

Spectral Properties of Dense Barabási–Albert Graphs

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Abstract

Preferential attachment graphs model networks whose growth produces highly uneven degree distributions, describing many real-world systems. Their adjacency spectra are important because they allow graph-theoretic questions to be studied through the eigenvalues of matrices. We analyze the adjacency matrix of a dense Barabási–Albert (B–A) multigraph, where the number of edges added at each step is proportional to the final number of vertices. First, we compute the large- n limit of the expected adjacency matrix and show that it is described by a rank-one limiting kernel, viewed as a continuous analogue of the adjacency matrix. After centering and scaling, the fluctuations form a random matrix with a computable variance profile. Using the quadratic vector equation approach, we derive the limiting bulk spectral distribution. We also determine the asymptotic location of the leading eigenvalue generated by the rank-one mean component.

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1 Introduction

1.1 Background and related work

Spectral properties of adjacency matrices of graphs, such as eigenvalue distributions and leading eigenvalues/eigenvectors, provide compact summaries of key graph properties, including degree-related information, walk counts, and connectivity patterns [8].

The Barabási–Albert (B–A) model [3] is a canonical random graph model for networks with heavy-tailed degree distributions. In this model, vertices arrive sequentially and attach to existing vertices with probability proportional to their current degree, producing the well-known scale-free degree distribution observed in many real-world networks, such as the network of scientific collaborations [4], the functional connections in the human brain [12], and the network of Wikipedia hyperlinks [9]. In its classical form, the process begins with a complete graph of m_0 vertices, and each new vertex attaches to $m \leq m_0$ existing vertices with probability proportional to their degrees, as described in [25, Section 3.1.1].

Empirical and theoretical investigations of spectral properties of scale-free networks have shown significant differences from classical random matrix behavior. Numerical and heuristic studies indicate that preferential attachment networks exhibit non-semicircular spectral densities and structural features strongly influenced by degree heterogeneity [11, 13].

Rigorous spectral results for preferential attachment graphs have largely focused on the sparse regime, where each vertex attaches to a fixed number of existing vertices. In this setting, high-degree vertices play a dominant role in determining extreme eigenvalues, and leading eigenvectors often localize around hubs [14]. Similar results exist for a wide variety of random trees [6].

The dense preferential attachment regime, in which the number of edges added at each step grows proportionally with the network size, is qualitatively different from the sparse setting. In the dense regime, adjacency matrices become dense weighted objects, and graph limit theory provides a natural framework for describing their asymptotic structure. Dense graph sequences can converge to graphons, which are measurable symmetric kernels encoding limiting edge densities [19].

Several works have established convergence of dense preferential attachment models to deterministic limits and have connected these limits to random graph constructions [18, 22, 23]. These results describe the limiting mean structure of dense preferential attachment networks. However, deterministic kernel convergence alone does not determine the limiting eigenvalue distribution of centered adjacency matrices, which typically requires random matrix analysis of fluctuations around the mean [26]. For such matrices, the limiting spectral distribution is determined by a self-consistent resolvent equation rather than by the semicircle law [15, 17]. The quadratic vector equation (QVE) framework developed by Ajanki, Erdős, and Krüger provides a general method for determining limiting spectral densities of Wigner-type matrices with arbitrary variance profiles under suitable moment and irreducibility conditions [2].

Additionally, finite-rank perturbation theory shows that adding a low-rank deterministic component to a random matrix can produce isolated eigenvalues separated from the bulk spectrum. Such outliers are well understood in the context of spiked random matrix models and can often be characterized explicitly [5].

Motivated by these developments, we study the spectral properties of dense preferential attachment multigraphs in a regime where the number of attachments grows linearly with the network size. We show that the expected adjacency matrix converges to an explicit kernel that generates a rank-one component. After centering and scaling, the fluctuation matrix forms a Wigner-type ensemble with an explicit variance kernel suitable for analysis under the QVE framework. This approach results in a deterministic limiting bulk spectral distribution and an explicit asymptotic

description of the leading eigenvalue generated by the rank-one mean structure. Our results complement existing work on sparse preferential attachment spectra and connect dense preferential attachment graph limits with modern random matrix theory.

1.2 Model and notation

Let n be the final number of vertices. Fix rational $c \in (0, \frac{1}{2})$, and consider only values of n for which $m := cn$ is an integer. All limits are taken as $n \rightarrow \infty$ along this admissible sequence. Time t indexes vertices: vertex t arrives at time t , except for the first $m + 1$ vertices.

Initialization: At time $t_0 := m + 1$, the graph is a simple clique K_{m+1} , a fully connected graph of $m + 1$ vertices. Let E_t be the number of edges at time t , and $D_t := 2E_t$ the total degree.

Preferential attachment: For each $t > t_0$ such that $t \leq n$, we add a new vertex to the graph – a total of $n - m - 1$ new vertices added to the initial clique. In the limit, the number of new vertices is on the order of m since $n - m - 1 \approx m/c$. Each new vertex t adds m edges independently to the existing vertices $\{1, \dots, t - 1\}$; the probability that a single edge connects vertex t to a vertex $j < t$ is

$$q_{t \rightarrow j} = \frac{d_j(t-1)}{\sum_{v=1}^{t-1} d_v(t-1)} = \frac{d_j(t-1)}{D_{t-1}},$$

where $d_j(s)$ is the degree of j at time s . Multiple edges for the same pair of vertices are allowed.

Total degree: At initialization, $E_{t_0} = \binom{m+1}{2}$, hence

$$D_{t_0} = m(m+1).$$

Each step $t > t_0$ adds exactly m edges, so

$$D_t = D_{t_0} + 2m(t - t_0) = 2mt - m(m+1) \tag{1.1}$$

for all $t \geq t_0$.

1.3 Main results

Theorem 1.1 (Limiting bulk spectrum). *Let M be the adjacency matrix of the Barabási–Albert multigraph with $m = cn$ and fixed $c \in (0, \frac{1}{2})$, and define the centered and rescaled matrix*

$$Y = \frac{M - \mathbb{E}[M]}{\sqrt{m}}.$$

Then the empirical spectral distribution of Y converges weakly in probability,¹ as $n \rightarrow \infty$, to a deterministic probability measure μ_c .

The Stieltjes transform

$$G_c(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\mu_c(\lambda), \quad z \in \mathbb{H},$$

¹A sequence of probability measures μ_n on $\mathbb{R}$ converges weakly in probability to a probability measure μ if, for all bounded continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and all $\varepsilon > 0$,

$$\mathbb{P} \left[\left| \int f d\mu_n - \int f d\mu \right| > \varepsilon \right] \rightarrow 0$$

as $n \rightarrow \infty$.

is given by

$$G_c(z) = cm_A(z) - \int_c^1 \frac{1}{z + u(x)K(z)} dx, \quad u(x) = (2x - c)^{-1/2},$$

where the pair $(K(z), L(z))$ is the unique solution in $\mathbb{H}^2$ of the system

$$\begin{aligned} K(z) &= L(z) - \frac{\sqrt{c}}{z + L(z)/\sqrt{c}}, \\ L(z) &= - \int_c^1 \frac{u(y)}{z + u(y)K(z)} dy, \end{aligned} \quad z \in \mathbb{H}, \quad (1.2)$$

and

$$m_A(z) = -\frac{1}{z + L(z)/\sqrt{c}}.$$

The limiting density ρ_c is recovered by Stieltjes inversion:

$$\rho_c(\lambda) = \frac{1}{\pi} \lim_{\eta \downarrow 0} \Im G_c(\lambda + i\eta).$$

Theorem 1.2 (Leading eigenvalue). *Let M be the adjacency matrix of the Barabási–Albert multi-graph with $m = cn$ and fixed rational $c \in (0, \frac{1}{2})$ such that n and m are integers. Then the largest eigenvalue of the scaled adjacency matrix satisfies*

$$\lambda_{\max} \left(\frac{M}{\sqrt{m}} \right) = \frac{\alpha(c)}{\sqrt{c}} \sqrt{n} + o_{\mathbb{P}}(\sqrt{n}), \quad n \rightarrow \infty,$$

with

$$\alpha(c) = c + \frac{c}{2} \log \frac{2-c}{c}.$$

See Figure 1 for an empirical verification of the bulk law obtained from numerically inverting the Stieltjes transform from Theorem 1.1 and the outlier eigenvalue asymptotics from Theorem 1.2.

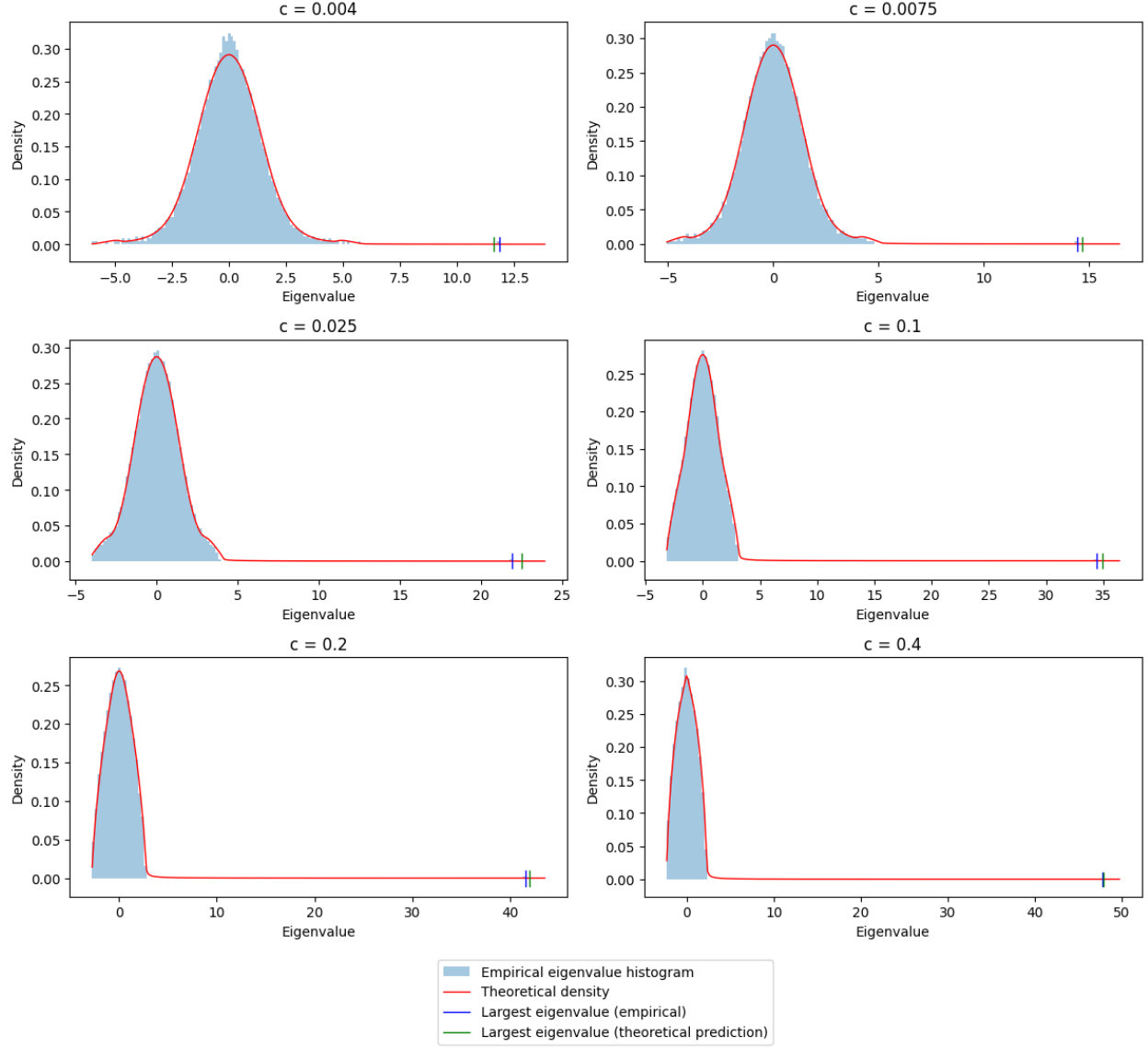


Figure 1: Spectrum of random B-A multigraphs. As c grows, the spectrum concentrates around the origin, approaching the spectrum of the fully connected graph. The shapes of the densities match previous results [13, Figure 1], which report triangle-like central parts and decaying tails.

