

Black Hole Zeckendorf Games

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ABSTRACT

Zeckendorf proved that every positive integer can be written as a decomposition of non-adjacent Fibonacci numbers. Baird-Smith, Epstein, Flint, and Miller converted the process of decomposing an integer n into a 2-player game, using the moves of $F_i + F_{i-1} = F_{i+1}$ and $2F_i = F_{i+1} + F_{i-2}$, where F_i is the i^{th} Fibonacci number. They showed non-constructively that for $n \neq 2$, Player 2 has a winning strategy; a constructive solution remains unknown.

We expand on this by investigating “black hole” variants of this game. The Black Hole Zeckendorf game on F_m is played with any n but solely in columns F_i for $i < m$. Gameplay is similar to the original Zeckendorf game, except any piece that would be placed on F_i for $i \geq m$ is locked out in a “black hole” and removed from play. With these constraints, we analyze the games with black holes on F_3 and F_4 and construct a solution for specific configurations, using a non-constructive proof to lead to a constructive one. We also examine a pre-game in which players take turns placing down n pieces in the outermost columns before the decomposition phase, and find constructive solutions for any n .

KEYWORDS

Zeckendorf decompositions; Fibonacci numbers; game theory; recreational mathematics

1. Introduction

1.1. Background

The beauty of the Fibonacci numbers is undeniable: a simple sequence, recursively defined by the sum of the two previous numbers, that has the tendency to show up in both natural and surprising places. Indexing so that $F_1 = 1$, $F_2 = 2$ and $F_{k+1} = F_k + F_{k-1}$, Zeckendorf proved a particularly interesting fact about the Fibonacci numbers, namely that any positive integer n can be written as the sum of non-adjacent Fibonacci numbers, known as the number’s Zeckendorf decomposition [?]. Baird-Smith, Epstein, Flint, and Miller [?, ?], created a game from the process of converting a positive integer into its Zeckendorf decomposition using the moves of $F_i + F_{i-1} = F_{i+1}$ and $2F_i = F_{i+1} + F_{i-2}$, where F_i is the i^{th} Fibonacci number. We outline the rules to the original Zeckendorf game as follows.

- (1) **Setup:** The game is played on a board with columns corresponding to each of the Fibonacci numbers, indexing so that the 1st column corresponds with $F_1 = 1$, the 2nd column

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corresponds with $F_2 = 2$ and the m^{th} column corresponds with F_m , the m^{th} Fibonacci number. All n pieces begin in the 1st column.

- (2) **Gameplay:** Players alternate, selecting their moves from the following.
- (a) Adding consecutive terms: If the board contains pieces in both F_i and F_{i-1} columns, players can remove one piece from each column to add as one piece in the F_{i+1} column.
 - (b) Merging 1's: If the board contains more than one piece in the F_1 column, players can remove two pieces from the F_1 column to merge as one piece in the F_2 column.
 - (c) Splitting: If the board contains more than one piece in the F_2 column, players can split two pieces from the F_2 column to place one piece in each F_1 and F_3 . For $i \geq 3$, players can split two pieces in the F_i column to place one in each F_{i-2} and F_{i+1} .
- (3) **Winning:** The last player to move wins.

They proved that the game is playable, meaning it always ends in finite time, and that the final board placed down will be equal to the Zeckendorf decomposition of n . Moreover, they showed that for all $n \neq 2$, Player 2 has a winning strategy. Notably, this is not a constructive winning strategy, and instead relies on a parity stealing argument. If one assumes that Player 1 has a winning strategy, Player 2 later has the opportunity to steal it, therefore Player 2 must have a winning strategy. With increasing n , the number of possible game positions grows exponentially, making the construction of a winning solution for Player 2 challenging. Multiple variations of this game have been studied; see [BCDD+15, BJMNSYY, CDHK+6i, CDHK+6ii, CMJDMMN, GMRVY, LLMMSXZ, MSY]. In order to develop a greater understanding of the original Zeckendorf game, we consider a variation occurring on a smaller board.

1.2. Main Results

We consider an “ F_m Black Hole” variation of the Zeckendorf game, where once a piece is placed on some F_i for $i \geq m$, it falls into the “Zeckendorf Black Hole” and is permanently removed from game play. This variant reduces the number of possible moves a player has, making the game easier to analyze. We combine this Black Hole variation with an Empty Board variation, where the game begins with an empty board, and players take turns placing down pieces in the outermost columns until the weighted sum equals the starting value n . The last player to place down a piece assumes the role of Player 2 from the original Zeckendorf game. Combining these two variations is interesting for a variety of reasons. The solution to the F_m Black Hole Zeckendorf game is heavily based on modular arithmetic, while the solution to the Empty Board Zeckendorf game uses a move mirroring strategy common in game theory. Considering these variations together actually simplifies our work, as it limits the possible initial setups for the decomposition phase of the game. We quickly consider the Empty Board F_m Black Hole Zeckendorf game with black holes on F_1 , which is not playable, and on F_2 , which is deterministic. We then shift our attention to the Empty Board F_m Black Hole Zeckendorf game with black holes on F_3 and on F_4 , determining which player has a winning strategy for any positive integer n , and providing a constructive solution.

1.2.1. Terminology

We first clarify some terminology. We refer to any column corresponding with the i^{th} Fibonacci number as the F_i column. The number of pieces in a column at any given game state is a for the F_1 column, b for the F_2 column, and c for the F_3 column, resulting in a game state (a, b, c) . Because our solutions are based on modular arithmetic, we also describe game states in terms

of α, β, γ and k_1, k_2, k_3 , where α and k_1 correspond with the F_1 column, β and k_2 correspond with the F_2 column, and γ and k_3 with the F_3 column.

1.2.2. F_3 Results

With a black hole on F_3 , pieces can only be placed in the F_1 and F_2 columns. The winner of all possible games can be determined based on the value of a and b modulo 3. We describe a board setup (a, b) as $(3\alpha + k_1, 3\beta + k_2)$, where $\alpha, \beta, k_1, k_2 \in \mathbb{Z}^{\geq 0}$, $0 \leq k_1, k_2 \leq 2$. We find that Player 2 wins (a, b) for all $a \equiv b \equiv 0$, $a \equiv 0, b \equiv 1$ or $a \equiv 1, b \equiv 0$. Player 1 wins for any other setup. It follows that when placing n pieces on an empty board, Player 1 has a constructive winning strategy for $n \equiv 1, 2, 3, 6, 8 \pmod{9}$ and Player 2 has a constructive winning strategy when $n \equiv 0, 4, 5, 7 \pmod{9}$.

1.2.3. F_4 Results

When the black hole moves to F_4 , the game immediately becomes more interesting. The game is relatively straightforward for $(a, 0, 0)$ with Player 2 winning for all $a \neq 2$. The game is also relatively straightforward for $(0, 0, c)$ with Player 1 winning for all $c \neq 0, 1, 5$.

However, once the board is in the position $(a, 0, c)$, the minor exceptions from the $(a, 0, 0)$ and the $(0, 0, c)$ cases become incredibly influential in the game reduction. Still, we are able to simplify the game, and determine winners as outlined in Figure 1. Then, we describe the board set up of $(a, 0, c)$ as $(3\alpha + k_1, 0, 4\gamma + k_3)$ where $\alpha, \gamma, k_1, k_3 \in \mathbb{Z}^{\geq 0}$, $0 \leq k_1 \leq 2$, and $0 \leq k_3 \leq 3$. We show the winners based on the values of α and γ below, denoting Player 2 wins in bold blue and Player 1 wins in red.

	$a \equiv 0 \pmod{3}$	$a \equiv 1 \pmod{3}$	$a \equiv 2 \pmod{3}$
$c \equiv 0 \pmod{4}$	$\alpha \geq \gamma$ $\alpha \leq \gamma - 1$	$\forall \alpha, \gamma$	$\alpha \geq \gamma + 1$ $\alpha \leq \gamma$
$c \equiv 1 \pmod{4}$	$\alpha \geq \gamma - 1$ $\alpha \leq \gamma - 2$	$\forall \alpha, \gamma$	$\alpha \geq \gamma$ $\alpha \leq \gamma - 1$
$c \equiv 2 \pmod{4}$	$\forall \alpha, \gamma$	$\alpha \geq \gamma + 1$ $\alpha \leq \gamma$	$\forall \alpha, \gamma$
$c \equiv 3 \pmod{4}$	$\forall \alpha, \gamma$	$\alpha \geq \gamma$ $\alpha \leq \gamma - 1$	$\forall \alpha, \gamma$

Figure 1.: Winners for board setups $(a, 0, c)$ in an F_4 Black Hole Zeckendorf Game. Player 2 wins are depicted in bold blue, and Player 1 wins are depicted in red.

With this knowledge, we continue onto the Empty Board F_4 Black Hole Zeckendorf game, where players may only place in the outermost columns; we define the game in this way as the general case gives players far more options, which poses a variety of challenges for determining a constructive solution. We find that Player 1 has a constructive winning solution for all $n \equiv 1, 3, 5, 7, 8, 10, 12, 14, 15 \pmod{16}$ such that $n \neq 2, 32$, for which Player 2 has a winning solution. Player 2 has a constructive winning solution for all $n \equiv 0, 2, 4, 6, 9, 11, 13 \pmod{16}$ such that $n \neq 17, 47$, for which Player 1 has a winning solution. For large values of n , $\alpha \geq \gamma + 1$, so it is possible to quickly determine winners by Figure 1. For smaller values of n , it is necessary to explicitly consider certain values of n , as the winner is then dependent on the values of k_1 and k_3 , which can lead to exceptions, as with $n = 2, 17, 47, 32$.

Moving the black hole to any F_m such that $m \geq 5$ limits both players' ability to reduce down

to a game with a known solution, meaning that a constructive solution is no longer immediately apparent. The fact that there exists no obvious constructive solution to a Zeckendorf game in as few as four columns supports the conjectured complexity of the original Zeckendorf game.

2. Rules

We define an Empty Board F_m Black Hole Zeckendorf game as follows.

- (1) **Setup:** The game begins on an empty board with $m - 1$ columns. The F_i column is weighted to correspond with the i^{th} Fibonacci number, indexing so that $F_1 = 1$ and $F_2 = 2$. Players start the game with any positive integer n pieces.
- (2) **Placing the pieces:** Players take turns placing one piece in the outermost columns of the board, namely F_1 and F_{m-1} , until the weighted sum equals n . For us, this is equivalent to placing in columns F_1 for an initial board setup (a) , F_1 and F_2 for an initial board setup (a, b) and in columns F_1 and F_3 for an initial board setup $(a, 0, c)$ ¹. Placing one piece in the F_i column removes F_i pieces from the pile of n , so players can only place in columns such that F_i is less than or equal to the number of pieces left. This stage ends when there are no pieces left to be placed.
- (3) **Decomposition:** The last player to place down a piece assumes the role of Player 2 from the original Zeckendorf game. Players now begin the decomposition phase of the game, with Player 1 moving first. Players alternate, selecting their moves from the following.
 - (a) *Add:* Add one piece from each F_i and F_{i+1} to combine as one piece on F_{i+2} .
 - (b) *Merge:* Merge two pieces from F_1 into one piece in column F_2 .
 - (c) *Split:* Split two pieces in column F_2 into one in each column F_1 and F_3 or for $i \geq 3$, split two pieces in column F_i into one in each column F_{i-2} and column F_{i+1} .
- (4) **Black Hole:** Note that in the moves above, it is possible for pieces to be placed in F_m . In this situation, they become trapped in the “Zeckendorf Black Hole”, where they are permanently removed from the board.
- (5) **Winning:** The last player to move wins the game.

Theorem 2.1. *The Empty Board Black Hole Zeckendorf game is playable and always ends at the Zeckendorf decomposition of $n \pmod{F_m}$.*

Proof. The Empty Board portion of the game does not affect whether the game is playable, as there are finite pieces to place, by definition always resulting in a setup with some integer pieces in the F_1 and F_{m-1} columns.

Since the Zeckendorf Game is playable and always results in the Zeckendorf decomposition of n , it is sufficient to show that any Black Hole Zeckendorf game with n pieces and a black hole on F_m reduces to a board such that the weighted sum of pieces is $n \pmod{F_m}$, as that is equivalent to a board in the original Zeckendorf game. From here, the game proceeds as the general Zeckendorf game does, so will reduce to the Zeckendorf decomposition of $n \pmod{F_m}$.

