OPTIMAL POINT SETS DETERMINING FEW DISTINCT ANGLES

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ABSTRACT. Let $P(k)$ denote the largest size of a non-collinear point set in the plane admitting at most k angles. We prove $P(1) = 3$, $P(2) = 5$ and $P(3) = 5$, and we characterize the optimal sets. We also leverage results from [FHJ+23] in order to provide the general bounds of $k+2 \leq P(k) \leq 6k$, although the upper bound may be improved pending progress toward the Weak Dirac Conjecture. It is surprising that $P(k) = \Theta(k)$ since, in the distance setting, the best known upper bound on the analogous quantity is quadratic and no lower bound is well-understood.

CONTENTS

1. INTRODUCTION

1.1. **Background.** In 1946, Erdős introduced the problem of finding asymptotic bounds on the minimum number of distinct distances among sets of n points in the plane [Erd46]. The Erdős distance problem, as it has become known, proved infamously difficult and was only finally (essentially) resolved by Guth and Katz in 2015 [GK15].

The Erdős distance problem has also spawned a wide variety of related questions, including the problem of finding maximal point sets with at most k distinct distances. Erdős and Fishburn determine maximal planar sets with at most k distinct distances [EF96]. Recent results by Szöllősi and Östergård classify the maximal 3-distance sets in $\mathbb{R}^4$, 4-distance sets in $\mathbb{R}^3$, and 6-distance sets in $\mathbb{R}^2$ [Xia12, SO20]. In [ELM+18, BDP+21, BDP+20] point sets with a low number of distinct triangles in Euclidean space are investigated. In [FHJ+23], a number of angle analogues of distinct distance problems are considered. Recently, new connections to frame theory and engineering have renewed interest in few-distance sets [SO20].

Characterizing the largest possible point sets satisfying a given property in this way is a classic problem in discrete geometry. As another example, Erdős introduced the problem of finding maximal point sets of all

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isosceles triangles in 1947 [EK47]. Ionin completely answers this question in Euclidean space of dimension at most 7 [Ion09].

We study one variation of a related problem of Erdős and Purdy [EP95]. They asked about $A(n)$, the minimum number of distinct angles formed by n not-all-collinear points in the plane. Recently, [FHJ$^+$23, FKM$^+$22] made partial progress on this problem, and the best known bounds are $n/6 \leq A(n) \leq n-2$. We consider the related problem of maximal planar point sets admitting at most k distinct angles in $(0, \pi)$. We ignore angles of 0 and π so as to align the convention in related research (see [PS92], for example), although we provide results including the 0 angle as corollaries. We completely answer this question for $k = 2$ and $k = 3$ and note that the work from [FHJ$^+$23] immediately implies asymptotically tight linear bounds for $k > 3$. In answering this question for $k = 2$ and $k = 3$, we systematically consider all possible triangles in such configurations and then reduce to adding points in a finite number of positions by geometric casework. We both find $P(2)$ and $P(3)$ and classify all optimal configurations.

1.2. **Definitions and Results.** By convention, we only count angles of magnitude strictly between 0 and π. Our computations still answer the related optimal point configuration questions including 0 angles (see Corollaries 3.1, 4.4). We begin by introducing convenient notation:

Definition 1.1. *Let $\mathcal{P} \subset \mathbb{R}^2$. Then*

$$A(\mathcal{P}) := \#\{|\angle abc| \in (0, \pi) \,:\, a, b, c \text{ distinct, } a, b, c \in \mathcal{P}\},$$

Now we define the quantity we are interested in studying.

Definition 1.2.

$$P(k) := \max\{\#\mathcal{P} \,:\, \mathcal{P} \subseteq \mathbb{R}^2, \text{ not all points in } \mathcal{P} \text{ are collinear, } A(\mathcal{P}) \leq k\}.$$

We first provide general linear lower and upper bounds for $P(k)$. In particular, we have the following theorem.

