GENERALIZING ZECKENDORF’S THEOREM TO HOMOGENEOUS LINEAR RECURRENCES

THOMAS C. MARTINEZ, STEVEN J. MILLER, CLAY MIZGERD, AND CHENYANG SUN

ABSTRACT. Zeckendorf’s theorem states that every positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers \( \{F_n\} \), where we take \( F_1 = 1 \) and \( F_2 = 2 \); in fact, it provides an alternative definition of the Fibonacci numbers. This has been generalized for any Positive Linear Recurrence Sequence (PLRS), which is, informally, a sequence satisfying a homogeneous linear recurrence with a positive leading coefficient and non-negative integer coefficients. Note these legal decompositions are generalizations of base \( B \) decompositions. We investigate linear recurrences with leading coefficient zero, followed by non-negative integer coefficients, with differences between indices relatively prime (abbreviated ZLRR), via two different approaches. The first approach involves generalizing the definition of a legal decomposition for a PLRS found in Koloğlu, Kopp, Miller and Wang. We prove that every positive integer \( N \) has a legal decomposition for any ZLRR using the greedy algorithm. We also show that \( D_n \), the number of decompositions of \( n \), grows faster than \( a_n \), implying the existence of decompositions for every positive integer \( N \), but uniqueness is lost. The second approach converts a ZLRR to a PLRR that has the same growth rate. We develop the Zeroing Algorithm, a powerful helper tool for analyzing the behavior of linear recurrence sequences. We use it to prove a very general result that guarantees the possibility of conversion between certain recurrences, and develop a method to quickly determine whether our sequence diverges to \(+ \infty\) or \(- \infty\), given any real initial values.

CONTENTS

1. Introduction and Definitions 2
   1.1. History and Past Results 2
   1.2. Main Results 3

2. Eventual Behavior of Linear Recurrence Sequences 6
   2.1. Properties of Characteristic Polynomials 6
   2.2. A Generalization of Binet’s Formula 7

3. ZLRS-Legal Decompositions 9
   3.1. Existence 9
   3.2. Loss of Uniqueness 10

4. The Zeroing Algorithm and Applications 14
   4.1. The Zeroing Algorithm 14
   4.2. A General Conversion Result 16
   4.3. Fast Determination of Divergence Using the Zeroing Algorithm 17

5. Conclusion and Future work 18

Appendix A. Some Examples of Running the Zeroing Algorithm 18
Appendix B. List of ZLRR’s and derived ZLRR’s 19
References 23

Date: January 22, 2020.
Key words and phrases. Characteristic polynomial, Fibonacci numbers, recurrence relation, Zeckendorf’s Theorem.
This work was partially supported by NSF grants DMS1659037 and DMS1561945, as well as the Finnerty Fund.
1. Introduction and Definitions

1.1. History and Past Results. The Fibonacci numbers are one of the most well-known and well-studied mathematical objects, and have captured the attention of mathematicians since their conception. This paper focuses on a generalization of Zeckendorf’s theorem, one of the many interesting properties of the Fibonacci numbers. Zeckendorf [Ze] proved that every positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers (called the Zeckendorf Decomposition), where the Fibonacci numbers are $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$. This results has been generalized to other types of recurrence sequences. We set some notation before describing these results.

**Definition 1.1.** We say a recurrence relation is a **Positive Linear Recurrence Relation (PLRR)** if there are non-negative integers $L, c_1, \ldots, c_L$ such that

$$H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L},$$

with $L, c_1$ and $c_L$ positive.

**Definition 1.2.** We say a sequence $\{H_n\}_{n=1}^{\infty}$ of positive integers arising from a PLRR is a **Positive Linear Recurrence Sequence (PLRS)** if $H_1 = 1$, and for $1 \leq n < L$ we have

$$H_{n+1} = c_1 H_n + c_2 H_{n-2} + \cdots + c_n H_1 + 1.$$ (1.2)

We call a decomposition $\sum_{i=1}^{m} a_i H_{m_i-1}$ of a positive integer $N$ (and the sequence $\{a_i\}_{i=1}^{m}$) legal if $a_1 > 0$, the other $a_i \geq 0$, and one of the following two conditions hold.

- **Condition 1:** We have $m < L$ and $a_i = c_i$ for $1 \leq i \leq m$.
- **Condition 2:** There exists $s \in \{1, \ldots, L\}$ such that

$$a_1 = c_1, a_2 = c_2, \ldots, a_{s-1} = c_{s-1}, a_s < c_s,$$

$$a_{s+1}, \ldots, a_{s+\ell} = 0 \text{ for some } \ell \geq 0, \text{ and } \{b_i\}_{i=1}^{m-s-\ell} (\text{with } b_i = a_{s+i}) \text{ is legal}.$$

Informally, a legal decomposition is one where we cannot use the recurrence relation to replace a linear combination of summands with another summand, and the coefficient of each summand is appropriately bounded; other authors [DG, Ste] use the phrase $G$-ary decomposition for a legal decomposition. For example, if $H_{n+1} = 3H_n + 2H_{n-1} + 4H_{n-2}$, then $H_5 + 2H_3 + 3H_2$ is legal, while $H_5 + 3H_4 + 2H_3 + 4H_2$ is not (we can replace $3H_4 + 2H_3 + 4H_2$ with $H_5$), nor is $6H_5 + 2H_4$ (the coefficient of $H_5$ is too large).

We now state an important generalization, and then describe what object we are studying and our results. See [BBGILMT] BM [BCCSW] CFHMN CFHMNPX DFFHMPP Ho MPX MX [Kd] Len for more on generalized Zeckendorf decompositions and [GT] MW for a proof of Theorem 1.3.

**Theorem 1.3 (Generalized Zeckendorf’s theorem for PLRS).** Let $\{H_n\}_{n=1}^{\infty}$ be a Positive Linear Recurrence Sequence. Then

1. there is a unique legal decomposition for each non-negative integer $N \geq 0$, and
2. there is a bijection between the set $S_n$ of integers in $[H_n, H_{n+1})$ and the set $D_n$ (of cardinality $D_n$) of legal decompositions $\sum_{i=1}^{m} a_i H_{n+1-i}$.

While this result is powerful and generalizes Zeckendorf’s theorem to a large class of recurrence sequences, it is restrictive in that the leading term must have a positive coefficient. We examine what happens in general to existence and uniqueness of legal decompositions if $c_1 = 0$. Special cases were studied in [CFHMN CFHMNPX], focusing on the Kentucky and $(s, b)$-Generacci Sequences; the first still had uniqueness of decomposition while the second did not.

**Definition 1.4.** We say a recurrence relation is an $s$-deep **Zero Linear Recurrence Relation (ZLRR)** if the following properties hold.

\[\text{If we use the standard initial conditions then } 1 \text{ appears twice and uniqueness is lost.}\]
(1) Recurrence relation: There are non-negative integers $s, L, c_1, \ldots, c_L$ such that
\[ G_{n+1} = c_1 G_n + \cdots + c_s G_{n+1-s} + c_{s+1} G_{n-s} + \cdots + c_L G_{n+1-L}, \]
with $c_1, \ldots, c_s = 0$ and $L, c_{s+1}, c_L$ positive.

(2) No degenerate sequences: Let $S = \{m \mid c_m \neq 0\}$ be the set of indices of positive coefficients. Then $\gcd(S) = 1$.

We impose the second restriction, because studying a sequence like $G_{n+1} = G_{n-1} + G_{n-3}$, where the odd terms and even terms do not interact, is not desirable as such a sequence naturally splits into two separate, independent sequences. Also note that 0-deep ZLRR’s are just PLRR’s, for which we can study their sequences very well. Notice that we do not define $s$-deep Zero Linear Recurrence Sequences (ZLRS), which requires the definition of initial conditions and legal decompositions because those depend on how we study ZLRR’s. This paper offers two methods: generalizing Zeckendorf’s theorem to $s$-deep ZLRS’s and converting $s$-deep ZLRR’s to PLRR’s.

However, before we can study the results of the two methods, we develop some important tools that are necessary for both. We do so in Section 2 mainly looking at characteristic polynomials of PLRR’s and $s$-deep ZLRR’s, and relating some properties to each other. We also look at a generalization of Binet expansions of recurrence sequences, which is more pertinent for the second method, that of converting $s$-deep ZLRR’s to PLRR’s.

1.2. Main Results. In Section 3 we study the first method, generalizing Zeckendorf’s theorem to $s$-deep ZLRS’s. We begin here the initial conditions and legal decompositions.

Definition 1.5. We say a sequence $\{G_n\}_{n=1}^\infty$ of positive integers arising from an $s$-deep ZLRR is an $s$-deep Zero Linear Recurrence Sequence (ZLRS) if $G_1 = 1, G_2 = 2, \ldots, G_{s+1} = s + 1$ and for $s + 2 \leq n \leq L$,
\[ G_n = \begin{cases} n, & c_{s+1} \leq s, \\ c_{s+1} G_{n-s+1} + c_{s+2} G_{n-s+2} + \cdots + c_{n-1} G_1 + 1 & c_{s+1} > s. \end{cases} \]

We call a decomposition $\sum_{i=1}^m a_i G_{m+1-i}$ of a positive integer $N$ (and the sequence $\{a_i\}_{i=1}^m$) legal if $a_i \geq 0$, and one of the following conditions hold.

- Condition 1: We have $a_1 = 1$ and $a_i = 0$ for $2 \leq i \leq m$.
- Condition 2: We have $s < m < L$ and $a_i = c_i$ for $1 \leq i \leq m$.
- Condition 3: There exists $t \in \{s + 1, \ldots, L\}$ such that

\[
a_1 = c_1, a_2 = c_2, \ldots, a_{t-1} = c_{t-1}, a_t < c_t, \quad a_{t+1}, \ldots, a_{t+\ell} = 0 \text{ for some } \ell \geq 0, \quad \text{and } \{b_i\}_{i=1}^{m-\ell} \text{ (with } b_i = a_{t+\ell+i}) \text{ is legal}.
\]

The idea behind Condition 1 is if $N$ appears in the sequence, say $N = G_n$, then we allow this to be a legal decomposition. This is necessary for there to be a legal decomposition for $N = 1$ for all $s$-deep ZLRS’s.