Both the add and merge options combine two pieces into one piece in a column corresponding to a greater F_m , therefore closer to the black hole. The split option moves one piece towards the black hole and one piece away from it. Since there are a finite number of pieces, all moves shift

¹We are not able to provide a solution for the general case, as during the decomposition phase neither player is able to reduce the value of the board without giving the other player the option to do so first.

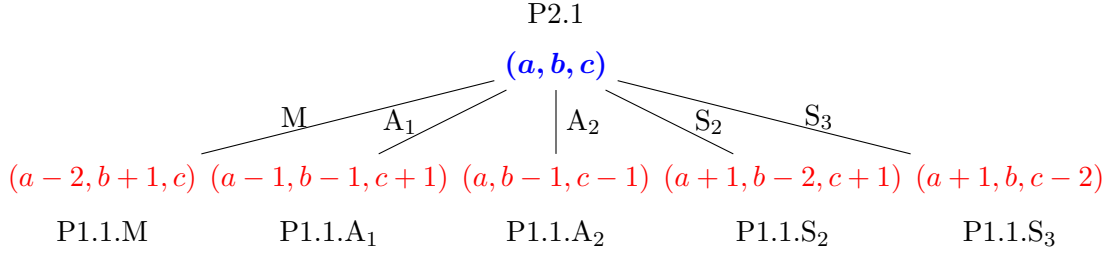


Figure 2.: Example Game Tree for a Setup (a, b, c) ,

at least the same amount of pieces towards the black hole as they do away from it, meaning that pieces must eventually be placed into the black hole if the weighted sum of pieces is greater than F_m . Every time a piece is placed in the black hole, it decreases the value of the board by F_m . Thus, the game must eventually reduced to a board such that the weighted sum of all pieces is $n \pmod{F_m}$. \square

Note that the weighted sum all pieces on the board is a non-increasing monovariant, as pieces must eventually be placed into the black hole and every time a piece is placed in the black hole, it reduces the value of the board by F_m . This is different from the original Zeckendorf game, in which the number of pieces on the board is a non-increasing monovariant, but the weighted sum of the board itself is constant.

We consider the games with black holes on F_2 , F_3 , and F_4 . For the game with a black hole on F_2 , possible outcomes are (0) and (1) . For the game with a black hole on F_3 , possible outcomes are $(0, 0)$, $(1, 0)$, and $(0, 1)$. For the game with a black hole on F_4 , possible outcomes are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 0, 1)$.

To play through games, we draw game trees, denoting a Player 2 move with bold blue text and a Player 1 move with red text. We label each edge with the corresponding move, labeling as follows:

- (1) M for merging,
- (2) A_1 for adding from columns F_1 and F_2 ,
- (3) A_2 for adding from columns F_2 and F_3 ,
- (4) S_2 for splitting from column F_2 , and
- (5) S_3 for splitting from column F_3 .

We organize our trees so that options 1 through 5 are considered from left to right. Note that not all moves are always possible. In the game with a black hole on F_2 , the only possible move is to merge, and in the game with a black hole on F_3 , only the moves M , A_1 and S_1 are possible. For game states that we reference later in the proof, we label them by the player that placed it, the round it was placed, and which move was used to place it, noting both if two moves can be used to reach that state. See Figure 2, as an example of the possible game moves from the setup (a, b, c) .

We determine winning strategies by proving that certain positions win during the decomposition phase of the game and then construct paths from winning state to winning state. As we do this, we only consider Player 1 and Player 2 within the context of the specific game state we are analyzing. Therefore, if we prove that Player 2 wins some setup (a_1, b_1, c_1) and that Player 1 can place the setup (a_1, b_1, c_1) from (a_2, b_2, c_2) , it follows that Player 1 wins (a_2, b_2, c_2) , as they assume the role of Player 2 at (a_1, b_1, c_1) . We use our knowledge of which player wins for given setups for the decomposition phase to determine winners for the Empty Board game.

3. Game with a Black hole on F_1 or F_2

We first consider the F_1 and F_2 Black Hole Zeckendorf games.

As we define it, the F_1 Black Hole Zeckendorf game is not possible, as every piece is immediately trapped in the black hole. There is also no Empty Board Black Hole Zeckendorf game to play, as by definition, players cannot place on the black hole.

In an F_2 Black Hole Zeckendorf game, the only possible move is to merge pieces into the black hole and there is only one column to place in for the Empty Board game, meaning it is also deterministic. We outline the winners for the F_2 Black Hole Zeckendorf game and the Empty Board F_2 Black Hole Zeckendorf game below.

Theorem 3.1. *Let (a) be a setup for an F_2 Black Hole Zeckendorf game. If $a \equiv 0, 1 \pmod{4}$, Player 2 has a winning strategy. If $a \equiv 2, 3 \pmod{4}$, Player 1 has a winning strategy. For the Empty Board F_2 Black Hole Zeckendorf game with n pieces, Player 1 wins for all $n \equiv 1, 2 \pmod{4}$ and Player 2 wins for all $n \equiv 0, 3 \pmod{4}$.*

Proof. We proceed by induction on a . As base cases, see that there are no moves from (0) or (1), so the Player 2 trivially wins, since Player 1 cannot move. Then, for induction's sake, assume Player 2 wins (a) for some $a \equiv 0, 1 \pmod{4}$ and consider $a + 4 \equiv 0, 1 \pmod{4}$. Player 1's next move is $(a + 2)$ and Player 2's next move is $(a) \equiv 0, 1 \pmod{4}$, which wins by hypothesis. Thus, Player 2 wins (a) for all $a \equiv 0, 1 \pmod{4}$. It follows that Player 1 wins all setups (a) such that $a \equiv 2, 3 \pmod{4}$, because they place some $(a - 2) \equiv 0, 1 \pmod{4}$, which we showed wins.

For the Empty Board F_2 Black Hole Zeckendorf game, players can only place pieces in the F_1 column. Player 2 concludes the setup phase when n is even, so they retain their role as Player 2. Therefore, Player 2 wins for all $n \equiv 0 \pmod{4}$, and Player 1 wins for all $n \equiv 2 \pmod{4}$.

When n is odd, Player 1 concludes the setup phase, so they assume the role of Player 2 from the original game. Therefore, Player 1 wins for all $n \equiv 1 \pmod{4}$, and Player 2 wins for all $n \equiv 3 \pmod{4}$. \square

4. Game with a Black Hole on F_3

We now consider the game with a black hole on $F_3 = 3$. Here players can choose how they move, making the game more interesting.

4.1. Single Column Winning Board Setups

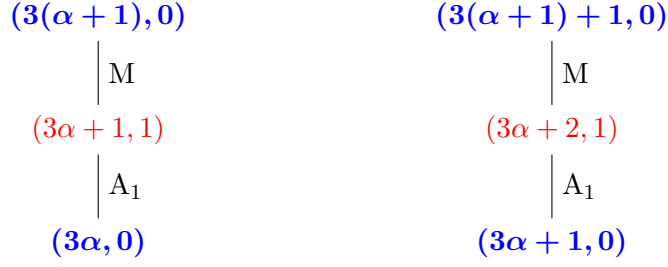
We first consider which setups result in wins for Player 2 during the decomposition stage of the game. We start by considering all pieces in one column.

Theorem 4.1. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf game. If the game starts with $a \equiv 0, 1 \pmod{3}$ and $b = 0$, Player 2 has a winning strategy. Similarly, Player 2 has a winning strategy if the game starts with $a = 0$ and $b \equiv 0, 1 \pmod{3}$.*

Proof. For games with $a \equiv 0, 1 \pmod{3}$ and $b = 0$, we proceed by induction on α . If the game starts with 0 or 1 pieces, in either the F_1 or F_2 column, then Player 1 cannot move, so Player 2 wins.

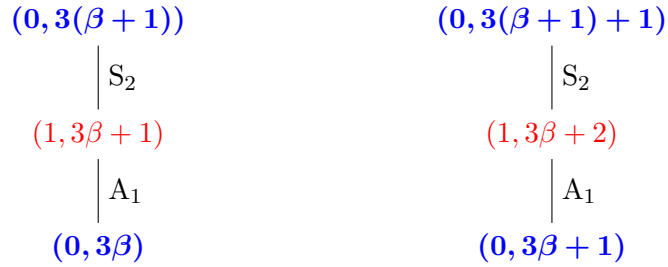
As our inductive hypothesis, assume that Player 2 wins $(3\alpha + k_1, 0)$ for a fixed α with $k_1 = 0, 1$. Then, consider the corresponding game trees below, with $k_1 = 0$ on the left and $k_1 = 1$ on the

right. Since Player 1's first move is forced, Player 2 will always have the possibility to follow the strategy outlined in these trees.



In both trees, Player 2 is able to reduce down to a setup that wins by the inductive hypothesis, proving the claim.

For games with $b \equiv 0, 1 \pmod{3}$ and $a = 0$, we proceed by induction on β . Assume as an inductive hypothesis that Player 2 wins $(0, 3\beta + k_2)$ for a fixed β with $k_2 = 0, 1$. Consider the corresponding game trees below, with $k_2 = 0$ on the left and $k_2 = 1$ on the right. Again, Player 1's first move is forced, so Player 2 can always use the strategy outlined.



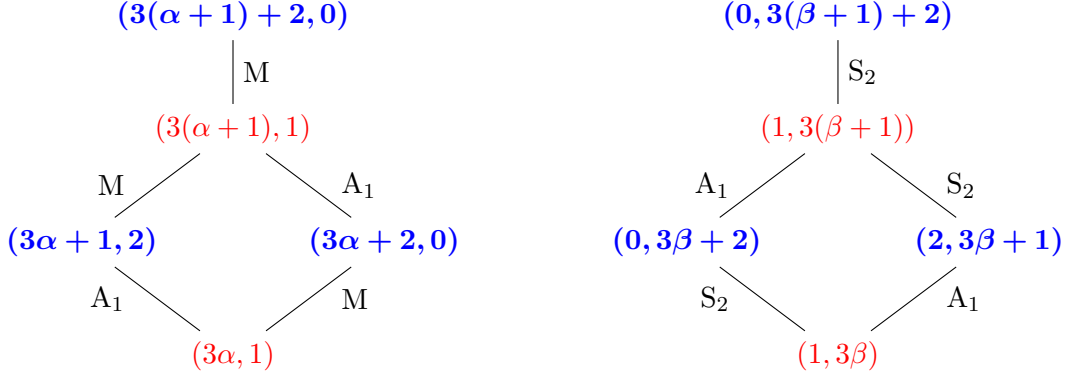
Again, Player 2 can reduce to a setup that wins by the inductive hypothesis, proving the claim.

Therefore, Player 2 wins the F_3 Black Hole Zeckendorf game for all board setups with a or $b \equiv 0, 1 \pmod{3}$, and the other equal to 0. \square

Theorem 4.2. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf game. If the game starts $a \equiv 2 \pmod{3}$ and $b = 0$ or starts with $a = 0$ and $b \equiv 2 \pmod{3}$, then Player 1 has a winning strategy.*

Proof. We proceed by induction. If the game starts with 2 pieces in the F_1 column and none in the other, Player 1 can merge to place $(0, 1)$ which wins. If the game starts with 2 pieces in the F_2 column, and none in the F_1 column, Player 1 can split to place $(1, 0)$ which wins.

For induction's sake, suppose Player 1 wins for a setup $(3\alpha + 2, 0)$ or $(0, 3\beta + 2)$. Since Player 1's only possible moves from here are $(3\alpha, 1)$ and $(1, 3\beta)$ respectively, this assumption also implies that Player 1 can win by placing $(3\alpha, 1)$ or $(1, 3\beta)$. Then, consider the game trees below, inducting on α in the left tree and β in the right tree.



