

ON THE WITHIN-PERFECT NUMBERS

CHUNG-HANG KWAN AND STEVEN J. MILLER

ABSTRACT. Motivated by the works of Erdős, Pomerance, Wolke and Harman on the sum-of-divisor function $\sigma(n)$, we study the distribution of a special class of natural numbers closely related to (multiply) perfect numbers which we term ‘ $(\ell; k)$ -within-perfect numbers’, where $\ell > 1$ is a real number and $k : [1, \infty) \rightarrow (0, \infty)$ is an increasing and unbounded function.

1. INTRODUCTION

A natural number n is said to be *perfect* if $\sigma(n) = 2n$, ℓ -*perfect* if $\sigma(n) = \ell n$ (with $\ell > 1$ being rational), and *multiply perfect* if $n \mid \sigma(n)$, where $\sigma(n)$ represents the sum of all positive divisors of n . An outstanding conjecture, originating from the ancient Greeks (300 BC), asserts that there are infinitely many even perfect numbers but no odd perfect numbers (see Euclid [Eu, Book VII, IX], Dickson [Di, Chapter I], Guy [Gu, Chapter B1]). This conjecture is well-supported by probabilistic heuristics due to Pomerance [Pol, pp. 249, 258-259]. For $\ell \in \{2, 3, \dots, 11\}$, there are known examples of ℓ -perfect numbers (see [Gu, Chapter B1]); however, for other values of ℓ , the (non)existence of ℓ -perfect numbers remains entirely open.

Starting in the mid-20th century, considerable interest emerged in understanding the statistical distribution of perfect numbers. These numbers are particularly *rare*, as demonstrated by the works of Erdős [Er56], Volkmann [Vo56], Kanold [Ka54, Ka57], Hornfeck [Ho55] and Hornfeck-Wirsing [HW57], culminating in the sharpest known *upper bound* for the number of perfect numbers up to x due to Wirsing [Wi59]. The bound obtained in [Wi59] is of the order $\exp(O(\frac{\log x}{\log \log x}))$ as $x \rightarrow \infty$, which possesses the pleasant feature of *uniform* applicability to ℓ -perfect numbers for any rational ℓ . When restricting to the class of odd perfect numbers, there is the celebrated *Dickson’s finiteness theorem* [Di13] which asserts the following: given a natural number k , there are only finitely many odd perfect numbers with exactly k distinct prime divisors. This was later refined by Pomerance [Pom77] and Heath-Brown [HB94].

Subsequently, it evolved into an active research area to investigate special classes of natural numbers closely linked to perfect numbers, see [Gu, Chapter B2-B3]. For instance, Sierpiński [Si65] introduced the notion of ‘*pseudo-perfect numbers*.’ In a companion article [CC+20], we studied a subclass of these numbers known as ‘*near-perfect numbers*’, proposed by Pollack-Shevelev [PS12]. In [CC+20], we strengthened the results and analysis of [PS12] by employing recursive partitions and sieve-theoretic techniques.

In this article, we investigate another type of ‘approximate’ perfect numbers of a somewhat different flavour which we term the $(\ell; k)$ -*within-perfect* numbers, where $\ell > 1$ is a

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real number and $k = k(y)$ is a certain threshold function. More precisely, a natural number n is said to be $(\ell; k)$ -*within-perfect* if the Diophantine inequality

$$|\sigma(n) - \ell n| < k(n) \quad (1.1)$$

holds. There are two distinct origins of these numbers. On one hand, Erdős ([Gu, pp. 46]) and Makowski [Ma79] were interested in the case when $\ell = 2$ and k is a constant. On the other hand, the inequality (1.1) arises naturally in the field of *Diophantine approximations* for arithmetic functions. Wolke [Wo77] studied (1.1) for any real $\ell > 1$ and function $k(y)$ of the form y^c . His result was improved by Harman [Ha10] and very recently by Järviemi [Ja22+]. In [Ja22+], it was shown that for any real $\ell > 1$ and any $c \in (0.45, 1)$, there exist *infinitely many* $(\ell; y^c)$ -within-perfect numbers. The range of c can be extended to $(0.39, 1)$ under the *Riemann Hypothesis* as indicated in [Ha10]. The results of [Ha10, Ja22+] rely on deep inputs from the distributions of primes in short intervals as well as that of the differences of consecutive primes, see [BHP01, HB21, St22+, Ja22+]. Interested readers are also referred to Alkan-Ford-Zaharescu [AFZ09a, AFZ09b] for settings more general than (1.1) and the related questions in Diophantine approximations.

The main results of this article concern the class of threshold functions $k = k(y)$ which are complementary to those considered in [Wo77, Ha10, Ja22+]. Moreover, we are also interested in estimating the size of the set

$$W(\ell; k; x) := \{n \leq x : |\sigma(n) - \ell n| < k(n)\}. \quad (1.2)$$

Consequently, our work employs a different set of techniques compared to the earlier mentioned works.

