HIGHER ORDER FIBONACCI SEQUENCES FROM GENERALIZED SCHREIER SETS

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ABSTRACT. A Schreier set S is a subset of the natural numbers with $\min S \geq |S|$. It has been known that the sequence $(a_{1,n})$, where

$$a_{1,n} \ := \ |\{S \subseteq \mathbb{N} : \max S = n \text{ and } \min S \geq |S|\}|,$$

is the Fibonacci sequence. Generalizing this result, we prove that for all $p \in \mathbb{N}$, the sequence $(a_{p,n})$, where

$$a_{p,n} \ := \ |\{S \subseteq \mathbb{N} : \max S = n \text{ and } \min S \geq p|S|\}|,$$

has a linear recurrence relation of higher order. We investigate further by requiring that $\min_2 S \geq q|S|$, where $\min_2 S$ is the second smallest element of S. We prove a linear recurrence relation for the sequence $(a_{p,q,n})$, where

$$a_{p,q,n} \ := \ |\{S \subseteq \mathbb{N} : \max S = n, \min S \geq p|S| \text{ and } \min_2 S \geq q|S|\}|,$$

and discuss a curious relationship between $(a_{q,n})$ and $(a_{p,q,n})$.

1. INTRODUCTION

A Schreier set S is a subset of the natural numbers with $\min S \geq |S|$, and the Schreier family containing all Schreier sets is denoted by $\mathcal{S}_1$. Schreier defined them to solve a problem in Banach space theory in 1930 [Sch]. These sets were also independently discovered in combinatorics and are connected to Ramsey-type theorems for subsets of $\mathbb{N}$. An online post [UA] proved that the Fibonacci sequence appears if we count Schreier sets under certain conditions.

Define

$$M_{1,n} \ := \ \{S \in \mathcal{S}_1 \ : \ \max S = n\}.$$

Then $|M_{1,1}| = 1$, $|M_{1,2}| = 1$ and $|M_{1,n+2}| = |M_{1,n+1}| + |M_{1,n}|$ for all $n \geq 1$ [UA]. The proof uses two one-to-one mappings to argue about cardinalities of sets. We generalize this result by defining, for $p \in \mathbb{N}$,

$$\mathcal{S}_p \ := \ \{S \subseteq \mathbb{N} : \min S \geq p|S|\}, \quad \text{and} \quad M_{p,n} \ := \ \{S \in \mathcal{S}_p \ : \ \max S = n\},$$

and prove the following[1].

Theorem 1.1. *Given $p \in \mathbb{N}$, consider the sequence $(|M_{p,n}|)_{n=1}^{\infty}$. We have*

1. $|M_{p,n+p}| = \sum_{k=1}^{n+p-1} \sum_{j=0}^{k/p-2} \binom{n+p-k-1}{j} + 1$, *and*
2. *for $n \geq 1$, $|M_{p,n+p+1}| = |M_{p,n+p}| + |M_{p,n}|$.*

We call $(|M_{p,n}|)_{n=1}^{\infty}$ the generalized Schreier-Fibonacci sequence of order p.

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[1] Our definition of $\mathcal{S}_p$ is not the same as what used in Banach space theory to indicate the Schreier sets of order p [AA].

Another natural extension is to put an additional restriction on our set S; in particular, we require that $\min_2 S \geq q|S|$, where $\min_2 S$ is the second smallest element in S. We define

$$\mathcal{S}_{p,q} \ := \ \{S \subseteq \mathbb{N} \,:\, \min S \geq p|S| \text{ and } \min_2 S \geq q|S|\}.$$

For a given n, we consider the family of sets $M_{p,q,n} = \{S \in \mathcal{S}_{p,q} \,:\, \max S = n\}$. When a set has exactly one element, we take the element to be both the smallest and the second smallest. The following theorem gives an explicit formula to calculate $|M_{p,q,n}|$.