Since Player 1 is able to place a position that wins by inductive hypothesis, these trees prove the inductive step. Therefore, Player 1 wins any board of the form $(3\alpha + 2, 0)$ or $(0, 3\beta + 2)$. \square

Corollary 4.3. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf game. Player 2 has a winning strategy for the setups $(3\alpha, 1)$ and $(1, 3\beta)$.*

Proof. As base cases $(1, 0)$ and $(0, 1)$ both win trivially. As shown in the proof of Theorem 4.2, any player who places $(3\alpha, 1)$ or $(1, 3\beta)$ can reduce to $(3(\alpha - 1), 1)$ and $(1, 3(\beta - 1))$ respectively, so $(3\alpha, 1)$ or $(1, 3\beta)$ win by induction. \square

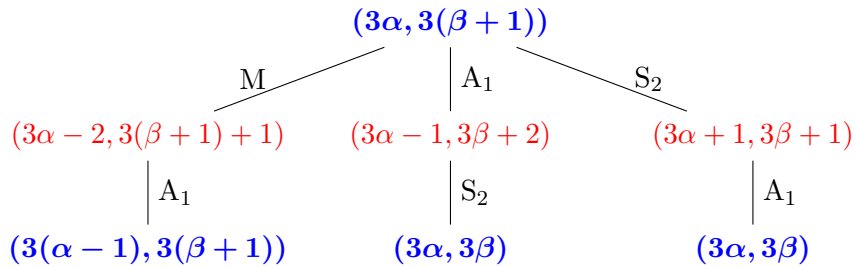
4.2. General Winning Setups

We now consider general winning setups of the form (a, b) . We show any case can be reduced modulo 3, and since both players have winning strategies in the lower cases, both players have winning strategies for higher cases.

4.2.1. Winning Setups for Player 2

Theorem 4.4. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf game. If $a \equiv b \equiv 0 \pmod{3}$, then Player 2 has a winning strategy.*

Proof. We proceed by induction. First, with the base case of $\beta = 0$, Player 2 has a winning strategy for $(3\alpha, 0)$ by Theorem 4.1. Then, let our inductive hypothesis be that Player 2 wins any game of the form $(3\alpha, 3\beta)$ for all α and some fixed β . Then, consider the following game tree on $(3\alpha, 3(\beta + 1))$.

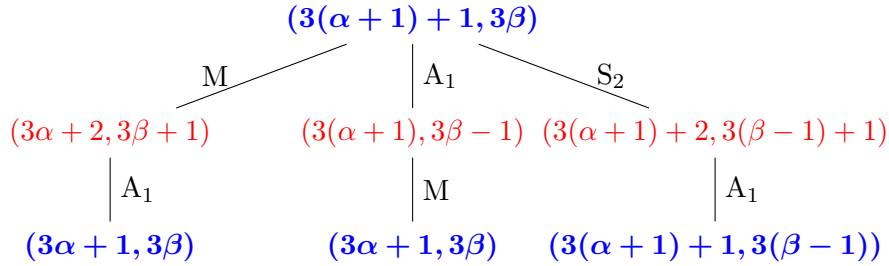


In the right and center columns, Player 2 is able to reduce to a game of the form $(3\alpha, 3\beta)$, thereby winning by the inductive hypothesis. In the left column, Player 2 does not immediately win, but the game options are the same as before, so will win by inductive hypothesis any time

Player 1 adds or splits. If Player 1 always chooses to merge, then Player 2 will eventually place $(0, 3(\beta + 1))$ which wins by Theorem 4.1. Thus, Player 2 has a winning strategy for the F_3 Black Hole Zeckendorf game if $a \equiv b \equiv 0 \pmod{3}$. \square

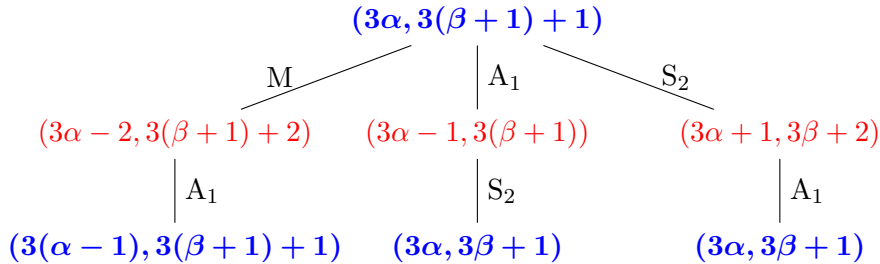
Theorem 4.5. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf Game. If $a \equiv 1 \pmod{3}$ and $b \equiv 0 \pmod{3}$, or vice versa, Player 2 has a winning strategy.*

Proof. We proceed by induction, first considering the game $(3\alpha + 1, 3\beta)$. First, fixing $\alpha = 0$, Player 2 has a winning strategy for $(1, 3\beta)$ by Corollary 4.3. Then, for our inductive hypothesis, suppose Player 2 wins any game of the form $(3\alpha + 1, 3\beta)$ for fixed α and all β . Then, consider the following game tree on $(3(\alpha + 1) + 1, 3\beta)$.



In the left and center columns, Player 2 is able to reduce to a game of the form $(3\alpha + 1, 3\beta)$, therefore winning by the inductive hypothesis. In the right column, Player 2 does not immediately win, but the game options are the same as before so will win by inductive hypothesis any time Player 1 merges or adds. If Player 1 always splits, then Player 2 will eventually place $(3(\alpha + \beta) + 1, 0)$ which wins by Theorem 4.1.

Similarly, consider the initial setup $(3\alpha, 3\beta + 1)$. Fixing $\beta = 0$, Player 2 wins $(3\alpha, 1)$ by Corollary 4.3. Then, as our inductive hypothesis, we assume Player 2 wins any game of the form $(3\alpha, 3\beta + 1)$ for some fixed β and all α . We consider the game on $(3\alpha, 3(\beta + 1) + 1)$.



In the center and right columns, Player 2 wins by inductive hypothesis. Again, Player 2 does not immediately win $(3(\alpha - 1), 3(\beta + 1) + 1)$ but either wins once Player 1 chooses to add or split, or once the game is reduced to $(0, 3(\beta + \alpha) + 1)$, as a result of Theorem 4.1.

Hence, Player 2 has a winning strategy if $a \equiv 1 \pmod{3}$ and $b \equiv 0 \pmod{3}$ or vice versa. \square

4.2.2. Winning Setups for Player 1

Theorem 4.6. *Let (a, b) be a setup for an F_3 Black Hole Zeckendorf game. Player 1 has a winning strategy for all possibilities listed below.*

- (1) $a \equiv b \equiv 1 \pmod{3}$,
- (2) $a \equiv b \equiv 2 \pmod{3}$,
- (3) $a \equiv 2 \pmod{3}$ and $b \equiv 1 \pmod{3}$,
- (4) $a \equiv 1 \pmod{3}$ and $b \equiv 2 \pmod{3}$,
- (5) $a \equiv 0 \pmod{3}$ and $b \equiv 2 \pmod{3}$,
- (6) $a \equiv 2 \pmod{3}$ and $b \equiv 0 \pmod{3}$.

Proof. For each statement, we show that Player 1 can place a move that we showed to win for “Player 2”, thereby assuming the role of Player 2.

First, suppose the board is set as $(3\alpha + 1, 3\beta + 1)$. Then, Player 1 can add from columns F_1 and F_2 to place $(3\alpha, 3\beta)$. As shown in Theorem 4.4, which ever player places this board has a winning strategy.

Similarly, suppose the board is set as $(3\alpha + 2, 3\beta + 2)$. Then Player 1 can merge from F_1 to place the board as $(3\alpha, 3(\beta + 1))$, and so has a winning strategy by Theorem 4.4 again.

Then, suppose the board is set as $(3\alpha + 2, 3\beta + 1)$ or $(3\alpha + 1, 3\beta + 2)$. Player 1 can add from columns F_1 and F_2 to place $(3\alpha + 1, 3\beta)$ and $(3\alpha, 3\beta + 1)$ respectively, which by Theorem 4.5 are both winning states.

Then, suppose the board is set as $(3\alpha, 3\beta + 2)$. Player 1 can split from column F_2 , to place the board as $(3\alpha + 1, 3\beta)$, which is a winning state by Theorem 4.5.

Lastly, suppose the board is set as $(3\alpha + 2, 3\beta)$. Player 1 can merge from column F_1 to place the board as $(3\alpha, 3\beta + 1)$, which is a winning state by Theorem 4.5. \square

4.3. Empty Board Game

We now continue to the Empty Board F_3 Black Hole Zeckendorf game and determine which player has a winning strategy for any given $n \in \mathbb{Z}^{>0}$. We find that Player 1 has a constructive winning strategy for $n \equiv 1, 2, 3, 6, 8 \pmod{9}$ and Player 2 has a constructive winning strategy for $n \equiv 0, 4, 5, 7 \pmod{9}$.

Theorem 4.7. *Let $(0, 0)$ be the beginning board for an Empty Board F_3 Black Hole Zeckendorf game with $n \in \mathbb{Z}^{>0}$ pieces. Players can force certain game setups as outlined below.*

- (1) For any $n \equiv \pm 3 \pmod{9}$, Player 1 can force the game into a setup $(n/3, n/3)$.
- (2) For any $n \equiv 0 \pmod{9}$, Player 2 can force the game into a setup $(n/3, n/3)$.
- (3) For any $n \equiv 1 \pmod{9}$, Player 1 can force the game into a setup $((n-1)/3+1, (n-1)/3)$.
- (4) For any $n \equiv 4, 7 \pmod{9}$, Player 2 can force the game into a setup $((n-1)/3+1, (n-1)/3)$.
- (5) For any $n \equiv 2 \pmod{9}$, Player 1 can force the game into a setup $((n-2)/3, (n-2)/3+1)$.
- (6) For any $n \equiv 8 \pmod{9}$, Player 1 can force the game into a setup $((n-2)/3+2, (n-2)/3)$.
- (7) For any $n \equiv 5 \pmod{9}$, Player 2 can force the game into a setup that is either $((n-2)/3+2, (n-2)/3)$ or $((n-2)/3, (n-2)/3+1)$.

Proof. First, consider $n \equiv \pm 3 \pmod{9}$. Player 1 should place their first piece in the F_2 column. Then, they should act opposite of Player 2, until there is one piece left, which Player 2 will be

forced to place in the F_1 column, setting up the board as $(n/3, n/3)$.

If $n \equiv 0 \pmod{9}$, Player 2 can force the setup $(n/3, n/3)$ by placing opposite Player 1, resulting in Player 2 placing down $(n/3, n/3)$.

Then, consider $n \equiv 1 \pmod{9}$. Player 1 should place their first piece in the F_1 column. For every round following, Player 1 should place in the opposite column of Player 2. Since $n \equiv 1 \pmod{3}$, this means that Player 1 will place the last piece, setting the board as $((n-1)/3 + 1, (n-1)/3)$. Therefore, they assume the role of Player 2 during the decomposition phase of the game.

For $n \equiv 4, 7 \pmod{9}$, Player 2 should place opposite of Player 1. Again, this means that Player 1 will place the last piece, setting the board as $((n-1)/3 + 1, (n-1)/3)$. Therefore, they assume the role of Player 2 during the decomposition phase of the game.

Next, consider $n \equiv 2 \pmod{9}$. Player 1 should place their first piece in the F_1 column, and then play opposite Player 2. Then, every time after Player 1 places down, the number of pieces left to place will be some $p \equiv 1 \pmod{3}$, meaning Player 2 will eventually be forced to set $((n-2)/3 + 2, (n-2)/3)$.

Next, consider $n \equiv 8 \pmod{9}$. Player 1 should place their first piece in the F_1 column, and then play opposite Player 2. Then, every time after Player 1 places down, the number of pieces left to place will be some $p \equiv 1 \pmod{3}$, meaning Player 2 will eventually be forced to set $((n-2)/3 + 2, (n-2)/3)$.

Lastly, for $n \equiv 5 \pmod{9}$, Player 2 should play opposite all of Player 1's moves, so that Player 2 eventually place the board $((n-2)/3, (n-2)/3)$. This forces Player 1 to either place $((n-2)/3, (n-2)/3 + 1)$ or $((n-2)/3 + 1, (n-2)/3)$, the latter allowing Player 2 to place the setup $((n-2)/3 + 2, (n-2)/3)$. \square

Note that these are the only setups that can be forced, as Player 1 has at most one more than half of the moves, while Player 2 has at most half of the moves. This strategy of move mirroring, where one player always plays opposite the other, is common in recreational math; see [?].

