AN IDENTITY FOR SUMS OF POLYLOGARITHM FUNCTIONS

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ABSTRACT. We derive an identity for certain linear combinations of polylogarithm functions with negative exponents, which implies relations for linear combinations of Eulerian numbers. The coefficients of our linear combinations are related to expanding moments of Satake parameters of holomorphic cuspidal newforms in terms of the moments of the corresponding Fourier coefficients, which has applications in analyzing lower order terms in the behavior of zeros of L-functions near the central point.

1. INTRODUCTION

The polylogarithm function $\mathrm{Li}_s(x)$ is

$$\mathrm{Li}_s(x) \ = \ \sum_{k=1}^{\infty} k^{-s} x^k. \tag{1.1}$$

If s is a negative integer, say $s = -r$, then the polylogarithm function converges for $|x| < 1$ and equals

$$\mathrm{Li}_{-r}(x) \ = \ \frac{\sum_{j=0}^{r} \left\langle {r \atop j} \right\rangle x^{r-j}}{(1-x)^{r+1}}, \tag{1.2}$$

where the $\left\langle {r \atop j} \right\rangle$ are the Eulerian numbers. The Eulerian number $\left\langle {r \atop j} \right\rangle$ is the number of permutations of $\{1, \dots, r\}$ with j permutation ascents. One has

$$\left\langle {r \atop j} \right\rangle \ = \ \sum_{\ell=0}^{j+1} (-1)^{\ell} \binom{r+1}{\ell} (j - \ell + 1)^r. \tag{1.3}$$

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We record $\mathrm{Li}_{-r}(x)$ for some r:

$$\begin{aligned}
\mathrm{Li}_0(x) &= \frac{x}{1-x} \\
\mathrm{Li}_{-1}(x) &= \frac{x}{(1-x)^2} \\
\mathrm{Li}_{-2}(x) &= \frac{x^2+x}{(1-x)^3} \\
\mathrm{Li}_{-3}(x) &= \frac{x^3+4x^2+x}{(1-x)^4} \\
\mathrm{Li}_{-4}(x) &= \frac{x^4+11x^3+11x^2+x}{(1-x)^5} \\
\mathrm{Li}_{-5}(x) &= \frac{x^5+26x^4+66x^3+26x^2+x}{(1-x)^6}.
\end{aligned} \tag{1.4}$$

From (1.2) we immediately deduce that, when s is a negative integer, $\mathrm{Li}_s(x)$ is a rational function whose denominator is $(1-x)^{|s|}$. Thus an appropriate integer linear combination of $\mathrm{Li}_0(x)$ through $\mathrm{Li}_{-n}(x)$ should be a simple rational function. In particular, we prove

Theorem 1.1. *Let $a_{\ell,i}$ be the coefficient of k^i in $\prod_{j=0}^{\ell-1}(k^2-j^2)$, and let $b_{\ell,i}$ be the coefficient of k^i in $(2k+1)\prod_{j=0}^{\ell-1}(k-j)(k+1+j)$. Then for $|x|<1$ and $\ell \geq 1$ we have*

$$\begin{aligned}
a_{\ell,2\ell}\mathrm{Li}_{-2\ell}(x) + \cdots + a_{\ell,0}\mathrm{Li}_0(x) &= \frac{(2\ell)!}{2}\,\frac{x^\ell(1+x)}{(1-x)^{2\ell+1}} \\
b_{\ell,2\ell+1}\mathrm{Li}_{-2\ell-1}(x) + \cdots + b_{\ell,0}\mathrm{Li}_0(x) &= (2\ell+1)!\,\frac{x^\ell(1+x)}{(1-x)^{2\ell+2}}.
\end{aligned} \tag{1.5}$$

We prove Theorem 1.1 in §2. While Theorem 1.1 only applies to linear combinations of polylogarithm functions with s a negative integer, it is interesting to see how certain special combinations equal a very simple rational function. One application is to use this result to deduce relations among the Eulerian numbers (possibly by replacing x with $1-x$ when expanding); another is of course to write $\mathrm{Li}_{-n}(x)$ in terms of $\mathrm{Li}_{-n+1}(x)$ through $\mathrm{Li}_0(x)$. The coefficients $a_{\ell,i}$ and $b_{\ell,i}$ which occur in our linear combinations also arise in expressions involving the Fourier coefficients of cuspidal newforms. We describe this connection in greater detail in §3; these expansions are related to understanding the lower order terms in the behavior of zeros of L-functions of cuspidal newforms near the central point. (see [Mil3] for a complete analysis).

