SUBSETS OF $\mathbb{F}_q[x]$ FREE OF 3-TERM GEOMETRIC PROGRESSIONS

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ABSTRACT. Several recent papers have considered the Ramsey-theoretic problem of how large a subset of integers can be without containing any 3-term geometric progressions. This problem has also recently been generalized to number fields, determining bounds on the greatest possible density of ideals avoiding geometric progressions. We study the analogous problem over $\mathbb{F}_q[x]$, first constructing a set greedily which avoids these progressions and calculating its density, and then considering bounds on the upper density of subsets of $\mathbb{F}_q[x]$ which avoid 3-term geometric progressions. This new setting gives us a parameter q to vary and study how our bounds converge to 1 as it changes, and finite characteristic introduces some extra combinatorial structure that increases the tractibility of common questions in this area.

CONTENTS

1. INTRODUCTION

In a 1961 paper Rankin [Ran] introduced the idea of considering how large a set of integers can be without containing terms which are in geometric progression. He constructed a subset of the integers which avoids 3-term geometric progressions and has asymptotic density approximately 0.719745. Brown and Gordon [BG] noted

that the set Rankin considered was the set obtained by greedily including integers subject to the condition that such integers do not create a progression involving integers already included in the set.

Other authors, including Riddell [Rid], Biegelböck, Bergelson, Hindman and Strauss [BBHS], Nathanson and O'Bryant [NO], and McNew [Mc], have refined bounds for the upper density of a set which avoids geometric progressions. Best, Huan, McNew, Miller, Powell, Tor and Weinstein [BHMMPTW] generalized the problem to quadratic number fields. Using many of the techniques from these other works, they obtained similar results for the density of the ideals in the ring of integers which similarly avoid geometric progressions.

The purpose of [BHMMPTW] was to see how changing from subsets of $\mathbb{Z}$ to subsets of number fields affected the answer. In this and another work in progress, [AFGMMMM], we consider the same problem in other situations. Here we investigate what happens over function fields of finite characteristic, and in [AFGMMMM] we explore what happens in non-commutative settings. In particular, using combinatorial tools as well as the methods of Rankin, McNew, and Best et al., we consider the size of the largest subset of the polynomial ring $\mathbb{F}_q[x]$ which avoids geometric progressions whose common ratio is a non-unit polynomial in this ring.

Remark 1.1. *It is worth remarking on the choice of problem. In the integer case it is interesting to study sets which avoid 3-term geometric progressions with integral ratio as these sets have a very different flavor from those sets (called primitive sets) which avoid 2-term progressions with integral ratio. The situation is richer over $\mathbb{F}_q[x]$, for example we now have many more units. In what follows we will require that our sets avoid 3-terms in a non-unit progression.*

We begin by constructing and characterizing the set constructed greedily (with respect to norm) which avoids 3-term non-unit geometric progressions. In Theorem 3.7 we show that this set has asymptotic density

$$\left(1 - \frac{1}{q}\right) \prod_{i=1}^{\infty} \prod_{n=1}^{\infty} \left(1 + q^{-3^i n}\right)^{m(n,q)} = \left(1 - \frac{1}{q}\right) \prod_{i=1}^{\infty} \frac{1 - q^{1-2\cdot 3^i}}{1 - q^{1-3^i}}, \qquad (1.1)$$

where $m(n, q)$ denotes the number of monic irreducible polynomials over $\mathbb{F}_q$ of degree n. We then study bounds on the upper density of sets avoiding 3-term non-unit progressions, and give the numerical values obtained for certain specific values of q.

2. Construction of the Greedy Set

Our goal is to construct a subset of significant density whose elements do not contain a three-term non-unit geometric progression. That is, a set that does not contain all of b, br and br^2 for some b and non-unit r. Observe that $\mathbb{F}_q[x]$ is a unique factorization domain. For each $f(x) \in \mathbb{F}_q[x]$, we can write it as

$$f(x) \;=\; u P_1^{e_1}(x) \cdots P_k^{e_k}(x), \qquad (2.1)$$

where $P_1(x), \cdots, P_k(x)$ are prime (monic irreducible polynomials) and $u \in \mathbb{F}_q$ is a unit. Since the primes are chosen to be monic, this factorization is unique.