Theorem 4.8. *Let $(0, 0)$ be the beginning board for an Empty Board F_3 Black Hole Zeckendorf Game with n pieces. Player 1 has a constructive strategy for winning for any $n \equiv 1, 2, 3, 6, 8 \pmod{9}$. Player 2 has a constructive strategy for winning for any $n \equiv 0, 4, 5, 7 \pmod{9}$.*

Proof. We first consider when $n \equiv 3, 6 \pmod{9}$. By Theorem 4.7, Player 1 can force Player 2 to set the board as $(n/3, n/3)$. Then, $n \equiv 0 \pmod{3}$ but $n/3 \not\equiv 0 \pmod{3}$. Since $a, b \not\equiv 0 \pmod{3}$, Player 1 has a constructive solution by Theorem 4.6.

Then, consider $n \equiv 0 \pmod{9}$, where Player 2 can force the game so that they place the setup $(n/3, n/3)$. Since $n \equiv 0 \pmod{9}$ implies $n/3 \equiv 0 \pmod{3}$ then Player 2 has a constructive winning strategy by Theorem 4.4.

Next, we consider when $n \equiv 1 \pmod{9}$, meaning Player 1 sets the board as $((n-1)/3 + 1, (n-1)/3)$ by Theorem 4.7. Since $(n-1)/3 + 1 \equiv 1 \pmod{3}$ and $(n-1)/3 \equiv 0 \pmod{3}$, then Player 1 has a constructive strategy for winning by Theorem 4.5, since they have assumed the role of Player 2.

Then, suppose $n \equiv 4, 7 \pmod{9}$, meaning Player 2 can force Player 1 to place the setup $((n-1)/3 + 1, (n-1)/3)$ by Theorem 4.7; this means that Player 2 assumes the role of Player

1 during the decomposition phase of the game. For $n \equiv 4 \pmod{9}$, $(n-1)/3 + 1 \equiv 2 \pmod{3}$ and $(n-1)/3 \equiv 1 \pmod{3}$, so by Theorem 4.6, Player 2 has a winning strategy. Similarly, if $n \equiv 7 \pmod{9}$, then $(n-1)/3 + 1 \equiv 0 \pmod{3}$ and $(n-1)/3 \equiv 2 \pmod{3}$ so Player 2 has a winning strategy by Theorem 4.6.

Then, we consider when $n \equiv 2 \pmod{9}$, meaning that Player 1 is able to place the setup $((n-2)/3, (n-2)/3 + 1)$ by Theorem 4.7, thereby assuming the role of Player 2. Since $(n-2)/3 + 1 \equiv 1 \pmod{3}$ and $(n-2)/3 \equiv 0 \pmod{3}$, Player 1 wins by Theorem 4.5.

If $n \equiv 8 \pmod{9}$, then Player 1 can force Player 2 to place the setup $((n-2)/3 + 2, (n-2)/3)$. Since $(n-2)/3 + 2 \equiv 1 \pmod{3}$ and $(n-2)/3 \equiv 2 \pmod{3}$, then Player 1 wins by Theorem 4.6.

If $n \equiv 5 \pmod{9}$, then Player 2 can force the game so that either they place the setup $((n-2)/3 + 2, (n-2)/3)$ or Player 1 places the setup $((n-2)/3, (n-2)/3 + 1)$. We note that $(n-2)/3 \equiv 1 \pmod{3}$, $(n-2)/3 + 1 \equiv 2 \pmod{3}$ and $(n-2)/3 + 2 \equiv 0 \pmod{3}$. It follows that Player 2 wins the setup $((n-2)/3, (n-2)/3 + 1)$ by Theorem 4.6, since here they assume the role of Player 1. Player 2 also wins the setup $((n-2)/3 + 2, (n-2)/3)$ by Theorem 4.5. \square

This concludes our analysis of the F_3 Black Hole Zeckendorf game, both with and without the Empty Board phase of the game.

5. Game with a Black Hole on F_4

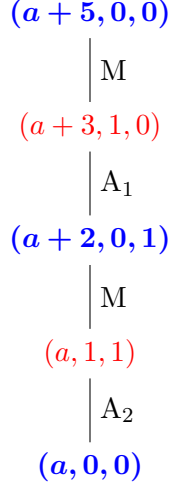
We now expand to the game with a black hole on $F_4 = 5$ with the setup (a, b, c) . This game becomes more interesting for a variety of reasons. There are more possible moves for each player and less symmetry, leading to a solution that is constructive but more intricate. Here, the winner is not solely based on the equivalence classes of a and c but also their value in relation to each other.

5.1. Single Column Winning Board Setups

We first consider winning strategies when all pieces are in either in column F_1 or column F_3 . We do not consider when all are in column F_2 for two reasons. First and foremost, we define the Empty Board game so that players are only able to place in the outermost columns, as this greatly simplifies players abilities to force certain setups. Second, a significant strategy in the solution is forcing all pieces to the outer columns, to prevent the other player from using the ‘‘Add’’ move, thereby limiting their options. This strategy clearly fails when all pieces are in the F_2 column, making analysis of the game challenging. A complete analysis of the F_4 Black Hole Zeckendorf game on (a, b, c) such that $b \neq 0$ would prove interesting in future work, as would analyzing the Empty Board game without restrictions on placing in the F_2 column.

Theorem 5.1. *Let $(a, 0, 0)$ be an initial setup for an F_4 Black Hole Zeckendorf game. For any $a \neq 2$, Player 2 has a constructive solution.*

Proof. We proceed by induction on a . See in Appendix A that Player 2 has a constructive winning strategy for all base cases $a = 1, 3, 4, 5, 7$. Then, for our inductive hypothesis suppose the starting position $(a, 0, 0)$ wins and the consider following game tree.



Since any $(a, 0, 0)$ wins by inductive hypothesis, and as shown above, it is always possible for Player 2 to reduce a modulo 5 until it reaches one of the base cases, then Player 2 has a winning strategy for $(a, 0, 0)$ for all $a \neq 2$. \square

Corollary 5.2. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. Player 2 has a winning strategy at $(a, 0, 1)$ for all $a \in \mathbb{Z}^{\geq 0}$. Player 1 has a winning strategy at $(a, 0, 2)$ for $a \neq 1 \in \mathbb{Z}^{\geq 0}$, and a winning strategy at $(a, 0, 3)$, $(a, 1, 1)$ and $(a + 1, 1, 2)$ for all $a \in \mathbb{Z}^{\geq 0}$.*

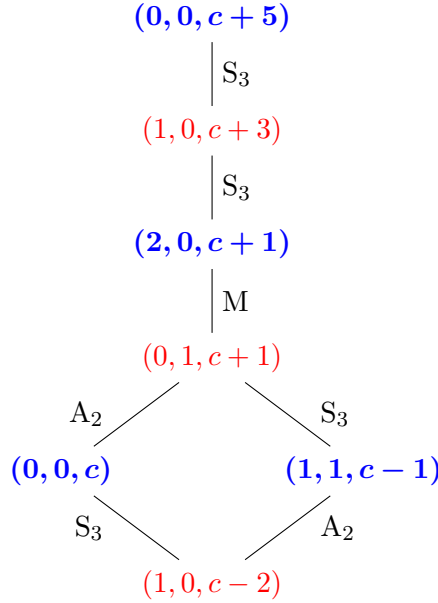
Proof. In the game tree above, $(a, 0, 1)$ is a Player 2 move so therefore wins.

From $(a, 0, 2)$ and $(a, 0, 3)$, Player 1 can split in the F_3 column to place $(a + 1, 0, 0)$ and $(a + 1, 0, 1)$ respectively, which win as shown above. The only exception to this is when the board is set as $(1, 0, 2)$, because placing $(2, 0, 0)$ loses

From $(a, 1, 1)$ and $(a, 1, 2)$ Player 1 can add from columns F_2 and F_3 to place $(a, 0, 0)$ and $(a, 0, 1)$ respectively, which win as shown above. \square

Theorem 5.3. *Let $(0, 0, c)$ be an initial setup for an F_3 Black Hole Zeckendorf game. For any $c \neq 0, 1, 5 \in \mathbb{Z}^{\geq 0}$, Player 1 has a constructive winning strategy.*

Proof. We proceed by induction on c . See in Appendix B that Player 1 has a constructive winning strategy for all base cases $c = 2, 3, 4, 6, 10$. Then, consider the starting position $(0, 0, c)$, which forces Player 1 to place $(1, 0, c - 2)$. For induction's sake, suppose this position wins. Then, consider the following game tree.



Since $(1, 0, c - 2)$ wins by inductive hypothesis and as shown above it is always possible for Player 1 to reduce c modulo 5 until it reaches a base case, then the Player 1 has a constructive winning solution for all $(0, 0, c)$ such that $c \neq 0, 1, 5$. \square

Corollary 5.4. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. Player 2 has a winning strategy at $(1, 0, c)$ for all $c \neq 3 \in \mathbb{Z}^{\geq 0}$ and at $(0, 1, c)$ for all $c \neq 1, 2, 6 \in \mathbb{Z}^{\geq 0}$. Player 1 has a winning strategy at $(2, 0, c)$ for all $c \neq 1 \in \mathbb{Z}^{\geq 0}$ and at $(2, 1, c)$ for all $c \in \mathbb{Z}^{\geq 0}$.*

Proof. Theorem 5.3 showed that $(1, 0, c)$ wins for all $c \neq 5 - 2$, so if Player 2 places it, they assume the role of Player 1 and win for all $c \neq 3 \in \mathbb{Z}^{\geq 0}$.

Similarly, if Player 2 places $(0, 1, c)$, they assume the role of Player 1 and win. This is true for all $c \neq 1, 2, 6$ since placing $(0, 0, c - 1)$ only wins for $c - 1 = 0, 1, 5$.

Next, see that from $(2, 0, c)$ is a Player 2 move in the tree above, so loses for all $c \neq 5 - 4$.

Lastly, from $(2, 1, c)$, Player 1 can place $(1, 0, c + 1)$ which wins for all $c \neq 2$. When $c = 2$, Player 1 can place $(2, 0, 1)$ which we also showed to win in Corollary 5.2. \square

5.2. General Winning Setups

Our proof of general winning setups for a black hole on F_4 is more involved, as the minor exceptions in the cases above prevent us from creating a solution solely based on the equivalence classes of a and c for the setup $(a, 0, c)$. To motivate this section, we remind the reader of Figure 1, which outlines the winners for the board setup $(a, 0, c)$ in an F_4 Black Hole Zeckendorf Game.

We construct a solution by providing a path from any winning game state to another winning game state. However, it is not possible to do this until we can verify that winning states as outlined in the table are in fact winning states. We do this using a mixture of constructive and non-constructive methods.

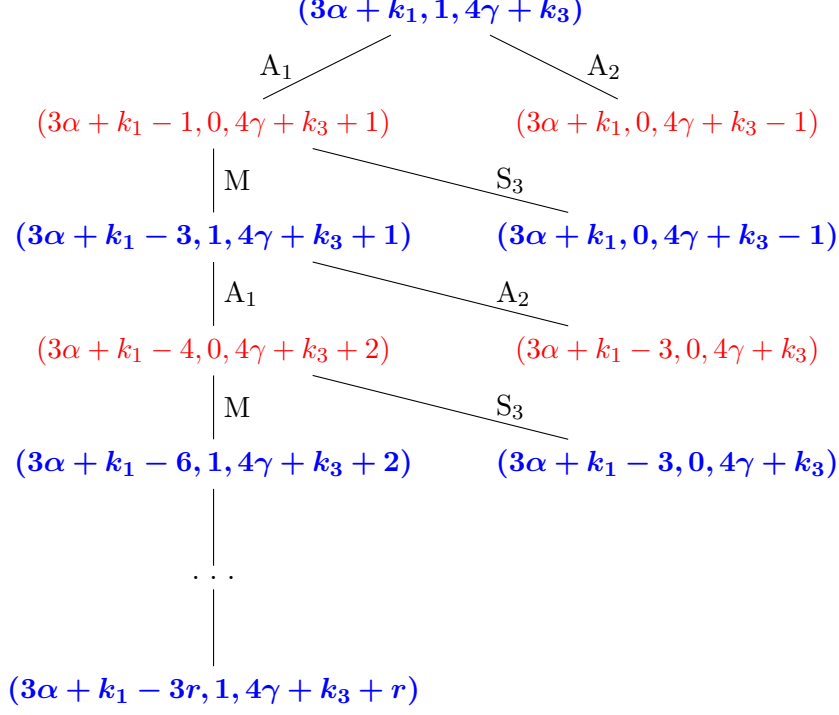


Figure 3.: Game Tree for the Game State $(3\alpha + k_1, 1, 4\gamma + k_3)$.