2 Finding the expected adjacency matrix.

2.1 Expected degrees of individual vertices.

Fix a vertex $j \leq t$. Define the random variable $\Delta d_j(t)$ to be the number of new edges from t to j at time t . Then,

$$\Delta d_j(t) \sim \text{Binomial}(m, q_{t \rightarrow j}), \quad \mathbb{E}[\Delta d_j(t)] = m q_{t \rightarrow j} = m \frac{d_j(t-1)}{D_{t-1}}.$$

Using (1.1), this gives a linear recursion for $\mu_j(t) := \mathbb{E}[d_j(t)]$. For $t > j$,

$$\mu_j(t) = \mu_j(t-1) \left(1 + \frac{m}{2m(t-1) - m(m+1)} \right) = \mu_j(t-1) \left(1 + \frac{1}{2(t-1) - (m+1)} \right). \quad (2.1)$$

At the time of arrival, each vertex has degree m , so $\mu_j(j) = m$.

Thus, with $s_0 := \max\{j, t_0\}$,

$$\mu_j(t) = m \prod_{s=s_0+1}^t \left(1 + \frac{1}{2(s-1) - (m+1)} \right) = m \prod_{s=s_0+1}^t \frac{s - \frac{m+2}{2}}{s - \frac{m+3}{2}}. \quad (2.2)$$

This product can be represented using Gamma-functions,

$$\mu_j(t) = m \frac{\Gamma(t - \frac{m}{2})}{\Gamma(s_0 - \frac{m}{2})} \frac{\Gamma(s_0 - \frac{m+1}{2})}{\Gamma(t - \frac{m+1}{2})}, \quad t \geq s_0, \quad (2.3)$$

which follows from Lemma 2.1.

Lemma 2.1. *Let $r_0 < r_1$ be integers and $a, b \in \mathbb{C}$ such that the Gamma terms below are defined. Then*

$$\prod_{r=r_0+1}^{r_1} \frac{r-a}{r-b} = \frac{\Gamma(r_1+1-a) \Gamma(r_0+1-b)}{\Gamma(r_0+1-a) \Gamma(r_1+1-b)}.$$

Proof. Using the standard result $\Gamma(z+1) = z\Gamma(z)$,

$$\begin{aligned} \Gamma(r_1+1-a) &= (r_1-a) \Gamma(r_1-a) \\ &= (r_1-a)(r_1-1-a) \Gamma(r_1-1-a) \\ &\vdots \\ &= \left(\prod_{r=r_0+1}^{r_1} (r-a) \right) \Gamma(r_0+1-a). \end{aligned}$$

Hence

$$\prod_{r=r_0+1}^{r_1} (r-a) = \frac{\Gamma(r_1+1-a)}{\Gamma(r_0+1-a)}.$$

The same argument applies to b and taking their ratio proves the lemma. $\square$

2.2 Expectation of the edge count $\mathbb{E}[M]$

For $k > j$, all edges between k and j are created at time k . Conditional on the state of the graph at time $k-1$, $\mathcal{F}_{k-1}$, we have

$$M_{kj} \mid \mathcal{F}_{k-1} \sim \text{Binomial} \left(m, \frac{d_j(k-1)}{D_{k-1}} \right),$$

since in the multigraph model, the m edges from k are placed independently, each choosing j with probability $d_j(k-1)/D_{k-1}$, making M_{kj} a binomial random variable.

Hence, using (1.1) and (2.3),

$$\mathbb{E}[M_{kj}] = m \frac{\mu_j(k-1)}{D_{k-1}} = \frac{m^2}{2m(k-1) - m(m+1)} \frac{\Gamma(k-1 - \frac{m}{2})}{\Gamma(s_0 - \frac{m}{2})} \frac{\Gamma(s_0 - \frac{m+1}{2})}{\Gamma(k-1 - \frac{m+1}{2})}, \quad (2.4)$$

where $s_0 := \max\{j, t_0\}$ and $t_0 = m+1$ is the initialization time.

For $1 \leq k, j \leq t_0$, the initial clique K_{m+1} yields $\mathbb{E}[M_{kj}] = \mathbf{1}_{k \neq j}$ and no later edges affect those pairs.

3 Kernel Approach to $\mathbb{E}[M]$

3.1 Definition of a kernel

Following Lovász [19], who introduced the concept to spectral graph theory, we define a **kernel** to be any bounded, symmetric, measurable function

$$W : [0, 1]^2 \rightarrow \mathbb{R},$$

which we can use to generalize adjacency matrices of graphs.

Given a finite graph H with adjacency matrix $A = (A_{kj})$, the associated step-function kernel W_H is defined by partitioning $[0, 1]$ into $|V(H)|$ equal intervals $I_1, \dots, I_{|V(H)|}$ and setting $W_H(x, y) = A_{kj}$ whenever $x \in I_k, y \in I_j$.

A graphon is a kernel whose values lie in the interval $[0, 1]$, corresponding to simple graphs [19]. In general, kernels serve as the limit objects for sequences of weighted or multigraph adjacency matrices.

3.2 The kernel of $\mathbb{E}[M]$

Now we calculate the kernel for the expected adjacency matrix of dense B–A multigraphs.

Theorem 3.1. *Let $c \in (0, \frac{1}{2})$ and $m = cn$. For indices $k > j$ with $k/n \rightarrow y \in (0, 1]$ and $j/n \rightarrow x \in (0, 1]$, set $t_0 = m + 1$ and $s_0 = \max\{j, t_0\}$. Then*

$$\mathbb{E}[M_{kj}] = m \frac{\mu_j(k-1)}{D_{k-1}} \rightarrow \frac{c}{\sqrt{(2 \max\{x, c\} - c)(2y - c)}}, \quad n \rightarrow \infty,$$

where $D_t = 2mt - m(m+1)$ and $\mu_j(\cdot) = \mathbb{E}[d_j(\cdot)]$. Moreover, $\mathbb{E}[M_{kj}] = O(1)$ uniformly in j, k, n .

Proof. We first prove the uniform bound. Set

$$s_0 := \max\{j, t_0\}.$$

From the degree evolution, for $t > s_0$,

$$\mathbb{E}_{\Delta d_j(t)}[d_j(t) \mid \mathcal{F}_{t-1}] = \left(1 + \frac{m}{D_{t-1}}\right) d_j(t-1).$$

Writing

$$\mu_j(t) := \mathbb{E}_{\mathcal{F}_t}[d_j(t)],$$

we get

$$\begin{aligned} \mu_j(t) &= \mathbb{E}_{\mathcal{F}_t}[d_j(t)] \\ &= \mathbb{E}_{\mathcal{F}_{t-1}}[\mathbb{E}_{\Delta d_j(t)}[d_j(t) \mid \mathcal{F}_{t-1}]] \\ &= \mathbb{E}_{\mathcal{F}_{t-1}}\left[\left(1 + \frac{m}{D_{t-1}}\right) d_j(t-1)\right] \\ &= \left(1 + \frac{m}{D_{t-1}}\right) \mathbb{E}_{\mathcal{F}_{t-1}}[d_j(t-1)] \\ &= \left(1 + \frac{m}{D_{t-1}}\right) \mu_j(t-1). \end{aligned}$$

Since

$$D_{t-1} \geq D_{t_0} = m(m+1),$$

we have

$$\frac{m}{D_{t-1}} \leq \frac{1}{m+1} \leq \frac{C_1}{n}$$

for some constant $C_1 > 0$. Since $\mu_j(s_0) = m$, iterating gives

$$\begin{aligned} \mu_j(k-1) &\leq \left(1 + \frac{C_1}{n}\right) \mu_j(k-2) \\ &\leq \left(1 + \frac{C_1}{n}\right)^2 \mu_j(k-3) \\ &\leq \dots \\ &\leq \left(1 + \frac{C_1}{n}\right)^{k-1-s_0} \mu_j(s_0) \\ &\leq \left(1 + \frac{C_1}{n}\right)^n m \\ &\leq C_2 n \end{aligned}$$

for some constant $C_2 > 0$. Therefore

$$\mathbb{E}[M_{kj}] = m \frac{\mu_j(k-1)}{D_{k-1}} \leq m \frac{C_2 n}{m(m+1)} \leq C,$$

which proves the uniform bound.

From the Gamma-function representation (2.3),

$$\frac{\mu_j(k-1)}{n} = \frac{m}{n} \cdot \frac{\Gamma(k-1-\frac{m}{2})}{\Gamma(k-1-\frac{m+1}{2})} \cdot \frac{\Gamma(s_0-\frac{m+1}{2})}{\Gamma(s_0-\frac{m}{2})}.$$

Set $z_0 := s_0 - \frac{m+1}{2}$ and $z_1 := k-1-\frac{m+1}{2}$.

For fixed $a, b \in \mathbb{C}$, we know from [21] that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \rightarrow z^{a-b} \left(1 + O\left(\frac{1}{z}\right)\right), \quad z \rightarrow \infty.$$

Therefore we obtain

$$\frac{\Gamma(z_1 + \frac{1}{2})}{\Gamma(z_1)} \rightarrow z_1^{1/2} (1 + O(z_1^{-1})), \quad \frac{\Gamma(z_0)}{\Gamma(z_0 + \frac{1}{2})} \rightarrow z_0^{-1/2} (1 + O(z_0^{-1})).$$

With $m = cn$ and $k = \lfloor yn \rfloor$, we have $k - yn \in (-1, 0]$, hence

$$z_1 = k - 1 - \frac{m+1}{2} = n \left(y - \frac{c}{2}\right) + O(1).$$

Similarly, with $s_0 = \max\{j, t_0\}$, where $j = \lfloor xn \rfloor$ and $t_0 = m+1 = cn+1$, we have $j - xn \in (-1, 0]$, $s_0/n \rightarrow \max\{x, c\}$, and

$$z_0 = s_0 - \frac{m+1}{2} = n \left(\max\{x, c\} - \frac{c}{2}\right) + O(1).$$

Next, from $D_t = 2mt - m(m+1)$,

$$D_{k-1} = 2m(k-1) - m(m+1) = 2m\left(z_1 + \frac{m+1}{2}\right) - m(m+1) = 2mz_1.$$

Hence

$$\frac{D_{k-1}}{n^2} = \frac{2mz_1}{n^2} = 2\frac{m}{n}\frac{z_1}{n} = c(2y-c) + O(n^{-1}).$$

