HIGHER ORDER FIBONACCI SEQUENCES FROM GENERALIZED SCHREIER SETS

HUNG VIET CHU, STEVEN J. MILLER, AND ZIMU XIANG

Abstract. A Schreier set $S$ is a subset of the natural numbers with $\min S \geq |S|$. It has been known that the sequence $(a_{1,n})$, where

$$a_{1,n} := |\{S \subseteq \mathbb{N} : \max S = n \text{ and } \min S \geq |S|\}|,$$

is the Fibonacci sequence. Generalizing this result, we prove that for all $p \in \mathbb{N}$, the sequence $(a_{p,n})$, where

$$a_{p,n} := |\{S \subseteq \mathbb{N} : \max S = n \text{ and } \min S \geq p|S|\}|,$$

has a linear recurrence relation of higher order. We investigate further by requiring that $\min_2 S \geq q|S|$, where $\min_2 S$ is the second smallest element of $S$. We prove a linear recurrence relation for the sequence $(a_{p,q,n})$, where

$$a_{p,q,n} := |\{S \subseteq \mathbb{N} : \max S = n, \min S \geq p|S| \text{ and } \min_2 S \geq q|S|\}|,$$

and discuss a curious relationship between $(a_{q,n})$ and $(a_{p,q,n})$.

1. Introduction

A Schreier set $S$ is a subset of the natural numbers with $\min S \geq |S|$, and the Schreier family containing all Schreier sets is denoted by $S_1$. Schreier defined them to solve a problem in Banach space theory in 1930 [Sc]. These sets were also independently discovered in combinatorics and are connected to Ramsey-type theorems for subsets of $\mathbb{N}$. An online post [UA] proved that the Fibonacci sequence appears if we count Schreier sets under certain conditions.

Define

$$M_{1,n} := \{S \in S_1 : \max S = n\}.$$ 

Then $|M_{1,1}| = 1$, $|M_{1,2}| = 1$ and $|M_{1,n+2}| = |M_{1,n+1}| + |M_{1,n}|$ for all $n \geq 1$ [UA]. The proof uses two one-to-one mappings to argue about cardinalities of sets. We generalize this result by defining, for $p \in \mathbb{N}$,

$$S_p := \{S \subseteq \mathbb{N} : \min S \geq p|S|\}, \quad \text{ and } \quad M_{p,n} := \{S \in S_p : \max S = n\},$$

and prove the following.

Theorem 1.1. Given $p \in \mathbb{N}$, consider the sequence $(|M_{p,n}|)_{n=1}^{\infty}$. We have

1. $|M_{p,n+p}| = \sum_{k=1}^{n+p-1} \sum_{j=0}^{k/p-2} \binom{n+p-k-1}{j} + 1$, and
2. for $n \geq 1$, $|M_{p,n+p+1}| = |M_{p,n+p}| + |M_{p,n}|$.

We call $(|M_{p,n}|)_{n=1}^{\infty}$ the generalized Schreier-Fibonacci sequence of order $p$.

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Our definition of $S_p$ is not the same as what used in Banach space theory to indicate the Schreier sets of order $p$ [AA].
Another natural extension is to put an additional restriction on our set $S$; in particular, we require that $\min_2 S \geq q|S|$, where $\min_2 S$ is the second smallest element in $S$. We define

$$S_{p,q} := \{S \subseteq \mathbb{N} : \min S \geq p|S| \text{ and } \min_2 S \geq q|S|\}.$$ 

For a given $n$, we consider the family of sets $M_{p,q,n} = \{S \in S_{p,q} : \max S = n\}$. When a set has exactly one element, we take the element to be both the smallest and the second smallest. The following theorem gives an explicit formula to calculate $|M_{p,q,n}|$.