Lemma 5.5. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. For all $\alpha, \gamma, k_1, k_3 \in \mathbb{Z}^{\geq 0}$, such that $1 \leq k_1 \leq 2$, and $0 \leq k_3 \leq 3$, Player 1 has a winning strategy for $(3\alpha + k_1, 1, 4\gamma + k_3)$.*

Proof. We proceed by a non-constructive proof. For contradiction's sake, suppose Player 2 wins $(3\alpha + k_1, 1, 4\gamma + k_3)$ when $k_1 = 1, 2$. Then, consider the game tree in Figure 3, where r is the number of rounds in which Player 1 then the Player 2 plays. We note that Player 1 has other possible moves, but we only consider moves relevant to the proof.

By assumption, Player 2 has a winning strategy regardless of what the other player places. Suppose Player 1 places $(3\alpha + k_1 - 1, 0, 4\gamma + 1)$ for their first move. Then, Player 2 can place either $(3\alpha + k_1 - 3, 1, 4\gamma + 1)$ or $(3\alpha + k_1, 0, 4\gamma - 1)$. But as shown in the tree, Player 1 had the opportunity to place $(3\alpha + k_1, 0, 4\gamma - 1)$ in the round before; so by assumption, placing it is a losing move. It follows that in order to win, Player 2 must place $(3\alpha + k_1 - 3, 1, 4\gamma + 1)$. As shown in the tree, Player 1 has the same options as before, so if they place $(3\alpha + k_1 - 4, 0, 4\gamma + k_3 + 2)$, then by assumption, Player 2 must place $(3\alpha + k_1 - 6, 1, 4\gamma + k_3 + 2)$.

Then, the game eventually reduces down to Player 2 placing $(3\alpha + k_1 - 3r, 1, 4\gamma + k_3 + r)$ after r rounds of Player 1 then Player 2 placing. After the α^{th} round, Player 2 will place $(k_1, 1, 4\gamma + k_3 + \alpha)$. If $k_1 = 1$, then Player 1 can add from columns F_2 and F_3 to place $(1, 0, 4\gamma + \alpha + k_3 - 1)$ which wins by Corollary 5.4 for all $4\gamma + \alpha + k_3 - 1 \neq 3$. If $k_1 = 2$, then Player 1 can add from columns F_1 and F_2 to place $(1, 0, 4\gamma + \alpha + k_3 + 1)$ which wins by Corollary 5.4 for all $4\gamma + \alpha + k_3 + 1 \neq 3$. Additionally, we show in Appendix C, that Player 1 also has a winning strategy for the cases when $4\gamma + \alpha + k_3 \pm 1 = 3$. This is a contradiction to the assumption that Player 2 has a winning strategy, so then Player 1 must have some winning strategy when $(3\alpha + k_1, 1, 4\gamma + k_3)$ is placed for $k_1 = 1, 2$. \square

Remark 5.6. For the sake of conserving space and avoiding repetition, from this point forward we assume both players are seeking optimal strategies. Then, when $k_1 = 1, 2$, we omit any move of the form $(3\alpha + k_1, 1, 4\gamma + k_3)$, as it immediately loses.

From here, we show that when $k_1 = 0$, Player 1 wins $(3\alpha, 1, 4\gamma + 1)$ for $\alpha \geq \gamma$ and $(3\alpha, 1, 4\gamma)$ for $\alpha \geq \gamma + 1$.

Lemma 5.7. Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. For all $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$, Player 2 has a winning strategy for $(3\alpha, 0, 4\gamma)$ when $\alpha \geq \gamma$.

Proof. We proceed induction on γ , letting α be an arbitrary integer such that $\alpha \geq \gamma$. As a base case, Player 2 wins for all $\gamma = 0$ by Theorem 5.1, since $3\alpha \neq 2$ for any integer α . Then, consider the game on $(3\alpha, 0, 4(\gamma + 1))$ as played out in Figure 4, with $\alpha \geq \gamma + 1$, assuming for induction's sake that placing $(3\alpha, 0, 4\gamma)$ wins for all $\alpha \geq \gamma$.

Note $\alpha \geq \gamma + 1$ implies that $\alpha - 1 \geq \gamma$ and $\alpha + 2 \geq \gamma - 2$ so P2.4.A₂ and P2.4.S₃ both win by inductive hypothesis. Some special consideration is needed here for the case when $\gamma = 1$, since P2.4.S₃ = $(3(\alpha + 2), 0, -4)$ is not a valid game state. Here, Player 2 wins at P2.3.S₃ = $(3(\alpha + 1) + 1, 0, 0)$ by Theorem 5.1 because $3(\alpha + 1) + 1 \neq 2$ for $\alpha \in \mathbb{Z}^{\geq 0}$.

For the center branch of the tree, if both players always choose to split once the tree is of the form $(3\alpha + 1, 0, 4(\gamma - 1) + 1)$, then Player 2 always places a board of the form $(3\alpha + 1 + 2r, 0, 4(\gamma - 1 - r) + 1)$, where r is a round of Player 2 playing then the Player 1 playing. Then, after the $(\gamma - 2)^{\text{nd}}$ round, Player 2 will place down $(3\alpha + 1 + 2(\gamma - 1), 0, 1)$ which wins by Corollary 5.2.

However, it is necessary to consider that every time Player 2 places $(3\alpha + 1 + 2r, 0, 4(\gamma - 1 - r) + 1)$ with $2r \equiv 1 \pmod{3}$, Player 1 has the opportunity to place $(3\alpha + 2r - 1, 1, 4(\gamma - 1 - r) + 1)$ without immediately losing by Lemma 5.5, since here $3\alpha + 2r - 1 \equiv 0 \pmod{3}$. However, from here Player 2 can add from columns F_2 and F_3 to place $(3\alpha + 2r - 1, 0, 4(\gamma - 1 - r))$ which wins by inductive hypothesis, since $\alpha \geq \gamma - 1 - r$.

Therefore, Player 2 has a winning strategy for $(3\alpha, 0, 4\gamma)$, for all $\alpha \geq \gamma$. □

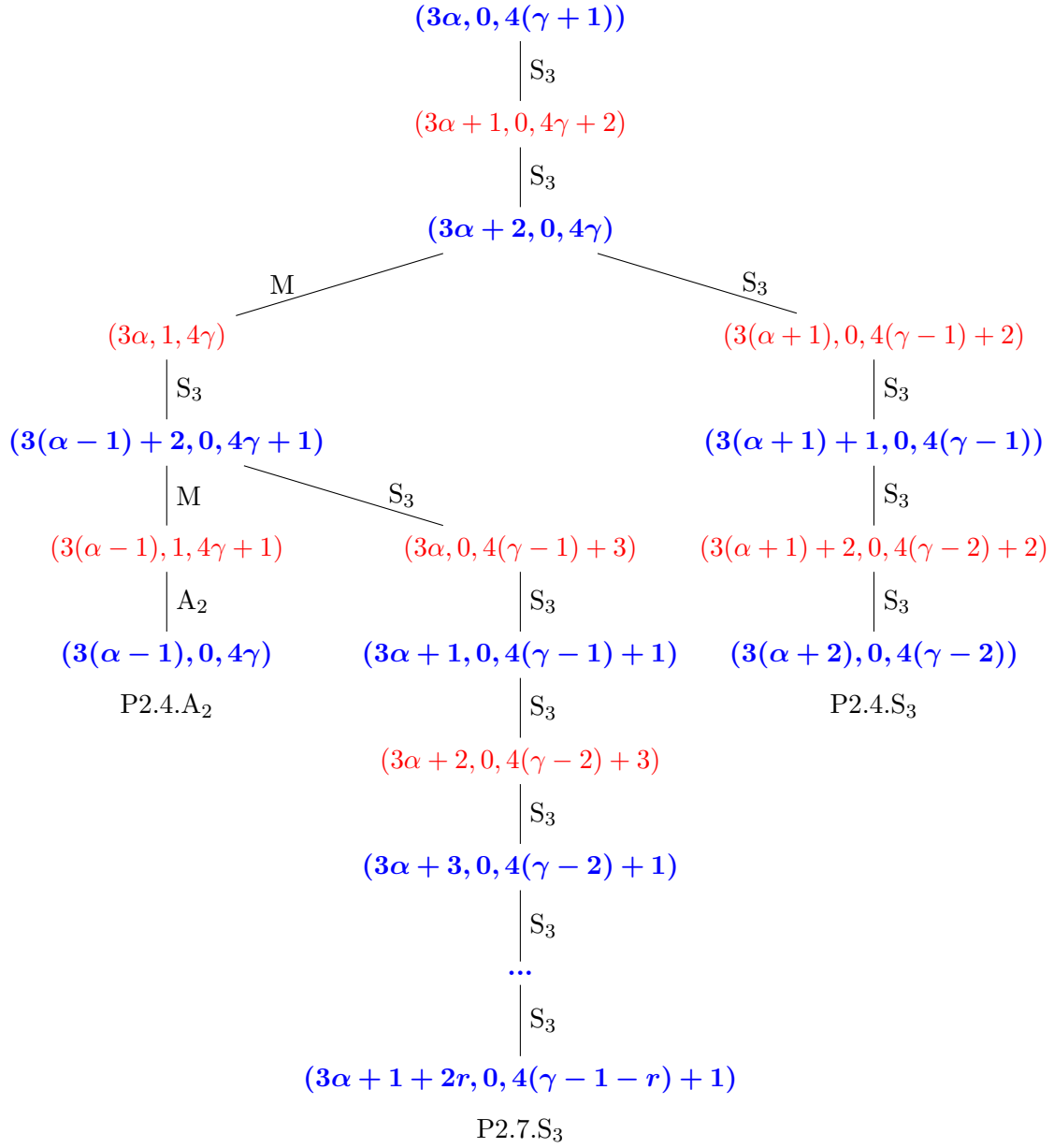
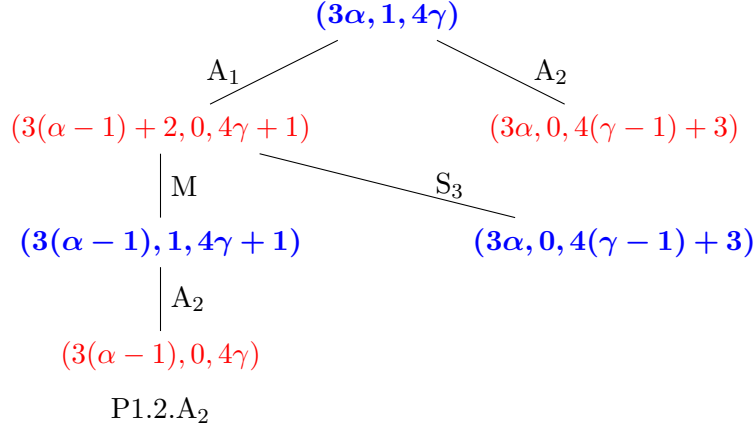


Figure 4.: Game Tree for the Game State $(3\alpha, 0, 4(\gamma + 1))$.

Corollary 5.8. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. For all $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$, Player 1 has a winning strategy for $(3\alpha, 1, 4\gamma + 1)$ when $\alpha \geq \gamma$ and a winning strategy for $(3\alpha, 1, 4\gamma)$ when $\alpha \geq \gamma + 1$.*

Proof. Player 1 can place $(3\alpha, 0, 4\gamma)$ from $(3\alpha, 1, 4\gamma + 1)$ which wins for all $\alpha \geq \gamma$ as shown in Lemma 5.7. Similarly, Player 1 wins any $(3\alpha, 1, 4\gamma)$ for all $\alpha \geq \gamma + 1$. For contradiction's sake, assume that Player 2 wins. Then, consider the game tree below.