Theorem 1.1. *Let $c \in (0, 1/3)$ be given. Suppose $k : [1, \infty) \rightarrow (0, \infty)$ is an increasing and unbounded function satisfying $k(y) \leq y^c$ for $y \geq 1$. Let Σ be the set $\{\sigma(m)/m : m \geq 1\}$. Then*

(a) *If $\ell = a/b \in (\mathbb{Q} \cap (1, \infty)) \setminus \Sigma$ with $(a, b) = 1$, then*

$$\#W(\ell; k; x) = O\left(ab^3 x^{2/3+c+o(1)}\right) \quad (1.3)$$

for $1 < \ell \leq x^c$ and $x \geq 1$, where the implicit constants are absolute.

(b) *If $\ell = a/b \in \Sigma$ with $(a, b) = 1$, and there exists $\delta > 0$ such that $k(y) \geq y^\delta$ for $y \geq 1$, then*

$$\lim_{x \rightarrow \infty} \frac{\#W(\ell; k; x)}{x/\log x} = \sum_{\sigma(m) = \ell m} \frac{1}{m}. \quad (1.4)$$

Remark 1.2. *We also have an analogous result for $c = 0$ and its proof is actually simpler; see Proposition 2.7 and Remark 3.1.*

Remark 1.3. *Firstly, the infinite series of (1.4) converges by Wirsing's Theorem ([Wi59]), which will be applied in various ways throughout the course of proving Theorem 1.1.*

Secondly, a key strength of our theorem is that all dependencies in (1.3) are made entirely explicit. Our bound remains non-trivial even if the numerator or denominator of ℓ grows with x at a controlled rate, though this necessitates appropriately shrinking the admissible range for c .

Thirdly, while it is possible that the dependence on ℓ (i.e., the factor ab^3) in (1.3) can be improved, it does not appear that this dependence can be removed. This is in contrast to Wirsing's Theorem.

1.1. **Notations.** We use the following notations throughout this article:

- $f(x) \asymp g(x)$ if there exist constants $c_1, c_2 > 0$ such that $c_1 g(x) < f(x) < c_2 g(x)$ for sufficiently large x ,
- $f(x) \sim g(x)$ if $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$,
- $f(x) = O(g(x))$ or $f(x) \ll g(x)$ if there exists a constant $C > 0$ such that $f(x) < Cg(x)$ for sufficiently large x ,
- $f(x) = o(g(x))$ if $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$,
- subscripts indicate dependence of implied constants on other parameters, and
- p always denotes a prime number.

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2. PRELIMINARY DISCUSSIONS AND PREPARATIONS

2.1. **Distribution Function and Phase Transition.** We begin by briefly explaining why the sublinear regime for $k = k(y)$ is of the greatest interest in the study of within-perfectness. For this, we recall the definition of a distribution function.

Definition 2.1. Let $-\infty \leq a < b \leq \infty$.

- (1) A function $F : (a, b) \rightarrow \mathbb{R}$ is a distribution function if F is increasing, right continuous, $F(a+) = 0$, and $F(b-) = 1$.
- (2) An arithmetic function $f : \mathbb{N} \rightarrow \mathbb{R}$ has a distribution function if there exists a distribution function F such that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \#\{n \leq x : f(n) \leq u\} = F(u)$$

at all points of continuity of F .

A celebrated theorem of Davenport [Da33] asserts that $\sigma(n)/n$ possesses a continuous and strictly increasing distribution function on $[1, \infty)$. Denote by $D(\cdot)$ the distribution function of $\sigma(n)/n$ and extend its definition to \mathbb{R} by setting $D(u) = 0$ for $u < 1$. More generally, a necessary and sufficient criterion for the existence of distribution functions for additive functions was established by Erdős-Wintner [EW39], see [Te, Chapter I.5, III.4] for further details.

The following result describes the phase transition for the asymptotic density for the set of all $(\ell; k)$ -within-perfect numbers which we denote by $W(\ell; k)$.

Proposition 2.2. Let $D(\cdot)$ be the distribution function of $\sigma(n)/n$. Then

- (a) If $k(n) = o(n)$, then $W(\ell; k)$ has asymptotic density 0.
- (b) If $k(n) \sim cn$ for some $c > 0$, then $W(\ell; k)$ has asymptotic density equal to $D(\ell + c) - D(\ell - c)$.
- (c) If $k(n) \asymp n$, then $W(\ell; k)$ has positive lower density and upper density strictly less than 1.
- (d) If $n = o(k(n))$, then $W(\ell; k)$ has asymptotic density 1.