Remark 1.6. We note one special case for the initial conditions. If $Z_{n+1} = Z_{n-1} + Z_{n-2}$ (a recurrence relation we call the “Lagonacci” as it has a similar recurrence relation to the Fibonacci, but the terms “lag” behind and grow slowly), then $Z_1 = 1, Z_2 = 2, Z_3 = 4, Z_4 = 3, Z_5 = 6, \text{ and so on.}^2$

Similarly to the initial conditions of a PLRS, we construct our initial conditions in such a way to guarantee existence of legal decompositions. The main idea behind the definition of legal decompositions is if $N$ does not appear in the sequence (i.e., $N \neq G_n$ for any $n \in \mathbb{N}_0$), then for some $m \in \mathbb{N}_0$, $G_m \leq N < G_m+1^3$ and we cannot use $G_m, G_m-1, \ldots, G_m-s+1$ in our decomposition of $N$. Let us illustrate this with an example.

---

2 We use $Z_n$ because the Lagonacci’s are easy to study, with interesting cases, usually requiring special attention. For an example of more standard behavior, consider $Y_{n+1} = 2Y_{n-1} + 2Y_{n-2}$, with $Y_1 = 1, Y_2 = 2, Y_3 = 3, Y_4 = 6, \ldots$

3 Note that if $4 \leq N < 3$, then $N$ is not an integer, so we reach no contradiction with our special initial condition case.
Example 1.7. Consider again the Lagonacci sequence $Z_{n+1} = Z_{n-1} + Z_{n-2}$, with the first terms

\[1, 2, 4, 3, 6, 7, 9, 13, 16, \ldots,\]

and let us decompose $N = 10$. Since $Z_7 = 9 \leq 10 < 13 = Z_8$, we cannot use $Z_7 = 9$ in our decomposition. So, we use the next largest number, $Z_6 = 7$, and get $10 = 7 + 3 = Z_6 + Z_4$. This is a legal 1-deep ZLRS decomposition; however, notice that we can also have $10 = 6 + 4 = Z_5 + Z_3$.

The above example suggests the following questions. Is uniqueness of decomposition lost for all ZLRS’s? If so, is it lost for finitely many numbers? For infinitely many numbers? For all numbers from some point onward?

Our main results for this method are

**Theorem 1.8** (Generalized Zeckendorf’s theorem for $s$-deep ZLRS’s). Let $\{G_n\}_{n=1}^\infty$ be an $s$-deep Zero Linear Recurrence Sequence. Then there exists a legal decomposition for each non-negative integer $N \geq 0$.

**Theorem 1.9** (Loss of Uniqueness of Decomposition for $s$-deep ZLRS’s). Let $\{G_n\}_{n=1}^\infty$ be an $s$-deep Zero Linear Recurrence Sequence. Uniqueness of decomposition is lost for at least one positive integer $N$. Further, the number of legal decompositions grows exponentially faster than the terms of our $s$-deep ZLRS.

The proof for Theorem 1.8 is a fairly straightforward strong induction proof. The difficulty arises with the initial conditions, which are split into two cases. The main idea behind proving Theorem 1.9 relies on comparing the number of legal decompositions that a ZLRS creates to that of a related PLRS (see Definition 3.1), and showing that the number of legal decompositions grows faster than the term of the ZLRS. We prove many auxiliary results regarding characteristic polynomials to prove Theorem 1.8.

We now state the main results of the second method, converting ZLRR’s to PLRR’s. We develop a powerful helper tool in analyzing linear recurrences, the **Zeroing Algorithm**; we give a full introduction of how it works in §4. It is worth noting that this method has more uses than that of generalizing Zeckendorf’s theorem. As the first method required specific initial conditions, converting ZLRR’s to PLRR’s requires no specificity of initial conditions. We have yet to formally describe a manner to use this method to obtain meaningful results about decompositions, but our hope is that others can use the Zeroing Algorithm to do so. Before going further, we introduce an object crucial in the study of recurrence relations.

**Definition 1.10.** Given a recurrence relation

\[a_{n+1} = c_1 a_n + \cdots + c_k a_{n+1-k}, \tag{1.5}\]

we call the polynomial

\[P(x) = x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k \tag{1.6}\]

the characteristic polynomial of the recurrence relation. The degree of $P(x)$ is known as the order of the recurrence relation.

We now state results relating to the second approach, which is converting any ZLRR into a PLRR derived from it in the following sense:

**Definition 1.11.** We say that a recurrence relation $R_b$ is derived from another recurrence relation $R_a$ if

\[P_b(x) = P_a(x)Q(x), \tag{1.6}\]

where $P_a(x)$ and $P_b(x)$ are the characteristic polynomials of $R_a$ and $R_b$ respectively, as defined by equation (1.6), and $Q(x)$ is some polynomial with integer coefficients with $Q(x)$ not being the zero polynomial.
Since the roots of $P_a$ are contained in $P_b$, any sequence satisfying the recurrence relation $R_a$ also satisfies $R_b$, which means that the two recurrence relations yield the same sequence if the initial values of $\{b_n\}_{n=1}^\infty$ satisfy the recurrence relation $R_a$. This provides motivation for why the idea of a derived PLRR is relevant.

To continue, it is pertinent to state an important result, which we prove in Section 2 specifically Lemma 2.1: the characteristic polynomial of any PLRR or ZLRR has a unique positive root of multiplicity 1 and magnitude greater than that of any other root. We call this the principal root of the characteristic polynomial, and denote it as $r$.

We now state a main result, which has two important corollaries that guarantee the possibility of conversion between certain linear recurrences; the Zeroing Algorithm itself provides a constructive way to do so.

**Theorem 1.12.** Given some PLRR/ZLRR, let $P(x)$ denote its characteristic polynomial, and $r$ its principal root. Suppose we are given an arbitrary sequence of real numbers $\gamma_1, \gamma_2, \ldots, \gamma_m$, and define, for $t \leq m$,

$$\Gamma_t(x) := \gamma_1 x^{t-1} + \gamma_2 x^{t-2} + \cdots + \gamma_{t-1} x + \gamma_t.$$  

(1.7)

If $\Gamma_m(r) > 0$, there exists a polynomial $p(x)$, divisible by $P(x)$, whose first coefficients are $\gamma_1$ through $\gamma_m$, with no positive coefficients thereafter.

**Corollary 1.13.** Given arbitrary integers $\gamma_1$ through $\gamma_m$ with $\Gamma_m(r) > 0$, there is a recurrence derived from $P(x)$ which has first coefficients $\gamma_1$ through $\gamma_m$ with no negative coefficients thereafter.

**Corollary 1.14.** Every ZLRR has a derived PLRR.

A natural question of interest that arises in the study of recurrences is the behavior of the size of terms in a recurrence sequence. The Fibonacci sequence behaves like a geometric sequence whose ratio is the golden ratio, and there is an analogous result for general linear recurrence sequences, proven in [BBGILMT]:

**Theorem 1.15.** Let $P(x)$ be the characteristic polynomial of some linear recurrence relation, and let the roots of $P(x)$ be denoted as $r_1, r_2, \ldots, r_j$, with multiplicities $m_1, m_2, \ldots, m_j \geq 1$, respectively.

Consider a sequence $\{a_n\}_{n=1}^\infty$ of complex numbers satisfying the recurrence relation. Then there exist polynomials $q_1, q_2, \ldots, q_j$, with $\text{deg}(q_i) \leq m_i - 1$, such that

$$a_n = q_1(n) r_1^n + q_2(n) r_2^n + \cdots + q_j(n) r_j^n.$$  

(1.8)

**Definition 1.16.** We call (1.8) the Binet expansion of the sequence $\{a_n\}_{n=1}^\infty$, in analogy to the Binet Formula that provides a closed form for Fibonacci numbers.

One might ask that given a PLRR/ZLRR with some real initial values, do the terms eventually diverge to positive infinity or negative infinity? One approach is to compute as many terms as needed for the eventual behavior to emerge; unfortunately, this could be very time-consuming. One could alternately solve for the Binet expansion, which often requires an excessive amount of computation.

The fact that the characteristic polynomials for PLRR/ZLRR’s have a principal root $r$ allows us a shortcut. Consider the Binet expansion of a ZLRS/PLRS; the coefficient attached to the $r^n$ term, whenever nonzero, indicates the direction of divergence. We develop the following method to determine the sign of this coefficient from the initial values of the recurrence sequence:

**Theorem 1.17.** Given a ZLRS/PLRS $\{a_n\}_{n=1}^\infty$ with characteristic polynomial $P(x)$ and real initial values $a_1, a_2, \ldots, a_k$, consider the Binet expansion of $\{a_n\}_{n=1}^\infty$. The sign of the coefficient attached to $r^n$ agrees with the sign of

$$Q(x) := a_1 x^{k-1} + (a_2 - d_2) x^{k-2} + (a_3 - d_3) x^{k-3} + \cdots + (a_k - d_k),$$  

(1.9)
evaluated at $x = r$, where

$$d_i = a_1 c_{i-1} + a_2 c_{i-2} + \cdots + a_{i-1} c_1 = \sum_{j=1}^{i-1} a_j c_{i-j}.$$  \hspace{1cm} (1.10)

We conclude in §5 with some open questions for future research.

2. Eventual Behavior of Linear Recurrence Sequences

In this section, we prove important lemmas related to the roots of characteristic polynomials that are used with both methods. In the celebrated Binet’s Formula for Fibonacci numbers, the principal root of its characteristic polynomial (i.e., the golden ratio) determines the behavior of the sequence as nearly geometric, with the golden ratio being the common ratio. We generalize this characterization of near-geometric behavior to more general linear recurrences.

2.1. Properties of Characteristic Polynomials. We first prove a lemma regarding recurrence relations of the form \((1.5)\), with $c_i$ non-negative integers for $1 \leq i \leq k$ and $c_k > 0$. We first justify the definition of the principal root.

**Lemma 2.1.** Consider $P(x)$ as in \((1.6)\) and let $S := \{m \mid c_m \neq 0\}$. Then

1. there exists exactly one positive root $r$, and this root has multiplicity 1,
2. every root $z \in \mathbb{C}$ satisfies $|z| \leq r$, and
3. if $\gcd(S) = 1$, then $r$ is the unique root of greatest magnitude.

**Proof.** By Descartes’s Rule of Signs, $P(x)$ has exactly one positive root of multiplicity one, completing the proof of Part (1).