By assumption, placing $(3\alpha, 0, 4(\gamma - 1) + 3)$ loses, so if Player 1 places $(3(\alpha - 1) + 2, 0, 4\gamma + 1)$, Player 2 must place $(3(\alpha - 1), 1, 4\gamma + 1)$. Since $\alpha \geq \gamma + 1$, Player 1 wins at P1.2.A₂ by Lemma 5.7. Therefore, it must also be true that Player 1 has a winning strategy for $(3\alpha, 1, 4\gamma)$ for all $\alpha \geq \gamma + 1$. \square

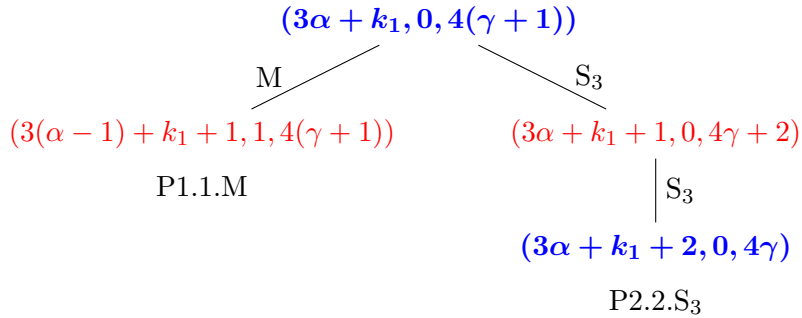
Corollary 5.9. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game, with $\alpha, \gamma, k_1 \in \mathbb{Z}^{\geq 0}$ such that $0 \leq k_1 \leq 2$. Then, Player 2 has a winning strategy for*

- (1) $(3\alpha + k_1, 0, 4\gamma)$ such that $\alpha \geq \gamma + 1$ and
- (2) $(3\alpha + k_1, 0, 4\gamma + 1)$ such that $\alpha \geq \gamma$.

Player 1 has a winning strategy for

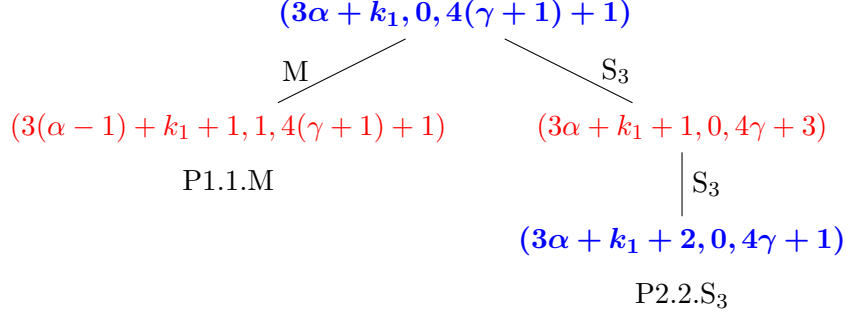
- (3) $(3\alpha + k_1, 0, 4\gamma + 2)$ such that $\alpha \geq \gamma + 1$ and
- (4) $(3\alpha + k_1, 0, 4\gamma + 3)$ such that $\alpha \geq \gamma$.

Proof. We prove Part 1 by inducting on γ , letting $\alpha \geq \gamma + 1$ be arbitrary. As a base case, we see that for $\gamma = 0$, Player 2 wins $(3\alpha + k_1, 0, 0)$ by Theorem 5.1. As our inductive hypothesis, assume Player 2 wins $(3\alpha + k_1, 0, 4\gamma)$ for all $\alpha \geq \gamma + 1$ and consider the tree below, on $(3\alpha + k_1, 0, 4(\gamma + 1))$ letting $\alpha \geq \gamma + 2$.



Player 2 wins at P2.2.S₃ by inductive hypothesis, $\alpha \geq \gamma + 2$ implies $\alpha \geq \gamma + 1$. If $k_1 = 0, 1$ then Player 1 loses at P1.1.M by Lemma 5.5. If $k_1 = 2$, then P1.1.M is $(3\alpha, 1, 4(\gamma + 1))$ which loses by Corollary 5.8 since $\alpha \geq \gamma + 2$.

Similarly, we prove Part 2 by inducting on γ , letting $\alpha \geq \gamma$ be arbitrary. As a base case, for $\gamma = 0$, placing $(3\alpha + k_1, 0, 1)$ wins by Corollary 5.2. Then, assume Player 2 wins for $(3\alpha + k_1, 0, 4\gamma + 1)$ when $\alpha \geq \gamma$ and consider the game tree on $(3\alpha + k_1, 0, 4(\gamma + 1) + 1)$, with $\alpha \geq \gamma + 1$.



Player 2 wins at P2.2.S₃ by inductive hypothesis, since $\alpha \geq \gamma + 1$ implies $\alpha \geq \gamma$. Again, if $k_1 = 0, 1$ then Player 1 loses at P1.1.M by Lemma 5.5. If $k_1 = 2$, then P1.1.M is $(3\alpha, 1, 4(\gamma + 1) + 1)$ which loses by Corollary 5.8 since $\alpha \geq \gamma + 1$.

As a direct result, Player 1 wins $(3\alpha + k_1, 0, 4\gamma + 2)$ when $\alpha \geq \gamma + 1$ as claimed in Part 3. From here, they can place $(3\alpha + k_1 + 1, 0, 4\gamma)$ which wins as proven above.

Likewise, Player 1 wins $(3\alpha + k_1, 0, 4\gamma + 3)$ for all $(3\alpha + k_1, 0, 4\gamma + 3)$ when $\alpha \geq \gamma$ as claimed in Part 4. From here, they can place $(3\alpha + k_1 + 1, 0, 4\gamma + 1)$ which wins as proven above. \square

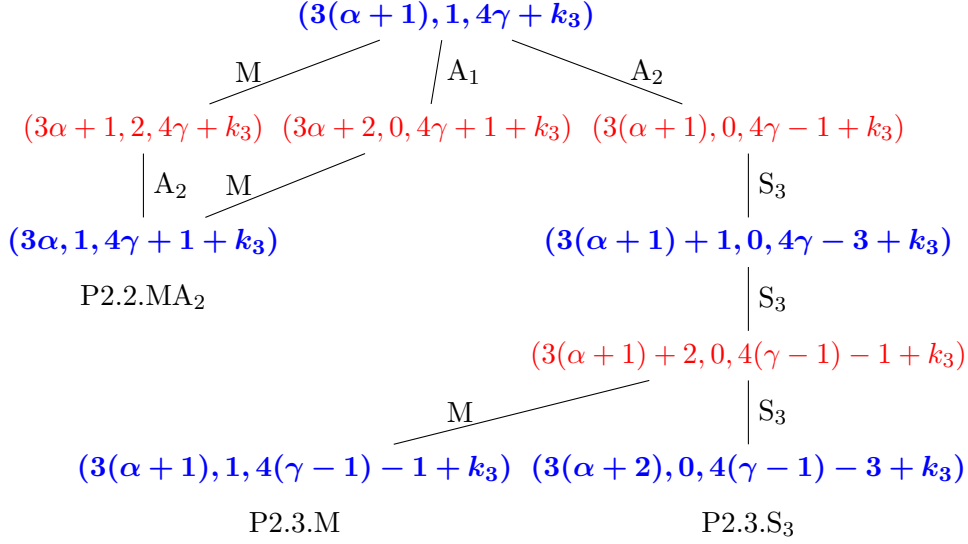
Corollary 5.10. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. Player 1 has a winning strategy for $(3\alpha, 1, 4\gamma + 2)$ when $\alpha \geq \gamma$. Player 1 has a winning strategy for $(3\alpha, 1, 4\gamma + 3)$ when $\alpha \geq \gamma + 3$.*

Proof. If Player 2 places $(3\alpha, 1, 4\gamma + 2)$, Player 1 can add the second two columns to place $(3\alpha, 0, 4\gamma + 1)$ which wins by Corollary 5.9 Part 2 for all $\alpha \geq \gamma$.

If Player 2 places $(3\alpha, 1, 4\gamma + 3)$, Player 1 can add the first two columns to place $(3(\alpha - 1) + 2, 0, 4(\gamma + 1))$, which wins by Corollary 5.9 Part 1 for all $\alpha - 1 \geq \gamma + 2$ which is equivalent to $\alpha \geq \gamma + 3$. \square

Lemma 5.11. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game, with $\alpha, \gamma, k_3 \in \mathbb{Z}^{\geq 0}$ such that $0 \leq k_3 \leq 3$. Player 2 has a winning strategy for $(3\alpha, 1, 4\gamma + k_3)$ for all $\alpha \leq \gamma - 2$.*

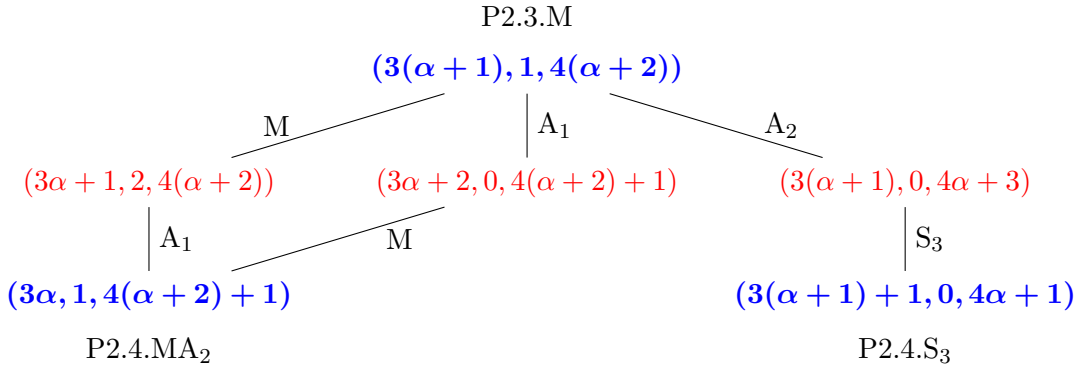
Proof. We proceed by induction on α . We first note that for $\alpha = 0, \gamma \geq 2$, Player 2 has a winning strategy for $(0, 1, 4\gamma + k_3)$ by Corollary 5.4. We then assume that Player 2 wins $(3\alpha, 1, 4\gamma + k_3)$ for all $\alpha \leq \gamma - 2$ and consider the game on $(3(\alpha + 1), 1, 4\gamma + k_3)$ letting $\alpha \leq \gamma - 3$.



It is easy to see that Player 2 wins at P2.2.MA₂ by the inductive hypothesis, since $\alpha \leq \gamma - 3$. Proving that Player 2 wins at P2.3.M and P2.3.S₃ takes a little more effort as Player 2 should choose different options based on the values of α, γ and k_3 . We evaluate these different possibilities below.

First, suppose $k_3 = 0$. Then, P2.3.S₃ = $(3(\alpha + 2), 0, 4(\gamma - 2) + 1)$ and P2.3.M = $(3(\alpha + 1), 1, 4(\gamma - 2) + 3)$. P2.3.M wins by inductive hypothesis for all $\alpha + 1 \leq (\gamma - 2) - 2$, and we assume $\alpha \leq \gamma - 3$. Therefore, the only values of α we need to prove Player 2 wins for are $\alpha = \gamma - 3$ and $\alpha = \gamma - 4$. Note that when $\gamma = \alpha + 3$, P2.3.S₃ = $(3(\alpha + 2), 0, 4(\alpha + 1) + 1)$. Then, the position P2.3.S₃ wins when $\alpha = \gamma - 3$ by Corollary 5.9 Part 2. Similarly, when $\gamma = \alpha + 4$, P2.3.S₃ = $(3(\alpha + 2), 0, 4(\alpha + 2) + 1)$, which also wins by Corollary 5.9 Part 2.