Sketch. The proof is elementary and we will only indicate certain details for part (d). Let's label the elements of $W(\ell; k)$ by n_j in increasing order. For any $j \in \mathbb{N}$, there exists $n_j \in \mathbb{N}$ such that $n/k(n) < 1/j$ for any $n \geq n_j$. Whenever $x \geq n_j$, we have

$$\frac{1}{x} \#\{n \leq x : |\sigma(n) - \ell n| < jn\} \leq \frac{n_j}{x} + \frac{1}{x} \#\{n \leq x : |\sigma(n) - \ell n| < k(n)\}.$$

Upon taking \liminf to both sides, it follows that $D(\ell + j) \leq 1$. By Davenport's theorem, $D(\cdot)$ is continuous and $D(\infty) = 1$. Letting $j \rightarrow \infty$ yields

$$\liminf_{x \rightarrow \infty} \frac{\#W(\ell; k; x)}{x} = 1.$$

□

2.2. On Congruences involving $\sigma(n)$. Henceforth, our focus shifts to the sublinear thresholds, where we crucially make use of the techniques developed by Pomerance and his co-authors over the years. In 1975, Pomerance [Pom75] initiated the study of the equation

$$\sigma(n) = \ell n + k \tag{2.1}$$

with ℓ, k being integers and $\ell > 1$. Central to [Pom75] is the following important concept, which proves very useful in many Erdős-style problems (see [PP16]):

Definition 2.3 (Regular/ Sporadic). *The solutions to the congruence $\sigma(n) \equiv k \pmod{n}$ of the form*

$$n = pm, \quad \text{where } p \nmid m, \quad m \mid \sigma(m), \quad \sigma(m) = k, \tag{2.2}$$

are called regular. All other solutions are called sporadic.

The main observation is that sporadic solutions occur much less frequently than regular solutions. In a series of works [Pom75, Pom76a, Pom76b, PS12, APP12, PP13], this was quantified with various degrees of precision and uniformity. We summarize the progress made in this direction.

Let $\text{Spor}_k(x)$ be the set of sporadic solutions of $\sigma(n) \equiv k \pmod{n}$ up to x . In [Pom75], [PS12], [APP12] and [PP13], it was shown, respectively, that the following bounds hold as $x \rightarrow \infty$:

- $\#\text{Spor}_k(x) = O_{\beta, k}(x \exp(-\beta \sqrt{\log x \log \log x}))$ for fixed k and fixed $\beta < 1/\sqrt{2}$,
- $\#\text{Spor}_k(x) = O(x^{2/3+o(1)})$ uniformly in $|k| < x^{2/3}$,
- $\#\text{Spor}_k(x) = O(x^{1/2+o(1)})$ uniformly in $|k| < x^{1/4}$, and
- $\#\text{Spor}_k(x) \cap \{\sigma(n) \text{ is odd}\} = O(|k|x^{1/4+o(1)})$ uniformly in $0 < |k| \leq x^{1/4}$.

When $k = 0$, we have a much stronger bound for (2.1) due to Wirsing ([Wi59]).

Unfortunately, the aforementioned results are not sufficient for our study of within-perfect numbers. We must consider a more general congruence

$$b\sigma(n) \equiv k \pmod{n} \tag{2.3}$$

where b and k are integers with $b \geq 1$. Accordingly, the definitions of regular and sporadic solutions should be extended as follows:

Definition 2.4. *Suppose $b \mid k$. Then n is said to be a regular solution to (2.3) if*

$$n = pm, \quad \text{where } p \nmid m, \quad m \mid b\sigma(m), \quad \sigma(m) = \frac{k}{b}. \tag{2.4}$$

All other solutions are called sporadic. In the case when $b \nmid k$, all solutions to (2.3) are considered sporadic.

We record the following generalization of [PS12, Lemma 8], which will be used in the proof of Theorem 1.1 and may also be of independent interest. For the convenience of the readers, we include a sketch of proof of this proposition here.

Proposition 2.5. *The number of sporadic solutions $n \leq x$ to the congruence $b\sigma(n) \equiv k \pmod{n}$ is $O(b^2 x^{2/3+o(1)})$ for any $x \geq 1$ and integer k satisfying $|k| < bx^{2/3}$. The implicit constants are absolute.*

Our intended applications take into account of the uniformity of the range and the strength of the upper bound in counting sporadic solutions. In light of Wirsing's Theorem ([Wi59]) and Remark 1.3, it is also desirable to maintain all implicit constants absolute but this can be somewhat subtle, see [APP12, Remark 1] and [PP13]. After a careful examination of existing strategies, the authors believe that the approach of [PS12] is the most suitable for attaining our desired generalization (e.g., ℓ can be rational), incorporating all the favourable features mentioned above. Their method softly utilizes the unique factorization of a natural number into its square-free and square-full parts, the anatomy of unitary divisors¹, and the following result from [Pol11, Theorem 1.3]:

$$\sum_{x^{1/3} < m \leq x^{2/3}} \frac{(m, \sigma(m))}{m^2} \leq 3x^{-1/3+o(1)}, \quad (2.5)$$

which turns out to be a nice application of Wirsing's Theorem!