Theorem 1.3. *For all $k \geq 1$,*

$$\begin{aligned} 2k+3 \;&\leq\; P(2k) \qquad\;\; \leq\; 12k \\ 2k+3 \;&\leq\; P(2k+1) \leq\; 12k+6. \end{aligned}$$

In the distance setting, the best known upper bound on the analogous parameter is the quadratic $(2 + k)(1 + k)$, and no lower bound is well-understood [SO20]. It is therefore interesting and surprising that we find $P(k) = \Theta(k)$ in the angle setting. We prove Theorem 1.3 in Section 2.

Furthermore, we explicitly compute $P(1)$, $P(2)$, and $P(3)$ and exhaustively identify all extremal point configurations for each.

Proposition 1.4. *We have $P(1) = 3$, and the equilateral triangle is the unique extremal configuration.*

In order to have only a single angle, every triangle of three points in the configuration must be equilateral. As this is impossible for point configurations that are not the vertices of an equilateral triangle, $P(1) = 3$. $P(2)$ and $P(3)$ are considerably less trivial quantities. We calculate $P(2)$ and $P(3)$ via exhaustive casework, simultaneously characterizing all of the unique optimal point configurations up to rigid motion transformations and dilation about the center of the configuration. We proceed by first considering sets of three points and then characterize the additional points that may be added without determining too many angles. We prove Theorem 1.5 in Section 3 and Theorem 1.6 in Section 4.

Theorem 1.5. *We have $P(2) = 5$. Moreover, the unique optimal point configuration is four vertices in a square with a fifth point at its center (see* **I** *in Figure 1).*

Theorem 1.6. *We have $P(3) = 5$. There are 5 unique optimal configurations, shown in Figure 1.*

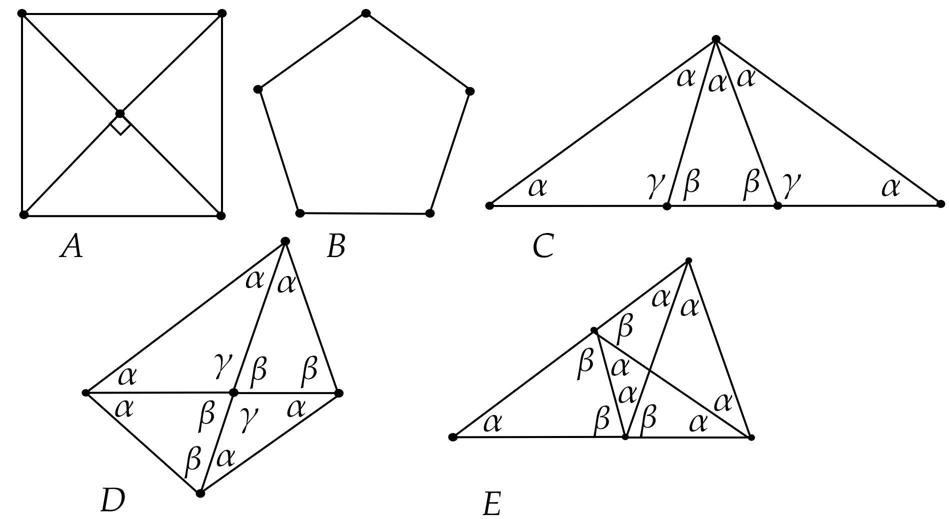

FIGURE 1. Optimal Two and Three Angle Configurations. $\alpha = \frac{\pi}{5}, \beta = \frac{2\pi}{5}, \gamma = \frac{3\pi}{5}$.