Finally, we derive

$$\begin{aligned} \mathbb{E}[M_{kj}] &= m \frac{\mu_j(k-1)}{D_{k-1}} = \frac{m}{n} \frac{\mu_j(k-1)}{n} \frac{n^2}{D_{k-1}} \\ &= c \left(c \sqrt{\frac{z_1}{z_0}} + O(n^{-1}) \right) \left(\frac{1}{c(2y-c)} + O(n^{-1}) \right) \\ &= c \frac{c \sqrt{z_1/z_0}}{c(2y-c)} + O(n^{-1}) = c \frac{\sqrt{z_1/z_0}}{2y-c} + O(n^{-1}) \\ &= c \frac{1}{2y-c} \left(\sqrt{\frac{n(y - \frac{c}{2}) + O(1)}{n(\max\{x, c\} - \frac{c}{2}) + O(1)}} + O(n^{-1}) \right) + O(n^{-1}) \\ &= c \frac{1}{2y-c} \left(\sqrt{\frac{y - \frac{c}{2} + O(n^{-1})}{\max\{x, c\} - \frac{c}{2} + O(n^{-1})}} + O(n^{-1}) \right) + O(n^{-1}) \\ &= \frac{c}{\sqrt{(2\max\{x, c\} - c)(2y-c)}} + O(n^{-1}) \\ &\rightarrow \frac{c}{\sqrt{(2\max\{x, c\} - c)(2y-c)}}, \quad n \rightarrow \infty. \end{aligned}$$

Solving for different cases results in $F : [0, 1]^2 \rightarrow [0, \infty)$, which corresponds to the expected edge count for a given vertex pair in the limit:

$$F(x, y) := \lim_{n \rightarrow \infty} \mathbb{E}[M_{[yn], [xn]}] = \begin{cases} 1, & x, y \in A, x \neq y, \\ \sqrt{\frac{c}{2y-c}}, & x \in A, y \in B, x \neq y, \\ \sqrt{\frac{c}{2x-c}}, & x \in B, y \in A, x \neq y, \\ \frac{c}{\sqrt{(2x-c)(2y-c)}}, & x, y \in B, x \neq y, \\ 0, & x = y; \end{cases} \quad (3.1)$$

see Figure 2. □

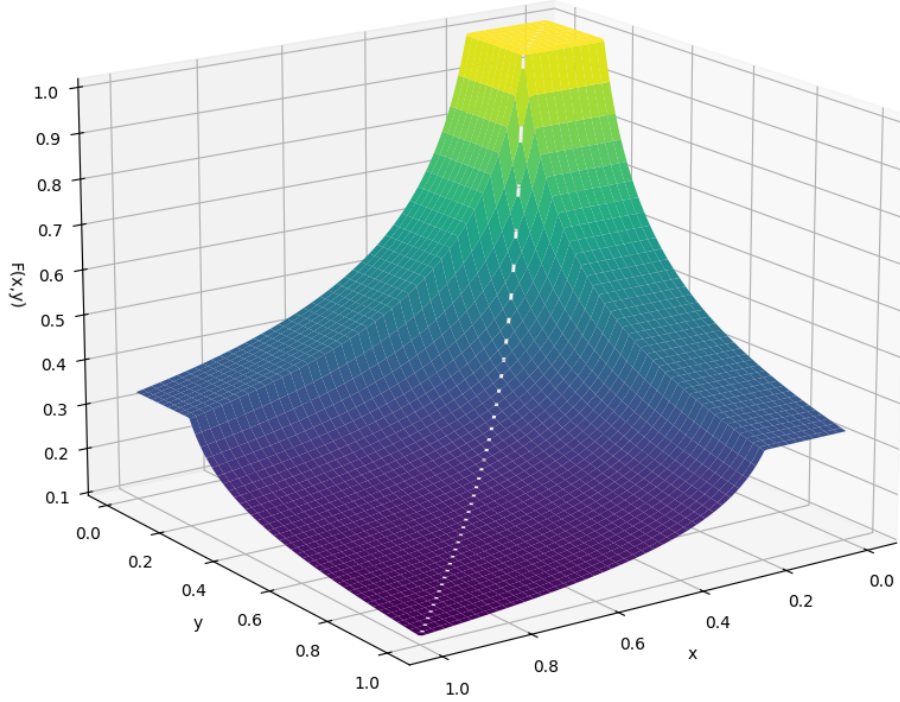


Figure 2: Plot of $F(x, y)$.

4 Bulk density

4.1 Centering and isolation of the bulk

Definition. Let M be the (symmetric) adjacency matrix of the B–A multigraph and set

$$Y := \frac{M - \mathbb{E}[M]}{\sqrt{m}}, \quad m = cn, \quad 0 < c < \frac{1}{2}.$$

We use Y to focus on the bulk of the spectrum.

Lemma 4.1 (Rank-one structure of $\mathbb{E}[M]$). *Let $A = [0, c]$, $B = (c, 1]$, and $u(t) = (2t - c)^{-1/2}$. Define the function $\phi : [0, 1] \rightarrow \mathbb{R}$ by*

$$\phi(x) := \mathbf{1}_{\{x \in A\}} + \sqrt{c} \mathbf{1}_{\{x \in B\}} u(x),$$

where $\mathbf{1}_{x \in S}$ is the indicator function defined as

$$\mathbf{1}_{x \in S} = \begin{cases} 1, & x \in S, \\ 0, & x \notin S, \end{cases}$$

for some set S . Then the limiting mean kernel F from (3.1) satisfies

$$F(x, y) = \phi(x) \phi(y).$$

Consequently, if one forms the $n \times n$ matrix

$$F_{kj}^{(n)} := F\left(\frac{k}{n}, \frac{j}{n}\right), \quad 1 \leq k, j \leq n,$$

then $F^{(n)}$ is rank one, since

$$F^{(n)} = v^{(n)} \left(v^{(n)}\right)^\top, \quad v_k^{(n)} := \phi\left(\frac{k}{n}\right).$$

In particular, by Theorem 3.1, the expectation matrix $\mathbb{E}[M]$ is asymptotically a rank-one matrix.

Proof. By definition of F and of ϕ ,

$$\phi(x) \phi(y) = \begin{cases} 1, & x, y \in A, \\ \sqrt{c} u(y), & x \in A, y \in B, \\ \sqrt{c} u(x), & x \in B, y \in A, \\ cu(x) u(y), & x, y \in B, \end{cases}$$

which matches $F(x, y)$ case-by-case (the diagonal $x = y$ is negligible for our purposes and can be set to 0). Hence

$$F(x, y) = \phi(x) \phi(y)$$

for all $(x, y) \in [0, 1]^2$.

Now define the vector $v^{(n)} \in \mathbb{R}^n$ by

$$v_k^{(n)} := \phi\left(\frac{k}{n}\right), \quad 1 \leq k \leq n.$$

Then

$$F\left(\frac{k}{n}, \frac{j}{n}\right) = \phi\left(\frac{k}{n}\right) \phi\left(\frac{j}{n}\right) = v_k^{(n)} v_j^{(n)},$$

so

$$F\left(\frac{k}{n}, \frac{j}{n}\right) = v^{(n)} \left(v^{(n)}\right)^\top,$$

and therefore $F^{(n)}$ has rank one.

Finally, Theorem 3.1 gives

$$\mathbb{E}[M_{kj}] = F\left(\frac{k}{n}, \frac{j}{n}\right) + O(n^{-1}),$$

so $\mathbb{E}[M]$ is asymptotically rank-one, with leading term $F\left(\frac{k}{n}, \frac{j}{n}\right) = v^{(n)} \left(v^{(n)}\right)^\top$. $\square$

Lemma 4.2 (From Tao [24]). *Let $\{X_n\}$ be a sequence of random Hermitian matrix ensembles such that $\left\{\mu_{\frac{1}{\sqrt{n}} X_n}\right\}$ converges almost surely to a limit μ . Let $\{Z_n\}$ be another sequence of random Hermitian matrix ensembles such that $\frac{1}{n} \text{rank}(Z_n)$ converges almost surely to zero. Then $\left\{\mu_{\frac{1}{\sqrt{n}}(X_n + Z_n)}\right\}$ converges almost surely to μ .*

Lemma 4.2 applies almost immediately to our case, where scaling by $\sqrt{n}$ is equivalent to scaling by $\sqrt{c}\sqrt{m}$, where $\sqrt{c}$ is constant, so Y and $M/\sqrt{m}$ have the same limiting spectral distribution. Since $\mathbb{E}[M]$ is rank-one by Lemma 4.1, the Corollary 4.3.9 from [16] applies: $M/\sqrt{m}$ has at most one outlier. Lemma 4.12 shows further why Y has no other outlier eigenvalues.

4.2 Poissonization

Now we use standard results (Lemma 4.3, Propositions 4.4, 4.5) to construct a matrix $\widetilde{M}$ whose entries are conditionally independent Poisson random variables.

Lemma 4.3 (Multinomial as conditional Poisson). *Let $X_1, \dots, X_k$ be independent random variables with $X_j \sim \text{Poisson}(\lambda_j)$ for $j = 1, \dots, k$, and set $\Lambda := \sum_{j=1}^k \lambda_j$. Then, for any $m \in \mathbb{N}$ and any $(x_1, \dots, x_k) \in \mathbb{N}^k$ satisfying $\sum_{j=1}^k x_j = m$,*

$$\mathbb{P} \left[X_1 = x_1, \dots, X_k = x_k \mid \sum_{j=1}^k X_j = m \right] = \frac{m!}{x_1! \cdots x_k!} \prod_{j=1}^k \left(\frac{\lambda_j}{\Lambda} \right)^{x_j}.$$

More concisely, conditional on $\sum_{j=1}^k X_j = m$, the random vector $(X_1, \dots, X_k)$ has the probability distribution described by Multinomial $\left(m; \frac{\lambda_1}{\Lambda}, \dots, \frac{\lambda_k}{\Lambda}\right)$.

Proposition 4.4 (Perturbation bound for Stieltjes transforms; Lu and Yao [20, Lemma 18]). *Let A, B be $n \times n$ Hermitian matrices, and let $z = x + i\eta \in \mathbb{C}$ with $\eta > 0$. Denote their resolvents by $G_A(z) = (A - zI)^{-1}$ and $G_B(z) = (B - zI)^{-1}$, and their Stieltjes transforms by*

$$s_A(z) = \frac{1}{n} \text{Tr } G_A(z), \quad s_B(z) = \frac{1}{n} \text{Tr } G_B(z).$$

Then

$$|s_A(z) - s_B(z)| \leq \frac{1}{(\Im z)^2} \cdot \frac{\|A - B\|_F}{\sqrt{n}}.$$

Proposition 4.5. *Let $S \sim \text{Poisson}(\mu)$. Then*

$$\mathbb{E}[|S - \mu|] \leq \sqrt{\mu}.$$

Proof. For any random variable X with mean μ , by the Cauchy-Schwarz inequality we have

$$\mathbb{E}[|X - \mu|] = \mathbb{E}[|X - \mu| \cdot 1] \leq \sqrt{\mathbb{E}[(X - \mu)^2]} \sqrt{\mathbb{E}[1^2]} = \sqrt{\text{Var}(X)}.$$

Since $\text{Var}(S) = \mu$ for a Poisson random variable, the claim follows. $\square$

Proposition 4.6. *Let $n \rightarrow \infty$ and $m = cn$ with fixed $c \in (0, \frac{1}{2})$. Let M denote the symmetric B - A multigraph adjacency matrix, and for each $k > t_0$ let*

$$\mathbf{d}(k-1) := (d_1(k-1), \dots, d_{k-1}(k-1)), \quad p_{kj} := \frac{d_j(k-1)}{D_{k-1}}, \quad j < k.$$

Conditionally on $\mathbf{d}(k-1)$, the row-vector $(M_{k1}, \dots, M_{k,k-1})$ is multinomial with m trials and probabilities $(p_{k1}, \dots, p_{k,k-1})$. Define the symmetric matrix $\widetilde{M}$ by

$$\widetilde{M}_{kj} \mid \mathbf{d}(k-1) \sim \text{Poisson}(\lambda_{kj}), \quad \lambda_{kj} := mp_{kj}, \quad j < k,$$

and set $\widetilde{M}_{jk} = \widetilde{M}_{kj}$, $\widetilde{M}_{kk} = M_{kk} = 0$.