**Theorem 1.2.** Given $p < q \in \mathbb{N}$, for the sequence $(|M_{p,q,n}|)_n^\infty$, we have $|M_{p,q,n}| = 0$ if $n \leq q - 1$, $|M_{p,q,n}| = 1$ if $q \leq n \leq 2q - 1$ and

$$|M_{p,q,n}| = 1 + (n - 2p) + \sum_{k=3}^{n+2-k} \sum_{i=qk}^{n+2-k} (i - pk) \binom{n-i-1}{k-3} \text{ if } n \geq 2q.$$ 

**Theorem 1.3.** Fix $p < q \in \mathbb{N}$. Consider $(M_{q,n})_n^\infty$ and $(M_{p,q,n})_n^\infty$. For each $n \in \mathbb{N}$, define $a_n := |M_{p,q,n+q}|$. We have

$$a_{n+q+1} = a_{n+q} + (q - p)|M_{q,n}|.$$ 

Note that when $p = q$, we have Theorem 1.1. We have the following corollary that shows a recurrence relation for the sequence $(|M_{p,q,n}|)_n^\infty$.

**Corollary 1.4.** Fix $p < q$ in $\mathbb{N}$. For $n \in \mathbb{N}$, define $a_n := |M_{p,q,n+q}|$. We have

$$a_{n+2q+2} = 2a_{n+2q+1} - a_{n+2q} + 2a_{n+q+1} - 2a_{n+q} - a_n.$$ 

**Proof.** Fix $n \in \mathbb{N}$. By Theorem 1.3 we have

$$a_{n+q+1} - a_{n+q} = a_n + (q - p)|M_{q,n}| \quad \text{(1.1)}$$

$$a_{n+2q+1} - a_{n+2q} = a_{n+q} + (q - p)|M_{q,n+q}| \quad \text{(1.2)}$$

$$a_{n+2q+2} - a_{n+2q+1} = a_{n+q+1} + (q - p)|M_{q,n+q+1}|. \quad \text{(1.3)}$$

By Theorem 1.1, we know that $|M_{q,n+q+1}| = |M_{q,n+q}| + |M_{q,n}|$. Subtract Equation (1.1) and Equation (1.2) from Equation (1.3) to finish the proof. 

**Remark 1.5.** For fixed $p, q$, Theorem 1.4 gives a recurrence relation of depth $2q + 2$; interestingly, the depth is independent of $p$.

2. **Proof of Theorem 1.1**

Given a set $S$ and a number $a$, define

$$a + S := \{a + s : s \in S\}.$$ 

In our proof, we partition $M_{p,n+p+1}$ into two disjoint sets $A$ and $B$ then use bijective maps to show that $|A| = |M_{p,n+p}|$ and $|B| = |M_{p,n}|$. This is the same technique used in [UA].

**Proof of Theorem 1.1**

(1) To find an explicit formula for $|M_{p,n+p}|$, we use the following simple counting argument. Let $k$ be the minimum element of our set $S \subseteq M_{p,n+p}$. If $k = n + p$, then $S = \{n + p\}$. If $k < n + p$, then we can choose it to be any number between 1 and $n + p - 1$. For each of
these choices, we have fixed the maximum and the minimum of our set and so, we can choose
j elements between k + 1 and n + p − 1, where j ≤ k/p − 2. Therefore

\[ |M_{p,n+p}| = \sum_{k=1}^{n+p-1} \sum_{j=0}^{k/p-2} \binom{n+p-k-1}{j} + 1, \]

which is the desired formula.

(2) The set \( M_{p,n+p+1} \) is the union of
(a) \( A = \{ S \in M_{p,n+p+1} : n + p \notin S \} \),
(b) \( B = \{ S \in M_{p,n+p+1} : n + p \in S \} \).

We compute \( |A| \) by considering the map \( R_1 : M_{p,n+p} \rightarrow A \) with \( R_1(S) = (S \setminus \{n + p\}) \cup \{n + p + 1\} \). The map is well-defined because it preserves the cardinality of the set and does not decrease the minimum element of a set. Injectivity of \( R_1 \) is clear. The map is also onto because given \( U \in A \), \( R_1(U \setminus \{n + p + 1\}) \cup \{n + p\} = U \). So, \( |A| = |M_{p,n+p+1}| \).

Next, we determine \( |B| \) by considering the map \( R_2 : M_{p,n} \rightarrow B \) with \( R_2(S) = (S + p) \cup \{n + p + 1\} \). Since \( \min S \geq p|S| \), \( \min(S + p) \geq p(|S| + 1) \). This shows that \( R_2 \) is well-defined. Injectivity is clear. The map is also onto because given \( U \in B \), \( R_2((U \setminus \{n + p + 1\}) - p) = U \). So, \( |B| = |M_{p,n}| \).