2. GENERAL BOUNDS

Although one may in principle calculate $P(k)$ for any k by extensive casework (as we later calculate $P(2), P(3)$), it quickly becomes overwhelming. As such, we instead provide general bounds on $P(k)$. In [FHJ$^+$23] the authors study the quantity $A(n)$, the minimum number of angles admitted by a non-collinear point set of n points in the plane. They show in Lemma 2.2 and Theorem 2.5 that $n/6 \leq A(n) \leq n-2$, noting that the lower bound may be improved up to as much as $n/4-1$, pending progress on the Weak Dirac Conjecture. Since $A(n) \leq n-2$, then $n \geq A(n)+2$, and so we deduce that $P(k) \geq k+2$. Similarly, we have $P(k) \leq 6k$. Combining these bounds gives the desired result

Proposition 2.1. $k+2 \leq P(k) \leq 6k$.

Notably, it is surprising that $P(k) = \Theta(k)$ since, in the distance setting, the best known upper bound on the analogous quantity is quadratic, and no lower bound is well-understood.

3. PROOF OF THEOREM 1.5

Proof. In any point configuration with at least three points, there are triangles. For any point configuration with at most two angles, all triangles must be isosceles. We divide into two cases based on whether or not there is an equilateral triangle. Unless otherwise specified, when considering points belonging to some region, we consider the interior of that region. Oftentimes the boundaries must be treated separately.

3.1. **There is an equilateral triangle.** We consider adding a fourth point in cases (Figure 2).

***Case 1*:** $p \in A$.
Then $\angle acp < \pi/3$ and $\angle cap > \pi/3$, leading to more than two angles.

***Case 2*:** $p \in \overline{ab}$.
Then $\angle bcp < \pi/3$ and one of $\angle cpb$ and $\angle apc \geq \pi/2$, leading to more than two angles.

***Case 3*:** $p \in \overleftrightarrow{ac}$ to the upper-right of a.
Then $\angle cbp > \pi/3$ and $\angle cpb < \pi/3$, again leading to more than two angles. The case for $p \in \overleftrightarrow{bc}$ to the right of b follows by symmetry.

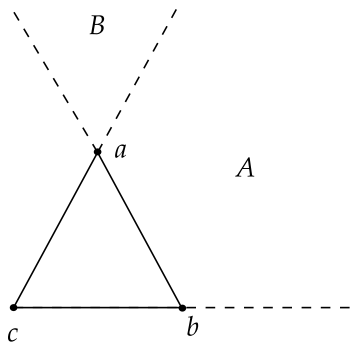

FIGURE 2. Equilateral Triangle Regions

***Case 4*:** $p \in B$.

In this case, $\angle cbp > \pi/3$ and $\angle cpb < \pi/3$, leading to more than two angles.

***Case 5*:** p is in the interior of $\triangle abc$.

In this case, one of $\angle apb, \angle bpc, \angle cpa \geq 2\pi/3$ and $\angle acp < \pi/3$, leading to more than two angles.

Up to symmetry, these cases are exhaustive. Thus if there is an equilateral triangle in the configuration, there can only be at most three points.

3.2. **There is no equilateral triangle.** Now, let a, b, and c be the vertices of an isosceles triangle with base angle β and α the apex vertex. We reduce the number of possibilities for additional points by partitioning the plane into regions A_i (Figure 3). Note that we may without loss of generality assume that no fourth point

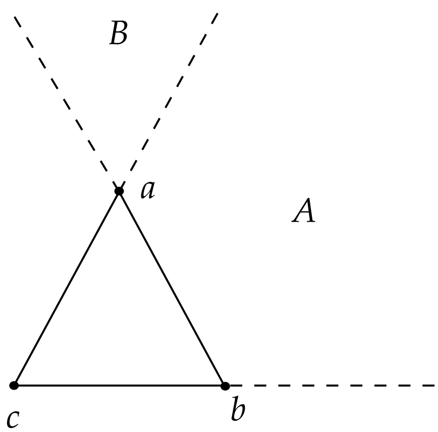

FIGURE 3. Isosceles Triangle Regions.

is added within the interior of $\triangle abc$ as we could then choose one of the resultant interior triangles as our initial triangle. Also note that A_1 and A_1', A_3 and A_3', and $\overleftrightarrow{ac}$ and $\overleftrightarrow{ab}$ are equivalent up to symmetry.