Let $A_3^*(\mathbb{Z}) = \{0, 1, 3, 4, 9, 10, 12, 13, \dots\}$ denote the set of non-negative integers formed by greedily constructing a set free of 3-term arithmetic progressions (see [Ran]). We wish to adapt the method of Rankin to construct a greedy set for

$\mathbb{F}_q[x]$ whose elements are able to be characterized as those with prime exponents $e_1, \ldots, e_k \in A_3^*(\mathbb{Z})$. We will refer to this greedy set $G_3^*(\mathbb{F}_q[x])$ as $G_{3,q}^*$ for brevity. Remember that for our construction, unit progressions are ignored.

Emulating Rankin's construction first requires dealing with units. Consider

$$f(x), \ ur(x)f(x), \ vr^2(x)f(x) \tag{2.2}$$

where u, v are units. Suppose $f(x), \ ur(x)f(x) \in G_{3,q}^*$. The exponents of the prime factors of $f(x), \ ur(x)f(x), \ vr^2(x)f(x)$ contain an arithmetic progression, and thus we wish to exclude $vr^2(x)f(x)$ from the greedy set. However, it is not clear that we can find another ratio $\rho(x)$ such that there exists $g(x), \rho(x)g(x) \in G_{3,q}^*$ and $\rho^2(x)^2 g(x) = vr^2(x)f(x)$, so $vr^2 f(x)$ may possibly not create an actual geometric progression. We show it does through the following lemma, allowing us to characterize our greedy set $G_{3,q}^*$ by its elements' prime factors.

Lemma 2.1. *Let $G_{3,q}^*$ be a subset of $\mathbb{F}_q[x]$ formed by taking the set $\mathbb{F}_q$, and greedily adding in elements by increasing degree, subject to the condition that included elements are not part of a 3-term geometric progression involving elements of smaller degree already included in $G_{3,q}^*$.*

Then, at the n-th stage of the construction of $G_{3,q}^*$ – that is, when we are choosing what degree n polynomials to add to the set – the elements of $\mathbb{F}_q[x]$ that are included in $G_{3,q}^*$ are exactly the polynomials

$$f(x) = uP_1(x)^{e_1} \cdots P_k(x)^{e_k} \tag{2.3}$$

where each $P_i(x)$ is prime, $u \in F_q$ is a unit, $e_i \in A_3^*(\mathbb{Z})$ for all i, and

$$n = \sum_{i=1}^{k} e_i \cdot \deg(P_i(x)). \tag{2.4}$$

Proof. We proceed by induction on n. Note that for $n \in \{0, 1\}$, the statement is true since there are no non-unit geometric progressions among the polynomials of degree at most 1. Now suppose that for all $n \leq N$, polynomials of degree $\leq N$ satisfy the inductive hypothesis. Consider $g(x) = uP_1(x)^{e_1} \cdots P_k(x)^{e_k} \in \mathbb{F}_q[x]$ a polynomial of degree n with each $P_i(x)$ prime and with some e_i not in $A_3^*(\mathbb{Z})$.

For each $e_i \notin A_3^*(\mathbb{Z})$ there exist $a_i, \ b_i \in A_3^*(\mathbb{Z})$ such that a_i, b_i, e_i forms an arithmetic progression with common difference d_i. That is $b_i = a_i + d_i, e_i = a_i + 2d_i$. Let

$$r(x) = \prod_{\substack{i \leq k \\ e_i \notin A_3^*(\mathbb{Z})}} P_i(x)^{d_i}.$$

Now, consider the geometric progression,

$$\frac{g(x)}{r(x)^2}, \ \frac{g(x)}{r(x)}, \ g(x). \tag{2.5}$$

Note that each of these terms is contained in $\mathbb{F}_q[x]$, and, by construction, all of the exponents appearing in the prime factorizations of the first two terms are in $A_3^*(\mathbb{Z})$. They also have degree less than or equal to n and so by the inductive hypotheses, are already contained in $G_{3,q}^*$.

Now consider $g(x) = uP_1^{e_1}(x) \cdots P_k^{e_k}(x) \in \mathbb{F}_q[x]$ with $e_i \in A_3^*(\mathbb{Z})$. Note that $g(x)$ cannot form a geometric progression with the elements of degree $\leq N$ in $G_{3,q}^*$, because the exponents on the prime factors of terms in geometric progression will

form an arithmetic progression and these polynomials are constructed so that is not possible. □

Theorem 2.2. *The elements of $G_{3,q}^*$ are precisely those elements with prime factorization $f(x) = uP_1^{e_1}(x) \cdots P_k^{e_k}(x)$ such that $e_i \in A_3^*(\mathbb{Z})$ for all i.*

Proof. Lemma 2.1 allows us to ignore units and characterize the elements in the greedy set by their primes. This theorem then follows immediately. □

3. COMPUTING DENSITIES

3.1. **Set-Up.** We will study the density using ideals like [BHMMPTW], with analogous functions for the density of elements with acceptable prime factors, and similar expressions for the density of the set $G_{3,q}^*$.