We conclude that
\[ |M_{p,n+p+1}| = |M_{p,n+p}| + |M_{p,n}|. \]

\[ \square \]

3. Proof of Theorem 1.2 and Theorem 1.3

Our proof of Theorem 1.2 employs straightforward counting arguments. For Theorem 1.3 we partition \( M_{p,q,n+2q+1} \) into three subsets and use bijective maps to argue that the cardinalities of these three subsets are equal to \( a_{n+q}, a_n \) and \( (q - p)|M_{q,n}| \), respectively.

Proof of Theorem 1.2 Fix \( p < q \in \mathbb{N} \). We prove the theorem by considering different ranges for \( n \). For \( n \leq q - 1 \), if \( |S| > 0 \) we have the contradiction
\[ q \leq q|S| \leq \min_2 S \leq n \leq q - 1. \]

For \( q \leq n \leq 2q - 1 \), we have \( |S| = 1 \) since otherwise we have the contradiction
\[ 2q \leq q|S| \leq \min_2 S \leq n \leq 2q - 1. \]

If \( n \geq 2q \), we prove that
\[ |M_{p,q,n}| = 1 + (n - 2p) + \sum_{k=3}^{n+2} \sum_{i=\frac{n-k+2}{q}}^{n} i \binom{n-i-1}{k-3}. \]

- The 1 on the right hand side comes from the set \( \{n\} \).
- For a two-element set \( S \), the maximum element \( n \) is also the second smallest element. Because \( \min_2 S = n \geq 2q \), \( \min_2 S/q = n/q \geq 2q/q = 2 = |S| \). Let \( m = \min S \). As we need \( m/p \geq |S| = 2 \), we must have \( m \geq 2p \). Therefore \( m \) can be any value from \( 2p \) to \( n - 1 \). Hence we have \( n - 2p \) sets of \( 2 \) elements.
\begin{itemize}
  \item For sets with at least three elements, we first find the range for the second smallest element. Let \( \min S = i \) and \( |S| = k \). Since there are \( k - 2 \) elements bigger than \( i \), \( i \leq n - k + 2 \). Because \( \min S/q \geq |S| \), we have \( i \geq qk \). So, \( qk \leq i \leq n - k + 2 \). Next, we find the upper bound for \( k \). It follows from the fact that \( qk \leq n - k + 2 \), and thus we obtain \( k \leq \frac{n+2}{q+1} \). With \( i \) and \( k \) fixed, there are \( i - pk \) choices for \( \min S \) because \( i = \min_2 S > \min S \geq pk \). Finally, we have \( \binom{n-i-1}{k-3} \) choices to pick \( k - 3 \) elements between \( \min_2 S = i \) and \( n \), so our formula is correct.
\end{itemize}

\[ \square \]

**Proof of Theorem 1.3** For a nonempty, finite set \( S \), define \( S' := S \setminus \{ \max S \} \). Clearly \( M_{p,q,n+2q+1} \) is the union of three following disjoint sets:

\begin{enumerate}
  \item \( A := \{ S \in M_{p,q,n+2q+1} : n + 2q \notin S \} \),
  \item \( B := \{ S \in M_{p,q,n+2q+1} : n + 2q \in S, S' - q \notin M_{p,q,n+q} \} \), and
  \item \( C := \{ S \in M_{p,q,n+2q+1} : n + 2q \in S, S' - q \notin M_{p,q,n+q} \} \).
\end{enumerate}

Let \( \tau(S) = (S \setminus \{ \max S \}) \cup \{ n + 2q + 1 \} \). We compute \( |A| \) by considering the map \( \tau : M_{p,q,n+2q} \to A \). The map is well-defined because

1. for all \( S \in M_{p,q,n+2q} \), \( \tau(S) \) does not contain \( n + 2q \),
2. \( \tau \) does not change the cardinality of a set, while both the smallest and the second smallest of the set do not decrease.