Now, consider any root $z \in \mathbb{C}$ of $P(x)$; we have $z^k = c_1 z^{k-1} + c_2 z^{k-2} + \cdots + c_k$. Taking the magnitude, we have

$$|z|^k = |z^k| = |c_1 z^{k-1} + c_2 z^{k-2} + \cdots + c_k| \leq |c_1 z^{k-1}| + |c_2 z^{k-2}| + \cdots + |c_k| = c_1 |z|^{k-1} + c_2 |z|^{k-2} + \cdots + c_k,$$

which means $P(|z|) \leq 0$. Since $P(x)$ becomes arbitrarily large with large values of $x$, we see that there is a positive root at or above $|z|$ by the Intermediate Value Theorem, which completes Part (2).

Finally, suppose $\gcd(S) = 1$. Suppose for sake of contradiction that a non-positive root $z$ satisfies $|z| = r$; we must have $P(|z|) = 0$, which means

$$|z|^k = |c_1 z^{k-1} + c_2 z^{k-2} + \cdots + c_k| = |c_1 z^{k-1}| + |c_2 z^{k-2}| + \cdots + |c_k|.$$  \hspace{1cm} (2.1)

This equality holds only if the complex numbers $c_1 z^{k-1}, c_2 z^{k-2}, \ldots, c_k$ share the same argument; since $c_k > 0$, $z^{k-j}$ must be positive for all $c_j \neq 0$. This implies $z^k$, as a sum of positive numbers, is positive as well. Writing $z = |z| e^{i\theta}$, we see that the positivity of $z^k = |z|^k e^{ik\theta}$ implies $k\theta$ is a multiple of $2\pi$, and consequently, $\theta = 2\pi d/k$ for some integer $d$. We may reduce this to $2\pi d'/k'$ for relatively prime $d', k'$.

Let $J := S \cup \{0\}$. Since $z^{k-j}$ is positive for all $j \in J$, we see that $2\pi d' (k - j)/k'$ is an integer multiple of $2\pi$, so $k'$ divides $d' (k - j)$; as $d'$ and $k'$ are relatively prime we have $k'$ divides $k - j$. Since the elements of $J$ have greatest common divisor 1, so do the elements of $K := \{k - j \mid j \in J\}$. Since $k'$ divides every element of $K$, we must have $k' = 1$, so $\theta = 2\pi d'$ and thus $z$ is a positive root. This is a contradiction, completing the proof of Part (3). \hfill \Box

\footnote{Note that this is Condition 2 from Definition 1.4 thus met by all $s$-deep ZLRS’s.}

\footnote{Observe that $k$ is in both $J$ and $K$. Suppose, for contradiction, that some $q > 1$ divides every element of $K$; then, every element of $\{k - \kappa \mid \kappa \in K\} = J$ is divisible by $q$, which is impossible.}
Next, we prove a lemma that sheds light on the growth rate of the terms of a ZLRR/PLRR with a specific set of initial values.

**Lemma 2.2.** For a PLRR/ZLRR, let $r$ be the principal root of its characteristic polynomial $P(x)$. Then, given initial values $a_i = 0$ for $0 \leq i \leq k - 2$, $a_{k-1} = 1$, we have

$$\lim_{n \to \infty} \frac{a_n}{r^n} = C,$$

(2.3)

where $C > 0$. Furthermore, the sequence $\{a_n\}_{n=1}^\infty$ is eventually monotonically increasing.

**Proof.** Since $r$ has multiplicity 1, $q_1$ is a constant polynomial. To see geometric behavior, we note that

$$\lim_{n \to \infty} \frac{a_n}{r^n} = \lim_{n \to \infty} q_1(n) \left(\frac{r^n}{r^n}\right) + \lim_{n \to \infty} q_2(n) \left(\frac{r^2}{r}\right)^n + \cdots + \lim_{n \to \infty} q_j(n) \left(\frac{r_j}{r}\right)^n.$$

(2.4)

Since $|r| > |r_i|$ for all $2 \leq i \leq j$, each limit with a $(r_i/r)^n$ term disappears, leaving just $q_1$, which must be positive, since the sequence $a_n$ does not admit negative terms.

To see that $a_n$ is eventually increasing, consider the sequence

$$A_n := a_{n+1} - a_n = (q_1 r_1 - q_1) r_1^n + (q_2 (n+1) r_2 - q_2(n)) r_2^n + \cdots + (q_j (n+1) r_j - q_j(n)) r_j^n. \quad (2.5)$$

A similar analysis shows

$$\lim_{n \to \infty} \frac{(q_2 (n+1) r_2 - q_2(n)) r_2^n + \cdots + (q_j (n+1) r_j - q_j(n)) r_j^n}{(q_1 r_1 - q_1) r_1^n} = 0, \quad (2.6)$$

meaning that the term $(q_1 r_1 - q_1) r_1^n$ grows faster than the sum of the other terms; thus $A_n$ is eventually positive as desired. \qed

**Corollary 2.3.** For a PLRR/ZLRR, let $r$ be the principal root of its characteristic polynomial $P(x)$. Then, given initial values satisfying $a_i \geq 0$ for $0 \leq i \leq k - 1$ and $a_i > 0$ for some $0 \leq i \leq k - 1$, we have

$$\lim_{n \to \infty} \frac{a_n}{r^n} = C,$$

(2.7)

where $C > 0$. Furthermore, the sequence $\{a_n\}_{n=1}^\infty$ is eventually monotonically increasing. That is, Lemma 2.2 can be generalized to any set of non-negative initial conditions that are not all zero.

**Proof.** We first note that the derivation of (1.8) does not rely on the initial values; any sequence satisfying the recurrence takes on this form.

Since one of the initial values $a_0, a_1, \ldots, a_{k-1}$ is a positive integer, we know that one of $a_k, a_{k+1}, \ldots, a_{2k-1}$ is also a positive integer by the recurrence relation, which forces $a_n$ to be at least $a_{n-k}$. Let $k \leq i \leq 2k - 1$ be such that $a_i$ is positive. Consider the sequence $b_n = a_{n+i-k+1}$, which has $b_{k-1} = a_i > 0$. By the recurrence relation, we have $b_n \geq a_n$ for all $n$, which would be impossible if the Binet expansion of $b_n$ had a non-positive coefficient attached to the $r^n$ term. Eventual monotonicity thus follows. \qed

### 2.2. A Generalization of Binet’s Formula.

In general, the Binet expansion of a recurrence sequence is quite unpleasant to compute or work with. However, things become much simpler when the characteristic polynomial has no multiple roots. In that case, we may construct an explicit formula for the $n$th term of the sequence, given a nice set of initial values. Keeping in mind that linear combinations of sequences satisfying a recurrence also satisfy the recurrence, one could construct a formula for the $n$th term given arbitrary initial values.
Theorem 2.4. Consider a ZLRR with characteristic polynomial $P(x)$ that does not have multiple roots, and initial values $a_i = 0$ for $0 \leq i \leq k - 2$, $a_{k-1} = 1$. Then each term of the resulting sequence may be expressed as

$$a_n = c_1 x_1^n + c_2 x_2^n + \cdots + c_k x_k^n,$$

(2.8)

where the $r_i$ are the distinct roots of $P(x)$, and $c_i = 1/P'(r_i)$.

Proof. Since each root has multiplicity 1, the existence of such explicit form follows from the Binet expansion (see Theorem 1.15), so we are left to prove that $c_i = 1/P'(r_i)$. Using the initial values, we see that the $c_i$ are solutions to the linear system

$$
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
r_1 & r_2 & r_3 & \cdots & r_k \\
r_1' & r_2' & r_3' & \cdots & r_k' \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1 \\
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
\vdots \\
c_k \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
\end{pmatrix}.

(2.9)

Denote the matrix by $A$; by Cramer’s rule, we have $c_i = \det(A_i)/\det(A)$, where $A_i$ is the matrix formed by replacing column $i$ of $A$ with the column vector of zeroes and a single 1. Using Laplace expansion, we see that $\det(A_i) = (-1)^{k+i} \det(M_{ki})$, where $M_{ki}$ is the $k; i$ minor matrix of $A$ formed by deleting row $k$ and column $i$. Notice that both $A$ and $M_{ki}$ are Vandermonde matrices, which means we have

$$
\det(A) = \prod_{1 \leq a < b \leq k} (r_b - r_a), \quad \det(M_{ki}) = \prod_{1 \leq a < b \leq k \atop a, b \neq i} (r_b - r_a).
$$

(2.10)

We may thus simplify and find

$$
c_i = (-1)^{k+i} \left( \prod_{1 \leq a < b \leq k \atop a, b \neq i} (r_b - r_a) \right) \left( \prod_{1 \leq a < b \leq k \atop a = i \text{ or } b = i} (r_b - r_a) \right)^{-1} \left( \prod_{1 \leq a < b \leq k} (r_b - r_a) \right) \left( r_i - r_1 \right) \left( r_i - r_2 \right) \cdots \left( r_i - r_{i-1} \right) \left( r_i - r_k \right) \left( r_i - r_{k-1} \right) \cdots \left( r_i - r_1 \right) \left( r_i - r_k \right) \\
= (-1)^{k+i} \left( \prod_{1 \leq a < b \leq k \atop a = i \text{ or } b = i} (r_b - r_a) \right)^{-1} \left( \prod_{1 \leq j \leq k \atop j \neq i} (r_i - r_j) \right) \\
= 1 \left( \prod_{1 \leq j \leq k \atop j \neq i} (r_i - r_j) \right).
$$

(2.11)

Note that the product is simply the function

$$
f(x) = \prod_{1 \leq j \leq k \atop j \neq i} (x - r_j)
$$

(2.12)
evaluated at $x = r_i$. To evaluate this, we may rewrite

$$f(r_i) = \lim_{x \to r_i} f(x) = \lim_{x \to r_i} \prod_{1 \leq j \leq k, j \neq i} (x - r_j)$$

$$= \lim_{x \to r_i} \frac{(x - r_i)}{(x - r_i)} \prod_{1 \leq j \leq k, j \neq i} (x - r_j) = \lim_{x \to r_i} \frac{\prod_{1 \leq j \leq k} (x - r_j)}{x - r_i}$$

$$= \lim_{x \to r_i} \frac{P(x)}{x - r_i}, \quad (2.13)$$

which equals $P'(r_i)$ by l’Hôpital’s rule. We thus have $c_i = 1/f(r_i) = 1/P'(r_i)$, completing the proof. □

3. ZLRS-LEGAL DECOMPOSITIONS

We prove Theorems 1.8 and 1.9 in sections §3.1 and §3.2, respectively.