Since $\mathbb{F}_q[x]$ is a Euclidean domain and primes $P(x)$ (monic irreducible polynomials) are in correspondence with prime ideals (via $P(x) \mapsto \langle P(x) \rangle$), the language of divisibility and norms of an element $f(x)$ are largely interchangeable with that of the corresponding ideal $\langle f(x) \rangle$. Note that the $q - 1$ associates of $f(x)$ all have the same corresponding ideal, as $\langle f(x) \rangle = \langle uf(x) \rangle$.

An introduction to Euclidean domains and the steps to get from the existence of a division algorithm to unique factorization and PID, which is used in the integer and $\mathbb{F}_q[x]$ case, can be found in [Con]. The problem of constructing a greedy set is tractable in any Euclidean or Dedekind domain R with the property that R/I is finite for all ideals $I \neq \langle 0 \rangle$, as both cases will have some sort of unique factorization and finite norms for all elements. A more in-depth introduction to ideals and Dedekind domains can be found in [AM].

Definition 3.1. *The norm of an element $f(x) \in \mathbb{F}_q[x]$ is*

$$N(f(x)) = N(\langle f(x) \rangle) = |\mathbb{F}_q[x]/\langle f(x) \rangle| = q^{\deg(f(x))}. \tag{3.1}$$

In particular, the norm of an element is the norm of the ideal it generates. Note that the norm is multiplicative.

Definition 3.2. *Let $S(n) := \{f(x) \in \mathbb{F}_q[x] : N(f(F)) \leq n\}$. Similarly, for $X \subseteq \mathbb{R}_{\geq 0}$ we have $S(X) = \{f(x) \in \mathbb{F}_q[x] : N(f(x)) \in X\}$.*

Definition 3.3. *The (asymptotic) density of a set $A \subseteq \mathbb{F}_q$ is defined as*

$$d(A) = \lim_{n \to \infty} \frac{|A \cap S(n)|}{|S(n)|} \tag{3.2}$$

(provided that this limit exists).

For our density formulas, we require a Riemann zeta-type function for $\mathbb{F}_p[x]$.

Definition 3.4. *For $\mathbb{F}_q[x]$, define*

$$\zeta_q(s) := \sum_{\substack{f(x) \in \mathbb{F}_q[x] \\ f(x) \text{ monic}}} \frac{1}{N(f(x))^s}. \tag{3.3}$$

This is an analogue of the Riemann zeta function, and likewise has its own Euler product:

$$\zeta_q(s) = \prod_{P(x) \text{ prime}} \frac{1}{1 - N(P(x))^{-s}} = \prod_{P(x) \text{ prime}} \frac{1}{1 - q^{-\deg(P)s}}. \tag{3.4}$$

We can rewrite $\zeta_q(s)$ using a fact from elementary number theory.

Lemma 3.5. *The number of irreducible monic polynomials of degree n over $\mathbb{F}_q[x]$ is*

$$\frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)q^d, \tag{3.5}$$

where μ is the Möbius function.

See Proposition 2.1 in [Ros] for a proof.

Lemma 3.6. *Let $m(n,q) = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)q^d$. We have*

$$\zeta_q(s) = \prod_{n=1}^{\infty}\frac{1}{(1-q^{-ns})^{m(n,q)}} = \frac{1}{1-q^{1-s}}.$$

Proof. The first expression follows from collecting the degree n primes in the Euler product expression in Definition 3.4. The second follows from (3.4) and the fact that there are exactly q^d monic polynomials of norm q^d. $\qquad\square$

3.2. **Density of the Greedy Set.** Let $P(x) \in \mathbb{F}_q[x]$ be prime, or equivalently a monic, irreducible polynomial. Then, as described in [BHMMPTW], the density of those polynomials in $\mathbb{F}_q[x]$ exactly divisible by $P(x)^k$ with $k \in A_3^*(\mathbb{Z})$ is

$$\left(\frac{N(P(x))-1}{N(P(x))}\right)\sum_{i\in A_3^*(\mathbb{Z})}\frac{1}{N(P(x))^i} = \left(1-\frac{1}{N(P(x))}\right)\prod_{i=0}^{\infty}\left(1+\frac{1}{N(P(x))^{3^i}}\right). \tag{3.6}$$