Let

$$Y := \frac{M - \mathbb{E}[M]}{\sqrt{m}}, \quad \widetilde{Y} := \frac{\widetilde{M} - \mathbb{E}[\widetilde{M}]}{\sqrt{m}}.$$

Then, for each fixed $z \in \mathbb{H}$, as $n \rightarrow \infty$

$$\left| \frac{1}{n} \mathbb{E}[\text{Tr}(Y - zI)^{-1}] - \frac{1}{n} \mathbb{E}[\text{Tr}(\widetilde{Y} - zI)^{-1}] \right| = o(1).$$

Consequently, Y and $\widetilde{Y}$ have the same limiting spectral measure via Stieltjes inversion.

Proof. Fix $k > t_0$ and let $\mathbf{d}(k-1)$ denote the vector of degrees at time $k-1$. In the multigraph model, conditional on $\mathbf{d}(k-1)$ the m attachments of vertex k are i.i.d. draws to $\{1, \dots, k-1\}$ with probabilities

$$p_{kj} := \frac{d_j(k-1)}{D_{k-1}}, \quad \sum_{j=1}^{k-1} p_{kj} = 1,$$

hence

$$(M_{k1}, \dots, M_{k,k-1}) \mid \mathbf{d}(k-1) \sim \text{Multinomial}(m; p_{k1}, \dots, p_{k,k-1}).$$

By Lemma 4.3, there exist independent random variables

$$\widetilde{M}_{kj} \mid \mathbf{d}(k-1) \sim \text{Poisson}(\lambda_{kj}), \quad \lambda_{kj} := mp_{kj}, \quad j < k,$$

such that, with $S_k := \sum_{j < k} \widetilde{M}_{kj}$,

$$(\widetilde{M}_{k1}, \dots, \widetilde{M}_{k,k-1}) \mid (\mathbf{d}(k-1), S_k = m) \stackrel{d}{=} (M_{k1}, \dots, M_{k,k-1}) \mid \mathbf{d}(k-1).$$

Here $\stackrel{d}{=}$ denotes equality in distribution, that is, two random objects X and Y satisfy $X \stackrel{d}{=} Y$ if they have the same probability law.

Conditionally on $\mathbf{d}$, define $\overline{M}$ row-wise by

$$(\overline{M}_{k1}, \dots, \overline{M}_{k,k-1}) \stackrel{d}{=} (\widetilde{M}_{k1}, \dots, \widetilde{M}_{k,k-1}) \mid (S_k = m, \mathbf{d}(k-1)),$$

and set $\overline{M}_{jk} = \overline{M}_{kj}$, $\overline{M}_{kk} = 0$. Then $(\overline{M} \mid \mathbf{d}) \stackrel{d}{=} (M \mid \mathbf{d})$, and

$$\mathbb{E}[\overline{M}_{kj} \mid \mathbf{d}] = \mathbb{E}[\widetilde{M}_{kj} \mid \mathbf{d}] = mp_{kj}, \quad k \neq j.$$

We introduce $\overline{M}$ because it conditionally matches the original matrix in distribution, $(\overline{M} \mid \mathbf{d}) \stackrel{d}{=} (M \mid \mathbf{d})$, while being constructed from $\widetilde{M}$ in the same probability space, so that their difference is an explicit small perturbation we can bound.

Unconditionally, $\mathbb{E}[M] = \mathbb{E}[\overline{M}] = \mathbb{E}[\widetilde{M}]$ by the tower property.

Let $\overline{Y} := (\overline{M} - \mathbb{E}[\overline{M}]) / \sqrt{m}$ and $\widetilde{Y} := (\widetilde{M} - \mathbb{E}[\widetilde{M}]) / \sqrt{m}$. Since $\mathbb{E}[M] = \mathbb{E}[\widetilde{M}]$, we have

$$\mathbb{E} \left[\frac{1}{n} \text{Tr}(Y - zI)^{-1} \mid \mathbf{d} \right] = \mathbb{E} \left[\frac{1}{n} \text{Tr}(\overline{Y} - zI)^{-1} \mid \mathbf{d} \right].$$

Define the “edit” difference

$$\Delta := \overline{Y} - \widetilde{Y} = \frac{\overline{M} - \widetilde{M}}{\sqrt{m}}.$$

Denote $s_{\overline{Y}}(z) := \frac{1}{n} \text{Tr}(\overline{Y} - zI)^{-1}$ and $s_{\tilde{Y}}(z) := \frac{1}{n} \text{Tr}(\tilde{Y} - zI)^{-1}$. By the perturbation bound for Stieltjes transforms (Proposition 4.4),

$$|s_{\overline{Y}}(z) - s_{\tilde{Y}}(z)| \leq \frac{1}{(\Im z)^2} \cdot \frac{\|\Delta\|_F}{\sqrt{n}}. \quad (4.1)$$

We only have to bound $\|\Delta\|_F$. Let $K_k := S_k - m$. Enforcing the constraint $S_k = m$ in row k can be achieved by at most $|K_k|$ “edits”, where each changes a single off-diagonal pair $(k, j), (j, k)$ by ± 1 , which adjusts the row sum by ± 1 while preserving symmetry.

Under the scaling by $\sqrt{m}$, one “edit” contributes

$$\left(\frac{1}{\sqrt{m}}\right)^2 + \left(\frac{1}{\sqrt{m}}\right)^2 \leq \frac{2}{m}$$

to $\|\Delta\|_F^2$. Summing over all rows $k > t_0$ and their at most $|K_k|$ “edits” gives

$$\|\Delta\|_F^2 \leq \frac{2}{m} \sum_{k > t_0} |K_k|.$$

Taking expectations, we first note that $\sum_{j < k} \lambda_{kj} = m$ and that

$$S_k \mid \mathbf{d}(k-1) \sim \text{Poisson}(m) \implies S_k \sim \text{Poisson}(m),$$

since

$$\mathbb{P}(S_k = k) = \mathbb{E}[\mathbb{P}(S_k = k \mid \mathbf{d}(k-1))] = \mathbb{E}\left[e^{-m} \frac{m^k}{k!}\right] = e^{-m} \frac{m^k}{k!}.$$

With $K_k := S_k - m$ and applying Proposition 4.5,

$$\mathbb{E}[|K_k|] = \mathbb{E}[|S_k - m|] \leq \sqrt{m}.$$

Therefore

$$\mathbb{E}[\|\Delta\|_F^2] \leq \frac{2}{m} \sum_{k > t_0} \mathbb{E}[|K_k|] \leq \frac{2n}{\sqrt{m}} = \frac{2}{\sqrt{c}} \sqrt{n},$$

and thus $\mathbb{E}[\|\Delta\|_F] = O(n^{1/4})$. For fixed $z \in \mathbb{H}$, taking expectations in (4.1) results in

$$\mathbb{E}[|s_{\overline{Y}}(z) - s_{\tilde{Y}}(z)|] \leq \frac{1}{(\Im z)^2} \cdot \frac{\mathbb{E}[\|\Delta\|_F]}{\sqrt{n}} = O(n^{-1/4}) \rightarrow 0, \quad n \rightarrow \infty.$$

Since $Y \stackrel{d}{=} \overline{Y}$, we conclude

$$\mathbb{E}[|s_Y(z) - s_{\tilde{Y}}(z)|] = \mathbb{E}[|s_{\overline{Y}}(z) - s_{\tilde{Y}}(z)|] \rightarrow 0, \quad n \rightarrow \infty.$$

Moreover, from

$$\|\Delta\|_F^2 \leq \frac{2}{m} \sum_{k > t_0} |K_k|, \quad \mathbb{E}[\|\Delta\|_F^2] = O(\sqrt{n}),$$

we have

$$\mathbb{E}\left[\frac{\|\Delta\|_F^2}{n}\right] = O(n^{-1/2}),$$

and hence by Markov's inequality, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{\|\Delta\|_F}{\sqrt{n}} \geq \varepsilon\right) = \mathbb{P}\left(\frac{\|\Delta\|_F^2}{n} \geq \varepsilon^2\right) \leq \frac{1}{\varepsilon^2} \mathbb{E}\left[\frac{\|\Delta\|_F^2}{n}\right] = O\left(n^{-1/2}\right) \rightarrow 0, \quad n \rightarrow \infty.$$

Combining this with the perturbation bound (4.1) gives that, for every $\varepsilon > 0$,

$$\mathbb{P}\left(|s_{\bar{Y}}(z) - s_{\tilde{Y}}(z)| > \varepsilon\right) \rightarrow 0, \quad n \rightarrow \infty.$$

Since $Y \stackrel{d}{=} \bar{Y}$, it follows likewise that for every $\varepsilon > 0$,

$$\mathbb{P}\left(|s_Y(z) - s_{\tilde{Y}}(z)| > \varepsilon\right) \rightarrow 0, \quad n \rightarrow \infty.$$

By definition, $s_Y(z)$ and $s_{\tilde{Y}}(z)$ are the Stieltjes transforms of the empirical eigenvalue distributions of Y and $\tilde{Y}$, respectively. Since $\tilde{Y}$ has a deterministic limiting Stieltjes transform (given by Proposition 4.10), the convergence in probability of their Stieltjes transforms implies that Y and $\tilde{Y}$ have the same limiting spectral measure. $\square$

Proposition 4.7 (Independent Poisson comparison). *For $k > t_0$ and $j < k$, define*

$$\bar{\lambda}_{kj} := \mathbb{E}\left[\widetilde{M}_{kj}\right] = \mathbb{E}[M_{kj}].$$

Let P_{kj} be independent random variables with

$$P_{kj} \sim \text{Poisson}(\bar{\lambda}_{kj}), \quad k > t_0, \quad j < k.$$

Define the symmetric random matrix H as

$$H_{kj} := \frac{P_{kj} - \bar{\lambda}_{kj}}{\sqrt{m}}, \quad H_{jk} := H_{kj}, \quad k > t_0, \quad j < k,$$

and set $H_{kj} = 0$ for entries in the initial clique, i.e. when $k \leq t_0$ and $j \leq t_0$. Then, for every fixed $z \in \mathbb{H}$,

$$|s_{\tilde{Y}}(z) - s_H(z)| \rightarrow 0$$

in probability. Consequently, $\tilde{Y}$ and H have the same limiting spectral measure.

Proof. For $k > t_0$ and $j < k$, write

$$\lambda_{kj} := mp_{kj} = m \frac{d_j(k-1)}{D_{k-1}}, \quad \bar{\lambda}_{kj} := \mathbb{E}[\lambda_{kj}].$$

It is enough to compare H with a random matrix with entries having the same distribution as the entries of $\tilde{Y}$, because the empirical spectral distribution depends only on the law of the matrix. Let

$$\widehat{Y} := \frac{\widehat{M} - \mathbb{E}[\widehat{M}]}{\sqrt{m}}.$$

We construct $\widehat{M}$ and P jointly so that $\widehat{M}$ has the same law as $\widetilde{M}$, while P has independent Poisson entries.