3.1. Existence. To prove the existence of legal decompositions for every integer $N \geq 0$ given any $s$-deep ZLRS, we show that the greedy algorithm always terminates in a legal decomposition through strong induction. At each step the greedy algorithm uses the largest element from a sequence, index-wise. For example, if $G_{10} < N < G_{11}$, then we use $G_{10}$ in our decomposition of $N$, even if $G_9 \geq G_{10}$. We also need to make sure our decomposition is legal. At each step, we use the largest coefficient possible, depending on the coefficients of our $s$-deep ZLRS, and make sure we do not have more terms than is legal. We show how we do this in the proof of Theorem 1.8.

**Proof of Theorem 1.8.** Recall that our $s$-deep ZLRS has the form of Equation (1.3). We first prove that the greedy algorithm terminates in a legal decomposition for all integers $N$ up to and including the last initial condition. We let the empty decomposition be legal for $N = 0$. These are the base cases. There are two cases and a special third case, as it only applies to a specific sequence.

Case 1: If $c_s+1 \leq s$, then note the initial conditions are the first $L$ integers. So, by Condition (1), we trivially have a legal decomposition for all of our initial conditions.

Case 2: If $c_s+1 > s$, then our initial conditions are specially constructed so that we guarantee existence of legal decompositions. We do so by adding the smallest integer that cannot be legally decomposed by the previous terms. We illustrate this with an example. Let us take 1-deep ZLRS’s of the form $G_{n+1} = c_1 G_{n-1} + c_2 G_{n-2}$, where $c_1 > 1$ and $c_2 > 0$. Then, our initial conditions start with $G_1 = 1$ and $G_2 = 2$. Assuming $G_3 > G_2$, we know all $N$ with $G_2 < N < G_3$ cannot use $G_2 = 2$ in their decomposition, so we can only use $G_1 = 1$. We also have a restriction of only being able to use $G_1 = 1$ at most $c_1$ times. So, the first number we cannot legally decompose is $c_1 + 1$, thus, $G_3 = c_1 + 1$, which comes from our construction as well. By a similar argument, $G_4 = 2c_1 + c_2 + 1$.

Case 3 (Special): If our ZLRS is the Lagonaccis, then we must consider the first four terms in our sequence instead of the first three terms. However, since all four integers appear in our sequence ($Z_1 = 1, Z_2 = 2, Z_3 = 4,$ and $Z_4 = 3$), we still get a trivial legal decomposition for the first four positive integers.

This is now our inductive step of the induction proof. We now assume that all integers up to and including $N - 1$ has a legal decomposition. We now show that $N$ must have a legal decomposition. Let $G_t \leq N < G_{t+1}$. There are two cases to consider.
Case 1: Suppose $N = G_t$. Then, trivially, we have a legal decomposition.

Case 2: Suppose $N > G_t$ and let $m \leq N$ be the largest integer created using a legal decomposition involving only summands drawn from $G_t, G_{t-1}, \ldots, G_{t-L}$. Suppose $m = a_1 G_{t-s-1}$, with $a_1 < c_{s+1}$. We want to show that $N - m$ can be expressed with the remaining terms. To do so, we need $N - m < G_{t-s-1}$. Suppose not. Then $N - m \geq G_{t-s-1}$. However, this implies that we have not used the maximum number of $G_{t-s-1}$’s in our greedy decomposition, which is a contradiction. So, we now have that $N - m < G_{t-s-1}$.

By the strong inductive hypothesis, there exists a legal decomposition of $N - m$. We then add $m$ to this legal decomposition to obtain our decomposition of $N$. Since the decomposition for $N - m$ is legal, adding $m$ is keeps our decomposition legal, by Condition (3) of Definition 1.5. So, we have a legal decomposition for $N$.

Let $c_i$ be the next non-zero constant in our recurrence relation. We then let $m = c_{s+1} G_{t-s-1} + a_i G_{t-s-i}$ with $a_i < c_{s+i}$. We want to show that $N - m$ can be expressed with the remaining terms. To do so, we need $N - m < G_{t-s-i}$. Suppose not. Then $N - m \geq G_{t-s-i}$. However, this implies that we have not used the maximum number of $G_{t-s-i}$’s in our greedy decomposition, which is a contradiction. So, we have that $N - m < G_{t-s-i}$. By the same reasoning as the previous case, we have a legal decomposition for $N$.

We continue this argument, taking the next non-zero constant, adding that on to $m$, until we reach this final case.

Let $m = c_1 G_t + c_2 G_{t-1} + \cdots + c_{L-1} G_{t+2-L} + (c_L - 1) G_{t+1-L}$. This is the largest possible value $m$ can attain while still being having a legal decomposition. We want to show that $N - m < G_{t-L+1}$. Noting $N < G_{t+1}$, we see that

$$N - m = N - (c_1 G_t + \cdots + c_{L-1} G_{t+2-L} + (c_L - 1) G_{t+1-L}) < G_{t+1} - (c_1 G_t + \cdots + c_{L-1} G_{t+2-L} + (c_L - 1) G_{t+1-L}) = (c_1 G_t + \cdots + c_{L-1} G_{t+2-L} + c_L G_{n+1-L}) - (c_1 G_t + \cdots + c_{L-1} G_{t+2-L} + (c_L - 1) G_{t+1-L}) = G_{t+1-L}.$$  

(3.1)

Thus $N - m < G_{t+1-L}$, and in every case we attain a legal decomposition for $N$, as desired. Therefore, by strong induction, we attain a legal decomposition for all positive integers $N$ and all $s$-deep ZLRS’s. \[\square\]

3.2. Loss of Uniqueness. We now explore the loss of uniqueness of legal decompositions in $s$-deep ZLRS’s, where $s \geq 1$. We prove Theorem 1.9 after introducing some notation.

**Definition 3.1.** Let $\{G_n\}_{n=1}^\infty$ be an $s$-deep ZLRS, with recurrence relation

$$G_{n+1} = c_1 G_n + c_2 G_{n-1} + \cdots + c_s G_{n+1-s} + c_{s+1} G_{n-s} + \cdots + c_L G_{n+1-L}. \quad (3.2)$$

We say a sequence $\{H_n\}_{n=1}^\infty$ is a fostered PLRS of $\{G_n\}_{n=1}^\infty$ if $\{H_n\}_{n=1}^\infty$ is a PLRS of the form

$$H_{n+1} = c_{s+1} H_n + c_{s+2} H_{n-1} + \cdots + c_L H_{n+s+1-L}. \quad (3.3)$$

The following lemmas prove results concerning the characteristic polynomials of our $s$-deep ZLRS and its fostered PLRS. We define the characteristic polynomial of our $s$-deep ZLRS as

$$P_Z(x) := x^L - c_{s+1} x^{L-s-1} - c_{s+2} x^{L-s-2} - \cdots - c_{L-1} x - c_L, \quad (3.4)$$

and of our fostered PLRS as

$$P_P(x) := x^L - c_{s+1} x^{L-s-1} - c_{s+2} x^{L-s-2} - \cdots - c_{L-1} x - c_L. \quad (3.5)$$

Note that all results of Lemma 2.1 apply to $P_Z(x)$ and $P_P(x)$, because these polynomials meet the necessary conditions. We now prove a lemma relating the two positive roots of $P_Z(x)$ and $P_P(x)$.

**Lemma 3.2.** Let $r$ be the root of greatest magnitude of $P_Z(x)$ and $w$ be the root of greatest magnitude of $P_P(x)$, defined in equations (3.4) and (3.5), respectively. Then $w > r > 1$. 

10
Lemma 3.4. Let \( w > r > 1, \) which is equivalent to

\[
L - w \geq c_{s+1} L^{s-1} + c_{s+2} x L^{s-2} + \cdots + c_{L-1} x + c_L > 1,
\]

which implies \( r > 1. \) Notice that

\[
0 = P_L(r) = r^L - c_{s+1} r^{L-s-1} - \cdots - c_{L-1} r - c_L
\]

\[
= r^L + (r^{L-s} - r^{L-s}) - c_{s+1} r^{L-s-1} - \cdots - c_{L-1} r - c_L
\]

\[
= r^L - r^{L-s} + P_L(r).
\]

Since \( r > 1, r^L - r^{L-s} > 0, \) which means \( P_L(r) < 0. \) Since \( \lim_{n \to \infty} P_L(x) = \infty, \) \( P_L \) must have a root greater than \( r \) by the Intermediate Value Theorem. By Lemma 3.2, we also know that the root of greatest magnitude, \( w, \) is positive. So, we find \( w > r > 1. \)

We now prove lemmas giving stronger relations on the roots \( w \) and \( r. \)

**Lemma 3.3.** Let \( w \) and \( r \) be defined as in Lemma 3.2. Then \( w^n > r^{n+1}, \) for \( n \geq \log_{w/r} w. \)

**Proof.** By Lemma 3.2, we know that \( w > r > 1. \) Thus \( r/w < 1, \) and there exists an \( n \in \mathbb{Z}^+ \) such that

\[
\left( \frac{r}{w} \right)^n w < 1,
\]

which is equivalent to \( w^n > r^{n+1}, \) as desired. Simple algebra yields \( n \geq \log_{w/r} w. \)

**Lemma 3.4.** Let \( w \) and \( r \) be as defined in Lemma 3.2. Then, \( w^{L-s} > r^L. \)

**Proof.** Equivalently, we prove \( w^{L-s} - r^L > 0. \) We see that

\[
w^{L-s} - r^L = (c_{s+1} w^{L-s-1} + \cdots + c_{L-1} w + c_L) - (c_{s+1} r^{L-s-1} + \cdots + c_{L-1} r + c_L)
\]

\[
= c_{s+1} (w^{L-s-1} - r^{L-s-1}) + c_{s+2} (w^{L-s-2} - r^{L-s-2}) + \cdots + c_{L-1} (w - r).
\]

By Lemma 3.2, we know that \( w > r > 1, \) so \( w^t > r^t \) for any \( t \in \mathbb{N}. \) Recall that \( c_i \geq 0 \) for all \( i, \) and \( c_{s+1}, c_L > 0. \) Thus

\[
c_{s+1} (w^{L-s-1} - r^{L-s-1}) + c_{s+2} (w^{L-s-2} - r^{L-s-2}) + \cdots + c_{L-1} (w - r) > 0,
\]

as desired.