Theorem 1.2. *Given $p < q \in \mathbb{N}$, for the sequence $(|M_{p,q,n}|)_{n\,=\,1}^{\infty}$, we have $|M_{p,q,n}| = 0$ if $n \leq q-1$, $|M_{p,q,n}| = 1$ if $q \leq n \leq 2q-1$ and*

$$|M_{p,q,n}| \ = \ 1 + (n-2p) + \sum_{k\,=\,3}^{\frac{n+2}{q+1}} \sum_{i\,=\,qk}^{n+2-k} (i - pk)\binom{n-i-1}{k-3} \text{ if } \ n \geq 2q.$$

Theorem 1.3. *Fix $p < q \in \mathbb{N}$. Consider $(M_{q,n})_{n=1}^{\infty}$ and $(M_{p,q,n})_{n=1}^{\infty}$. For each $n \in \mathbb{N}$, define $a_n := |M_{p,q,n+q}|$. We have*

$$a_{n+q+1} \ = \ a_{n+q} + a_n + (q-p)|M_{q,n}|.$$

Note that when $p = q$, we have Theorem 1.1. We have the following corollary that shows a recurrence relation for the sequence $(|M_{p,q,n}|)_{n=1}^{\infty}$.

Corollary 1.4. *Fix $p < q$ in $\mathbb{N}$. For $n \in \mathbb{N}$, define $a_n := |M_{p,q,n+q}|$. We have*

$$a_{n+2q+2} \ = \ 2a_{n+2q+1} - a_{n+2q} + 2a_{n+q+1} - 2a_{n+q} - a_n.$$

Proof. Fix $n \in \mathbb{N}$. By Theorem 1.3, we have

$$a_{n+q+1} - a_{n+q} \ = \ a_n + (q-p)|M_{q,n}| \tag{1.1}$$
$$a_{n+2q+1} - a_{n+2q} \ = \ a_{n+q} + (q-p)|M_{q,n+q}| \tag{1.2}$$
$$a_{n+2q+2} - a_{n+2q+1} \ = \ a_{n+q+1} + (q-p)|M_{q,n+q+1}|. \tag{1.3}$$

By Theorem 1.1, we know that $|M_{q,n+q+1}| = |M_{q,n+q}| + |M_{q,n}|$. Subtract Equation (1.1) and Equation (1.2) from Equation (1.3) to finish the proof. □

Remark 1.5. For fixed p, q, Theorem 1.4 gives a recurrence relation of depth $2q+2$; interestingly, the depth is independent of p.

2. Proof of Theorem 1.1

Given a set S and a number a, define

$$a + S \ := \ \{a + s \,:\, s \in S\}.$$

In our proof, we partition $M_{p,n+p+1}$ into two disjoint sets A and B then use bijective maps to show that $|A| = |M_{p,n+p}|$ and $|B| = |M_{p,n}|$. This is the same technique used in [UA].

Proof of Theorem 1.1.

(1) To find an explicit formula for $|M_{p,n+p}|$, we use the following simple counting argument. Let k be the minimum element of our set $S \in M_{p,n+p}$. If $k = n+p$, then $S = \{n+p\}$. If $k < n+p$, then we can choose it to be any number between 1 and $n+p-1$. For each of

these choices, we have fixed the maximum and the minimum of our set and so, we can choose j elements between $k+1$ and $n+p-1$, where $j \leq k/p-2$. Therefore

$$|M_{p,n+p}| = \sum_{k=1}^{n+p-1} \sum_{j=0}^{k/p-2} \binom{n+p-k-1}{j} + 1,$$

which is the desired formula.

(2) The set $M_{p,n+p+1}$ is the union of

(a) $A = \{S \in M_{p,n+p+1} : n+p \notin S\}$,
(b) $B = \{S \in M_{p,n+p+1} : n+p \in S\}$.

We compute $|A|$ by considering the map $R_1 : M_{p,n+p} \to A$ with $R_1(S) = (S \backslash \{n+p\}) \cup \{n+p+1\}$. The map is well-defined because it preserves the cardinality of the set and does not decrease the minimum element of a set. Injectivity of R_1 is clear. The map is also onto because given $U \in A$, $R_1(U \backslash \{n+p+1\} \cup \{n+p\}) = U$. So, $|A| = |M_{p,n+p}|$.