Note that if A and B are independent Poisson random variables with means α and β , then $A + B$ is Poisson with mean $\alpha + \beta$.

Fix the collection (λ_{ab}) . For all k, j , define

$$\alpha_{kj} := \min \{ \lambda_{kj}, \bar{\lambda}_{kj} \}, \quad \beta_{kj} := |\lambda_{kj} - \bar{\lambda}_{kj}|.$$

Let A_{kj} and B_{kj} be independent Poisson random variables with means α_{kj} and β_{kj} , respectively, independently over all k, j .

If $\lambda_{kj} \geq \bar{\lambda}_{kj}$, define

$$P_{kj} := A_{kj}, \quad \widehat{M}_{kj} := A_{kj} + B_{kj}.$$

If $\lambda_{kj} < \bar{\lambda}_{kj}$, define

$$\widehat{M}_{kj} := A_{kj}, \quad P_{kj} := A_{kj} + B_{kj}.$$

In either case, conditional on (λ_{ab}) ,

$$\widehat{M}_{kj} \sim \text{Poisson}(\lambda_{kj}), \quad P_{kj} \sim \text{Poisson}(\bar{\lambda}_{kj}).$$

Since $\bar{\lambda}_{kj}$ is deterministic, the law of P_{kj} does not depend on (λ_{ab}) , so P_{kj} are independent Poisson random variables.

Also, conditionally on (λ_{ab}) , the variables $\widehat{M}_{kj}$ have the same law as the Poissonized entries $\widetilde{M}_{kj}$, so $\widehat{Y}$ has the same law as $\widetilde{Y}$.

Now consider that

$$\widehat{M}_{kj} - P_{kj} = \pm B_{kj},$$

so

$$\left(\widehat{M}_{kj} - P_{kj} \right)^2 = B_{kj}^2.$$

Since $B_{kj} \sim \text{Poisson}(\beta_{kj})$ after conditioning on the collection (λ_{ab}) , we have

$$\mathbb{E}_{\widehat{M}_{kj}, P_{kj}} \left[\left(\widehat{M}_{kj} - P_{kj} \right)^2 \mid (\lambda_{ab}) \right] = \beta_{kj} + \beta_{kj}^2.$$

Therefore

$$\mathbb{E}_{\widehat{M}_{kj}, P_{kj}} \left[\left(\widehat{M}_{kj} - P_{kj} \right)^2 \mid (\lambda_{ab}) \right] = |\lambda_{kj} - \bar{\lambda}_{kj}| + |\lambda_{kj} - \bar{\lambda}_{kj}|^2.$$

Take expectations of both sides with respect to (λ_{ab}) to get

$$\begin{aligned} \mathbb{E}_{(\lambda_{ab})} \left[\left(\widehat{M}_{kj} - P_{kj} \right)^2 \right] &= \mathbb{E}_{(\lambda_{ab})} \left[\mathbb{E}_{\widehat{M}_{kj}, P_{kj}} \left[\left(\widehat{M}_{kj} - P_{kj} \right)^2 \mid (\lambda_{ab}) \right] \right] \\ &= \mathbb{E}_{(\lambda_{ab})} \left[|\lambda_{kj} - \bar{\lambda}_{kj}| + |\lambda_{kj} - \bar{\lambda}_{kj}|^2 \right] \\ &= \mathbb{E}_{(\lambda_{ab})} [|\lambda_{kj} - \bar{\lambda}_{kj}|] + \mathbb{E}_{(\lambda_{ab})} [|\lambda_{kj} - \bar{\lambda}_{kj}|^2]. \end{aligned}$$

From the proof of Proposition 4.8,

$$\text{Var}(d_j(k-1)) = O(n).$$

Since D_{k-1} is deterministic and $D_{k-1} = \Theta(n^2)$,

$$\text{Var}(\lambda_{kj}) = \text{Var} \left(m \frac{d_j(k-1)}{D_{k-1}} \right) = \frac{(cn)^2}{D_{k-1}^2} \text{Var}(d_j(k-1)) = O\left(\frac{1}{n}\right).$$

Therefore

$$\mathbb{E} \left[|\lambda_{kj} - \bar{\lambda}_{kj}|^2 \right] = \text{Var}(\lambda_{kj}) = O\left(\frac{1}{n}\right),$$

and by Cauchy-Schwarz,

$$\mathbb{E} \left[|\lambda_{kj} - \bar{\lambda}_{kj}| \right] \leq \sqrt{\mathbb{E} \left[|\lambda_{kj} - \bar{\lambda}_{kj}|^2 \right]} = O\left(\frac{1}{\sqrt{n}}\right).$$

Hence

$$\mathbb{E} \left[\left(\widehat{M}_{kj} - P_{kj} \right)^2 \right] = O\left(\frac{1}{\sqrt{n}}\right).$$

Since

$$\widehat{Y} - H = \frac{\widehat{M} - P}{\sqrt{m}}$$

on the non-deterministic off-diagonal entries, and since deterministic entries contribute zero, we have

$$\mathbb{E} \left[\left\| \widehat{Y} - H \right\|_F^2 \right] \leq \frac{2}{m} \sum_{k>t_0} \sum_{j<k} \mathbb{E} \left[\left(\widehat{M}_{kj} - P_{kj} \right)^2 \right] \leq \frac{2}{cn} \cdot n^2 \cdot O\left(\frac{1}{\sqrt{n}}\right) = O(\sqrt{n}).$$

Therefore

$$\mathbb{E} \left[\left\| \widehat{Y} - H \right\|_F \right] \leq \sqrt{\mathbb{E} \left[\left\| \widehat{Y} - H \right\|_F^2 \right]} = O\left(n^{1/4}\right).$$

By Proposition 4.4,

$$|s_{\widehat{Y}}(z) - s_H(z)| \leq \frac{1}{(\Im z)^2} \frac{\left\| \widehat{Y} - H \right\|_F}{\sqrt{n}}.$$

Taking expectations gives

$$\mathbb{E} \left[|s_{\widehat{Y}}(z) - s_H(z)| \right] \leq \frac{1}{(\Im z)^2} \frac{O(n^{1/4})}{\sqrt{n}} = O\left(n^{-1/4}\right) \rightarrow 0.$$

Hence

$$|s_{\widehat{Y}}(z) - s_H(z)| \rightarrow 0$$

in probability. Since $\widehat{Y}$ has the same law as $\widetilde{Y}$, we have that $\widetilde{Y}$ and H have the same limiting spectral distribution. $\square$

4.3 Variance for the centered matrix

Proposition 4.8 (Variance kernel for the independent centered matrix). *Let H be the independent Poisson matrix from Proposition 4.7, with $m = cn$ and fixed $c \in (0, \frac{1}{2})$. For $k > t_0 := m + 1$ and $j < k$, set*

$$p_{kj} := \frac{d_j(k-1)}{D_{k-1}}, \quad D_{k-1} := \sum_{v=1}^{k-1} d_v(k-1).$$

Then

$$\text{Var}(H_{kj}) = \mathbb{E}[p_{kj}], \quad n \text{Var}(H_{kj}) = n\mathbb{E}[p_{kj}].$$

Consequently, if $j/n \rightarrow x$ and $k/n \rightarrow y$, then

$$\sigma^2(x, y) := \lim_{n \rightarrow \infty} n \operatorname{Var}(H_{kj}) = \frac{1}{c} F(x, y). \quad (4.2)$$

Equivalently, with $A = [0, c]$, $B = (c, 1]$, and $u(t) = (2t - c)^{-1/2}$,

$$\sigma^2(x, y) = \begin{cases} 0, & x, y \in A, \\ \frac{1}{\sqrt{c}} u(y), & x \in A, y \in B, \\ \frac{1}{\sqrt{c}} u(x), & x \in B, y \in A, \\ u(x) u(y), & x, y \in B, \end{cases} \quad u(t) = (2t - c)^{-1/2}. \quad (4.3)$$

Proof. Let H be the independent Poisson matrix from Proposition 4.7. For $k > t_0$ and $j < k$, write

$$\bar{\lambda}_{kj} := \mathbb{E}[M_{kj}].$$

By definition,

$$H_{kj} = \frac{P_{kj} - \bar{\lambda}_{kj}}{\sqrt{m}}, \quad P_{kj} \sim \text{Poisson}(\bar{\lambda}_{kj}).$$

Therefore

$$\operatorname{Var}(H_{kj}) = \frac{1}{m} \operatorname{Var}(P_{kj}) = \frac{\bar{\lambda}_{kj}}{m} = \frac{\mathbb{E}[M_{kj}]}{m}.$$

Since $m = cn$, this gives

$$n \operatorname{Var}(H_{kj}) = \frac{n}{m} \mathbb{E}[M_{kj}] = \frac{1}{c} \mathbb{E}[M_{kj}].$$

If $k, j \leq t_0$, then H_{kj} is deterministic and equal to 0, so

$$\operatorname{Var}(H_{kj}) = 0.$$

Then the variance kernel is zero on $A \times A$.

Now assume that the vertices are not in the initial clique. By symmetry, it is sufficient to consider $k > t_0$ and $j < k$. By Theorem 3.1, with $x = j/n$ and $y = k/n$,

$$\mathbb{E}[M_{kj}] \rightarrow F(x, y).$$

Therefore

$$n \operatorname{Var}(H_{kj}) = \frac{1}{c} \mathbb{E}[M_{kj}] \rightarrow \frac{1}{c} F(x, y).$$

Hence

$$\lim_{n \rightarrow \infty} n \operatorname{Var}(H_{kj}) = \frac{1}{c} F(x, y)$$

outside $A \times A$.

So the variance kernel is

$$\sigma^2(x, y) = \begin{cases} 0, & x, y \in A, \\ \frac{1}{c} F(x, y), & \text{otherwise.} \end{cases}$$

Then (4.3) follows from the definition of F in (3.1). $\square$

Proposition 4.9. *We have σ^2 is rank-two.*

Proof. Define

$$\phi_1(x) = \mathbf{1}_{x \in A}, \quad \phi_2(x) = u(x) \mathbf{1}_{x \in B}.$$

Then σ^2 from (4.3) can be expressed using only ϕ_1 and ϕ_2 :

$$\sigma^2(x, y) = \frac{1}{\sqrt{c}} \phi_1(x) \phi_2(y) + \frac{1}{\sqrt{c}} \phi_2(x) \phi_1(y) + \phi_2(x) \phi_2(y).$$

Thus $\sigma^2(x, y)$ lies in the span of the two functions ϕ_1, ϕ_2 , so it has rank at most 2. If it were rank-one, where $\sigma^2(x, y) = \psi(x) \psi(y)$ for some ψ , then $\sigma^2 = 0$ on $A \times A$ forces $\psi = 0$ on A , contradicting $\sigma^2(x, y) > 0$ on $A \times B$, so the operator has to be rank-two. $\square$

4.4 Quadratic vector equations (QVE)

Proposition 4.10 (From [2]). *Let $H = H^{(N)}$ be a self-adjoint $N \times N$ random matrix with centered, independent entries (up to symmetry). Set*

$$s_{kj} := \mathbb{E}[|h_{kj}|^2], \quad S = (s_{kj})_{k,j=1}^N.$$

Assume the variance matrices $S^{(N)}$ satisfy the following.