2. Proof of Theorem 1.1

Before proving Theorem 1.1 we introduce some useful expressions.

Definition 2.1. *Let*

$$c_{2\ell} = \prod_{j=0}^{\ell-1}(\ell^2-j^2) = (2\ell)!/2, \quad c_{2\ell+1} = (2\ell+1)\prod_{j=0}^{\ell-1}(\ell-j)(\ell+1+j) = (2\ell+1)!. \tag{2.1}$$

Define constants $c_{m,r}$ as follows: $c_{m,r}=0$ if $m \not\equiv r \bmod 2$, and

(1) *for r even,* $c_{0,0} = 0$, $c_{2k,0} = (-1)^k 2$ *for* $k \geq 1$, *and for* $1 \leq \ell \leq k$ *set*

$$c_{2k,2\ell} \ = \ \frac{(-1)^{k+\ell}}{c_{2\ell}} \prod_{j=0}^{\ell-1} (k^2 - j^2) \ = \ \frac{(-1)^{k+\ell}}{c_{2\ell}} \frac{k \cdot (k+\ell-1)!}{(k-\ell)!}; \tag{2.2}$$

(2) *for r odd and* $0 \leq \ell \leq k$ *set*

$$c_{2k+1,2\ell+1} \ = \ \frac{(-1)^{k+\ell}}{c_{2\ell+1}} (2k+1) \prod_{j=0}^{\ell-1} (k-j)(k+1+j) \ = \ \frac{(-1)^{k+\ell}(2k+1)}{c_{2\ell+1}} \frac{(k+\ell)!}{(k-\ell)!}. \tag{2.3}$$

Note $c_{m,r} = 0$ *if* $m < r$. *Finally, set* $B_r(x) = \sum_{m=0}^{\infty} c_{m,r} (-x)^{m/2}$ *for* $|x| < 1$. *Thus for* $r = 2\ell \geq 2$ *we have*

$$B_{2\ell}(x) \ = \ \sum_{m=0}^{\infty} c_{m,2\ell} (-x)^{m/2} \ = \ \sum_{k=1}^{\infty} \left(\frac{(-1)^{k+\ell}}{c_{2\ell}} \prod_{j=0}^{\ell-1} (k^2 - j^2) \right) (-x)^k. \tag{2.4}$$

Immediately from the definition of c_r we have

$$c_{2\ell-1} \ = \ \frac{c_{2\ell}}{\ell} \ = \ \frac{c_{2\ell+1}}{2\ell(2\ell+1)}, \tag{2.5}$$

as well as

$$c_{2\ell+2} \ = \ (2\ell+2)(2\ell+1) c_{2\ell}, \quad c_{2\ell+3} \ = \ (2\ell+3)(2\ell+2) c_{2\ell+1}. \tag{2.6}$$

While the definition of the $c_{m,r}$'s above may seem arbitrary, these expressions arise in a very natural manner in number theory. See [Mil3] for applications of these coefficients in understanding the behavior of zeros of $\mathrm{GL}(2)$ L-functions; we briefly discuss some of these relations in §3.

Proof of Theorem 1.1. We first consider the case of $r = 2\ell$ even. We proceed by induction. We claim that

$$B_{2\ell}(x) \quad = \quad \sum_{k=1}^{\infty} \left(\frac{(-1)^{k+\ell}}{c_{2\ell}} \prod_{j=0}^{\ell-1} (k^2 - j^2) \right) (-x)^k \ = \ (-1)^{\ell} \frac{x^{\ell}(1+x)}{(1-x)^{2\ell+1}} \tag{2.7}$$

for all ℓ.