**Corollary 3.5.** Let \( w \) and \( r \) be as defined in Lemma 3.2. Then

\[
w^{(n(L-s)/L-1)} > r^n.
\]

**Proof.** By Lemma 3.4, which states \( w^{L-s} > r^L, \) and Lemma 3.2, which states \( w > r > 1, \) we have

\[
w^{(L-s)/L} > r.
\]

Thus \( w^{(L-s)/L}/r > 1, \) which implies there exists \( n \) such that\(^6\)

\[
\left( \frac{w^{(L-s)/L}}{r} \right)^n > w.
\]

Through algebraic manipulation, the above is equivalent to (3.11), as desired.

We introduce some more notation before combining these results into the proof of Theorem 1.9. As usual, we have \( \{G_n\}_{n=1}^{\infty} \) as an \( s \)-deep ZLRS, with \( s > 1, \) and \( \{H_n\}_{n=1}^{\infty} \) as our fostered PLRS.

**Definition 3.6.** We define the five objects that will be studied in the following lemmas and in the proof of Theorem 1.9.

\(^6\)In fact, this statement is true for \( n > \log(w)/\log(w^{(L-s)/L}/r). \)
(1) $D_n$: The set of $s$-deep ZLRS legal decompositions for all integers $N < G_{n+1}$. Note that these decompositions use elements of $\{G_1, G_2, \ldots, G_n\}$, and we include the empty decomposition in this count.

(2) $E_n$: The set of PLRS legal decompositions for all integers $N < H_{n+1}$. Note that these decompositions use elements of $\{H_1, H_2, \ldots, H_n\}$, and we include the empty decomposition in this count.

(3) A decomposition arising from the recurrence relation $R$ is denoted by

$$ (a_na_{n-1} \ldots a_2a_1)_R = a_nR_n + a_{n-1}R_{n-1} + \cdots + a_2R_2 + a_1R_1. \tag{3.14} $$

For example, the decomposition $(a_na_{n-1} \ldots a_2a_1)_G$ denotes a decomposition in $D_n$.

(4) $f_G(N)$: the number of legal decompositions of the positive integer $N$ from the ZLRS $\{G_n\}_{n=1}^\infty$.

(5) $f_{G,\text{ave}}(n) = \frac{1}{G_{n+1}} \sum_{m=0}^{G_{n+1}-1} f_G(m)$, the average number of decompositions for all integers $N < G_{n+1}$.

As previously proved in [Ho, Ke, KKMW, Len], we know $E_n$ very well. In fact, we know $|E_n| = H_{n+1}$, since there is a unique decomposition for each integer $N < H_{n+1}$, and we count the empty legal decomposition in this. We do not know $D_n$ very well, but we can bound it using relationships to $E_n$. We now provide some relationships between the sizes of $D_n$ and $E_n$.

**Lemma 3.7.** Let $D_n$ and $E_n$ be as defined in Definition 3.6 Then

1. $|E_n| \geq |D_n|$ for $n \geq 0$, and
2. $|D_n| \geq |E_{n(L-s)/L}|$ for $n \geq L$.

**Proof:**

We first prove (1). Recall that we are considering an $s$-deep ZLRS $\{G_n\}_{n=1}^\infty$ and its fostered PLRS $\{H_n\}_{n=1}^\infty$. Consider a decomposition $(a_na_{n-1} \ldots a_2a_1)_G \in D_n$. We show $(a_na_{n-1} \ldots a_2a_1)_H \in E_n$ by showing $(a_na_{n-1} \ldots a_2a_1)_H$ satisfies the legal PLRS decomposition conditions and represents an integer $N < H_{n+1}$. We first illustrate this with an example. Let us consider the 2-deep ZLRS with $G_{n+1} = 4G_n - 2 + 5G_{n-1} + 7G_{n-2}$, which has fostered PLRS $H_{n+1} = 4H_n + 5H_{n-1} + 7H_{n-2}$. Consider the decomposition $(0453000440)_G \in D_{10}$. We wish to show $(0453000440)_H \in E_{10}$. As shown in previous papers, such as [MW], we know that $(0453000440)_H$ represents the PLRS legal decomposition for $N$, then $N < H_{11}$. We also see that this decomposition follows all conditions laid out in Definition 3.2 as all coefficients are appropriately bounded. So, $(0453000440)_H \in E_{10}$.

We now show $(a_na_{n-1} \ldots a_2a_1)_H \in E_n$ by showing $(a_na_{n-1} \ldots a_2a_1)_H$ satisfies the legal PLRS decomposition conditions and represents an integer $N < H_{n+1}$. The latter is simple. Suppose $N \geq H_{n+1}$, then we must use $H_{n+1}$ (or a larger term) in our decomposition; however, our decomposition $(a_na_{n-1} \ldots a_2a_1)_H$ does not use $H_{n+1}$ (or any larger term) in its decomposition, so we reach a contradiction. Now suppose $(a_na_{n-1} \ldots a_2a_1)_H$ did not satisfy the legal PLRS decomposition conditions. Then, for some $i$ and $j$, we have $a_i > c_j$, where $c_j$ is the corresponding non-negative coefficient. However, if this is true, then it is also the case for $(a_na_{n-1} \ldots a_2a_1)_G$, meaning $(a_na_{n-1} \ldots a_2a_1)_G$ is not an $s$-deep ZLRS legal decomposition, which is a contradiction. Thus $(a_na_{n-1} \ldots a_2a_1)_H \in E_n$, implying $|E_n| \geq |D_n|$ for $n \geq 0$.

We now prove (2). We wish to create in injective function $f : E_{n(L-s)/L} \rightarrow D_n$. We define $f$ as follows: take a decomposition $(a[n(L-s)/L]a[n(L-s)/L-1] \ldots a_2a_1)_H \in E_{n(L-s)/L}$ and add $s$ zeros in front of the first positive $a_i$, starting from the left. Then move down the decomposition until Condition 1 or the first portion of Condition 2 of Definition 3.2 is met. Then move to the next positive $a_i$, and add $s$ zeros, and repeat. Once we finish this process, we add the sufficient number of zeros to the front of the decomposition, such that we have a total of $n$ coefficients. Note that this guarantees an $s$-deep ZLRS legal decomposition, since we always have $s$ zeros between each ‘chunk’ and the coefficients will be appropriately bounded.
We illustrate this with a specific example. Take the 4-deep ZLRS \( G_{n+1} = 2G_{n-4} + 3G_{n-5} + 5G_{n-6} \), which has fostered PLRS \( H_{n+1} = 2H_n + 3H_{n-1} + 5H_{n-2} \), and take \( n = 24 \). Note that \( L = 7 \), so \( [n(L-s)/L] = [24(7-4)/7] = 10 \). Finally, consider the decomposition \((2302320022)_H \in E_{10} \). We see that

\[
 f((2302340022)_H) = (000000230000023400000022)_G \in D_{24}.
\]

We illustrate this procedure with a general example. Take the decomposition in \( E_{[n(L-s)/L]} \) that uses the most coefficients. This process is shown in Figure 1.

![Figure 1. General example of \( f \).](image)

Now that we have explained the function, and we see that if a decomposition \( x \in E_{[n(L-s)/L]} \), then \( f(x) \in D_n \), we now show this function is injective. Once we show the function is injective, we know that \( |D_n| \geq |E_{[n(L-s)/L]}| \).

Consider decompositions \( a, b \in E_{[n(L-s)/L]} \) where \( a = (a_1 |_{n(L-s)/L} a_2 |_{n(L-s)/L} \ldots a_2 a_1)_H \) and \( b = (b_1 |_{n(L-s)/L} b_2 |_{n(L-s)/L} \ldots b_2 b_1)_H \) such that \( f(a) = f(b) \). We wish to show that \( a = b \). Suppose, to the contrary, that \( a \neq b \). Then \( a_i \neq b_i \) for some \( 1 \leq i \leq [n(L-s)/L] \). However, if this is the case then \( f(a) \neq f(b) \), because \( f \) does not change the value of \( a_i \) or \( b_i \). If the decomposition begins with seven 4’s, then it ends with seven 4’s. The only change \( f \) makes to the decomposition is the addition of a number of zeros. Next, the relative positioning of each number in the decomposition is left unchanged. For example, if there is a 5 in the decomposition followed by a 4, this will still be true, albeit there may be \( s \) zeros between the 5 and 4 once \( f \) is applied. From this, we see that \( f \) is injective.

Now, we have all of the ingredients to prove Theorem 1.9.

**Proof of Theorem 1.9** We note, by definition

\[
 f_{G, \text{ave}}(n) = \frac{1}{G_{n+1}} \sum_{m=0}^{G_{n+1}-1} f_G(m) = \frac{|D_n|}{G_{n+1}}. \quad (3.15)
\]

We first prove the upper bound. By Lemma 3.7 and the definition of \( E_n \), we find

\[
 \frac{|D_n|}{G_{n+1}} \leq \frac{|E_n|}{G_{n+1}} = \frac{H_{n+1}}{G_{n+1}} \approx \left( \frac{w}{r} \right)^{n+1}, \quad (3.16)
\]

where the approximation is justified by Corollary 2.3. Note that by Lemma 3.2, \( w > r \), so \( f_{G, \text{ave}}(n) \) is bounded above by \( \lambda_1^{n+1} \) where \( \lambda_1 = w/r > 1 \).
4. The Zeroing Algorithm and Applications

An alternate approach to understanding decompositions arising from ZLRR’s is to see if for every ZLRR one could associate a PLRR with similar behavior: a derived PLRR. In this section, we develop the machinery of the Zeroing Algorithm, which is an extremely powerful tool for understanding recurrence sequences analytically. We prove a very general result about derived recurrences that implies every ZLRS has a derived PLRS.