- (A) *Flatness:* $s_{kj} \leq C/N$ for all k, j , with a constant $C > 0$ that is independent of N .
- (B) *Uniform primitivity:* there exist $p > 0$ and $L \in \mathbb{N}$ with $(S^L)_{kj} \geq p/N$ for all k, j .
- (C) *Bounded solution:* the solution below obeys $|m_k(z)| \leq P$ for all z in the upper half-plane $\mathbb{H}$ and all k .
- (D) *Moment bounds:* $\mathbb{E}[|h_{kj}|^b] \leq \mu_b s_{kj}^{b/2}$ for all $b \in \mathbb{N}$, all k, j .

Define the quadratic vector equation (QVE): for each $z \in \mathbb{H}$ there is a unique vector $\mathbf{m}(z) = (m_1(z), \dots, m_N(z)) \in (\mathbb{H})^N$ solving

$$-\frac{1}{m_k(z)} = z + \sum_{j=1}^N s_{kj} m_j(z), \quad k = 1, \dots, N, \quad (4.4)$$

and $\Im m_k(z) > 0$ for $\Im z > 0$.

Define

$$\rho^{(N)}(\tau) := \frac{1}{\pi N} \sum_{k=1}^N \Im m_k(\tau + i0).$$

As $N \rightarrow \infty$, the empirical eigenvalue distribution of H converges to the deterministic measure with density $\rho^{(N)}$.

Continuous version: if in addition

$$s_{kj}^{(N)} = \frac{1}{N} f\left(\frac{k}{N}, \frac{j}{N}\right)$$

for a bounded, symmetric, nonnegative $f : [0, 1]^2 \rightarrow \mathbb{R}_+$, then as $N \rightarrow \infty$ the discrete system converges to the continuous QVE

$$-\frac{1}{m(x; z)} = z + \int_0^1 f(x, y) m(y; z) dy, \quad x \in [0, 1], \quad z \in \mathbb{H}, \quad (4.5)$$

and the limiting eigenvalue density is

$$\rho(\tau) = \frac{1}{\pi} \int_0^1 \Im m(x; \tau + i0) dx, \quad \tau \in \mathbb{R}. \quad (4.6)$$

Proposition 4.11 (Verification of the AEK assumptions for the B–A multigraph variance profile). *Let $H^{(n)} = (H_{kj})_{1 \leq k, j \leq n}$ be the independent Poisson matrix from Proposition 4.7, and let*

$$s_{kj}^{(n)} := \mathbb{E} [|H_{kj}|^2]$$

be its variance profile. Set $N := n$, and let $S^{(n)} := (s_{kj}^{(n)})_{k, j=1}^N$.

Then, for all sufficiently large n , the matrices $S^{(n)}$ satisfy the assumptions (A)–(D) of Proposition 4.10 with constants independent of n , and the corresponding limiting variance kernel is σ^2 from (4.3).

Proof. This is verified in Appendix A. □

Lemma 4.12 (From [2]). *Let $H^{(n)}$ be a centered Wigner-type matrix – a real symmetric matrix with entries h_{kj} independent for $k \leq j$ and $\mathbb{E}[h_{kj}] = 0$ for all k, j . Let the variance profile of $H^{(n)}$ satisfy assumptions (A)–(D) from Proposition 4.10, and let ρ be the deterministic limiting density obtained from the corresponding QVE. Let $\text{supp } \rho = \bigcup_{r=1}^R [\alpha_r, \beta_r]$. Then, it holds asymptotically with overwhelming probability that all eigenvalues of $H^{(n)}$ lie inside $\text{supp } \rho$. If E_n denotes the stated event, then E_n holds asymptotically with overwhelming probability if for every $D > 0$ there exists $N_D \in \mathbb{N}$ such that for all $n \geq N_D$,*

$$\mathbb{P}(E_n) \geq 1 - n^{-D}.$$

Equivalently, $H^{(n)}$ has no outlier eigenvalues.

Proposition 4.13. *Let $Y = (M - \mathbb{E}[M]) / \sqrt{m}$ with $m = cn$ and fixed $c \in (0, \frac{1}{2})$, and let $\sigma^2 : [0, 1]^2 \rightarrow \mathbb{R}_+$ be the variance kernel in (4.3). Then, as $n \rightarrow \infty$, the empirical spectral distribution of Y converges weakly in probability to the deterministic probability measure with density*

$$\rho(\tau) = \frac{1}{\pi} \int_0^1 \Im m(x; \tau + i0) dx,$$

where $m(\cdot; z)$ is the unique solution of the continuum QVE

$$-\frac{1}{m(x; z)} = z + \int_0^1 \sigma^2(x, y) m(y; z) dy, \quad x \in [0, 1], \quad z \in \mathbb{H}.$$

Proof. Let H be the independent Poisson matrix from Proposition 4.7. Its entries are centered, independent up to symmetry, and have variance profile $S^{(n)} = (s_{kj}^{(n)})$ with

$$s_{kj}^{(n)} = \mathbb{E}[|H_{kj}|^2] = \frac{1}{n} \sigma^2\left(\frac{k}{n}, \frac{j}{n}\right) + o\left(\frac{1}{n}\right),$$

as shown in Proposition 4.8.

Thus the variance matrices $S^{(n)}$ satisfy assumptions (A)–(D) of [2]. By Proposition 4.10, the empirical spectral distribution of H converges in probability to the deterministic measure whose density is

$$\rho(\tau) = \frac{1}{\pi} \int_0^1 \Im m(x; \tau + i0) dx,$$

where m solves the continuum QVE with kernel σ^2 .

By Proposition 4.7, the Stieltjes transforms of H and $\tilde{Y}$ differ by $o(1)$ in probability at each fixed $z \in \mathbb{H}$. Therefore $\tilde{Y}$ has the same limiting spectral distribution as H .

Finally, by Proposition 4.6, the Stieltjes transforms of $\tilde{Y}$ and Y differ by $o(1)$ in probability at each fixed $z \in \mathbb{H}$. Therefore Y has the same limiting spectral distribution as $\tilde{Y}$, and hence the same limiting spectral distribution as H . $\square$

4.5 Reduction of the continuum QVE

Recall that the limiting variance kernel σ^2 from (4.3) is of the form

$$\sigma^2(x, y) = \begin{cases} 0, & x, y \in A = [0, c], \\ \frac{1}{\sqrt{c}}u(y), & x \in A, y \in B, \\ \frac{1}{\sqrt{c}}u(x), & x \in B, y \in A, \\ u(x)u(y), & x, y \in B, \end{cases} \quad u(t) = (2t - c)^{-1/2}.$$

By Proposition 4.13, the continuum QVE associated with σ^2 is

$$-\frac{1}{m(x; z)} = z + \int_0^1 \sigma^2(x, y) m(y; z) dy, \quad x \in [0, 1], z \in \mathbb{H}. \quad (4.7)$$

Since σ^2 is rank-two by Proposition 4.9, the integral equation collapses to a closed system of two scalar quantities. We carry out this reduction explicitly.

Define

$$M_A(z) := \int_0^c m(y; z) dy, \quad L(z) := \int_c^1 u(y) m(y; z) dy.$$

These two scalar quantities capture all dependence of (4.7) on $m(\cdot; z)$, because σ^2 only couples m through the functions $\mathbf{1}_A$ and $\mathbf{1}_B u$. We can now prove Theorem 1.1.

Proof. For $x \in A$, $\sigma^2(x, y) = 0$ for $y \in A$ and $\sigma^2(x, y) = (1/\sqrt{c})u(y)$ for $y \in B$. Hence

$$\int_0^1 \sigma^2(x, y) m(y; z) dy = \frac{1}{\sqrt{c}} \int_c^1 u(y) m(y; z) dy = \frac{L(z)}{\sqrt{c}}.$$

Substituting into (4.7) gives

$$-\frac{1}{m_A(z)} = z + \frac{L(z)}{\sqrt{c}}, \quad m_A(z) = -\frac{1}{z + L(z)/\sqrt{c}}.$$

Integrating over $A = [0, c]$ yields $M_A(z) = c m_A(z)$.

For $x \in B$,

$$\sigma^2(x, y) = \begin{cases} \frac{1}{\sqrt{c}}u(x), & y \in A, \\ u(x)u(y), & y \in B, \end{cases}$$

so that

$$\int_0^1 \sigma^2(x, y) m(y; z) dy = \frac{1}{\sqrt{c}}u(x) M_A(z) + u(x) L(z) = u(x) \left(L(z) + \frac{M_A(z)}{\sqrt{c}} \right).$$

The QVE becomes

$$-\frac{1}{m(x; z)} = z + u(x) \left(L(z) + \frac{M_A(z)}{\sqrt{c}} \right),$$

hence

$$m(x; z) = -\frac{1}{z + u(x) K(z)}, \quad K(z) := L(z) + \frac{M_A(z)}{\sqrt{c}}.$$

Using $M_A(z) = c m_A(z) = -c / (z + L(z) / \sqrt{c})$, we obtain

$$K(z) = L(z) + \frac{c}{\sqrt{c}} \left(-\frac{1}{z + L(z) / \sqrt{c}} \right) = L(z) - \frac{\sqrt{c}}{z + L(z) / \sqrt{c}}.$$

By definition of L ,

$$L(z) = \int_c^1 u(y) m(y; z) dy = - \int_c^1 \frac{u(y)}{z + u(y) K(z)} dy.$$

This proves the reduced system (1.2). □

The system (1.2) is a pair of equations that implicitly defines the pair $(L(z), K(z))$, which uniquely determines the solution $m(\cdot; z)$ of the continuum QVE. In particular, once $L(z)$ and $K(z)$ are known, the Stieltjes transform of the limiting eigenvalue distribution is

$$G_c(z) = \int_0^1 m(x; z) dx = c m_A(z) + \int_c^1 -\frac{1}{z + u(x) K(z)} dx.$$

Stieltjes inversion then yields the limiting spectral density.

5 The outlier eigenvalue

In this section we describe the leading eigenvalue of the adjacency matrix.

Proposition 5.1 (Weyl's perturbation theorem; [7, III.2.6]). *Let A and B be Hermitian $n \times n$ matrices, and let $\lambda_1(\cdot) \geq \dots \geq \lambda_n(\cdot)$ denote their eigenvalues. Then*

$$\max_{1 \leq j \leq n} |\lambda_j(A) - \lambda_j(B)| \leq \|A - B\|.$$

5.1 Top eigenvalue of $\mathbb{E}[M]$

Recall from Lemma 4.1 that the limiting mean kernel satisfies $F(x, y) = \phi(x) \phi(y)$, where

$$\phi(x) = \begin{cases} 1, & x \in [0, c], \\ \sqrt{c} (2x - c)^{-1/2}, & x \in (c, 1]. \end{cases}$$

Define the vector $v^{(n)} \in \mathbb{R}^n$ by

$$v_i^{(n)} := \phi\left(\frac{i}{n}\right), \quad 1 \leq i \leq n.$$

Lemma 5.2. *Let $\lambda_1(\mathbb{E}[M]) \geq \dots \geq \lambda_n(\mathbb{E}[M])$ be the eigenvalues of $\mathbb{E}[M]$. Then*

$$\lambda_1(\mathbb{E}[M]) = \alpha(c)n + O(1),$$

where

$$\alpha(c) := \int_0^1 \phi(x)^2 dx = c + \frac{c}{2} \log \frac{2-c}{c}. \quad (5.1)$$

Proof. By Theorem 3.1, for each fixed k, j ,

$$\mathbb{E}[M_{kj}] = F\left(\frac{k}{n}, \frac{j}{n}\right) + O\left(\frac{1}{n}\right) = v_k^{(n)} v_j^{(n)} + O\left(\frac{1}{n}\right).$$

Set

$$R_{kj}^{(n)} := \mathbb{E}[M_{kj}] - v_k^{(n)} v_j^{(n)}.$$

Then

$$\|R^{(n)}\|_F^2 = \sum_{k,j=1}^n O\left(\frac{1}{n^2}\right) = O(1),$$

hence $\|R^{(n)}\| \leq \|R^{(n)}\|_F = O(1)$.