***Case 1*:** $p \in A_1$.

In this case, $\angle pab > \alpha$ and $\angle pcb > \beta$. So, regardless of whether α or β is greater, adding p introduces an additional angle. So, no additional points can be in A_1 or A_1'.

***Case 2*:** $p \in A_2$.

In this case, $\angle pcb$ and $\angle pbc$ are greater than β, so both must be α to not add additional angles. But then $\angle cpb = \pi - 2\alpha \neq \beta$. Then, in order to not add additional angles, we must have $3\alpha = \pi$. But, this implies $\triangle pcb$ is an equilateral triangle. Thus no points may be added in A_2.

Case 3: $p \in A_3$.

In this case, $\angle bap > \alpha$ and $\angle abp > \beta$, so there is an additional angle added regardless and no additional points are possible.

Case 4: $p \in A_4$.

In this case, $\angle cap, \angle bap < \alpha$, so both must equal β. Therefore, $2\beta = \alpha$, which implies $\beta = \pi/4$ and $\alpha = \pi/2$. Moreover, since $\angle acp$ and $\angle abp$ are greater than β, they must both equal $\alpha = \pi/2$. So, the only possibility for an addable point in this case is for p to be the fourth vertex of the square $acpb$.

Case 5: $p \in \overleftrightarrow{bc}$.

If p is on $\overleftrightarrow{bc}$ between b and c, then $\angle cap, \angle bap < \alpha$. In order for these not to introduce additional angles, they must both be equal to β. This implies $\beta = \pi/4$ and $\alpha = \pi/2$ and p is the center of the side bc. If $p \in \overleftrightarrow{bc}$ to the left of c (or by symmetry, right of b), $\angle bap > \alpha$ and thus $\angle bap = \beta$. Since $2\beta + \alpha = \pi$, $\beta < \pi/2$. But then $\angle acp > \pi/2 > \beta > \alpha$, and hence we would have more than two angles. Thus there is exactly one point possible on line $\overleftrightarrow{bc}$, the centerpoint of the edge between b and c.

Case 6: $p \in \overleftrightarrow{ac}$.

If p is between a and c, then $\angle cbp < \beta$ and thus $\angle cbp = \alpha$. But, as before, $\beta < \pi/2$. Moreover, one of $\angle bpc$ or $\angle bpa$ is at least $\pi/2 > \beta > \alpha$. Thus there are too many angles in this case. If p is to the bottom left of c, $\angle apb < \beta$ and thus $\angle apb = \alpha$. But, again, either $\angle bca$ or $\angle bcp > \pi/2 > \beta$, creating too many angles in this case. If p is on $\overleftrightarrow{ac}$ to the upper right of a, $\angle pbc > \beta$ and thus equals α. Then $\angle pba < \alpha$ and must equal β and thus $2\beta = \alpha$. This implies $\beta = \pi/4$ and $\alpha = \pi/2$ and $\triangle cbp$ is an isosceles right triangle with b the apex vertex, p on $\overleftrightarrow{ac}$ to the upper right of a, and a at the center of side $\overline{pc}$.

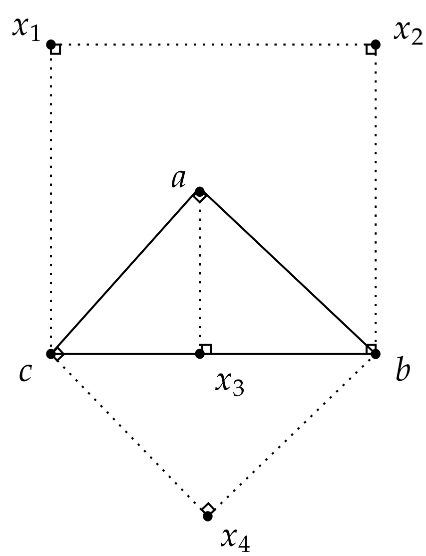

FIGURE 4. Compatible Points with the Right Triangle.