Clearly \( \tau \) is one-to-one. We show that it is also onto. Let \( U \in A \). If \( |U| = 1 \), that is \( U = \{ n + 2q + 1 \} \), then \( \tau(\{ n + 2q \}) = U \). If \( |U| = 2 \), we have \( U = \{ m, n + 2q + 1 \} \) for some \( 2p \leq m < n + 2q \). Then \( \tau(\{ m, n + 2q \}) = U \). If \( |U| \geq 3 \), then

\[ \tau(\{ n + 2q \} \cup U \setminus \{ n + 2q + 1 \}) = U. \]

Therefore \( \tau \) is onto and thus, bijective. So, \( |A| = |M_{p,q,n+2q}| = a_{n+q} \).

Let \( \psi(S) = (S + q) \cup \{ n + 2q + 1 \} \). We compute \( |B| \) by considering the map \( \psi : M_{p,q,n+q} \to B \). Note that \( \psi \) is well-defined because while \( \psi \) makes the cardinality of a set increase by 1, both the smallest and the second smallest increase by \( q \). Clearly \( \psi \) is one-to-one, and by the definition of \( B \), it is also onto. Therefore \( |B| = |M_{p,q,n+q}| = a_n \).

Finally, we compute \( |C| \). Partition \( C \) into \( C_i \), where

\[ C_i = \{ S \in M_{p,q,n+2q+1} : n + 2q \in S \text{ and } p|S| + i = \min S \}; \]

for \( 0 \leq i \leq q - p - 1 \). We show that \( C = \bigcup_{i=0}^{q-p-1} C_i \). Let \( F \in C_i \) for some \( 0 \leq i \leq q - p - 1 \). We have

\[ \min(F' - q) = \min F - q = p|F| + i - q < p|F| - p = p|F' - q|. \]

So, \( F' - q \notin M_{p,q,n+q} \). Hence, \( F \in C \). We have shown that \( \bigcup_{i=0}^{q-p-1} C_i \subseteq C \). Now, let \( E \in C \). Because \( E \in M_{p,q,n+2q+1} \) and \( E' - q \notin M_{p,q,n+q} \), it is straightforward to deduce that \( \min(E' - q) < p|E' - q| \), which implies that \( p|E| \leq \min E < p|E| + (q - p) \). Therefore, \( \min E = p|E| + i \) for some \( 0 \leq i \leq q - p - 1 \). This shows that \( C \subseteq \bigcup_{i=0}^{q-p-1} C_i \). We conclude that \( C = \bigcup_{i=0}^{q-p-1} C_i \).

It remains to prove that \( |C_i| = |M_{q,n}| \). Consider the map

\[ \phi_i : C_i \to M_{q,n} \]

\[ S \mapsto (S \setminus \{ \min S \}) - 2q. \]
We show that $\phi_i$ is well-defined as follows. Let $F \in C_i$. Observe that
\[
q|\phi_i(F)| = q|(F \setminus \{\min F\}) - 2q = q|F| - 2q \\
\leq \min_2 F - 2q = \min((F \setminus \{\min F\}) - 2q) = \min \phi_i(F).
\]
To see that $\phi_i$ is onto, let $G \in M_{q,n}$ and $H = \{p(|G|+2) + i\} \cup (G + 2q) \cup \{n + 2q + 1\}$. We have $\min H = p(|G|+2) + i$ since
\[
p(|G|+2) + i \leq p(|G|+2) + (q - p) < p|G|+2q \leq \min(G + 2q).
\]
It follows that $H \in C_i$ because
\[
p|H| = p(|G|+2) \leq p(|G|+2) + i = \min H,
\]
and
\[
q|H| = q(|G|+2) \leq \min G + 2q = \min_2 H.
\]
Clearly $\phi_i(H) = G$ and thus $\phi_i$ is onto. Since injectivity of $\phi_i$ is clear, $\phi_i$ is bijective. This shows that $|C_i| = |M_{q,n}|$ and so,
\[
|C| = (q - p)|C_i| = (q - p)|M_{q,n}|.
\]
We conclude that
\[
|M_{p,q,n+2q+1}| = |A| + |B| + |C| = |a_{n+q}| + |a_n| + (q - p)|M_{q,n}|.
\]

References


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Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61820
Email address: chuh19@mail.wlu.edu

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267
Email address: sjm1@williams.edu

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213
Email address: zimux@andrew.cmu.edu