Write

$$\mathbb{E}[M] = v^{(n)} (v^{(n)})^\top + R^{(n)}.$$

The rank-one matrix $v^{(n)} (v^{(n)})^\top$ has eigenvalues $\|v^{(n)}\|_2^2$ and 0 with multiplicity $n-1$. Applying Proposition 5.1 results in

$$\left| \lambda_1(\mathbb{E}[M]) - \|v^{(n)}\|_2^2 \right| \leq \left\| \mathbb{E}[M] - v^{(n)} (v^{(n)})^\top \right\| = \|R^{(n)}\| = O(1).$$

Thus

$$\lambda_1(\mathbb{E}[M]) = \|v^{(n)}\|_2^2 + O(1). \quad (5.2)$$

Next we compute $\|v^{(n)}\|_2^2$. By definition,

$$\|v^{(n)}\|_2^2 = \sum_{k=1}^n \phi\left(\frac{k}{n}\right)^2 = \sum_{k=1}^{cn} 1 + \sum_{k=cn+1}^n \frac{c}{\frac{2k}{n} - c}.$$

The first sum equals $cn + O(1)$. For the second, write it as a Riemann sum:

$$\sum_{k=cn+1}^n \frac{c}{\frac{2k}{n} - c} = n \int_c^1 \frac{c}{2x - c} dx + O(1) = n \cdot \frac{c}{2} \log \frac{2-c}{c} + O(1).$$

Therefore

$$\|v^{(n)}\|_2^2 = n \left(c + \frac{c}{2} \log \frac{2-c}{c} \right) + O(1) = \alpha(c)n + O(1).$$

Substituting into (5.2) yields

$$\lambda_1(\mathbb{E}[M]) = \alpha(c)n + O(1),$$

as claimed. □

5.2 Outlier eigenvalue of M

Proposition 5.3 (Asymptotic outlier eigenvalue). *Let M be the adjacency matrix of the B - A multigraph with $m = cn$ and fixed $c \in (0, \frac{1}{2})$. Then*

$$\lambda_{\max}(M) = \alpha(c)n + o_{\mathbb{P}}(n),$$

where $\alpha(c)$ is given by (5.1).

Proof. Set

$$Y := \frac{M - \mathbb{E}[M]}{\sqrt{m}}, \quad m = cn.$$

By Proposition 4.6, there exists a matrix $\bar{Y}$ such that $\bar{Y} \stackrel{d}{=} Y$ and

$$\frac{\|\bar{Y} - \tilde{Y}\|_F}{\sqrt{n}} \rightarrow 0$$

in probability. By Proposition 4.7, there is a matrix $\hat{Y}$ such that $\hat{Y} \stackrel{d}{=} \tilde{Y}$ and

$$\frac{\|\hat{Y} - H\|_F}{\sqrt{n}} \rightarrow 0$$

in probability.

Since $\hat{Y} \stackrel{d}{=} \tilde{Y}$,

$$\frac{\|\bar{Y} - H\|_F}{\sqrt{n}} \leq \frac{\|\bar{Y} - \hat{Y}\|_F}{\sqrt{n}} + \frac{\|\hat{Y} - H\|_F}{\sqrt{n}} \rightarrow 0$$

in probability. Hence

$$\|\bar{Y} - H\| \leq \|\bar{Y} - H\|_F = o_{\mathbb{P}}(\sqrt{n}).$$

By Lemma 4.12, applied to the independent matrix H , the limiting measure obtained from the QVE is supported on a finite union of compact intervals, hence has compact support. Write

$$\text{supp } \mu \subset [-C, C]$$

for some constant $C < \infty$.

Since H is a real symmetric matrix, all of its eigenvalues are real, and

$$\|H\| = \max_{1 \leq i \leq n} |\lambda_i(H)|.$$

By Lemma 4.12, asymptotically with overwhelming probability all eigenvalues of H lie in $\text{supp } \mu$. Hence, asymptotically with overwhelming probability,

$$\lambda_i(H) \in [-C, C]$$

for all $i = 1, \dots, n$. Then

$$\mathbb{P}[\|H\| > C] = \mathbb{P}\left[\max_{1 \leq i \leq n} |\lambda_i(H)| > C\right] \rightarrow 0$$

and

$$\|H\| = O_{\mathbb{P}}(1).$$

It follows that

$$\|\bar{Y}\| \leq \|H\| + \|\bar{Y} - H\| = O_{\mathbb{P}}(1) + o_{\mathbb{P}}(\sqrt{n}) = o_{\mathbb{P}}(\sqrt{n}).$$

Since $\bar{Y} \stackrel{d}{=} Y$, we also have

$$\|Y\| = o_{\mathbb{P}}(\sqrt{n}).$$

Therefore

$$\|M - \mathbb{E}[M]\| = \sqrt{m} \|Y\| = O(\sqrt{n}) o_{\mathbb{P}}(\sqrt{n}) = o_{\mathbb{P}}(n).$$

Now apply Proposition 5.1. Since both M and $\mathbb{E}[M]$ are Hermitian,

$$|\lambda_{\max}(M) - \lambda_1(\mathbb{E}[M])| \leq \|M - \mathbb{E}[M]\| = o_{\mathbb{P}}(n).$$

By Lemma 5.2,

$$\lambda_1(\mathbb{E}[M]) = \alpha(c)n + O(1).$$

Combining the last two expressions gives

$$\lambda_{\max}(M) = \alpha(c)n + O(1) + o_{\mathbb{P}}(n).$$

Finally, since $O(1) = o(n)$ deterministically, it follows that

$$O(1) + o_{\mathbb{P}}(n) = o_{\mathbb{P}}(n).$$

Hence

$$\lambda_{\max}(M) = \alpha(c)n + o_{\mathbb{P}}(n).$$

□

Theorem 1.2 is a corollary of Proposition 5.3.

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A QVE conditions verification

We verify (A)-(D) in order.

(A) Flatness. First note that diagonal entries and entries inside the initial clique are deterministic after centering, so their variances are zero. Thus it remains to consider entries with $k > t_0$ and $j < k$; by symmetry this covers all off-diagonal entries.

For the independent Poisson matrix H from Proposition 4.7, we have

$$H_{kj} = \frac{P_{kj} - \bar{\lambda}_{kj}}{\sqrt{m}}, \quad P_{kj} \sim \text{Poisson}(\bar{\lambda}_{kj}), \quad \bar{\lambda}_{kj} = \mathbb{E}[M_{kj}].$$

Hence

$$s_{kj}^{(n)} := \mathbb{E}[|H_{kj}|^2] = \text{Var}(H_{kj}) = \frac{1}{m} \text{Var}(P_{kj}) = \frac{\bar{\lambda}_{kj}}{m} = \frac{\mathbb{E}[M_{kj}]}{m}.$$

By Theorem 3.1,

$$\mathbb{E}[M_{kj}] = O(1)$$

uniformly in j, k . Since $m = cn$, it follows that

$$s_{kj}^{(n)} = \frac{\mathbb{E}[M_{kj}]}{m} = O\left(\frac{1}{n}\right)$$

uniformly in j, k . Hence, for all k, j ,

$$0 \leq s_{kj}^{(n)} \leq \frac{C}{n}$$

for some constant C independent of n .

(B) Uniform primitivity. Fix any $\delta \in (0, 1 - c)$ and set

$$J := [c + \delta, 1], \quad \mathcal{J}_n := \left\{ \ell \in \{1, \dots, n\} : \frac{\ell}{n} \in J \right\}.$$

Then $|\mathcal{J}_n| \geq n|J|/2$ for all sufficiently large n .

We claim that there is a constant $a > 0$, independent of n , such that

$$s_{i\ell}^{(n)} \geq \frac{a}{n}$$

for all sufficiently large n , all i , and all $\ell \in \mathcal{J}_n$ with $i \neq \ell$.

Let $r = \max\{i, \ell\}$ and $q = \min\{i, \ell\}$. Since $\ell/n \in J \subset (c, 1]$, we have $r > t_0$ for all sufficiently large n . For i, ℓ , the corresponding independent Poisson entry in H has variance

$$s_{i\ell}^{(n)} = \text{Var}(H_{i\ell}) = \frac{\mathbb{E}[M_{rq}]}{m}.$$

Since

$$\mathbb{E}[M_{rq}] = m\mathbb{E}[p_{rq}], \quad p_{rq} = \frac{d_q(r-1)}{D_{r-1}},$$

we have

$$s_{i\ell}^{(n)} = \mathbb{E}[p_{rq}].$$

Every existing vertex has degree at least m , so $d_q(r-1) \geq m$. Also $D_{r-1} \leq D_n \leq 2mn$. Hence

$$p_{rq} \geq \frac{m}{D_n} \geq \frac{1}{2n}.$$

Therefore

$$s_{i\ell}^{(n)} = \mathbb{E}[p_{rq}] \geq \frac{1}{2n}.$$

So the claim holds with $a = 1/2$.

Therefore, for $L = 2$,

$$\left(S^{(n)}\right)_{ij}^2 = \sum_{\ell=1}^n s_{i\ell}^{(n)} s_{\ell j}^{(n)} \geq \sum_{\substack{\ell \in \mathcal{J}_n \\ \ell \neq i, j}} s_{i\ell}^{(n)} s_{\ell j}^{(n)}.$$

For all sufficiently large n ,

$$|\{\ell \in \mathcal{J}_n : \ell \neq i, j\}| \geq \frac{n|J|}{4}.$$

Hence

$$\left(S^{(n)}\right)_{ij}^2 \geq \frac{n|J|}{4} \cdot \frac{1}{4n^2} = \frac{|J|}{16n} \quad \forall i, j$$

for all sufficiently large n . Thus (B) holds with $L = 2$ and $p = |J|/16$.