Proof of Proposition 2.5. We can certainly assume that $x \geq b$, otherwise the count is trivially bounded by b , which is acceptable in view of Theorem 1.1. Let $|k| < bx^{2/3}$. Suppose $n \leq x$ is a sporadic solution to the congruence $b\sigma(n) \equiv k \pmod{n}$. One can simultaneously assume that $n > x^{2/3}$ and the square-full part of n is bounded by $x^{2/3}$. Indeed, the contribution from the complement can easily be seen to be $O(x^{2/3})$. We then consider the following two cases.

- (1) Suppose $p := P^+(n) > x^{1/3}$. By the assumption made, we have $n = pm$ with $p \nmid m$ and $m < x^{2/3}$. The congruence can be written as $b\sigma(n) = qn + k$ for some integer $q \geq 0$. It follows that

$$b(p+1)\sigma(m) = b\sigma(n) = qpm + k,$$

and

$$p(b\sigma(m) - qm) = k - b\sigma(m).$$

If $k - b\sigma(m) = 0$, then n is a regular solution, which is a contradiction! So, $k - b\sigma(m) \neq 0$. For each m , the number of such p is $O(\log |k - b\sigma(m)|) = O(\log(bx^{2/3} \log \log x)) = O(\log bx)$ because of $p \mid (k - b\sigma(m))$. Therefore, the number of such n is $O(x^{2/3} \log bx) = O(x^{2/3} \log x)$, which is acceptable.

- (2) Suppose $p := P^+(n) \leq x^{1/3}$. Such n must have a *unitary divisor* m in the interval $(x^{1/3}, x^{2/3}]$, see [PS12]. We have $\sigma(n) \equiv 0 \pmod{\sigma(m)}$ and $b\sigma(n) \equiv k \pmod{m}$. By the Chinese Remainder Theorem, we have $b\sigma(n) \equiv a_n \pmod{[m, \sigma(m)]}$ for some unique $0 \leq a_n < [m, \sigma(m)]$. Given $m \in (x^{1/3}, x^{2/3}]$, the number of possible values for $b\sigma(n)$ is $\leq 1 + (2bx \log x)/[m, \sigma(m)]$. Summing over

¹Note: m is a unitary divisor of n if n has a decomposition of the form $n = mm'$, where $(m, m') = 1$.

$m \in (x^{1/3}, x^{2/3}]$, the total number of possible values of $b\sigma(n)$ is

$$\begin{aligned} &\leq \sum_{x^{1/3} < m \leq x^{2/3}} \left(\frac{2bx \log x}{[m, \sigma(m)]} + 1 \right) \\ &\leq x^{2/3} + (2bx \log x)(3x^{-1/3+o(1)}) \leq 7bx^{2/3+o(1)} \end{aligned}$$

from Lemma 2.5. Moreover, the size of $q = (b\sigma(n) - k)/n$ is clearly

$$\ll b \log \log x + \frac{|k|}{n} < b \log \log x + b \ll b \log \log x$$

by the assumptions $n > x^{2/3}$ and $|k| < bx^{2/3}$. Since $b\sigma(n) = qn + k$, the number of possible values of n is at most

$$(7bx^{2/3+o(1)})(b \log \log x) \leq 7b^2x^{2/3+o(1)}.$$

The desired result follows by putting the conclusions of the two cases together. \square

2.3. A simple application. Let k, a, b be integers such that $k \neq 0, a > b \geq 1$, and $(a, b) = 1$. Denote by $S(a, b; k)$ the set of all solutions to the Diophantine equation $b\sigma(n) = an + k$ that generalizes (2.1). Let $S(a, b; k; x) := S(a, b; k) \cap [1, x]$. The main result of [Pom75] is stated as follows.

Theorem 2.6. *As $x \rightarrow \infty$, we have*

$$\#S(a, 1; k; x) \ll_{k,a} \frac{x}{\log x}. \quad (2.6)$$

Motivated by Pomerance's theorem, Davis-Klyve-Kraght [DKK13] recently performed an extensive numerical investigation on the true size of $S(a, 1; k; x)$. As a first application of Proposition 2.5, we sharpen Pomerance's theorem and confirm some of the observations and speculations made in [DKK13]. This will also serve as the base case for Theorem 1.1, see Section 3.2.

Proposition 2.7.

(a) *If $k \geq 1, ab \mid k$, and $\sigma(k/a) = k/b$, then*

$$\#S(a, b; k; x) \sim \frac{a}{k} \frac{x}{\log(ax/k)} \quad (2.7)$$

as $x \rightarrow \infty$.