4.1. The Zeroing Algorithm. Consider some ZLRS/PLRS with characteristic polynomial

\[ P(x) := x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k, \]

and choose a sequence of \( k \) real numbers \( \beta_1, \beta_2, \ldots, \beta_k \); the \( \beta_i \) are considered the input of the algorithm. For nontriviality, the \( \beta_i \) are not all zero. We define the Zeroing Algorithm to be the following procedure. First, create the polynomial

\[ Q_0(x) := \beta_1 x^{k-1} + \beta_2 x^{k-2} + \cdots + \beta_{k-1} x + \beta_k. \]

Next, for \( t \geq 1 \), define a sequence of polynomials

\[ Q_t(x) := x Q_{t-1}(x) - q(1, t - 1) P(x), \]

indexed by \( t \), where \( q(1, t) \) is the coefficient of \( Q_t(x) \) at the \( x^{k-1} \) term. We terminate the algorithm at step \( t \) if \( Q_t(x) \) does not have positive coefficients.

To understand the algorithm through linear recurrences, we denote by \( q(n, t) \) the coefficient of \( Q_t(x) \) at the term \( x^{k-n} \), where \( n \) ranges from 1 to \( k \). We unravel the recurrence relation on the polynomials, and obtain the following system of recurrence relations

\[
\begin{align*}
q(1, t) &= q(2, t - 1) + c_1 q(1, t - 1), \\
q(2, t) &= q(3, t - 1) + c_2 q(1, t - 1), \\
& \quad \vdots \\
q(k - 1, t) &= q(k, t - 1) + c_{k-1} q(1, t - 1), \\
q(k, t) &= c_k q(1, t - 1),
\end{align*}
\]

with initial values

\[ q(1, 0) = \beta_1, \quad q(2, 0) = \beta_2, \quad \cdots, \quad q(k, 0) = \beta_k. \]

Note that if \( q(1, t) \) through \( q(k, t) \) are all non-positive, then so are \( q(1, t + 1) \) through \( q(k, t + 1) \); the same holds for nonnegativity.
Lemma 4.1. The sequence \( q(1,t) \) satisfies the recurrence specified by the characteristic polynomial \( P(x) \). For each \( 1 \leq n \leq k \), \( q(n,t) \) is a positive linear combination of \( q(1,t) \) at various stages:

\[
q(n,t) = c_n q(1,t-1) + c_{n+1} q(1,t-2) + \cdots + c_k q(1,t-(k+1-n))
\]

\[
= \sum_{i=0}^{k-n} c_{n+i} q(1,t-(i+1)).
\]  

(4.5)

Proof. We first examine the sequence \( q(1,t) \). For \( t \geq k \), we have

\[
q(1,t) = c_1 q(1,t-1) + q(2,t-1)
= c_1 q(1,t-1) + c_2 q(1,t-2) + q(3,t-2)
\]

\[
\vdots
= c_1 q(1,t-1) + c_2 q(1,t-2) + \cdots + c_{k-1} q(1,t-(k-1)) + q(k,t-(k-1))
= c_1 q(1,t-1) + c_2 q(1,t-2) + \cdots + c_{k-1} q(1,t-(k-1)) + c_k q(1,t-k),
\]  

(4.6)

which is what we want.

The latter part can also be proven by unraveling the system of recurrences: we have

\[
q(n,t) = c_n q(1,t-1) + q(n+1,t-1)
= c_n q(1,t-1) + c_{n+1} q(1,t-2) + q(n+2,t-2)
= c_n q(1,t-1) + c_{n+1} q(1,t-2) + c_{n+2} q(1,t-3) + q(n+3,t-3)
\]

\[
\vdots
= c_n q(1,t-1) + c_{n+1} q(1,t-2) + \cdots + q(n+(k-n),t-(k-n))
= c_n q(1,t-1) + c_{n+1} q(1,t-2) + \cdots + q(k,t-(k-n))
= c_n q(1,t-1) + c_{n+1} q(1,t-2) + \cdots + c_k q(1,t-(k-n+1)),
\]  

(4.7)

as desired.

Now we may prove a very useful result.

Lemma 4.2. Let \( r \) be the principal root of \( P(x) \). Consider the Binet expansion of the sequence \( q(n,t) \) (indexed by \( t \)) for each \( n \). The sign of the coefficient attached to the term \( r^t \) agrees with the sign of \( Q_0(r) \).

Proof. Recall the recurrence relation \( Q_t(x) = x Q_{t-1}(x) - q(1,t-1) P(x) \). Evaluating at \( x = r \), the \( P(x) \) term drops out and we have \( Q_t(r) = r Q_{t-1}(r) \), and iterating this procedure gives \( r^t Q_0(r) \).

Recalling that \( q(n,t) \) is defined to be the coefficient of \( Q_t(x) \) at the term \( x^{k-n} \), we have

\[
r^t Q_0(r) = Q_t(r) = r^{k-1} q(1,t) + r^{k-2} q(2,t) + \cdots + r q(k-1,t) + q(k,t).
\]  

(4.8)

Note that this means the sequence \( Q_t(r) \) satisfies the recurrence specified by \( P(x) \) as well. Since each \( q(n,t) \) is a positive linear combination of \( q(1,t) \) at various stages, they all have the same sign on the coefficient of the \( r^t \) term in their explicit expansion as a sum of geometric sequences, and this sign agrees with the coefficient of \( r^t \) in the expansion of \( Q_t(r) \). Now we just need to show the sign in \( Q_t(r) \) agrees with the sign of \( Q_0(r) \).

Consider the quantity \( \lim_{t \to \infty} Q_t(r) / r^t \), which gives the coefficient of the \( r^t \) term in \( Q_t(r) \). Since \( Q_t(r) = r^t Q_0(r) \), we have

\[
\lim_{t \to \infty} \frac{Q_t(r)}{r^t} = \lim_{t \to \infty} \frac{r^t Q_0(r)}{r^t} = Q_0(r)
\]  

(4.9)

as desired.
We can now establish an exact condition on when the Zeroing Algorithm terminates.

**Theorem 4.3.** Let \( Q_0(x) \) be as defined in (4.2) and let \( r \) be the principal root of \( P(x) \). The Zeroing Algorithm terminates if and only if \( Q_0(r) < 0 \).

**Proof of Theorem 4.3** If \( Q_0(r) < 0 \), then the coefficient of \( r^t \) in the expansion of \( q(n, t) \) is also negative for each \( n \); this means \( q(n, t) \) diverges to negative infinity, and that there must be some \( t \) when \( q(n, t) \) is non-positive for each \( n \).

For the other direction, if \( Q_0(r) \geq 0 \) then suppose, for contradiction, that there is some \( t_0 \) where \( q(n, t_0) \leq 0 \) for all \( n \). Then we would have

\[
x^{t_0} Q_0(r) = Q_{t_0}(r) = r^{k-1} q(1, t_0) + r^{k-2} q(2, t_0) + \cdots + r q(k-1, t_0) + q(k, t_0) \leq 0,
\]

which implies \( Q_0(r) \leq 0 \), forcing \( Q_0(r) = 0 \).

Notice that this equality only occurs when \( q(1, t_0) = q(2, t_0) = \cdots = q(k, t_0) = 0 \). This means for each \( n, q(n, t) = 0 \) for all \( t > t_0 \), so each \( q(n, t) \) is identically zero, which contradicts our assumption of non-triviality.

### 4.2. A General Conversion Result

Now that we have developed the main machinery of the Zeroing Algorithm, we could prove a very general result on converting between linear recurrences.

**Proof of Theorem 1.12** For ease of notation, extend the \( \gamma \) sequence by setting \( \gamma_i = 0 \) for \( i > m \). We modify the Zeroing Algorithm slightly to produce the desired \( p(x) \).

Consider a sequence of polynomials \( Q_t(x) \) of degree at most \( k - 1 \), with

\[
Q_1(x) = \gamma_1 (P(x) - x^k),
\]

\[
Q_t(x) = x Q_{t-1}(x) - (q(1, t-1) - \gamma_t) P(x) - \gamma_t x^k,
\]

where again, \( q(n, t) \) denotes the coefficient of \( Q_t(x) \) at \( x^{k-n} \). Note that after iteration \( m \), \( \gamma_t = 0 \) and we have the unmodified Zeroing Algorithm again.

**Lemma 4.4.** Define \( p_t(x) := x^k \Gamma_t(x) + Q_t(x) \). At each iteration \( t \), we have the following:

1. \( P(x) \) divides \( p_t(x) \),
2. the first \( t \) coefficients of \( p_t(x) \) are \( \gamma_1 \) through \( \gamma_t \), and
3. \( Q_t(r) = -r^k \Gamma_t(r) \).

**Proof.** A straightforward induction argument suffices for all of them.

1. We have

\[
p_1(x) = x^k \gamma_1(x) + Q_1(x) = x^k \gamma_1 + \gamma_1 (P(x) - x^k) = \gamma_1 P(x).
\]

Assuming \( P(x) \) divides \( p_t(x) \), we have

\[
p_{t+1}(x) = x^k \Gamma_{t+1}(x) + Q_{t+1}(x)
= x^k (\gamma_1 x^t + \gamma_2 x^{t-1} + \cdots + \gamma_{t+1}) + Q_{t+1}(x)
= x \cdot x^k (\gamma_1 x^{t-1} + \gamma_2 x^{t-2} + \cdots + \gamma_t) + \gamma_{t+1} x^k + x Q_t(x) - (q(1, t) - \gamma_{t+1}) P(x) - \gamma_{t+1} x^k
= x (x^k (\gamma_1 x^{t-1} + \gamma_2 x^{t-2} + \cdots + \gamma_t) + Q_t(x)) - (q(1, t) - \gamma_{t+1}) P(x)
= x p_t(x) - (q(1, t) - \gamma_{t+1}) P(x),
\]

which is divisible by \( P(x) \) by the inductive hypothesis.
(2) We first prove that \( Q_t(x) \) has degree at most \( k - 1 \). This is certainly true for \( Q_1(x) = \gamma_1(P(x) - x^k) \). Assume \( Q_t(x) \) as degree at most \( k - 1 \); we then have

\[
Q_{t+1}(x) = x Q_t(x) - (q(1, t) - \gamma_{t+1}) P(x) - \gamma_{t+1} x^k. \tag{4.14}
\]

It is evident that the highest power of \( x \) to appear is \( x^k \), which has coefficient

\[
q(1, t) - (q(1, t) - \gamma_{t+1}) = 0. \tag{4.15}
\]

From the construction \( p_t(x) := x^k \Gamma_t(x) + Q_t(x) \), now it is evident that the first \( t \) coefficients are just those of \( \Gamma_t(x) \).