Next, we determine $|B|$ by considering the map $R_2 : M_{p,n} \to B$ with $R_2(S) = (S+p) \cup \{n+p+1\}$. Since $\min S \geq p|S|$, $\min(S+p) \geq p(|S|+1)$. This shows that R_2 is well-defined. Injectivity is clear. The map is also onto because given $U \in B$, $R_2((U \backslash \{n+p+1\}) - p) = U$. So, $|B| = |M_{p,n}|$. We conclude that

$$|M_{p,n+p+1}| \;=\; |M_{p,n+p}| + |M_{p,n}|.$$

□

3. Proof of Theorem 1.2 and Theorem 1.3

Our proof of Theorem 1.2 employs straightforward counting arguments. For Theorem 1.3, we partition $M_{p,q,n+2q+1}$ into three subsets and use bijective maps to argue that the cardinalities of these three subsets are equal to a_{n+q}, a_n and $(q-p)|M_{q,n}|$, respectively.

Proof of Theorem 1.2. Fix $p < q \in \mathbb{N}$. We prove the theorem by considering different ranges for n. For $n \leq q-1$, if $|S| > 0$ we have the contradiction

$$q \;\leq\; q|S| \leq\; \min{}_2 S \;\leq\; n \;\leq\; q-1.$$

For $q \leq n \leq 2q-1$, we have $|S| = 1$ since otherwise we have the contradiction

$$2q \;\leq\; q|S| \leq\; \min{}_2 S \;\leq\; n \;\leq\; 2q-1.$$

If $n \geq 2q$, we prove that

$$|M_{p,q,n}| = 1 + (n-2p) + \sum_{k=3}^{\frac{n+2}{q+1}} \sum_{i=qk}^{n-k+2} (i-pk)\binom{n-i-1}{k-3}.$$

- The 1 on the right hand side comes from the set $\{n\}$.
- For a two-element set S, the maximum element n is also the second smallest element. Because $\min_2 S = n \geq 2q$, $\min_2 S/q = n/q \geq 2q/q = 2 = |S|$. Let $m = \min S$. As we need $m/p \geq |S| = 2$, we must have $m \geq 2p$. Therefore m can be any value from $2p$ to $n-1$. Hence we have $n-2p$ sets of 2 elements.

- For sets with at least three elements, we first find the range for the second smallest element. Let $\min_2 S = i$ and $|S|= k$. Since there are $k-2$ elements bigger than i, $i \le n-k+2$. Because $\min_2 S/q \ge |S|$, we have $i \ge qk$. So, $qk \le i \le n-k+2$. Next, we find the upper bound for k. It follows from the fact that $qk \le n-k+2$, and thus we obtain $k \le \frac{n+2}{q+1}$. With i and k fixed, there are $i - pk$ choices for $\min S$ because $i = \min_2 S > \min S \ge pk$. Finally, we have $\binom{n-i-1}{k-3}$ choices to pick $k-3$ elements between $\min_2 S = i$ and n, so our formula is correct.

$\square$

Proof of Theorem 1.3. For a nonempty, finite set S, define $S' := S\backslash\{\max S\}$. Clearly $M_{p,q,n+2q+1}$ is the union of three following disjoint sets:

(a) $A := \{S \in M_{p,q,n+2q+1} : n + 2q \notin S\}$,
(b) $B := \{S \in M_{p,q,n+2q+1} : n + 2q \in S, S' - q \in M_{p,q,n+q}\}$, and
(c) $C := \{S \in M_{p,q,n+2q+1} : n + 2q \in S, S' - q \notin M_{p,q,n+q}\}$.

Let $\tau(S) = (S\backslash\{\max S\}) \cup \{n + 2q + 1\}$. We compute $|A|$ by considering the map $\tau : M_{p,q,n+2q} \to A$. The map is well-defined because

(1) for all $S \in M_{p,q,n+2q}$, $\tau(S)$ does not contain $n + 2q$,
(2) τ does not change the cardinality of a set, while both the smallest and the second smallest of the set do not decrease.