Theorem 3.7. *Define*

$$\mathcal{F}(x) := \left(1-\frac{1}{x}\right)\prod_{i=0}^{\infty}\left(1+\frac{1}{x^{3^i}}\right). \tag{3.7}$$

The asymptotic density of the greedy set $G_{3,q}^* \subseteq \mathbb{F}_q[x]$, denoted $d(G_{3,q}^*)$, is

$$d(G_{3,q}^*) = \prod_{\substack{P(x)\in\mathbb{F}_q[x] \\ P(x)\text{ prime}}}\mathcal{F}(N(P(x)))$$

$$= \frac{1}{\zeta_q(2)}\prod_{i=1}^{\infty}\frac{\zeta_q(3^i)}{\zeta_q(2\cdot 3^i)} \tag{3.8}$$

$$= \left(1-\frac{1}{q}\right)\prod_{i=1}^{\infty}\prod_{n=1}^{\infty}\left(1+q^{-3^i n}\right)^{m(n,q)} \tag{3.9}$$

$$= \left(1-\frac{1}{q}\right)\prod_{i=1}^{\infty}\frac{1-q^{1-2\cdot 3^i}}{1-q^{1-3^i}}. \tag{3.10}$$

Proof. The first equation follows from (3.6) and the Chinese Remainder Theorem. The second comes follows from some algebraic manipulation:

$$
\begin{aligned}
d(G_{3,q}^*) &= \prod_{\substack{P(x)\in\mathbb{F}_q[x]\\P(x)\text{ prime}}}\left[\left(1-\frac{1}{N(P(x))}\right)\prod_{i=0}^{\infty}\left(1+\frac{1}{N(P(x))^{3^i}}\right)\right]\\
&= \prod_{\substack{P(x)\in\mathbb{F}_q[x]\\P(x)\text{ prime}}}\left[\left(1-\frac{1}{N(P(x))^2}\right)\prod_{i=1}^{\infty}\left(\frac{1-\frac{1}{N(P(x))^{2\cdot3^i}}}{1-\frac{1}{N(P(x))^{3^i}}}\right)\right]\\
&= \frac{1}{\zeta_q(2)}\prod_{i=1}^{\infty}\prod_{\substack{P(x)\in\mathbb{F}_q[x]\\P(x)\text{ prime}}}\left(\frac{1-N(P(x))^{-2\cdot3^i}}{1-N(P(x))^{-3^i}}\right)\\
&= \frac{1}{\zeta_q(2)}\prod_{i=1}^{\infty}\frac{\zeta_q(3^i)}{\zeta_q(2\cdot3^i)}. \tag{3.11}
\end{aligned}
$$

The third and fourth equations follow by expanding (3.11) using the two representations for $\zeta_q(s)$ in Lemma 3.6, and the fact that $1/\zeta_q(2) = (1-q^{-1})$:

$$
\begin{aligned}
d(G_{3,q}^*) &= \frac{1}{\zeta_q(2)}\prod_{i=1}^{\infty}\frac{\zeta_q(3^i)}{\zeta_q(2\cdot3^i)}\\
&= \left(1-\frac{1}{q}\right)\prod_{i=1}^{\infty}\prod_{n=1}^{\infty}\left(\frac{1-q^{-2\cdot3^i n}}{1-q^{-3^i n}}\right)^{m(n,q)}\\
&= \left(1-\frac{1}{q}\right)\prod_{i=1}^{\infty}\prod_{n=1}^{\infty}\left(1+q^{-3^i n}\right)^{m(n,q)}. \tag{3.12}
\end{aligned}
$$

$\square$

We can truncate the product in the fourth expression at a suitable length to calculate $d(G_{3,q}^*)$ for various finite fields up to a few decimal places. See Table 1 and Figure 1.

Table 1. Density of $G_{3,q}^*$, calculated to 6 decimal places.

q	$d(G_{3,q}^*)$		q	$d(G_{3,q}^*)$		q	$d(G_{3,q}^*)$		q	$d(G_{3,q}^*)$
2	.648361		3	.747027		5	.833069		7	.874948
4	.799231		9	.899985		25	.961538		49	.980000
8	.888862		27	.964286		125	.992063		343	.997093

4. LOWER BOUNDS FOR THE SUPREMUM OF UPPER DENSITIES

We start with a definition of upper density (see Definition 3.2 for the definition of $S(n)$ and $S(X)$ for $X \subseteq \mathbb{R}_{\geq0}$).