(b) *Otherwise, we have*

$$\#S(a, b; k; x) = O(b^2x^{2/3+o(1)}) \quad (2.8)$$

for any $x \geq 1$, where the implicit constants are absolute. In particular, the bound (2.8) is uniform in a .

Proof. Suppose $n \in S(a, b; k)$. Then $b\sigma(n) \equiv k \pmod{n}$. If $b \nmid k$, then all solutions are sporadic by Definition 2.4 and (2.8) follows at once from Proposition 2.5.

Suppose $b \mid k$ and n is a regular solution. Then $k(1+p) = b(1+p)\sigma(m) = apm + k$. Hence, $b\sigma(m) = k = am$ and in particular, we have

$$a \mid k \quad \text{and} \quad \sigma(k/a) = k/b (= \sigma(m)). \quad (2.9)$$

In other words, if (2.9) is violated, then all solutions are sporadic and once again (2.8) holds. This proves part (b) of Proposition 2.7.

Suppose $ab \mid k$ and $\sigma(k/a) = k/b$. Then the set $\{n \in \mathbb{N} : n = p(k/a), p \nmid (k/a)\}$ consists of all regular solutions and is contained in $S(a, b; k)$. Using Proposition 2.5, the Prime Number Theorem, and the bound $\omega(n) = O(\log n)^2$, it follows that

$$\begin{aligned} \#S(a, b; k; x) &= \pi(ax/k) + O(\log |k|) + O(b^2 x^{2/3+o(1)}) \\ &\sim \frac{a}{k} \frac{x}{\log(ax/k)} \end{aligned}$$

as $x \rightarrow \infty$. Hence, part (a) follows and this completes the proof of Proposition 2.7. \square

3. PROOF OF THEOREM 1.1

3.1. Upper Bound. We begin by proving the harder part of Theorem 1.1, i.e., the upper bounds for $W(\ell; k; x)$. Fix $c \in (0, 1/3)$. Let $k_c(y) := y^c$ and $k(y) \leq k_c(y)$ for $y \geq 1$. Given a function f , we write $\widetilde{W}(\ell; f; x) := \{n \leq x : |\sigma(n) - \ell n| < f(x)\}$. The following inequality is apparent:

$$\#W(\ell; k; x) \leq \#\widetilde{W}(\ell; k; x) \leq \#\widetilde{W}(\ell; k_c; x). \quad (3.1)$$

Let $\ell = a/b > 1$ be in the lowest term. For $n \in \widetilde{W}(\ell; k_c; x)$, we have $b\sigma(n) - an = k$ for some integer k such that $|k| < bx^c$. In particular, we have

$$b\sigma(n) \equiv k \pmod{n} \quad \text{for } |k| < bx^c. \quad (3.2)$$

By Proposition 2.5, the number of $n \in \widetilde{W}(\ell; k_c; x)$ which is a sporadic solution (see Definition 2.4) is $O(bx^c) \cdot O(b^2 x^{2/3+o(1)}) = O(b^3 x^{2/3+c+o(1)})$, which is acceptable in view of Theorem 1.1. On the other hand, the number of regular solutions $n = pm \in \widetilde{W}(\ell; k_c; x)$ with $p \leq bx^c$ is $O(bx^{c+o(1)})$ by Wirsing's Theorem ([Wi59] or see Section 1). The contribution is clearly negligibly small.

Suppose $p > bx^c$ and $b\sigma(m) = rm$ for some integer $r \geq a + 1$. Then

$$\begin{aligned} |b\sigma(n) - an| &= |b(1+p)\sigma(m) - apm| = |(1+p)rm - apm| = m|r + p(r-a)| \\ &\geq m(r+p) > bx^c. \end{aligned}$$

This contradicts with (3.2)!

Suppose $p > bx^c$ and $b\sigma(m) = rm$ for some integer $r \leq a - 1$. Then

- If $r + p(r - a) \geq 0$ (which implies $p < a$), then a contradiction with $p > bx^c$ arises whenever $x^c > \ell$.
- If $r + p(r - a) < 0$, then $|b\sigma(n) - an| < bx^c \iff m[(a - r)p - r] < bx^c$. By Merten's Theorem, the number of such n is at most

$$\begin{aligned} \sum_{2 \leq r \leq a-1} \sum_{bx^c < p \leq x} \frac{bx^c}{(a-r)p-r} &\leq abx^c \sum_{bx^c < p \leq x} \frac{1}{p-(a-1)} \\ &\leq 2abx^c \sum_{p \leq x} \frac{1}{p} \ll abx^c \log \log x \end{aligned}$$

whenever $x^c \geq 2\ell$. The contribution is once again negligible.