As such, in order to add additional points to an isosceles triangle point configuration without adding additional angles, we must have $\alpha = \pi/2$ and $\beta = \pi/4$. The four additional possible points are marked in Figure 4.

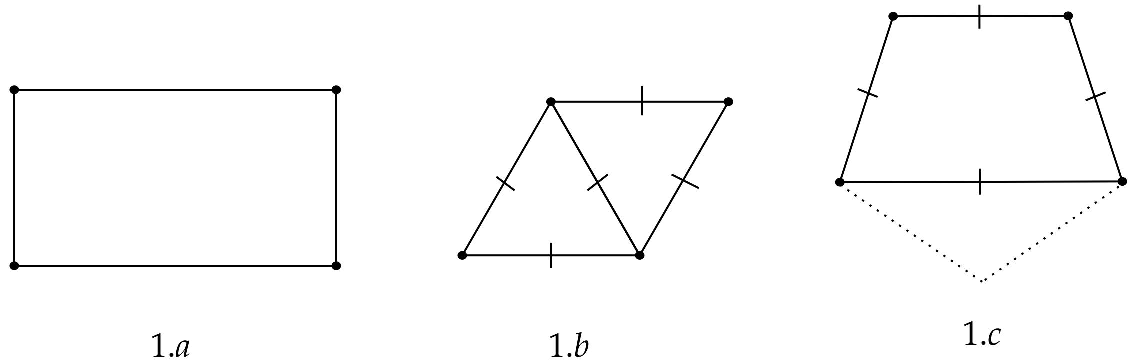

FIGURE 5. Configurations of points in a convex quadrilateral defining at most three distinct angles.

Note that $\angle x_4ax_1, \angle x_4ax_2 > \pi/2$. So, x_4 cannot be in the same point configuration as x_1 or x_2. The same follows for x_3. However, we may have both x_1 and x_2 or both x_3 and x_4, either of which give the unique extremal configuration **I** in Figure 1. □

Corollary 3.1. *One might also wish to include the trivial 0-angle in our count. In this case,* $P(2) = 4$*, and the unique configuration is the square.*

Proof. The only 5-point configuration no longer holds when we count the 0-angle. Figure 4 displays all valid four point configurations which define only 2 angles excluding 0, as detailed in the proof of $P(2)$. All the shown points but x_4 define a 0-angle, so the only valid 4 point configuration is the square. □

4. PROOF OF THEOREM 1.6

Lemma 4.1. *Let* $ABCD$ *be a convex quadrilateral defining three angles or fewer. Then, they form one of the three configurations of Figure 5, where* $1.a$ *is a rectangle,* $1.b$ *is two attached equilateral triangles, and* $1.c$ *is four of the five vertices of a regular pentagon.*

Proof. Assume we are not in case $1.a$, so that the angles of the quadrilateral are not all $\pi/2$. In particular, there is at least one obtuse angle, γ, and one acute angle, β. Any angle α formed by splitting β is less than β and thus must be exactly $\beta/2$ so as not to create two additional angles for a total of four. These three angles $\alpha = \beta/2, \beta, \gamma$ are then exactly the three angles in the configuration. Now we consider each of the four cases of placing β and γ about the quadrilateral, with the first listed angle corresponding to vertex A, the second to B, and so on and with A, B, C, and D in clockwise cyclic order.

Case $\gamma\beta\gamma\beta$**:** Equal opposite angles implies the quadrilateral is a parallelogram. The fact that BD bisects the two β angles implies that $ABCD$ is in fact a rhombus. Thus, AC also bisects the γ angles, implying that $\gamma/2 = \beta$. So, $6\beta = 2\pi$ and $\alpha = \pi/6, \beta = \pi/3$, and $\gamma = 2\pi/3$ in this case. Given that $ABCD$ is a rhombus, the configuration in this case is similar to $1.b$.