Definition 4.1. *The upper density of a set $A \subseteq \mathbb{F}_q$ is defined as*

$$
\overline{d}(A) = \limsup_{n\to\infty}\frac{|A\cap S(n)|}{|S(n)|}. \tag{4.1}
$$

Figure 1. Density of $G_{3,q}^*$ graphed against q for $q \leq 130$.
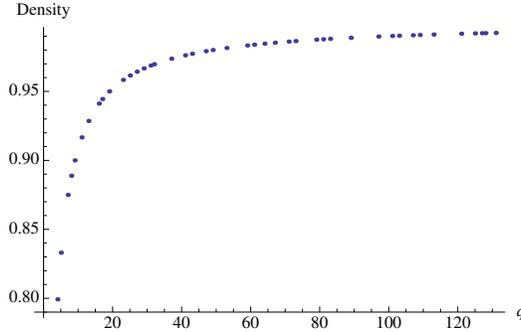
Since the sequence on the right hand side is bounded and monotonic, $\overline{d}(A)$ always exists.

As in prior work studying this problem over other rings, we wish to study the supremum of the upper densities of 3-term non-unit geometric progression-free sets. In particular, we would like to see how much larger the upper density of such a subset of $\mathbb{F}_q[x]$ can be made compared to the density of the set obtained greedily. By constructing a set of sizeable upper density we get lower bounds for this supremum. Our method is similar to that of the construction given first in [Mc] and generalized in [BHMMPTW]. However, due to the very nice multiplicative structure of the possible norms in $\mathbb{F}_q[x]$, (i.e., they are all of the form q^n) we can actually do slightly better than what is obtained by simply applying the technique used there to this setting.

The idea behind this construction is to leave certain gaps between norms of elements included in the set in order to guarantee that the set will be free of some geometric progressions.

Lemma 4.2. *The elements of $A_3^*(\mathbb{Z}) = \{0, 1, 3, 4, 9, 10, 12, 13, \ldots\}$, the set constructed by greedily avoiding 3-term arithmetic progressions in $\mathbb{Z}$, are exactly those containing no 2 in their ternary expansion*

This is well known, see for example [Ran].

Theorem 4.3. *Fix q and consider the set $S(T_{3,q})$ where*

$$T_{3,q} := \{q^n : n \in A_3^*(\mathbb{Z})\}. \tag{4.2}$$

This set is free of geometric progressions and has upper density

$$\overline{d}(S(T_{3,q})) = m_q := \left(1 - q^{-2}\right) \prod_{i=1}^{\infty} \left(1 + q^{-3^i}\right). \tag{4.3}$$

Furthermore, no geometric-progression-free subset of $\mathbb{F}_q[x]$ of the form $S(X)$ for some $X \subset \mathbb{R}$ has upper density greater than m_q.

Proof. If the elements b, rb, $r^2 b$ in $\mathbb{F}_q[x]$ are in geometric progression, then the degrees of these polynomials form an arithmetic progression $\deg(b)$, $\deg(b) + \deg(r)$, $\deg(b) + 2\deg(r)$ and, equivalently, their norms form a geometric progression, $N(b) = q^{\deg(b)}$, $N(rb) = q^{\deg(b)+\deg(r)}$, $N(r^2 b) = q^{\deg(b)+2\deg(r)}$ in $\mathbb{Z}$. Since the

set $T_{3,q}$ has no 3-term geometric progressions, $S(T_{3,q})$ cannot have any 3-term geometric progressions either.

