n) \geq 2$. Therefore, if we can prove $\pi(n)$ is even then the rest of the argument in Theorem 4.2 also follows. The given condition implies $\pi(2) = 2$, therefore, let us assume $n > 2$.

Define

$$\begin{aligned}
h(n) &:= 2ah(n-1) + h(n-2) \\
\text{with} \quad h(0) &= 0, \ h(1) \ = \ 1.
\end{aligned} \tag{7.13}$$

Furthermore, let $\pi_h(n)$ be the period of $h(n)$ modulo n. A simple proof by induction yields

$$\begin{pmatrix} 2a & 1 \\ 1 & 0 \end{pmatrix}^{n+1} = \begin{pmatrix} h(n+2) & h(n+1) \\ h(n+1) & h(n) \end{pmatrix},$$

and then the same argument as for the Pell numbers (Theorem 4.2) gives

$$\begin{pmatrix} 2a & 1 \\ 1 & 0 \end{pmatrix}^{\pi_h(n)} = I_n \mod n. \tag{7.14}$$

Also, induction on n gives

$$\begin{pmatrix} 2a & 1 \\ 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} 2ay + 2x & y \\ y & 2x \end{pmatrix} = \begin{pmatrix} f(n+2) & f(n+1) \\ f(n+1) & f(n) \end{pmatrix},$$

and

$$\det \begin{pmatrix} 2ay + 2x & y \\ y & 2x \end{pmatrix} = 4axy + 4x^2 - y^2 = -1 \neq 0.$$

We note that $4axy + 4x^2 - y^2 = \pm 1$, but since y is odd, we can write $y = 2k+1$, and quickly realize that $4axy + 4x^2 = (2k+1)^2 + 1$ has no solutions. Therefore, we only consider $4axy + 4x^2 - y^2 = -1$.

We now have

$$\begin{aligned}\begin{pmatrix}2a & 1\\ 1 & 0\end{pmatrix}^{\pi(n)}\begin{pmatrix}2ay+2x & y\\ y & 2x\end{pmatrix} &= \begin{pmatrix}2ay+2x & y\\ y & 2x\end{pmatrix}\\ \implies \begin{pmatrix}2a & 1\\ 1 & 0\end{pmatrix}^{\pi(n)} &= I_n \text{ over } \mathbb{Z}/n\mathbb{Z}.\end{aligned} \tag{7.15}$$

Now (7.14) and (7.15) imply that $\pi(n) = \pi_h(n)$. Similarly, Theorem 4.2 tells us that $\pi_h(n)$ is even. Lastly we note that when $C(N) = 1$ then $x, y \geq 0$ means that the terms of the sequence are non-negative, which leads to the following inequality via induction when y is odd.

$$f(n+N-1) \;<\; \sum_{i=0}^{N-1} f(n+i) \;<\; f(n+N). \tag{7.16}$$

This results in a contradiction for $N > 1$, which completes our proof. □

A Appendix

The computational experiments for the paper were carried out in the Wolfram and Python Languages. The GitHub repository can be accessed from https://github.com/navvye/Polymath-Pell-Numbers.

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