(C) Bounded solution of the QVE. Since boundedness can be checked only after solving for $m(\cdot, z)$, another approach is necessary. By Remark 1.1 of [2], condition (C) is equivalent to the conditions of Theorem 6.1 of [1], namely

A1 (Symmetry and positivity preserving). Note that S is an integral operator with a symmetric, non-negative kernel S_{xy} , i.e.,

$$S_{xy} = S_{yx} \geq 0.$$

Equivalently,

$$\langle u, Sw \rangle = \langle Su, w \rangle, \quad (Sp)(x) \geq 0$$

whenever

$$p(y) \geq 0.$$

A2 (Smoothing / $L^2 \rightarrow L^\infty$ boundedness). The operator S acts by

$$(Sw)(x) = \int_0^1 S_{xy} w(y) dy,$$

and for each fixed x the function $y \mapsto S_{xy}$ lies in L^2 . To see the concrete form of this condition, apply Cauchy-Schwarz at every x :

$$|(Sw)(x)| = \left| \int_0^1 S_{xy} w(y) dy \right| \leq \left(\int_0^1 S_{xy}^2 dy \right)^{1/2} \|w\|_{L^2}.$$

Taking the supremum over x yields

$$\|Sw\|_{L^\infty} \leq \left(\sup_{x \in [0,1]} \left(\int_0^1 S_{xy}^2 dy \right)^{1/2} \right) \|w\|_{L^2}.$$

Thus the requirement that S be a bounded operator $L^2 \rightarrow L^\infty$ is equivalent to demanding

$$\sup_{x \in [0,1]} \left(\int_0^1 S_{xy}^2 dy \right)^{1/2} < \infty.$$

B1 (Quantitative block fully indecomposability). There exist constants $\phi > 0$ and $K \in \mathbb{N}$, a fully indecomposable matrix $Z = (Z_{kj})_{k,j=1}^K$ with $Z_{kj} \in \{0,1\}$, and a measurable partition $\mathcal{I} := \{I_j\}_{j=1}^K$ of $[0,1]$ such that for every $1 \leq i, j \leq K$ the following hold:

$$|I_j| = \frac{1}{K}, \quad \sigma^2(x, y) \geq \phi Z_{ij}$$

whenever

$$(x, y) \in I_i \times I_j.$$

Note that the original condition (C) is framed in terms of a real-valued matrix, while the alternative conditions A1, A2, and B1 apply to a possibly infinite-dimensional operator. We then write $S^{(n)}$ using the conventions of [1], starting with defining

$$X_n := \{1, \dots, n\}, \quad \pi_n(\{j\}) := \frac{1}{n}, \quad B_n := \ell^\infty(X_n), \quad \|w\|_{B_n} := \max_{1 \leq j \leq n} |w_j|,$$

and

$$\langle u, w \rangle_n := \frac{1}{n} \sum_{j=1}^n u_j \overline{w_j}, \quad \langle w \rangle_n := \frac{1}{n} \sum_{j=1}^n w_j.$$

The operator S in the notation of [1] is the operator induced by the variance matrix $S^{(n)} = (s_{kj}^{(n)})$, namely

$$(Sw)_k := \sum_{j=1}^n s_{kj}^{(n)} w_j, \quad 1 \leq k \leq n.$$

Equivalently,

$$S_{kj} := n s_{kj}^{(n)}.$$

A1. Since $S^{(n)}$ is a variance matrix,

$$s_{kj}^{(n)} = s_{jk}^{(n)} \geq 0.$$

Therefore, for $u, w \in B_n$,

$$\langle u, Sw \rangle_n = \frac{1}{n} \sum_{k=1}^n u_k \sum_{j=1}^n s_{kj}^{(n)} w_j = \frac{1}{n} \sum_{j=1}^n w_j \sum_{k=1}^n s_{jk}^{(n)} u_k = \langle Su, w \rangle_n.$$

Thus S is symmetric. If $p_j \geq 0$ for all j , then

$$(Sp)_k = \sum_{j=1}^n s_{kj}^{(n)} p_j \geq 0$$

for all k . Hence A1 holds uniformly in n .

A2. In the finite setting, A2 requires

$$\sup_{1 \leq k \leq n} \left(\frac{1}{n} \sum_{j=1}^n S_{kj}^2 \right)^{1/2} < \infty$$

uniformly in n , with $S_{kj} = n s_{kj}^{(n)}$. So it is enough to show

$$\sup_{1 \leq k \leq n} \left(n \sum_{j=1}^n \left(s_{kj}^{(n)} \right)^2 \right)^{1/2} < \infty.$$

By (A), there is a constant $C_1 > 0$, independent of n , such that

$$0 \leq s_{kj}^{(n)} \leq \frac{C_1}{n}$$

for all k, j . Therefore

$$n \sum_{j=1}^n \left(s_{kj}^{(n)} \right)^2 \leq n \sum_{j=1}^n \frac{C_1^2}{n^2} = C_1^2.$$

Hence

$$\sup_{1 \leq k \leq n} \left(n \sum_{j=1}^n \left(s_{kj}^{(n)} \right)^2 \right)^{1/2} \leq C_1.$$

Thus A2 holds uniformly in n .

B1. Set $A = [0, c]$, $B = (c, 1]$, and $u(t) = (2t - c)^{-1/2}$. Since $c < 1/2$, fix δ such that

$$0 < \delta < \frac{1}{2} - c,$$

and let

$$B_\delta := (c + \delta, 1] \subset B.$$

Then

$$|B_\delta| = 1 - c - \delta > \frac{1}{2}.$$

Choose a sufficiently large $K \in \mathbb{N}$ and define

$$r := \lfloor K |B_\delta| \rfloor$$

such that

$$\frac{K}{2} < r \leq K |B_\delta|.$$

Now choose disjoint measurable sets

$$I_{K-r+1}, \dots, I_K \subset B_\delta$$

with

$$|I_j| = \frac{1}{K}, \quad K-r+1 \leq j \leq K.$$

Partition the complement

$$[0, 1] \setminus \bigcup_{j=K-r+1}^K I_j$$

arbitrarily into measurable sets $I_1, \dots, I_{K-r}$ of measure $1/K$ each. Thus $\{I_j\}_{j=1}^K$ is a measurable partition of $[0, 1]$ with

$$|I_j| = \frac{1}{K}$$

for all j .

Let

$$P := \{K-r+1, \dots, K\}.$$

Define $Z \in \{0, 1\}^{K \times K}$ as

$$Z_{ij} = \begin{cases} 1, & i \neq j \text{ and } (i \in P \text{ or } j \in P), \\ 0, & \text{otherwise.} \end{cases}$$

We first show that Z is fully indecomposable. The zero entries of Z lie either on the diagonal or in the submatrix indexed by $P^c \times P^c$. Since

$$|P^c| = K-r < \frac{K}{2},$$

any zero submatrix contained in $P^c \times P^c$ has size at most $(K-r) \times (K-r)$. If such a zero submatrix has size $a \times b$, then

$$a+b \leq 2(K-r) < K.$$

The remaining zero submatrices are single diagonal entries, and these also satisfy $a+b < K$ for $K \geq 3$. Therefore Z has no zero submatrix of size $a \times b$ with $a+b \geq K$, so Z is fully indecomposable.

Now define

$$m_1 := \min_{x \in [c+\delta, 1]} u(x) > 0, \quad m_2 := \min_{y \in [c, 1]} u(y) > 0.$$

For $x \in B_\delta$ and $y \in [0, 1]$, we have from (4.3) that

$$\sigma^2(x, y) \geq \min \left\{ \frac{1}{\sqrt{c}} m_1, m_1 m_2 \right\} =: \phi > 0.$$

By symmetry, the same bound holds if $y \in B_\delta$ and $x \in [0, 1]$.

If $Z_{ij} = 1$ and $x \in I_i$, $y \in I_j$, then either $i \in P$ or $j \in P$. Hence either $I_i \subset B_\delta$ or $I_j \subset B_\delta$. Therefore

$$\sigma^2(x, y) \geq \phi.$$

Since H has variance profile

$$s_{kj}^{(n)} = \frac{\mathbb{E}[M_{kj}]}{m},$$

Theorem 3.1 gives

$$ns_{kj}^{(n)} = \frac{n}{m} \mathbb{E}[M_{kj}] \rightarrow \frac{1}{c} F\left(\frac{k}{n}, \frac{j}{n}\right) = \sigma^2\left(\frac{k}{n}, \frac{j}{n}\right).$$

Hence, for all sufficiently large n ,

$$ns_{kj}^{(n)} \geq \frac{\phi}{2}$$

when $\frac{k}{n} \in I_i$, $\frac{j}{n} \in I_j$, and $Z_{ij} = 1$.

Then the partition $\{I_j\}_{j=1}^K$ and the fully indecomposable $K \times K$ matrix Z satisfy

$$|I_j| = \frac{1}{K}, \quad ns_{kj}^{(n)} \geq \frac{\phi}{2} Z_{ij}$$

whenever $\frac{k}{n} \in I_i$ and $\frac{j}{n} \in I_j$, satisfying B1.

(D) Bounded moments. Let $b \in \mathbb{N}$ be fixed. If $s_{kj}^{(n)} = 0$, then $H_{kj} = 0$, so

$$\mathbb{E}\left[|H_{kj}|^b\right] = 0 \leq \mu_b \left(s_{kj}^{(n)}\right)^{b/2}.$$

Thus, by symmetry, it is sufficient to consider entries with $k > t_0$ and $j < k$. Write

$$\bar{\lambda}_{kj} := \mathbb{E}[M_{kj}].$$

By definition of H ,

$$H_{kj} = \frac{P_{kj} - \bar{\lambda}_{kj}}{\sqrt{m}}, \quad P_{kj} \sim \text{Poisson}(\bar{\lambda}_{kj}),$$

and

$$s_{kj}^{(n)} = \text{Var}(H_{kj}) = \frac{\bar{\lambda}_{kj}}{m}.$$

We first note that $\bar{\lambda}_{kj}$ is bounded above and below uniformly. The upper bound

$$\bar{\lambda}_{kj} = \mathbb{E}[M_{kj}] = O(1)$$

follows from the uniform bound in Theorem 3.1. For the lower bound, since every existing vertex has degree at least m and $D_{k-1} \leq D_n \leq 2mn$, we have

$$p_{kj} = \frac{d_j(k-1)}{D_{k-1}} \geq \frac{m}{D_n} \geq \frac{1}{2n}.$$

Therefore

$$\bar{\lambda}_{kj} = \mathbb{E}[M_{kj}] = m\mathbb{E}[p_{kj}] \geq \frac{m}{2n} = \frac{c}{2}.$$

Hence there are constants $0 < a < A < \infty$ such that

$$a \leq \bar{\lambda}_{kj} \leq A$$

uniformly in j, k, n for all non-deterministic entries.

Now let $X \sim \text{Poisson}(\lambda)$. First,

$$\mathbb{E}\left[|X - \lambda|^b\right] \leq \mathbb{E}\left[(X + \lambda)^b\right] = \sum_{a=0}^b \binom{b}{a} \lambda^{b-a} \mathbb{E}[X^a].$$

The moments of a Poisson random variable satisfy

$$\mathbb{E}[X^a] = \sum_{r=0}^a S(a, r) \lambda^r,$$

where $S(a, r)$ are the Stirling numbers of the second kind (from [10, p.160]). Thus

$$\mathbb{E}[|X - \lambda|^b] \leq \sum_{a=0}^b \binom{b}{a} \lambda^{b-a} \sum_{r=0}^a S(a, r) \lambda^r = P_b(\lambda),$$

where P_b is a polynomial with positive coefficients depending only on b . Since $\bar{\lambda}_{kj} \in [a, A]$, we get

$$\mathbb{E}[|P_{kj} - \bar{\lambda}_{kj}|^b] \leq C_b \leq \mu_b \bar{\lambda}_{kj}^{b/2}$$

for some constant μ_b depending only on b .

Therefore

$$\mathbb{E}[|H_{kj}|^b] = m^{-b/2} \mathbb{E}[|P_{kj} - \bar{\lambda}_{kj}|^b] \leq \mu_b m^{-b/2} \bar{\lambda}_{kj}^{b/2} = \mu_b \left(\frac{\bar{\lambda}_{kj}}{m} \right)^{b/2} = \mu_b \left(s_{kj}^{(n)} \right)^{b/2}.$$

This proves (D).

Combining (A)–(D), we conclude that the variance profile $S^{(n)}$ associated with the independent Poisson matrix H satisfies all assumptions of Proposition 4.10, and the limiting variance kernel is σ^2 from (4.3).