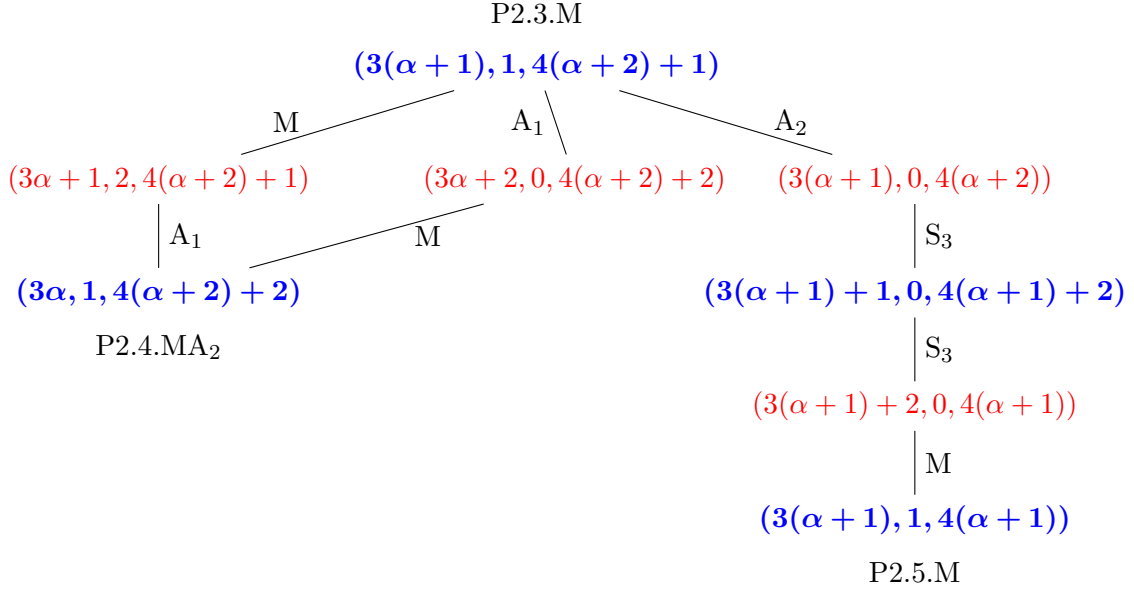
Next, suppose $k_3 = 1$. Then, P2.3.S₃ = $(3(\alpha + 2), 0, 4(\gamma - 2) + 2)$ and P2.3.M = $(3(\alpha + 1), 1, 4(\gamma - 1))$. P2.3.M wins by inductive hypothesis for all $\alpha \leq (\gamma - 2) - 2$, and we assume $\alpha \leq \gamma - 3$, so we only need to consider when $\alpha = \gamma - 3$, which is equivalent to $\gamma = \alpha + 3$. Here, P2.3.M = $(3(\alpha + 1), 1, 4(\alpha + 2))$. Consider the game on P2.3.M below.



Player 2 wins at P2.4.MA₂ by inductive hypothesis and wins at P2.4.S₃ by Corollary 5.9 Part 2. Thus, Player 2 wins when $k_3 = 1$.

Then, suppose $k_3 = 2$, meaning that P2.3.S₃ = $(3(\alpha + 2), 0, 4(\gamma - 2) + 3)$ and P2.3.M = $(3(\alpha + 1), 1, 4(\gamma - 1) + 1)$. Following the logic from the proof for $k_3 = 1$, Player 2 wins at P2.3.M

by inductive hypothesis for all $\alpha \leq \gamma - 4$, and we assume $\alpha \leq \gamma - 3$. Then, consider when $\alpha = \gamma - 3$, such that $P2.3.M = (3(\alpha + 1), 1, 4(\alpha + 2) + 1)$ as below.



Player 2 wins at $P2.4.MA_2$ by the inductive hypothesis. We continue the tree from $P2.5.M$ in Figure 5.

In Figure 5, Player 2 wins at $P2.7.MA_1$ by inductive hypothesis and wins at $P2.7.S_3$ by Corollary 5.9 Part 2. Otherwise, the game reduces to $(3(\alpha - r), 1, 4(\alpha - r))$ at $P2.9.M$, so the game tree repeats for r rounds, until Player 2 places a state which we have shown to win by our inductive hypothesis or places $(0, 1, 0)$ which trivially wins. Therefore, Player 2 wins when $k_3 = 2$.

Lastly, we consider when $k_3 = 3$ so that $P2.3.M = (3(\alpha + 1), 1, 4(\gamma - 1) + 2)$ and $P2.3.S_3 = (3(\alpha + 2), 0, 4(\gamma - 1))$. Following the logic from the proof for $k_3 = 1, 2$, Player 2 wins at $P2.3.M$ by inductive hypothesis for all $\alpha \leq \gamma - 4$. Hence, we only need to consider when $\alpha = \gamma - 3$. Here, $P2.3.S_3 = (3(\alpha + 2), 0, 4(\alpha + 2))$ which wins by Lemma 5.7.

Thus, Player 2 has a winning strategy for $(3\alpha, 1, 4\gamma + k_3)$ for all $\alpha \leq \gamma - 2$. \square

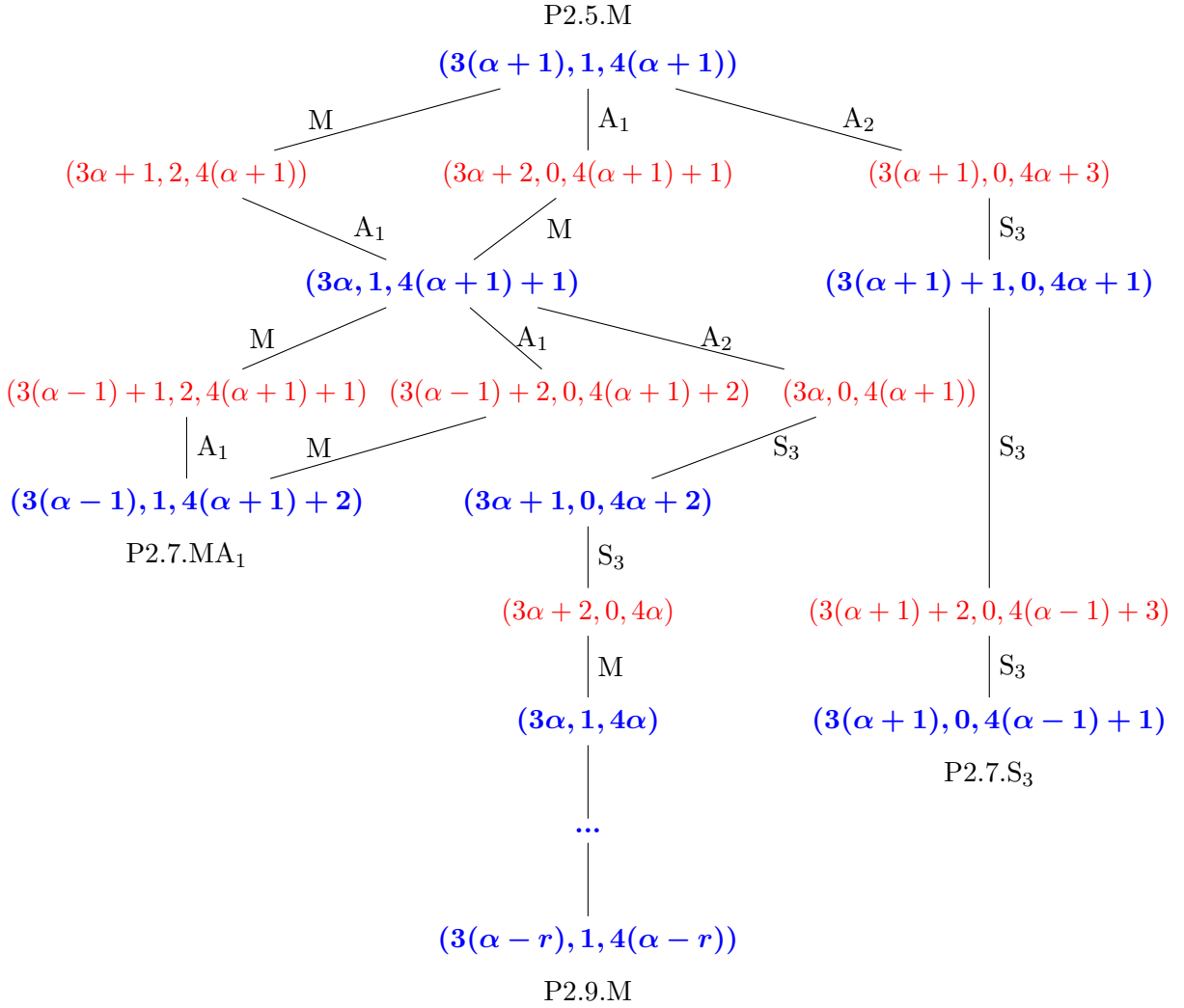


Figure 5.: Game Tree for the Game State $(3(\alpha + 1), 1, 4(\alpha + 1))$

Corollary 5.12. Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game, with $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$. Player 2 has a winning strategy for $(3\alpha, 1, 4\gamma)$ when $\alpha = \gamma$.

Proof. We showed in Lemma 5.11 that $(3\alpha, 1, 4\alpha)$ wins under the assumption that that $(3\alpha, 1, 4\gamma + k_3)$ wins for all $\alpha \leq \gamma - 2$. Since we proved the Lemma 5.11 is in fact true, then it follows that Player 2 wins $(3\alpha, 1, 4\gamma)$ for all $\alpha = \gamma$. \square

We now know that Player 2 wins all $(3\alpha, 1, 4\gamma + k_3)$ for all $\alpha \leq \gamma - 2$, and from Lemma 5.8 and Corollary 5.10, we know that Player 2 wins all $(3\alpha, 1, 4\gamma + k_3)$ for all $\alpha \geq \gamma + 3$. By considering each k_3 individually, we determine the winner for $(3\alpha, 1, 4\gamma + k_3)$ for any $\alpha, \gamma, k_3 \in \mathbb{Z}^{\geq 0}$ such that $0 \leq k_3 \leq 3$.

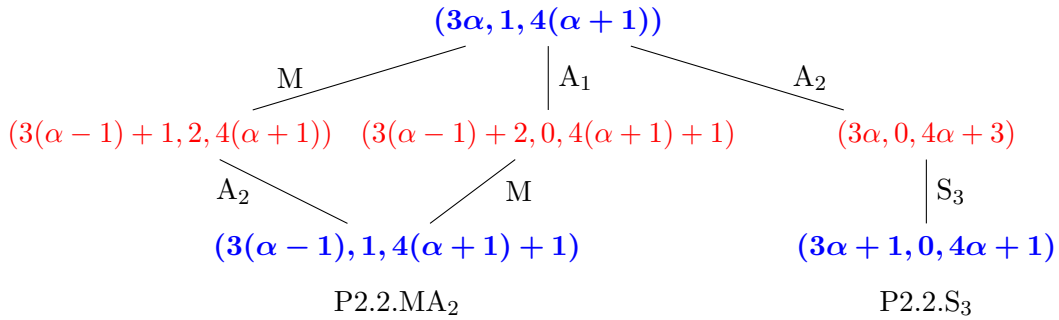
Theorem 5.13. Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf Game with $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$.

- (1) Player 2 wins $(3\alpha, 1, 4\gamma)$ for all $\alpha \leq \gamma$ and Player 1 wins $(3\alpha, 1, 4\gamma)$ for all $\alpha \geq \gamma + 1$.
- (2) Player 2 wins $(3\alpha, 1, 4\gamma + 1)$ for all $\alpha \leq \gamma - 1$ and Player 1 wins $(3\alpha, 1, 4\gamma + 1)$ for all $\alpha \geq \gamma$.
- (3) Player 2 wins $(3\alpha, 1, 4\gamma + 2)$ for all $\alpha \leq \gamma - 2$ and Player 1 wins $(3\alpha, 1, 4\gamma + 2)$ for all $\alpha \geq \gamma - 1$.
- (4) Player 2 wins $(3\alpha, 1, 4\gamma + 3)$ for all $\alpha \leq \gamma + 2$ and Player 1 wins $(3\alpha, 1, 4\gamma + 3)$ for all $\alpha \geq \gamma + 3$.

Proof. We proceed by proving each part of the theorem individually.

Proof when $k_3 = 0$.