Case $\gamma\gamma\beta\beta$**:** Note that we have $\gamma + \beta = \pi$ from the angle sum of the quadrilateral. This implies that AB and CD are parallel. So, by analyzing the alternate interior angles given by the transversal AC, we have $\gamma = \alpha + (\gamma - \alpha)$, where $\alpha = \angle CAB$ and $\gamma - \alpha = \angle CAD$. Thus, $\gamma - \alpha = \beta$ and $3\beta/2 = \gamma$, so $\alpha = \pi/5, \beta = 2\pi/5$, and $\gamma = 3\pi/5$. Then by considering isosceles triangles DAB and ABC, we see that segments $\overline{DA}, \overline{AB}$, and $\overline{BC}$ are all of equal length. Thus, the configuration is similar to $1.c$ in this case.

Case $\gamma\gamma\gamma\beta$**:** Diagonal BD bisects the angle β. Then, since the sum of the angles of $\triangle BCD$ and $\triangle ABD$ are both π and $\beta \neq \gamma$, we must have $\angle ABD = \angle DBC = \beta$. The diagonal AC must then also bisect angles $\angle DAB$ and $\angle DCB$ or else yield more than three distinct angles. But then, $3\beta = \pi = 4\beta$ from the angle sums of $\triangle ACD$ and $\triangle ABC$, a contradiction.

Case $\beta\beta\beta\gamma$: By an analogous argument as in the previous case, we must have that diagonal BD bisects the angle γ at D and $\angle ABD = \angle DBC = \beta$. But then, $4\alpha = \pi = 6\alpha$ by looking at the angle sums of $\triangle ABD$ and $\triangle BCD$, a contradiction.

□

To handle the configurations without convex quadrilaterals, we will make use of the following proposition.

Proposition 4.2. *Let A, B, C, D be points such that D is contained in the interior of $\triangle ABC$ and the configuration induces at most three distinct angles. Then, $\triangle ABC$ must be equilateral and D must be in the center of $\triangle ABC$.*

Proof. Note that $\angle ADB > \angle ACB > \angle ACD$. This is similarly true of $\angle BDC, \angle BAC, \angle BAD$ and of $\angle ADC, \angle ABC, \angle ABD$. Symmetry and the maximum of three distinct angles then allows the completion of all angles in the configuration, finishing the proof. See Figure 6. □

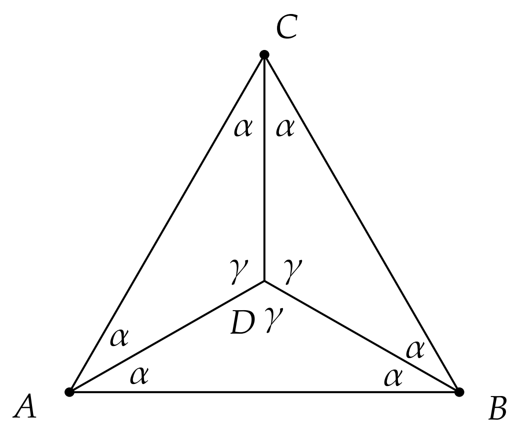

FIGURE 6. Resultant Triangular Configuration

Lemma 4.3. *Let A, B, C, D, and E be five points such that their convex hull is $\triangle ABC$, no four of them form a convex quadrilateral, and the configuration induces at most 3 distinct angles. Then, there is only one possible configuration.*

2.a) *The stereographic projection of the points of a regular pentagon onto a line,* **III** *of Figure 1.*

Proof. We proceed by casework on the number of points in the interior of $\triangle ABC$.

No points in the interior of $\triangle ABC$: If neither D nor E are in the interior of $\triangle ABC$ then, since the convex hull of the five points is $\triangle ABC$, D and E must both be on the edges of $\triangle ABC$. If they are not on the same side of the triangle, then the quadrilateral formed by D, E, and the ends of the edge which neither D nor E lie on is convex, yielding a contradiction.