(3) We have

\[
Q_1(r) = \gamma_1(P(r) - r^k) = -r^k \gamma_1. \tag{4.16}
\]

Suppose \( Q_t(r) = -r^k \Gamma_t(r) \); we have

\[
Q_{t+1}(r) = r Q_t(r) - (q(1, t) - \gamma_{t+1}) P(r) - \gamma_{t+1} r^k
\]

\[
= r (-r^k \Gamma_t(r)) - \gamma_{t+1} r^k
\]

\[
= -r^k (r \Gamma_t(r) + \gamma_{t+1})
\]

\[
= -r^k \Gamma_{t+1}(r). \tag{4.17}
\]

Now we have \( Q_m(r) = -r^m \Gamma_m(r) < 0 \), since \( \Gamma_m(r) > 0 \). Running the Zeroing Algorithm starting with \( Q_m(x) \) yields some \( Q_{m+t_0}(x) \) that does not have positive coefficients. We see that \( p_{m+t_0}(x) = x^k \Gamma_{m+t_0}(x) + Q_{m+t_0}(x) \) is divisible by \( P(x) \), has first \( m + t_0 \) coefficients \( \gamma_1 \) through \( \gamma_m \) followed by \( t_0 \) 0’s, and thus does not have positive coefficients after \( \gamma_m \); we may choose \( p(x) = p_{m+t_0}(x) \).

**Corollary 4.5.** Given \( \gamma_1 = 1 \) and arbitrary integers \( \gamma_2 \) through \( \gamma_m \) with \( \Gamma_m(r) > 0 \), there is a recurrence derived from \( P(x) \) whose characteristic polynomial has first coefficients \( \gamma_1 \) through \( \gamma_m \) with no positive coefficients thereafter.

**Proof.** Take \( p(x) \) from Theorem 1.12 which has first coefficients \( \gamma_1 \) through \( \gamma_m \). Since \( \gamma_1 = 1 \), \( p(x) \) is the characteristic polynomial of a linear recurrence. In fact, since \( \gamma_2 \) through \( \gamma_m \) are integers, \( p(x) \), and thus the recurrence, has integer coefficients.

**Corollary 4.6.** Every ZLRR has a derived PLRR.

**Proof.** Take \( m = 2 \), \( \gamma_1 = 1 \), \( \gamma_2 = -1 \). We thus have \( \Gamma_m(r) = r - 1 > 0 \), as shown in the section on characteristic polynomials. We can thus find \( p(x) \) with first two coefficients \( 1, -1 \) with no positive coefficients thereafter; this is the characteristic polynomial of a PLRR.

Note that a ZLRR does not have a unique derived PLRR; the Zeroing Algorithm simply produces a PLRR whose characteristic polynomial takes the coefficients \( 1, -1 \), a bunch of 0’s, and up to \( k \) nonzero terms at the end, where \( k \) is the degree of the characteristic polynomial of the ZLRR. In fact, for any positive integer \( n \) less than the principal root of a ZLRR, there exists a derived PLRR with leading coefficients \( 1, -n \); this is seen by taking \( \gamma_2 = -n \) in 4.6.

4.3. Fast Determination of Divergence Using the Zeroing Algorithm. Finally, we have all of the tools necessary to prove our final result, which predicts the direction of divergence of a PLRS/ZLRS using its initial values.
Proof of Theorem 1.17: We set $Q_0(x) = Q(x)$ and run the Zeroing Algorithm; we have proved that the sequence $q(1,t)$ follows the linear recurrence and has behavior determined by $Q_0(r)$. Thus, it suffices to show that $q(1,t)$ has the same initial values as $a_t$; explicitly, $q(1,t-1) = a_t$ for $1 \leq t \leq k$.

We first notice, from the recurrences on $q(n,t)$ (4.4), that

$$q(1,t) = c_1 q(1,t-1) + q(2,t-1)$$
$$= c_1 q(1,t-1) + c_2 q(1,t-2) + q(3,t-2)$$
$$\vdots$$
$$= c_1 q(1,t-1) + c_2 q(1,t-2) + \cdots + c_t q(1,0) + q(t+1,0)$$
$$= c_1 q(1,t-1) + c_2 q(1,t-2) + \cdots + c_t q(1,0) + (\alpha_{t+1} - d_{t+1}).$$

(4.18)

Now we proceed by strong induction. By construction, $q(1,0) = a_1$. For some $t$, assume $q(1,\tau-1) = a_{\tau}$ for all $1 \leq \tau < t$. We thus have

$$q(1,t) = c_1 q(1,t-1) + c_2 q(1,t-2) + \cdots + c_t q(1,0) + (\alpha_{t+1} - d_{t+1})$$
$$= (c_1 a_t + c_2 a_{t-1} + \cdots + c_t a_1) + a_{t+1} - d_{t+1}$$
$$= d_{t+1} + a_{t+1} - d_{t+1}$$
$$= a_{t+1}$$

(4.19)

as desired. □

5. Conclusion and Future Work

We have introduced two distinct ways to consider decompositions arising from ZLRS’s.

- As we saw from the first method, we can define decompositions in such a way that we have existence, but not uniqueness. Is there a different definition such that we have uniqueness, but not existence? Is it possible to have both existence and uniqueness, or can we prove that having both is impossible for ZLRS’s?

- Using the Zeroing Algorithm, we were able to convert any ZLRR into a PLRR. A natural question to ask is how long does the algorithm take to terminate (see appendices for painfully long conversions). The challenge of this question lies with the fact that every coefficient of $Q_0(x)$ needs to be taken into account; the degree itself is not enough information.

- Using the Zeroing Algorithm, how can one understand the nature of the Zeckendorf decompositions with ZLRS’s? Does there need to be specific initial conditions? Is there a definition that is at all meaningful?

Appendix A. Some Examples of Running the Zeroing Algorithm

Consider the recurrence relation

$$H_{n+1} = 2H_{n-1} + H_{n-2},$$

which has characteristic polynomial $P(x) = x^3 - 2x - 1$ (principal root $r = (1 + \sqrt{5})/2$), where we have the coefficients $c_1 = 0, c_2 = 2, c_3 = 1$. Suppose we are given $\beta_1 = 3, \beta_2 = -2, \beta_3 = -5$; we run the algorithm as follows:
We reach termination on step 4, since $Q_4$ does not have positive coefficients.

Suppose that given the same recurrence relation, and initial values $a_0 = 3, a_1 = -2, a_3 = 1$, we wish to determine whether the recurrence sequence diverges to negative infinity.

Using the method introduced in Theorem 1.17, we first determine the values of

$$d_2 = a_1 c_1 = 0, \quad d_3 = a_1 c_2 + a_2 c_1 = 6,$$

from which we construct

$$Q(x) = a_1 x^2 + (a_2 - d_2)x + (a_3 - d_3) = 3x^2 - 2x - 5.$$

We have $Q(r) = 3r^2 - 2r - 5 = 3(r + 1) - 2r - 5 = r - 2 < 0$, which predicts that $\{a_n\}$ diverges to negative infinity.

Manually computing the terms gives

$$3, -2, 1, -1, 0, -1, -2, -3, -5, -8, -13, \ldots,$$

which confirms our prediction.

**APPENDIX B. LIST OF ZLRR’S AND DERIVED ZLRR’S**

1. Recurrence: $G_{n+1} = G_{n-1} + G_{n-2}$, $P(x) = x^3 - 0x^2 - x - 1$.

$$\begin{matrix}
\gamma_1 &=& 1 & 0 & -1 & -1 \\
\gamma_2 &=& -1 & -1 & 0 & 1 \\
\gamma_3 &=& 0 & 0 & 0 & -1 \\
\end{matrix}$$

$Q_1(x) = 0x^2 - x - 1$

$Q_2(x) = -x^2 + 0x + 1$

$Q_3(x) = 0x^2 + 0x - 1$

Derived PLRR characteristic polynomial: $x^5 - x^4 - 0x^3 - 0x^2 - 0x - 1$, which corresponds to the recurrence $H_{n+1} = H_n + H_{n-4}$.

2. Current ZLRR: $G_{n+1} = G_{n-1} + G_{n-2} + G_{n-3}$. Current characteristic polynomial: $x^4 - x^2 - x - 1$.

Derived characteristic polynomial: $x^6 - x^5 - x^2 - 1$. Derived PLRR: $H_{n+1} = H_n + H_{n-3} + H_{n-5}$.


Derived characteristic polynomial: $x^5 - x^4 - 2x - 4$. Derived PLRR: $H_{n+1} = H_n + 2H_{n-3} + 4H_{n-4}$. 
4. Current ZLRR: \( G_{n+1} = 19 G_{n-1} + 38 G_{n-4} \). Current characteristic polynomial: \( x^5 - 19x^3 - 38 \).

Derived characteristic polynomial:
\[
x^{20} - x^{28} - 310601172680577x^4 - 40586681545596725x^3 - 4277914985538462x^2 - 170201741455942x - 81203021913963806.
\]

Derived PLRR: \( H_{n+1} = H_n + 310601172680577 H_{n-24} + 40586681545596725 H_{n-25} + 4277914985538462 H_{n-26} + 170201741455942 H_{n-27} + 81203021913963806 H_{n-28} \). 

5. Current ZLRR: \( G_{n+1} = 6 G_{n-1} + 3 G_{n-2} + 5 G_{n-3} \). Current characteristic polynomial: \( x^4 - 6x^2 - 3x - 5 \).

Derived characteristic polynomial:
\[
x^{10} - x^9 - 69x^3 - 1669x^2 - 722x - 1245.
\]

Derived PLRR: \( H_{n+1} = H_n + 69 H_{n-6} + 1669 H_{n-7} + 722 H_{n-8} + 1245 H_{n-9} \). 

6. Current ZLRR: \( G_{n+1} = G_{n-2} + G_{n-3} \). Current characteristic polynomial: \( x^4 - x - 1 \).

Derived characteristic polynomial:
\[
x^{20} - x^{19} - 4x^3 - x^2 - 1.
\]

Derived PLRR: \( H_{n+1} = H_n + 4H_{n-16} + H_{n-17} + H_{n-19} \). 