Clearly τ is one-to-one. We show that it is also onto. Let $U \in A$. If $|U|= 1$, that is $U = \{n + 2q + 1\}$, then $\tau(\{n + 2q\}) = U$. If $|U|= 2$, we have $U = \{m, n + 2q + 1\}$ for some $2p \le m < n + 2q$. Then $\tau(\{m, n + 2q\}) = U$. If $|U|\ge 3$, then

$$\tau(\{n + 2q\} \cup U\backslash\{n + 2q + 1\}) \;=\; U.$$

Therefore τ is onto and thus, bijective. So, $|A|= |M_{p,q,n+2q}|= a_{n+q}$.

Let $\psi(S) = (S+q)\cup\{n+2q+1\}$. We compute $|B|$ by considering the map $\psi : M_{p,q,n+q} \to B$. Note that ψ is well-defined because while ψ makes the cardinality of a set increase by 1, both the smallest and the second smallest increase by q. Clearly ψ is one-to-one, and by the definition of B, it is also onto. Therefore $|B|= |M_{p,q,n+q}|= a_n$.

Finally, we compute $|C|$. Partition C into C_i, where

$$C_i \;=\; \{S \in M_{p,q,n+2q+1} \;:\; n + 2q \in S \text{ and } p|S|+i = \min S\},$$

for $0 \le i \le q - p - 1$. We show that $C = \cup_{i=0}^{q-p-1} C_i$. Let $F \in C_i$ for some $0 \le i \le q - p - 1$. We have

$$\min(F' - q) \;=\; \min F - q \;=\; p|F|+i - q \;<\; p|F|-p \;=\; p|F' - q|.$$

So, $F' - q \notin M_{p,q,n+q}$. Hence, $F \in C$. We have shown that $\cup_{i=0}^{q-p-1} C_i \subseteq C$. Now, let $E \in C$. Because $E \in M_{p,q,n+2q+1}$ and $E' - q \notin M_{p,q,n+q}$, it is straightforward to deduce that $\min(E' - q) < p|E' - q|$, which implies that $p|E|\le \min E < p|E|+(q - p)$. Therefore, $\min E = p|E|+i$ for some $0 \le i \le q-p-1$. This shows that $C \subseteq \cup_{i=0}^{q-p-1} C_i$. We conclude that $C = \cup_{i=0}^{q-p-1} C_i$.

It remains to prove that $|C_i|= |M_{q,n}|$. Consider the map

$$\begin{aligned} \phi_i : C_i &\longrightarrow M_{q,n} \\ S &\longrightarrow (S'\backslash\{\min S\}) - 2q. \end{aligned}$$

We show that ϕ_i is well-defined as follows. Let $F \in C_i$. Observe that

$$\begin{aligned} q|\phi_i(F)| &= q|(F' \backslash \{\min F\}) - 2q| = q(|F|-2) = q|F|-2q \\ &\leq \min_2 F - 2q = \min((F' \backslash \{\min F\}) - 2q) = \min \phi_i(F). \end{aligned}$$

To see that ϕ_i is onto, let $G \in M_{q,n}$ and $H = \{p(|G|+2) + i\} \cup (G + 2q) \cup \{n + 2q + 1\}$. We have $\min H = p(|G|+2) + i$ since

$$p(|G|+2) + i \leq p(|G|+2) + (q-p) < p|G|+2q \leq \min(G + 2q).$$

It follows that $H \in C_i$ because

$$\begin{aligned} p|H| &= p(|G|+2) \leq p(|G|+2) + i = \min H, \text{ and} \\ q|H| &= q(|G|+2) \leq \min G + 2q = \min_2 H. \end{aligned}$$

Clearly $\phi_i(H) = G$ and thus ϕ_i is onto. Since injectivity of ϕ_i is clear, ϕ_i is bijective. This shows that $|C_i| = |M_{q,n}|$ and so, $|C| = (q-p)|C_i| = (q-p)|M_{q,n}|$.

We conclude that

$$|M_{p,q,n+2q+1}| = |A|+|B|+|C| = |a_{n+q}|+|a_n|+(q-p)|M_{q,n}|.$$

□

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