Hence, we are left to consider the case when $r = a$, i.e.,

$$n = pm \in \widetilde{W}(\ell; k_c; x) \quad \text{such that } p > bx^c \text{ and } \sigma(m) = \ell m. \quad (3.3)$$

²Denote by $\omega(n)$ the number of distinct prime factors of n .

If $\ell \notin \Sigma$, there is clearly no such n and thus $\#W(\ell; k; x) = O(ab^3x^{2/3+c+o(1)})$ by taking into account the paragraphs right above. This proves part (a) of Theorem 1.1.

Suppose $\ell \in \Sigma$. Firstly, observe from partial summation and Wirsing's Theorem that

$$\sum_{\sigma(m)=\ell m} \frac{\log m}{m} = \int_1^\infty \frac{\log t}{t} dP_\ell(t) = \lim_{t \rightarrow \infty} \frac{\log t}{t^{1-\sigma(1)}} + \int_1^\infty \frac{\log t}{t^{2-\sigma(1)}} dt \ll 1, \quad (3.4)$$

where $P_\ell(t) := \#\{m \leq t : \sigma(m) = \ell m\}$. As a result, both of the series

$$\sum_{\sigma(m)=\ell m} \frac{\log m}{m} \quad \text{and} \quad \sum_{\sigma(m)=\ell m} \frac{1}{m} \quad (3.5)$$

can readily seen to be convergent. Notice that the bound (3.4) is uniform in ℓ .

Secondly, we have $m < x^{1-c}$ since $x \geq n = pm > x^c m$.³ It follows that

$$0 < \frac{\log m}{\log x} \leq 1 - c < 1,$$

and

$$\left(1 - \frac{\log m}{\log x}\right)^{-1} = 1 + O_c\left(\frac{\log m}{\log x}\right). \quad (3.6)$$

Let $\beta > 1$ be given. The Prime Number Theorem asserts the existence of a constant $X_0 = X_0(\beta) > 0$ such that $\pi(x) < \beta x / \log x$ whenever $x \geq X_0$. For $x \geq X_0^{1/c}$, the number of n satisfying (3.3) is at most

$$\begin{aligned} \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^{1-c}}} \pi(x/m) &< \beta \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^{1-c}}} \frac{x/m}{\log(x/m)} \\ &< \frac{\beta x}{\log x} \sum_{\sigma(m)=\ell m} \frac{1}{m} + O_c\left(\frac{\beta x}{(\log x)^2} \sum_{\sigma(m)=\ell m} \frac{\log m}{m}\right) \\ &< \frac{\beta x}{\log x} \sum_{\sigma(m)=\ell m} \frac{1}{m} + O_c\left(\frac{\beta x}{(\log x)^2}\right) \end{aligned}$$

with the help of (3.6) and the convergence of the series in (3.5). Therefore, we have

$$\limsup_{x \rightarrow \infty} \frac{\#W(\ell; k; x)}{x / \log x} \leq \beta \sum_{\sigma(m)=\ell m} \frac{1}{m} \quad (3.7)$$

for any $\beta > 1$. Let $\beta \rightarrow 1+$ in (3.7), it follows that

$$\limsup_{x \rightarrow \infty} \frac{\#W(\ell; k; x)}{x / \log x} \leq \sum_{\sigma(m)=\ell m} \frac{1}{m}.$$

This proves the upper bound of Theorem 1.1.(b).

³Actually, $m \leq x^{\min\{c, 1-c\}}$.

3.2. Lower Bound. The proof of the lower bound for Theorem 1.1.(b) is relatively straightforward. Suppose $\ell \in \Sigma$. Based on the experience of Pomerance et. al. (see Section 2.2), a lower bound for $\#W(\ell; k; x)$ can be obtained by estimating the size of the set

$$\mathcal{L}_\ell(\delta) := \{n \leq x : n = pm, p \nmid m, \sigma(m) = \ell m, \ell m < (pm)^\delta\} \quad (3.8)$$

provided that $k(y) \geq y^\delta$ for any $y \geq 1$, where $\delta \in (0, 1)$. We have

$$\#\mathcal{L}_\ell(\delta) = \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta/\ell}} \sum_{\substack{(\ell m)^{1/\delta}/m < p \leq x/m \\ p \nmid m}} 1.$$

Using the bound $\omega(m) = O(\log m)$, Wirsing's theorem and partial summation, it follows that

$$\#\mathcal{L}_\ell(\delta) = \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta/\ell}} \sum_{(\ell m)^{1/\delta}/m < p \leq x/m} 1 + O_\delta(x^{o(1)}),$$

and

$$\sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta/\ell}} \sum_{p \leq (\ell m)^{1/\delta}/m} 1 \ll \ell^{1/\delta} \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta}} m^{1/\delta-1} \ll \ell^{1/\delta} x^{1-\delta+o(1)}.$$