We consider the basis case, when $\ell = 1$. Thus we must show for $|x| < 1$ that $B_2(x) = -x(1+x)/(1-x)^3$. As $r = 2$, the only non-zero terms are when $m = 2k > 0$ is even. As $c_2 = 2$ and $c_{2k,2} = (-1)^{k+1} k^2$ for $k \geq 1$, we find that

$$B_2(x) \quad = \quad \sum_{k=1}^{\infty} (-1)^{k+1} k^2 (-x)^{2k/2} \ = \ -\sum_{k=1}^{\infty} k^2 x^k \ = \ -\mathrm{Li}_{-2}(x) \ = \ -\frac{x(1+x)}{(1-x)^3}, \tag{2.8}$$

which completes the proof of the basis step. For the inductive step, we assume

$$\sum_{k=1}^{\infty} \left(\frac{(-1)^{k+\ell}}{c_{2\ell}} \prod_{j=0}^{\ell-1} (k^2 - j^2) \right) (-x)^k \ = \ (-1)^{\ell} \frac{x^{\ell}(1+x)}{(1-x)^{2\ell+1}}, \tag{2.9}$$

and we must show the above holds with ℓ replaced by $\ell + 1$. We apply the differential operator

$$\left(x \frac{d}{dx} \right)^2 - \ell^2 \tag{2.10}$$

to both sides of (2.9). After canceling the minus signs we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\left(c_{2\ell}^{-1}\prod_{j=0}^{\ell-1}(k^2-j^2)\right)(k^2-\ell^2)x^k &= \left(\left(x\frac{d}{dx}\right)^2-\ell^2\right)\left(\frac{x^\ell(1+x)}{(1-x)^{2\ell+1}}\right)\\ \sum_{k=1}^{\infty}c_{2\ell}^{-1}\left(\prod_{j=0}^{\ell}(k^2-j^2)\right)x^k &= (2\ell+2)(2\ell+1)\frac{x^{\ell+1}(1+x)}{(1-x)^{2(\ell+1)+1}}\\ \sum_{k=1}^{\infty}c_{2(\ell+1)}^{-1}\left(\prod_{j=0}^{\ell+1-1}(k^2-j^2)\right)x^k &= \frac{x^{\ell+1}(1+x)}{(1-x)^{2(\ell+1)+1}}, \qquad (2.11)\end{aligned}$$

where the last line follows from (2.6), which says $c_{2\ell+2}=(2\ell+2)(2\ell+1)c_{2\ell}$. Thus (2.7) is true for all ℓ.

As we have defined $a_{\ell,i}$ to be the coefficient of k^i in $\prod_{j=0}^{\ell-1}(k^2-j^2)$, (2.7) becomes

$$\sum_{k=1}^{\infty}\sum_{i=0}^{2\ell}a_{\ell,i}\,k^i\,x^k \;=\; c_{2\ell}\frac{x^\ell(1+x)}{(1-x)^{2\ell+1}}. \qquad (2.12)$$

The proof of Theorem 1.1 for r even is completed by noting that the left hand side above is just

$$a_{\ell,2\ell}\mathrm{Li}_{-2\ell}(x)+\cdots+a_{\ell,0}\mathrm{Li}_0(x). \qquad (2.13)$$

The proof for $r=2\ell+1$ odd proceeds similarly, the only significant difference is that now we apply the operator

$$\left(x\frac{d}{dx}\right)^2+\left(x\frac{d}{dx}\right)-\ell(\ell+1), \qquad (2.14)$$

which will bring down a factor of $(k-\ell)(k+1-\ell)$. □

3. CONNECTIONS WITH NUMBER THEORY

We now describe how our polylogarithm identity can be used to analyze zeros of L-functions near the central point. Katz and Sarnak [KaSa] conjecture that, in the limit as the conductors tend to infinity, the behavior of the normalized zeros near the central point agree with the $N\to\infty$ scaling limit of the normalized eigenvalues near 1 of a subgroup of $U(N)$ ($N\times N$ unitary matrices); see [DM, FI, Gü, HR, HM, ILS, KaSa, Mil1, Ro, Rub, Yo] for many examples. While the main terms for many families are the same as the conductors tend to infinity, a more careful analysis of the explicit formula allows us to isolate family dependent lower order terms.