We will first show $S(T_{3,q})$ has upper density at least m_q. For any $k \geq 1$ define

$$N_k := \sum_{i=0}^{k-1} 3^i = \frac{3^k - 1}{2}, \tag{4.4}$$

(note that each N_k is an integer which has only 1's in its ternary expansion) and consider the quantity $|S(T_{3,q}) \cap S(q^{N_k})|/|S(q^{N_k})|$:

$$\begin{aligned}
\frac{|S(T_{3,q}) \cap S(q^{N_k})|}{|S(q^{N_k})|} &= \frac{1}{q^{N_k+1}} \sum_{\substack{n \in A_3^*(\mathbb{Z}) \\ n \leq N_k}} |S(q^n)| \\
&= \frac{1}{q^{N_k+1}} \sum_{\substack{n \in A_3^*(\mathbb{Z}) \\ n \leq N_k}} \left(q^n - q^{n-1} \right) \\
&= \sum_{\substack{n \in A_3^*(\mathbb{Z}) \\ n \leq N_k}} \left(q^{-N_k+n} - q^{-N_k+n-1} \right). \tag{4.5}
\end{aligned}$$

Now, the key observation is that for these choices of N_k, the sets $A_3^*(\mathbb{Z}) \cap [0, N_k]$ have an important symmetry: $A_3^*(\mathbb{Z}) \cap [0, N_k] = N_k - (A_3^*(\mathbb{Z}) \cap [0, N_k])$. This follows from the characterization in Lemma 4.2. Thus

$$\begin{aligned}
\frac{|S(T_{3,q}) \cap S(q^{N_k})|}{|S(q^{N_k})|} &= \sum_{n \in A_3^*(\mathbb{Z}) \cap [0, N_k]} \left(q^{-N_k+n} - q^{-N_k+n-1} \right) \\
&= \sum_{n \in A_3^*(\mathbb{Z}) \cap [0, N_k]} \left(q^{-n} - q^{-n-1} \right). \tag{4.6}
\end{aligned}$$

Another way to view this is that the set $S(T_{3,q}) \cap S(q^{N_k})$ is equal to the set obtained by constructing a subset of the elements of $\mathbb{F}_q[x]$ of norm at most q^{N_k} by working backwards, starting with elements of largest possible norm and greedily including those that do not form a geometric progression.

Because $S(T_{3,q}) \cap S(q^{N_k})$ is an initial subset of the set $S(T_{3,q})$, we have that

$$\bar{d}(S(T_{3,q})) = \limsup \frac{|S(T_{3,q}) \cap S(N)|}{|S(N)|} \tag{4.7}$$

$$\geq \lim_{k \to \infty} \frac{|S(T_{3,q}) \cap S(q^{N_k})|}{|S(q^{N_k})|} = \sum_{n \in A_3^*(\mathbb{Z})} \left(q^{-n} - q^{-n-1} \right)$$

$$= \left(1 - \frac{1}{q} \right) \left(\sum_{n \in A_3^*(\mathbb{Z})} q^{-n} \right) = \left(1 - \frac{1}{q} \right) \prod_{i=0}^{\infty} \left(1 + q^{-3^i} \right)$$

$$= \left(1 - \frac{1}{q^2} \right) \prod_{i=1}^{\infty} \left(1 + q^{-3^i} \right) = m_q. \tag{4.8}$$

Now we show that the upper density of $S(T_{3,q})$ is at most m_q by showing that any geometric-progression free set of the form $S(X)$ has upper density at most m_q. Fix $X \subset \mathbb{R}_{\geq 0}$ such that $S(X)$ is free of geometric progressions, and let $A = \{n \in \mathbb{Z}_{\geq 0} \mid q^n \in X\}$ be the set of degrees of the polynomials included in $S(X)$. As

argued before, the set of integers in A must be free of arithmetic progressions in order for $S(X)$ to be free of geometric progressions.

For any integer $M > 0$ consider $A \cap [0, M]$: the set of integers in A up to M. Note that the number $(q^{n+1} - q^n)$ of polynomials of degree n in $\mathbb{F}_q[x]$ is greater than or equal to the number of polynomials of all lower degrees combined (q^n). Thus we see that $|S(X) \cap S(q^m)|$ is maximized when $A \cap [0, M] = M - (A_3^* \cap [0, M])$, because the largest possible set is obtained by including all polynomials of degree M, and subsequently greedily including polynomials of the largest possible degree that can be added without introducing a geometric progression. Thus

$$\frac{|S(X) \cap S(q^M)|}{|S(q^m)|} \leq \frac{1}{q^{M+1}} \sum_{\substack{i \in A_3^*(\mathbb{Z}) \\ i \leq M}} q^{m-i+1} - q^{m-i} = \sum_{\substack{i \in A_3^*(\mathbb{Z}) \\ i \leq M}} \left(q^{-i} - q^{-i-1} \right), \quad (4.9)$$

and so

$$\overline{d}(S(X)) \;=\; \limsup \frac{|S(X) \cap S(q^M)|}{|S(q^M)|} \;\leq\; \sum_{i \in A_3^*(\mathbb{Z})} \left(q^{-i} - q^{-i-1} \right) = m_q, \quad (4.10)$$

as seen in (4.7). Therefore, in particular, $\overline{d}(S(T_{3,q})) \leq m_q$. $\qquad\square$

As an immediate corollary, we get a lower bound on the upper densities of sets avoiding 3-term geometric progressions in $\mathbb{F}_q[x]$. Note, however, that any such set with higher upper density could not be of the form $S(X)$ for any $X \subset \mathbb{R}_{\geq 0}$.