Hence,

$$\#\mathcal{L}_\ell(\delta) = \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta/\ell}} \pi(x/m) + O_\delta\left(\ell^{1/\delta} x^{1-\delta+o(1)}\right). \quad (3.9)$$

Let $\alpha < 1$. By the Prime Number Theorem, there exists $x_0 = x_0(\alpha) > 0$ such that $\pi(x) > \alpha x / \log x$ whenever $x \geq x_0$. Thus, if $x > (x_0)^{1/(1-\delta)}$, then

$$\begin{aligned} \#\mathcal{L}_\ell(\delta) &> \frac{\alpha x}{\log x} \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta/\ell}} \frac{1}{m} + O_\delta\left(\ell^{1/\delta} x^{1-\delta+o(1)}\right) \\ &= \frac{\alpha x}{\log x} \left(\sum_{\sigma(m)=\ell m} \frac{1}{m} + O\left((\ell/x^\delta)^{1+o(1)}\right) \right) + O_\delta\left(\ell^{1/\delta} x^{1-\delta+o(1)}\right). \end{aligned}$$

From this, we may deduce that

$$\liminf_{x \rightarrow \infty} \frac{\#W(\ell; k; x)}{x/\log x} > \alpha \sum_{\sigma(m)=\ell m} \frac{1}{m}. \quad (3.10)$$

Since this holds for any $\alpha < 1$, the lower bound for Theorem 1.1.(b) follows.

Remark 3.1. Suppose $\ell \in \Sigma$. Proposition 2.7 implies that when $k \equiv k_0 \geq 1$ is a constant function, we have

$$\frac{\#W(\ell; k_0; x)}{x/\log x} = \sum_{|k| < bk_0} \frac{\#S(a, b; k; x)}{x/\log x} \sim \sum_{\substack{0 < m < k_0/\ell \\ b|m \\ \sigma(m)=\ell m}} \frac{1}{m} \quad (3.11)$$

as $x \rightarrow \infty$. The rightmost quantity of (3.11) converges to $\sum_{\substack{\sigma(m)=\ell m \\ b|m}} 1/m$ as $k_0 \rightarrow \infty$. In particular, when $\ell \in \mathbb{Z}$ and k is an increasing unbounded function, one really observes that

$$\frac{\#W(\ell; k; x)}{x/\log x} \sim \lim_{k_0 \rightarrow \infty} \frac{\#W(\ell; k_0; x)}{x/\log x} \quad (3.12)$$

as $x \rightarrow \infty$. However, the asymptotic (3.12) is not necessarily true when $\ell \notin \mathbb{Z}$ because of the restriction $b \mid m$ present in (3.11)!

4. CONCLUDING DISCUSSIONS, NUMERICS, & FURTHER DIRECTIONS

Building upon the method of [APP12], Pollack-Pomerance-Thompson [PPT18] recently proved a variant of the main result of [APP12], albeit with a weaker error term and uniformity. Specifically, they proved that if $\ell \in \mathbb{Z}$ is kept fixed, the number of sporadic solutions to the equation $\sigma(n) = \ell n + k$ up to x is $O(x^{3/5+o_\epsilon(1)})$ as $x \rightarrow \infty$ and for any integer k . To the best of the authors' knowledge, there seems to be a number of subtleties in generalizing the method of [PPT18] to the equation $b\sigma(n) = an + k$. Additionally, as noted in [APP12], it appears that obtaining an estimate that is fully uniform in all of a, b, k would require considerable effort.

If ℓ is restricted to be an integer and is kept fixed, the same argument from Theorem 1.1 along with the main theorem of [PPT18], should yield the slightly improved admissible range $c \in (0, 2/5)$. However, the barrier of our method seems to be $c \in (0, 1/2)$, see [PPT18, Conjecture 4.3]. When ℓ is not an integer, it is unclear what the barrier should be and likely somewhat smaller than $(0, 1/2)$.

Theorem 1.1 leads to several interesting consequences which are stated as follows. As usual, $k : [1, \infty) \rightarrow (0, \infty)$ is an increasing function such that

$$y^\delta \leq k(y) \leq y^c \quad \text{for } y \geq 1. \quad (4.1)$$

Firstly, it is natural to consider the following quantity

$$\mathcal{D}_c(\ell) := \lim_{x \rightarrow \infty} \frac{\#W(\ell; y^c; x)}{x/\log x} =: \lim_{x \rightarrow \infty} \mathcal{D}_c(\ell; x) \quad (4.2)$$

for $\ell \in [1, \infty)$ and $c \in (0, 1)$. In view of Proposition 2.2, this new 'distribution function' is arguably well-suited to study the within-perfect numbers with respect to sublinear threshold. However, this distribution function behaves quite differently from the ones described in Definition 2.1.