Our coefficients $c_{m,r}$ are related to writing the moments of Satake parameters of certain $\mathrm{GL}(2)$ L-functions in terms of the moments of their Fourier coefficients, which we briefly review. Let $H_k^\star(N)$ be the set of all holomorphic cuspidal newforms of weight k and level N; see [Iw2] for more details. Each $f\in H_k^\star(N)$ has a Fourier expansion

$$f(z) \;=\; \sum_{n=1}^{\infty}a_f(n)e(nz). \qquad (3.1)$$

Let $\lambda_f(n) = a_f(n)n^{-(k-1)/2}$. These coefficients satisfy multiplicative relations, and $|\lambda_f(p)| \le 2$. The L-function associated to f is

$$L(s,f) \ = \ \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \ = \ \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right)^{-1}, \tag{3.2}$$

where χ_0 is the principal character with modulus N. We write

$$\lambda_f(p) \ = \ \alpha_f(p) + \beta_f(p). \tag{3.3}$$

For $p \nmid N$, $\alpha_f(p)\beta_f(p) = 1$ and $|\alpha_f(p)| = 1$. If $p|N$ we take $\alpha_f(p) = \lambda_f(p)$ and $\beta_f(p) = 0$. Letting

$$L_\infty(s,f) \ = \ \left(\frac{2^k}{8\pi}\right)^{1/2} \left(\frac{\sqrt{N}}{\pi}\right)^s \Gamma\left(\frac{s}{2} + \frac{k-1}{4}\right) \Gamma\left(\frac{s}{2} + \frac{k+1}{4}\right) \tag{3.4}$$

denote the local factor at infinity, the completed L-function is

$$\Lambda(s,f) \ = \ L_\infty(s)L(s,f) \ = \ \epsilon_f \Lambda(1-s,f), \quad \epsilon_f = \pm 1. \tag{3.5}$$

The zeros of L-functions often encode arithmetic information, and their behavior is well-modeled by random matrix theory [CFKRS, KaSa, KeSn3]. The main tool in analyzing the behavior of these zeros is through an explicit formula, which relates sums of a test function at these zeros to sums of the Fourier transform of the test function at the primes, weighted by factors such as $\alpha_f(p)^m + \beta_f(p)^m$. For example, if ϕ is an even Schwartz function, $\widehat{\phi}$ its Fourier transform, and $\frac{1}{2} + i\gamma_f$ denotes a typical zero of $\Lambda(s,f)$ for $f \in H_k^\star(N)$ (the Generalized Riemann Hypothesis asserts each $\gamma_f \in \mathbb{R}$), then the explicit formula is

$$\begin{aligned} &\frac{1}{|H_k^*(N)|} \sum_{f \in H_k^*(N)} \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log N}{2\pi}\right) \\ &= \ \frac{A(\phi)}{\log N} + \frac{1}{|H_k^*(N)|} \sum_{f \in H_k^*(N)} \sum_{m=1}^{\infty} \sum_p \frac{\alpha_f(p)^m + \beta_f(p)^m}{p^{m/2}} \frac{\log p}{\log N} \widehat{\phi}\left(m \frac{\log p}{\log N}\right); \end{aligned} \tag{3.6}$$

see [ILS, Mil3] for details and a definition of $A(\phi)$. Similar expansions hold for other families of L-functions. Information about the distribution of zeros in a family of L-functions (the left hand side above) is obtained by analyzing the prime sums weighted by the moments of the Satake parameters (on the right hand side). Thus it is important to be able to evaluate quantities such as

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \left(\alpha_f(p)^m + \beta_f(p)^m\right) \tag{3.7}$$

for various families of L-functions.

For some problems it is convenient to rewrite $\alpha_f(p)^m + \beta_f(p)^m$ in terms of a polynomial in $\lambda_f(p)$. This replaces moments of the Satake parameters $\alpha_f(p)$ and $\beta_f(p)$ with moments of the Fourier coefficients $\lambda_f(p)$, and for many problems the Fourier coefficients are more tractable; we give two examples.

First, the p^{th} coefficient of the L-function of the elliptic curve $y^2 = x^3 + Ax + B$ is $p^{-1/2} \sum_{x \bmod p} \left(\frac{x^3+Ax+B}{p}\right)$; here $\left(\frac{x}{p}\right)$ is the Legendre symbol, which is 1 if x is a non-zero square modulo p, 0 if $x \equiv 0 \bmod p$, and -1 otherwise. Our sum equals the number of solutions to $y^2 \equiv x^3 + Ax + B \bmod p$, and thus these sums can be analyzed by using results on sums of Legendre symbols (see for example [ALM, Mil2]).