Corollary 4.4. *The supremum of the upper densities of sets avoiding 3-term geometric progressions in $\mathbb{F}_q[x]$ is greater than or equal to*

$$m_q \;=\; \left(1 - q^{-2}\right) \prod_{i=1}^{\infty} \left(1 + q^{-3^i}\right) \quad (4.11)$$

$$=\; \left(1 - q^{-2}\right) + \left(q^{-3} - q^{-5}\right) + \left(q^{-9} - q^{-11}\right) + \left(q^{-12} - q^{-14}\right) + \cdots.$$

Some values are computed in Table 2.

Table 2. Lower bounds m_q on the supremum of upper densities of subsets of $\mathbb{F}_q[x]$ avoiding 3-term progressions, calculated to 6 decimal places.

q	Lower Bd		q	Lower Bd		q	Lower Bd		q	Lower Bd
2	.845398		3	.921858		5	.96768		7	.982448
4	.952152		9	.989009		25	.998464		49	.999592
8	.986298		27	.998679		125	.999937		343	.999992

5. Upper Bounds for the Supremum of Upper Densities

In order to obtain upper bounds for the upper density of a set of polynomials avoiding 3 term geometric progressions, we again use techniques analogous to those developed by Riddell [Rid] over the integers. To compute an upper bound for this upper density, we show that a certain proportion of elements must be excluded in order to avoid three term progressions, using the fact that the minimum non-unit norm is q.

5.1. An Upper Bound for the Upper Density. Fix an element $r \in \mathbb{F}_q[x]$ of norm q. The proportion of elements of $\mathbb{F}_q[x]$ that are coprime to r is $(q-1)/q$. Now consider $S(q^n)$. For each $b \in S(q^n)$ such that b is coprime to r and $N(b) \leq q^{n-2}$, we have that b, br, br^2 forms a progression, and that all of these progressions $\{b, \ br, \ br^2\}$ are disjoint for different choices of b. Thus, for each such b, one of the elements b, br, br^2 must be excluded in order to avoid geometric progressions.

This can be extended further by including additional progressions. If b is coprime to r and $N(b) \leq q^{n-5}$, then $\{br^3, \ br^4, \ br^5\}$ forms another progression. These progressions are also disjoint for different choices of b as well as from the previously mentioned progressions. Taking these two sets of progressions into account and letting $n \to \infty$, we get an upper bound of

$$\lim_{n\to\infty} 1 - \left(\frac{q-1}{q}\right)\left(\frac{\left|S(q^{n-2})\right| + \left|S(q^{n-5})\right|}{\left|S(q^n)\right|}\right) \ = \ 1 - \left(\frac{q-1}{q}\right)(q^{-2} + q^{-5}) \quad (5.1)$$

for the supremum of upper densities of geometric progression free subsets of $\mathbb{F}_q[x]$. By continuing in this way, including additional non-overlapping progressions we can improve this bound to

$$1 - \left(\frac{q-1}{q}\right)\left(\frac{1}{q^2}\right)\left(\sum_{i=0}^{\infty} \frac{1}{q^{3i}}\right) \ = \ 1 - \left(\frac{q-1}{q^3}\right)\left(\frac{1}{1-\frac{1}{q^3}}\right) \ = \ 1 - \frac{q-1}{q^3-1}. \quad (5.2)$$

We remark that this can be improved further by applying the methods of Nathanson and O'Bryant [NO2]. In particular, one observes that removing one out of every three consecutive integers is still insufficient to avoid arithmetic progressions in the exponents of r, and thus to avoid progressions involving r. Taking into account all progressions involving powers of the element r as in [NO] we obtain the upper bound

$$(q-1)\sum_{n=1}^{\infty} \frac{1}{q^{r_n}}, \quad (5.3)$$

where r_n is the least integer such that there exists a subset of $[1, r_n]$ of size n. (The sequence of r_n begins $(r_n) = 1, 2, 4, 5, 9, 11, 13, 14, 20, \ldots$)

This results in a small numerical improvement in the upper bounds. Some values are computed in Table 3.