Proposition 4.1. *The function $\ell \mapsto \mathcal{D}_c(\ell)$ is discontinuous on a dense subset of $[1, \infty)$, for any $c \in (0, 1/3)$.*

Proof. It follows from a theorem of Anderson (see [Pol, pp. 270]) that $(\mathbb{Q} \cap [1, \infty)) \setminus \Sigma$ is dense in $[1, \infty)$. Observe that \mathcal{D}_c takes the value 0 on $(\mathbb{Q} \cap [1, \infty)) \setminus \Sigma$ but it takes positive values on Σ by Theorem 1.1. So, \mathcal{D}_c is discontinuous on Σ . It is a well-known theorem that Σ is again dense in $[1, \infty)$ (see [Pol, pp. 275]). This completes the proof. \square

Secondly, a real number $\ell > 1$ is said to be Σ -approximable if there exists a function $f(x) \rightarrow \infty$ and a sequence of positive integers $(m_i)_{i \geq 1}$ such that $|\ell - \sigma(m_i)/m_i| < 1/f(m_i)$ for any $i \geq 1$. It is clear that

Proposition 4.2. *If $\ell > 1$ is Σ -approximable, then $\#W(\ell; k; x) \gg x/\log x$ on an unbounded set of x .*

In fact, for any function $f(x) \rightarrow \infty$, there are *irrational* numbers $\ell > 1$ that are Σ -approximable by f . This follows from the standard nested interval argument and the theorems of Anderson used in the proof of Proposition 4.1.

We conclude this article with some numerics and open problems for further investigation.

We calculate the quotient of $D_c(2; x)$ for various values of $c \in (0, 1)$ and at $x = 1, 000, 000$, $x = 10, 000, 000$, and $x = 20, 000, 000$. Note: $\sum_{\sigma(m)=2m} \frac{1}{m} \approx 0.2045$.

$k(y)$	$x = 1, 000, 000$	$x = 10, 000, 000$	$x = 20, 000, 000$
$y^{0.9}$	3.661860	3.305180	3.196040
$y^{0.8}$	1.141480	0.945623	0.908751
$y^{0.7}$	0.494278	0.435395	0.426470
$y^{0.6}$	0.311567	0.274586	0.267904
$y^{0.5}$	0.276559	0.259482	0.255962
$y^{0.4}$	0.264968	0.252956	0.250063
$y^{0.3}$	0.225980	0.247837	0.247299
$y^{0.2}$	0.151238	0.195911	0.197430

TABLE 1. $\mathcal{D}_c(2; x)$ for various values of x and c .

It is natural to ask the following.

Problem 4.3. When $c \in (1/2, 1)$, does the correct order of magnitude for $\#W(\ell; k; x)$, with $\ell \in \Sigma$ and k satisfying (4.1), continue to be $x/\log x$ as $x \rightarrow \infty$?

In between the sublinear and linear regime, e.g., $k(y) = y/\log y$, Proposition 2.2 gives no conclusion. Consider the plot of $x \mapsto \#W(2; k; x)/(x/\log x)$ for such k from $x = 2$ to $x = 10, 000$:

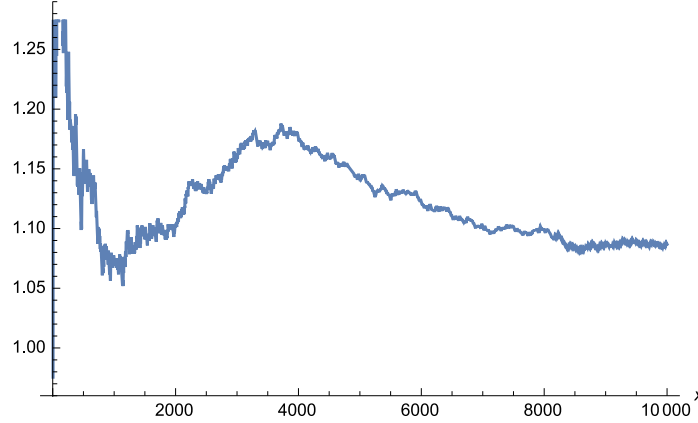


FIGURE 1. This plot shows the quantity $\#W(2; k; x)/(x/\log x)$ with $k(y) = y/\log y$ for $x = 2$ to $10, 000$.

Problem 4.4. What is the order of magnitude of $\#W(\ell; k; x)$ if the function k satisfies $y^c = o(k(y))$ for any $c \in (0, 1)$?

Problem 4.5. *What is the order of magnitude for $\#W(\ell; k; x)$ for irrational ℓ ?*

Problem 4.6. *Determine set of points of continuity for the distribution function $\ell \mapsto \mathcal{D}_c(\ell)$.*

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Email address: ucahckw@ucl.ac.uk

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON

Email address: sjml@williams.edu, Steven.Miller.MC.96@aya.yale.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267