Now, suppose without loss of generality that D and E lie on $\overline{AB}$ with the order of the points being A, D, E, and then B. Three distinct angles are immediately induced in this case. Namely, $\angle ACD = \alpha < \angle ACE = \beta < \angle ACB = \gamma$. Since the difference between each pair of angles is also induced by this configuration, we have that $\beta = 2\alpha$ and $\gamma = 3\alpha$. Since $\angle ADC > \angle AEC > \angle ABC$, we have $\angle ADC = \gamma, \angle AEC = \beta$, and $\angle ABC = \alpha$. This is similarly true of $\angle CEB, \angle CDB$, and

$\angle CAB$ by symmetry. Thus, the angle sum of $\triangle ACB$ implies $5\alpha = \pi$ and thus $\alpha = \pi/5, \beta = 2\pi/5$, and $\gamma = 3\pi/5$. So, in this case the points are configuration $2.a$).

One point in the interior of $\triangle ABC$: Suppose without loss of generality that D is the point along an edge of $\triangle ABC$, say $\overline{AB}$. Then, E is in the interior of $\triangle ABC$. Now E must be on $\overline{CD}$ or else one of $ADEC$ or $BCED$ is a convex quadrilateral.

Now, from Proposition 4.2, $\triangle ABC$ must be equilateral and E must be the center of the triangle. This induces angles of $\pi/6, \pi/3, 2\pi/3$. However, D and E form a right angle, yielding more than three distinct angles. Hence, there are no valid configurations in this case.

Both points in the interior of $\triangle ABC$: From Proposition 4.2, $\triangle ABC$ must be equilateral and both D and E must be the center of the triangle, a contradiction.

□

Now we exhaustively check the points that may be added to the configurations given by Lemmas 4.1 and 4.3. All valid configurations of five points inducing at most five distinct angles arise from either adding a point to a configuration from Lemma 4.1 or the configuration given in 4.3. This is because the convex hull of the configuration must have at least three vertices (by definition of $P(k)$) and, if the convex hull has five vertices, any four of the vertices forms a convex quadrilateral.

1.a): Consider adding a point to configuration $1.a$, with the angles formed by the vertices of the rectangle being $\alpha < \beta < \gamma$ with $\alpha + \beta = \gamma = \pi/2$. Label the vertices of the rectangle shown in 1.a of Lemma 4.1 as A, B, C, and D starting from the top left as A and proceeding clockwise. Then, if a point E is added in the exterior of $ABCD$, it will form an obtuse angle with one edge of the angle being a side of the rectangle. For example, if E is added below $\overline{CD}$, then $\angle BCE$ is obtuse. If E is added to edge AB, then $\angle DEB$ is obtuse. It will similarly induce an obtuse angle if it is added to any other edge. Finally, if E is added to the interior of $ABCD$, then the only way E may be added without inducing an obtuse angle is if all the segments from E to the vertices of the rectangle form angles of $\pi/2$ with each other at E. However, this would imply that the diagonals of $ABCD$ intersect at E at a right angle, implying that $ABCD$ is a square.

So, the only valid configurations require that $ABCD$ form a square. Moreover, if $ABCD$ form a square, we can still not induce any obtuse angles. This is because the other two angles in any triangle with an obtuse angle could not both be $\pi/4$ (and cannot be $\pi/2$), yielding more than three distinct angles. Thus, the only extremal configuration in this case is adding a fifth point E as the centerpoint of a square, **I** of Figure 1.

1.b): In configuration $1.b$), the angles are all determined: $\alpha = \pi/6, \beta = \pi/3$, and $\gamma = 2\pi/3$. Let the points A, B, C, and D be in clockwise order around the configuration such that $\overline{AC}$ is the segment dividing the two equilateral triangles. In order to not contradict Lemma 4.1, any added point must be in the interior of the rhombus (no point may be added to decrease the number of vertices in the convex hull since $ABCD$ is a parallelogram). In order for E to not yield any angles smaller than α, E must be in the center of $\triangle ABC$ or $\triangle CDA$. However, in either case, this yields a new angle of $\pi/2$. So, no points may be added in this case.