6. Future Work

Naturally, one would like to be able to further tighten these bounds by either finding sets avoiding geometric progressions with larger densities and upper densities or by improving the upper bounds. We end with a few ideas for how this might be done, and additional areas to explore.

In the integer case [Mc] it has been shown that it is possible to construct sets with higher upper density than the greedy set (but only when avoiding progressions with integer ratio). In particular, by looking at the small primes 2, 3 and 5, it is possible to obtain a small improvement on the greedy set by removing those elements with only a single power of 2, 3 or 5, which allows other elements to be included later on. It seems possible that a similar construction might yield a set with higher asymptotic density in $\mathbb{F}_q[x]$.

Table 3. Upper bounds obtained using (5.2) and (5.3) on the supremum of upper densities of subsets of $\mathbb{F}_q[x]$ avoiding 3-term progressions, rounded to 9 decimal places, and compared to the lower bounds from (4.11).

q	Upper Bd (5.2)	Upper Bound (5.3)	Lower Bound (4.11)
2	0.857142857	0.846375541	0.845397956
3	0.923076923	0.921925273	0.921857532
4	0.952380952	0.952160653	0.952152070
5	0.967741935	0.967682134	0.967680495
7	0.982456140	0.982447941	0.982447814
8	0.986301370	0.986297660	0.986297615
9	0.989010989	0.989009149	0.989009131
11	0.992481203	0.992480647	0.992480643
13	0.994535519	0.994535314	0.994535313
16	0.996336996	0.996336937	0.996336937
17	0.996742671	0.996742630	0.996742630
19	0.997375328	0.997375307	0.997375307
23	0.998191682	0.998191675	0.998191675
25	0.998463902	0.998463898	0.998463898

Question 6.1. *Can a set be constructed to avoid 3-term progressions in $\mathbb{F}_q[x]$ with density greater than $G_{3,q}^*$? If so, can this be done for all q or only for q small enough/large enough?*

In previous work, [Mc, Section 4], [BHMMPTW, Section 4.2] improved upper bounds for the upper density were obtained by considering progressions among the smooth integers. Thus far this technique has not proven to be as useful in this ring, however it may just be that more work and computation are required.

Question 6.2. *Can bounds for progressions among the smooth polynomials (those without any irreducible factors of high degree) be used to obtain better upper bounds for the upper density of a subset of $\mathbb{F}_q[x]$ avoiding 3-term geometric progressions?*

In general, one might even hope that the additional combinatorial structure of the ring $\mathbb{F}_q[x]$ might allow one to work out exactly what the supremum of the upper densities of sets avoiding 3-term progressions is, even though this appears to be far more elusive over the integers. From the tightness of our bounds and the natural construction of $S(T_{3,q})$, we propose:

Conjecture 6.3. *The supremum of upper densities of sets avoiding 3-term geometric progressions in $\mathbb{F}_q[x]$ is m_q.*

A similar trick to the second half of Theorem 4.3 proves useful: if the upper density up to some norm q^N of a set S is higher than m_q, then there must be some q^B such that all polynomials of norm q^B are in $S(T_{3,q})$ and some, but not all, are in S. Then there is some factorization pattern of elements of norm q^B that are all not excluded. Most likely, the divisors of the factorization will either have too many geometric progressions or there will be too few to begin with.

It appears likely that considering progressions among the smooth polynomials would be far more useful in $\mathbb{F}_q[x]$ for studying subsets that avoid progressions with

rational ratio (where one allows the geometric progressions to have common ratio in $\mathbb{F}_q(x)$ rather than just $\mathbb{F}_q[x]$). This version of the problem has also been well studied over the integers, however we have chosen not to consider such progressions here. This would be interesting to consider in further work as well.

Question 6.4. *How does the situation described here change if one allows geometric progressions with ratio in $\mathbb{F}_q(x)$? Note that the greedy set $G_{3,q}^*$ constructed here avoids such progressions as well, but the set $T_{3,q}$ does not. Can we find an example of a set avoiding such rational-ratio progressions with larger upper density than $G_{3,q}^*$?*

Question 6.5. *What happens if the field $\mathbb{F}_q$ is replaced by a different finite ring?*

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