7. Current ZLRR: \( G_{n+1} = 3 G_{n-2} + G_{n-3} + 3 G_{n-4} \). Current characteristic polynomial: \( x^5 - 3x^2 - x - 3 \).

Derived characteristic polynomial:
\[
x^{13} - x^{12} - 14x^4 - 3x^3 - 54x^2 - 4x - 39.
\]

Derived PLRR: \( H_{n+1} = H_n + 14H_{n-8} + 3H_{n-9} + 54H_{n-10} + 4H_{n-11} + 39H_{n-12} \). 

8. Current ZLRR: \( G_{n+1} = G_{n-2} + G_{n-19} \). Current characteristic polynomial: \( x^{20} - x^{17} - 1 \).

Derived characteristic polynomial:
\[
x^{358} - x^{357} - 4000705295x^{19} - 7080648306x^{18} - 575930712x^{17} - 1937068817x^{16} - 1082811308x^{15} - 92014103x^{14} - 2546102784x^{13} - 1062101754x^{12} - 372938426x^{11} - 3264026504x^{10} - 996542899x^9 - 834914708x^8 - 4089249024x^7 - 890353375x^6 - 1541366894x^5 - 5013188421x^4 - 759208181x^3 - 256764878x^2 - 6018966637x - 635668820.
\]

Derived PLRR: \( H_{n+1} = H_n + 4000705295 H_{n-38} + 7080648306 H_{n-39} + 575930712 H_{n-34} + 1937068817 H_{n-341} + 1082811308 H_{n-342} + 92014103 H_{n-343} + 2546102784 H_{n-344} + 1062101754 H_{n-345} + 372938426 H_{n-346} + 3264026504 H_{n-347} + 996542899 H_{n-348} + 834914708 H_{n-349} + 4089249024 H_{n-350} + 890353375 H_{n-351} + 1541366894 H_{n-352} + 5013188421 H_{n-353} + 759208181 H_{n-354} + 256764878 H_{n-355} + 6018966637 H_{n-356} + 635668820 H_{n-357} \). 

9. Current ZLRR: \( G_{n+1} = G_{n-2} + G_{n-19} + G_{n-20} \). Current characteristic polynomial: \( x^{21} - x^{18} - x - 1 \).

Derived characteristic polynomial:
\[
x^{156} - x^{155} - 16626x^{20} - 6x^{19} - 16814x^{18} - 4094x^{17} - 1037x^{16} - 6777x^{15} - 5088x^{14} - 1849x^{13} - 9106x^{12} - 6334x^{11} - 3060x^{10} - 109106x^9 - 7932x^8 - 4851x^7 - 16190x^6 - 10031x^5 - 7482x^4 - 21483x^3 - 12839x^2 - 11312x - 11809.
\]

Derived PLRR: \( H_{n+1} = H_n + 16626 H_{n-135} + 6 H_{n-136} + 16814 H_{n-137} + 4094 H_{n-138} + 1037 H_{n-139} + 6777 H_{n-140} + 5088 H_{n-141} + 1849 H_{n-142} + 9106 H_{n-143} + 6334 H_{n-144} + 3060 H_{n-145} + 12166 H_{n-146} + \)
7932 H_{n-147} + 4851 H_{n-148} + 16190 H_{n-149} + 10031 H_{n-150} + 7482 H_{n-151} + 21483 H_{n-152} + 12839 H_{n-153} + 11312 H_{n-154} + 11809 H_{n-155}.

10. Current ZLRR: G_{n+1} = G_{n-1} + 2 G_{n-2} + 2 G_{n-4} + 3 G_{n-5}. Current characteristic polynomial:
\[x^6 - x^4 - 2 x^3 - 2 x - 3.\]

Derived characteristic polynomial: \[x^{11} - x^{10} - 2 x^9 - 2 x^8 - 15 x^3 - x^2 - 7 x - 15.\]

Derived PLRR: \[H_{n+1} = H_n + 2 H_{n-5} + 2 H_{n-6} + 15 H_{n-7} + H_{n-8} + 7 H_{n-9} + 15 H_{n-10}.\]

11. Current ZLRR: G_{n+1} = 40 G_{n-3} + 52 G_{n-4}. Current characteristic polynomial: \[x^5 - 40 x - 52.\]

Derived characteristic polynomial: \[x^{25} - x^{24} - 555888384 x^4 + 1064960000 x^3 - 519168000 x^2 - 3308595200 x - 4535145472.\]

Derived PLRR: \[H_{n+1} = H_n + 555888384 H_{n-20} + 1064960000 H_{n-21} + 519168000 H_{n-22} + 3308595200 H_{n-23} + 4535145472 H_{n-24}.\]

12. Current ZLRR: G_{n+1} = G_{n-8} + G_{n-9}. Current characteristic polynomial: \[x^{10} - x - 1.\]

Derived characteristic polynomial: \[x^{488} - x^{487} - 7634770044678 x^9 - 16848326467063 x^8 - 25319805215106 x^7 - 29495744687667 x^6 - 27304765351108 x^5 - 19325535741204 x^4 - 8910253837548 x^3 - 1049595609091 x^2 - 321640563521 x - 1106933774826.\]

Derived PLRR: \[H_{n+1} = H_n + 7634770044678 H_{n-478} + 16848326467063 H_{n-479} + 25319805215106 H_{n-480} + 29495744687667 H_{n-481} + 27304765351108 H_{n-482} + 19325535741204 H_{n-483} + 8910253837548 H_{n-484} + 1049595609091 H_{n-485} + 321640563521 H_{n-486} + 1106933774826 H_{n-487}.\]

13. Current ZLRR: G_{n+1} = G_{n-2} + G_{n-4} + G_{n-6}. Current characteristic polynomial: \[x^7 - x^4 - x^2 - 1.\]

Derived characteristic polynomial: \[x^{23} - x^{22} - x^6 - 6 x^5 - x^4 - 6 x^3 - x^2 - 3 x - 2.\]

Derived PLRR: \[H_{n+1} = H_n + H_{n-16} + 6 H_{n-17} + H_{n-18} + 6 H_{n-19} + H_{n-20} + 3 H_{n-21} + 2 H_{n-22}.\]

14. Current ZLRR: G_{n+1} = 3 G_{n-1} + 5 G_{n-2}. Current characteristic polynomial: \[x^3 - 3 x - 5.\]

Derived characteristic polynomial: \[x^5 - x^4 - 2 x^2 - 4 x - 15.\]

Derived PLRR: \[H_{n+1} = H_n + 2 H_{n-2} + H_{n-3} + 15 H_{n-4}.\]

15. Current ZLRR: G_{n+1} = G_{n-6} + G_{n-12}. Current characteristic polynomial: \[x^{13} - x^6 - 1.\]

Derived characteristic polynomial: \[x^{572} - x^{571} - 141734291356872 x^{12} - 1386240086076478 x^{11} - 3383864145243271 x^{10} - 4628373080436668 x^9 - 4069191511013055 x^8 - 2094637579574813 x^7 - 395154232336030 x^6 - 528518791146011 x^5 - 1761055564629423 x^4 - 2792877805797871 x^3 - 2780671348399214 x^2 - 1681201891412681 x - 40187925813162.\]

Derived PLRR: \[H_{n+1} = H_n + 141734291356872 H_{n-559} + 1386240086076478 H_{n-560} + 3383864145243271 H_{n-561} + 4628373080436668 H_{n-562} + 4069191511013055 H_{n-563}.\]
16. Current ZLRR: $G_{n+1} = G_{n-9} + G_{n-10}$. Current characteristic polynomial: $x^{11} - x - 1$.

Derived characteristic polynomial: $x^{665} - x^{664} - 17581679276200473 x^{10} - 43065699679149511 x^9 - 70765959937154578 x^8 - 91624450164084254 x^7 - 98016133194347743 x^6 - 86803369058214690 x^5 - 61120624939489989 x^4 - 30036033003931493 x^3 - 5927897678515792 x^2 - 271244487735336 x - 1643001862841472$.

Derived PLRR: $H_{n+1} = H_n + 17581679276200473 H_{n-654} + 43065699679149511 H_{n-655} + 70765959937154578 H_{n-656} + 91624450164084254 H_{n-657} + 98016133194347743 H_{n-658} + 86803369058214690 H_{n-659} + 61120624939489989 H_{n-660} + 30036033003931493 H_{n-661} + 5927897678515792 H_{n-662} + 271244487735336 H_{n-663} + 1643001862841472 H_{n-664}$.

17. Current ZLRR: $G_{n+1} = G_{n-1} + G_{n-6}$. Current characteristic polynomial: $x^7 - x^5 - 1$.

Derived characteristic polynomial: $x^{37} - x^{36} - 18 x^6 - 2 x^5 - 9 x^4 - 2 x^3 - 7 x^2 - 9 x - 4$.

Derived PLRR: $H_{n+1} = H_n + 18 H_{n-30} + 2 H_{n-31} + 9 H_{n-32} + 2 H_{n-33} + 7 H_{n-34} + 9 H_{n-35} + 4 H_{n-36}$.

18. Current ZLRR: $G_{n+1} = 2 G_{n-2} + 3 G_{n-3} + 5 G_{n-5}$. Current characteristic polynomial: $x^6 - 2 x^3 - 3 x^2 - 5$.

Derived characteristic polynomial: $x^{19} - x^{18} - 75 x^5 - 207 x^4 - 708 x^3 - 384 x^2 - 370 x - 740$.

Derived PLRR: $H_{n+1} = H_n + 75 H_{n-13} + 207 H_{n-14} + 708 H_{n-15} + 384 H_{n-16} + 370 H_{n-17} + 740 H_{n-18}$.


Derived characteristic polynomial: $x^8 - x^7 - x^2 - x - 6$. Derived PLRR: $H_{n+1} = H_n + H_{n-5} + H_{n-6} + 6 H_{n-7}$.
REFERENCES


E-mail address: tmartinez@hmc.edu

DEPARTMENT OF MATHEMATICS, HARVEY MUDD COLLEGE, CLAREMONT, CA 91711

E-mail address: sjm1@williams.edu, Steven.Miller.MC.96@aya.yale.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267

E-mail address: cmml12@williams.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267

E-mail address: cs19@williams.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267