1.c): As in $1.b$), the angles in $1.c$) are all determined: $\alpha = \pi/5, \beta = 2\pi/5$, and $\gamma = 3\pi/5$. Label the points $ABCD$ clockwise starting from the top left as in the diagram of $1.c$ in Lemma 4.1. In order to not violate Lemma 4.1, any added point must be in the interior of $ABCD$, must result in a triangular convex hull, or must be outside of $ABCD$ and have every convex quadrilateral in the configuration an instance of $1.c$). In the former case, in order to not add an angle smaller than α, E must be added at the intersection of $\overline{AC}$ and $\overline{BD}$. In the second case, E must be added at the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$. In the last case, the configuration with the added point cannot have a convex hull of a quadrilateral, as that quadrilateral could not be an instance of $1.c$). Thus, it must be a pentagon. In order to guarantee that every convex quadrilateral in the configuration is a copy of $1.c$), it must be regular. All three configurations (**II**, **IV**, and **V** of Figure 1) are valid, but are not mutually compatible as adding multiple of these points would form an angle of magnitude less than α.

2.a): As in Lemma 4.3, suppose that the convex hull of the configuration is $\triangle ABC$ with D and E on $\overline{AB}$ such that the points are in the order A, D, E, and then B.

If another point were added to this configuration, either the convex hull would remain a triangle or there would be four points which form a convex quadrilateral. In the former case, no point could be in the interior of a triangle, as that would force the angles to be as in Proposition 4.2, which they are not. Thus, an additional added point would have to be placed on an existing edge. It could not be placed on $\overline{AB}$, as it would split an angle of α. If it were placed on $\overline{AC}$ or $\overline{BC}$ it would form a convex quadrilateral with C, D, and E. Given the induced values of the angles in this case, that quadrilateral would have to be similar to configuration $1.c)$. However, from the prior casework, no configuration containing a similar copy of $1.c)$ may have more than five points. Hence, the only extremal configuration in this case is **III** of Figure 1.

Therefore, $P(3) = P(2) = 5$, with five optimal configurations as in Figure 1.

Corollary 4.4. *One might also wish to include the trivial 0-angle in our count. In this case,* $P(3) = 5$*, but the square with the center-point and the pentagon are now the only valid configurations.*

Proof. Since this is a more restricted setting, the set of valid five-point configurations be a subset of the configurations identified above. By direct inspection, the square with the center-point and the pentagon are the only of the five in Figure 1 which define only three angles. All the others define three angles greater than zero and also the 0-angle by collinearity. □

5. Future Work

While it seems possible to compute $P(k)$ by exhaustive casework for higher values of k, the casework quickly becomes overwhelming. Additionally, while it is potentially possible to repeat such methods in higher dimensions, the visualization of the proofs played a crucial role in this analysis.

Future work may tighten our upper bound on $P(k)$. However, we make the following conjecture.

Conjecture 5.1. *The lower bound on* $P(k)$ *in Theorem 1.3 is tight. Namely,* $P(2k) = 2k+3$ *and* $P(2k+1) = 2k+3$ *for all* $k \geq 1$.

Therefore, we believe that future work should improve the upper bound of $P(n) \leq 6n$, either via progress towards the Weak Dirac Conjecture (which would still fall short of our conjecture) or by some other means. Alternatively, future research may find a more efficient method of constructing viable point sets without the need for the exhaustive search we perform.

It is also an open problem to investigate $P(k)$ with point sets in more than two dimensions. Low angle configurations using variations of Lenz's construction, as in [FHJ+23], may yield insight into optimal structures in higher dimensions.

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