Second, the Petersson formula (see Corollary 2.10, Equation (2.58) of [ILS]) yields, for $m, n > 1$ relatively prime to the level N,

$$\frac{1}{W_R(\mathcal{F})} \sum_{f \in H_k^*(N)} w_R(f)\lambda_f(m)\lambda_f(n) \ = \ \delta_{mn} \ + \ O\left((mn)^{1/4}\frac{\log 2mnN}{k^{5/6}N}\right), \tag{3.8}$$

where $\delta_{mn} = 1$ if $m = n$ and 0 otherwise. Here the $w_R(f)$ are the harmonic weights

$$w_R(f) \ = \ \zeta_N(2)/Z(1,f) \ = \ \zeta(2)/L(1, \mathrm{sym}^2 f). \tag{3.9}$$

They are mildly varying, with (see [Iw1, HL])

$$N^{-1-\epsilon} \ \ll_k \ \omega_R(f) \ \ll_k \ N^{-1+\epsilon}; \tag{3.10}$$

if we allow ineffective constants we can replace N^ϵ with $\log N$ for N large.

We can now see why our polylogarithm identity is useful. Using $\alpha_f(p)+\beta_f(p) = \lambda_p$, $\alpha_f(p)\beta_f(p) = 1$ and $|\alpha_f(p)| = |\beta_f(p)| = 1$, we find that

$$\begin{aligned}
\alpha_f(p) \ + \beta_f(p) \ &= \ \lambda_f(p) \\
\alpha_f(p)^2 + \beta_f(p)^2 \ &= \ \lambda_f(p)^2 - 2 \\
\alpha_f(p)^3 + \beta_f(p)^3 \ &= \ \lambda_f(p)^3 - 3\lambda_f(p) \\
\alpha_f(p)^4 + \beta_f(p)^4 \ &= \ \lambda_f(p)^4 - 4\lambda_f(p)^2 + 2 \\
\alpha_f(p)^5 + \beta_f(p)^5 \ &= \ \lambda_f(p)^5 - 5\lambda_f(p)^3 + 5\lambda_f(p) \\
\alpha_f(p)^6 + \beta_f(p)^6 \ &= \ \lambda_f(p)^6 - 6\lambda_f(p)^4 + 9\lambda_f(p)^2 - 2 \\
\alpha_f(p)^7 + \beta_f(p)^7 \ &= \ \lambda_f(p)^7 - 7\lambda_f(p)^5 + 14\lambda_f(p)^3 - 7\lambda_f(p) \\
\alpha_f(p)^8 + \beta_f(p)^8 \ &= \ \lambda_f(p)^8 - 8\lambda_f(p)^6 + 20\lambda_f(p)^4 - 16\lambda_f(p)^2 + 2.
\end{aligned} \tag{3.11}$$

Writing $\alpha_f(p)^m + \beta_f(p)^m$ as a polynomial in $\lambda_f(p)$, we find that

$$\alpha_f(p)^m + \beta_f(p)^m \ = \ \sum_{\substack{r=0 \\ r \equiv m \bmod 2}}^{m} c_{m,r}\lambda_f(p)^r, \tag{3.12}$$

where the $c_{m,r}$ are our coefficients from Definition 2.1. A key ingredient in the proof is noting that

(1) $c_{2k,2\ell} = c_{2k-1,2\ell-1} - c_{2k-2,2\ell}$ if $\ell \in \{1, \dots, k-1\}$ and $k \geq 2$;
(2) $c_{2k+1,2\ell+1} = c_{2k,2\ell} - c_{2k-1,2\ell+1}$ if $\ell < k$.

We briefly describe the application of our identity, ignoring the book-keeping needed to deal with $m \leq 2$. From the explicit formula (3.6), we see we must understand sums such as

$$\sum_p \sum_{m=3}^{\infty} \frac{1}{W_R(\mathcal{F})} \sum_{f \in \mathcal{F}} w_R(f) \frac{\alpha_f(p)^m + \beta_f(p)^m}{p^{m/2}} \frac{\log p}{\log R}\, \widehat{\phi}\left(\frac{\log p}{\log R}\right), \tag{3.13}$$

where $\mathcal{F}$ is a family of cuspidal newforms and $W_R(\mathcal{F}) = \sum_{f \in \mathcal{F}} w_R(f)$ (a simple Taylor series shows there is negligible contribution in replacing $\widehat{\phi}(m \log p/\log R)$ with $\widehat{\phi}(\log p/\log R)$). As the sums of powers of the Satake parameters are polynomials in $\lambda_f(p)$, we may rewrite this as

$$\sum_p \sum_{m=3}^{\infty} \sum_{\substack{r=0 \\ r\equiv m \bmod 2}}^{m} \frac{c_{m,r}A_{r,\mathcal{F}}(p)}{p^{m/2}} \frac{\log p}{\log R}\, \widehat{\phi}\left(\frac{\log p}{\log R}\right), \tag{3.14}$$

where $A_{r,\mathcal{F}}(p)$ is the r^{th} moment of $\lambda_f(p)$ in the family $\mathcal{F}$:

$$A_{r,\mathcal{F}}(p) \;=\; \frac{1}{W_R(\mathcal{F})} \sum_{\substack{f\in\mathcal{F} \\ f\in S(p)}} w_R(f)\lambda_f(p)^r. \tag{3.15}$$

We interchange the m and r sums (which is straightforward for $p \geq 11$, and follows by Abel summation for $p \leq 7$) and then apply our polylogarithm identity (Theorem 1.1) to rewrite the sum as

$$\sum_p \sum_{r=0}^{\infty} \frac{A_{r,\mathcal{F}}(p)p^{r/2}(p-1)\log p}{(p+1)^{r+1}\log R}\, \widehat{\phi}\left(\frac{\log p}{\log R}\right). \tag{3.16}$$

For many families we either know or conjecture a distribution for the (weighted) Fourier coefficients. If this were the case, then we could replace the $A_{r,\mathcal{F}}(p)$ with the r^{th} moment. In many applications (for example, using the Petersson formula for families of cuspidal newforms of fixed weight and square-free level tending to infinity) we know the moments up to a negligible correction (the distribution is often known or conjectured to be Sato-Tate, unless we are looking at families of elliptic curves with complex multiplication, where the distribution is known and slightly more complicated). Simple algebra yields

Lemma 3.1. *Assume for $r \geq 3$ that*

$$A_{r,\mathcal{F}}(p) \;=\; \begin{cases} M_\ell + O\left(\frac{1}{\log^2 R}\right) & \text{if } r = 2\ell \\ 0 & \text{otherwise,} \end{cases} \tag{3.17}$$

and that there is a nice function g_M such that

$$g_M(x) \;=\; M_2x^2 + M_3x^3 + \cdots \;=\; \sum_{\ell=2}^{\infty} M_\ell\, x^\ell. \tag{3.18}$$

Then the contribution from the $r \geq 3$ terms in the explicit formula is

$$-\frac{2\widehat{\phi}(0)}{\log R} \sum_p g_M\left(\frac{p}{(p+1)^2}\right) \cdot \frac{(p-1)\log p}{p+1} + O\left(\frac{1}{\log^3 R}\right). \tag{3.19}$$

Thus we can use our polylogarithm identity to rewrite the sums arising in the explicit formula in a very compact way which emphasizes properties of the known or conjectured distribution of the Fourier coefficients. One application of this is in analyzing the behavior of the zeros of L-functions near the central point. Many investigations have shown that, for numerous families, as the conductors tend to infinity the behavior of these zeros is the same as the $N \to \infty$ scaling limit of eigenvalues near 1 of subgroups of $U(N)$.

Most of these studies only examine the main term, showing agreement in the limit with random matrix theory (the scaling limits of eigenvalues of $U(N)$). In particular, all one-parameter families of elliptic curves over $\mathbb{Q}(T)$ with the same rank and same limiting distribution of signs of functional equation have the same main term for the behavior of their zeros. What is unsatisfying about this is that the arithmetic of the families is not seen; this is remedied, however, by studying the lower order terms in the 1-level density. There we *do* break the universality and see arithmetic dependent terms. In particular, our formula shows that we have different answers for families of elliptic curves with and without complex multiplication (as these two cases have different densities for the Fourier coefficients).

These lower order differences, which reflect the arithmetic structure of the family, are quite important. While the behavior of many properties of zeros of L-functions of height T are well-modeled by the $N \to \infty$ scaling limits of eigenvalues of a classical compact group, better agreement (taking into account lower order terms) is given by studying matrices of size $N = (\log T)/2\pi$ (see [KeSn1, KeSn2, KeSn3]). Recently it has been observed that even better agreement is obtained by replacing N with N_{eff}, where N_{eff} is chosen so that the main and first lower order terms match (see [BBLM, DHKMS]). Thus one consequence of our work is in deriving a tractable formula to identify the lower order correction terms, which results in an improved model for the behavior of the zeros.

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