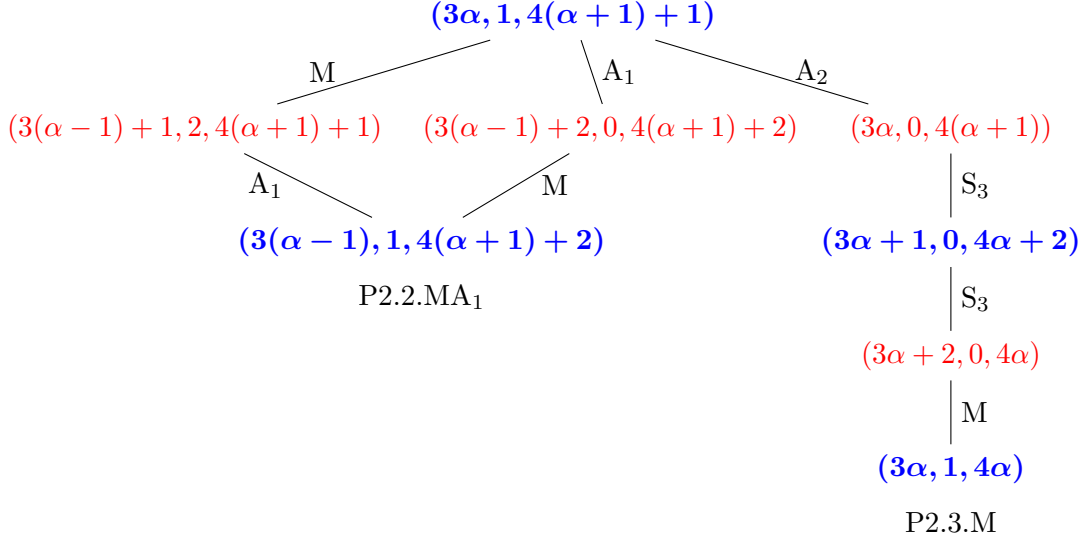
First, we consider when $k_3 = 0$. From Corollary 5.8, we know that Player 1 has a winning strategy for all $\alpha \geq \gamma + 1$, and from Lemma 5.11, Player 2 has a winning strategy for all $\alpha \leq \gamma - 2$. Moreover, we showed that Player 2 wins $(3\alpha, 1, 4\alpha)$ in Corollary 5.12. Therefore, it is only necessary to show that Player 2 also wins for $\gamma = \alpha + 1$, which we show in the game tree below.



Player 2 wins at P2.2.MA₂ by Lemma 5.11 and at P2.2.S₃ by Corollary 5.9 Part 2. Thus, Player 2 wins $(3\alpha, 1, 4\gamma)$ for all $\alpha \leq \gamma$ and Player 1 wins $(3\alpha, 1, 4\gamma)$ for all $\alpha \geq \gamma + 1$.

Proof when $k_3 = 1$.

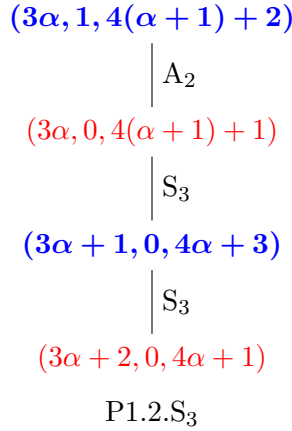
Then, we consider when $k_3 = 1$. From Corollary 5.8, we know that Player 1 has a winning strategy for all $\alpha \geq \gamma$, and from Lemma 5.11, Player 2 has a winning strategy for all $\alpha \leq \gamma - 2$. Thus, we only need to consider $\gamma = \alpha + 1$, which we do in the gameplay tree below.



Player 2 wins at P2.2.MA₁ by Lemma 5.11 and at P2.3.M by Corollary 5.12. Hence, Player 2 wins $(3\alpha, 1, 4\gamma + 1)$ for all $\alpha \leq \gamma - 1$ and Player 1 wins $(3\alpha, 1, 4\gamma + 1)$ for all $\alpha \geq \gamma$.

Proof when $k_3 = 2$.

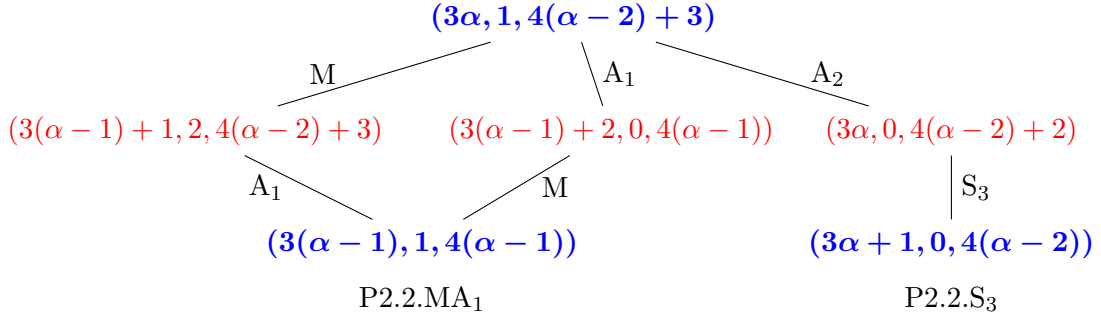
Then, we consider $k_3 = 2$. From Corollary 5.8, we know that Player 1 has a winning strategy for all $\alpha \geq \gamma$, and from Lemma 5.11 Player 2 has a winning strategy for all $\alpha \leq \gamma - 2$, so again, we only need to show that Player 1 wins $\gamma = \alpha + 1$, as we do in the game play tree below.



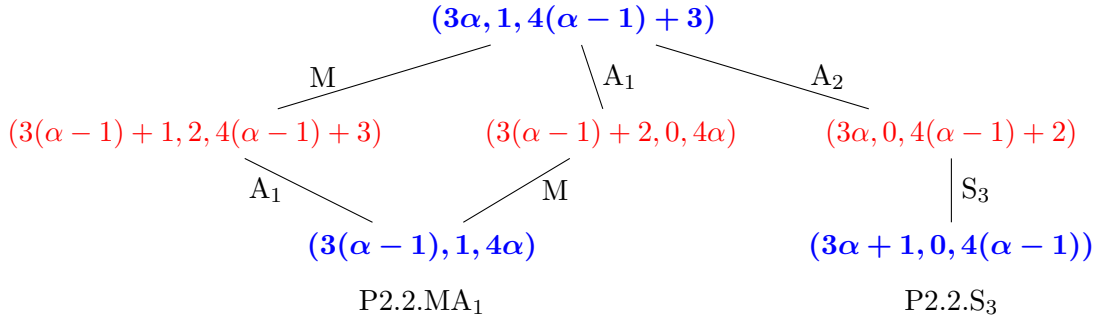
Player 1 wins at P1.2.S₃ by Corollary 5.9 Part 2, so Player 2 wins $(3\alpha, 1, 4\gamma + 2)$ for all $\alpha \leq \gamma - 2$ and Player 1 wins $(3\alpha, 1, 4\gamma + 2)$ for all $\alpha \geq \gamma - 1$.

Proof when $k_3 = 3$.

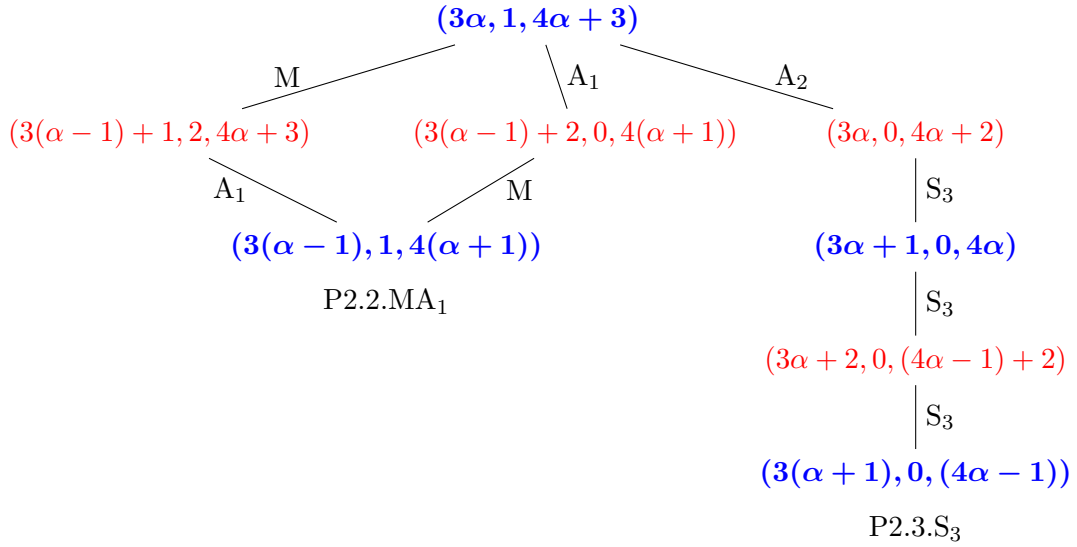
Finally, we consider when $k_3 = 3$. By Lemma 5.11, Player 2 wins for all $\alpha \leq \gamma - 2$ and by Corollary 5.10, Player 1 wins for all $\alpha \geq \gamma + 3$. Thus, we must consider when $\alpha - 2 \leq \gamma \leq \alpha + 1$. We begin with the game tree such that $\gamma = \alpha - 2$.



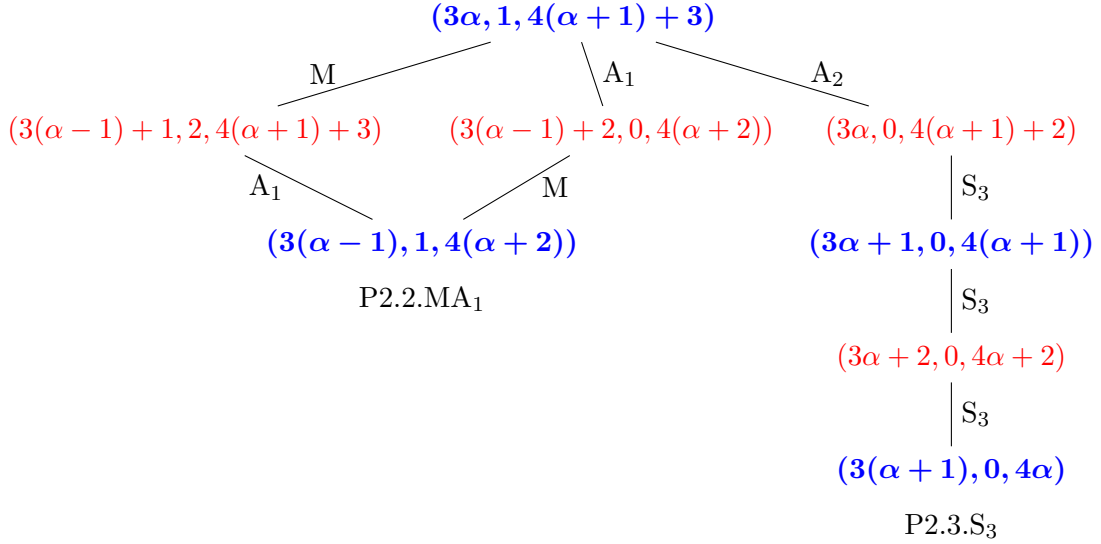
Player 2 wins at P2.2.MA₁ by the proof of Part 1 of this Theorem. Additionally, Player 2 wins at P2.2.S₃ by Corollary 5.9 Part 1. Next, we consider the game tree such that $\gamma = \alpha - 1$.



Player 2 wins at P2.2.MA₁ by the proof of Part 1 of this Theorem. Additionally, Player 2 wins at P2.2.S₃ by Corollary 5.9 Part 1. Then, we continue to $\gamma = \alpha$.



When $\alpha = \gamma = 0$, the game starts at $(0, 1, 3)$ which Player 2 wins by Corollary 5.4. For $\alpha \in \mathbb{Z}^{>0}$ Player 2 wins at P2.2.MA₁ by the proof of Part 1 of this Theorem and at P2.3.S₃ by Corollary 5.9 Part 1. Then, we conclude with the case where $\gamma = \alpha + 1$, which is essentially identical to the case for $\gamma = \alpha$.



Again, Player 2 wins at P2.2.MA₁ by the proof of Part 1 of this Theorem and at P2.3.S₃ by Corollary 5.9 Part 1.

This concludes our proof of Theorem 5.13 □

From here, we can begin to prove some of the statements claimed in Figure 1. For a solution to be truly constructive, we want a path from any winning state to another winning state. In Lemma 5.14 we prove that certain states are indeed winning states below, and then provide constructive solutions in Theorem 5.15.

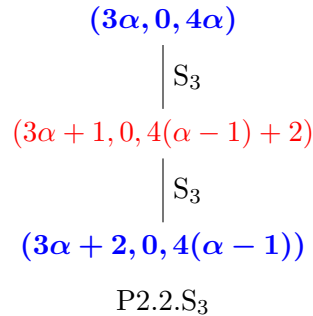
Lemma 5.14. *Let $(a, 0, c)$ be a game state for an the F_4 Black Hole Zeckendorf game $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$. Player 2 has a winning strategy for board setups*

- (1) $(3\alpha, 0, 4\gamma)$ such that $\alpha \geq \gamma$, and
- (2) $(3\alpha + 1, 0, 4\gamma)$ for all α, γ .

Proof. We prove each statement individually.

Proof of Part 1.

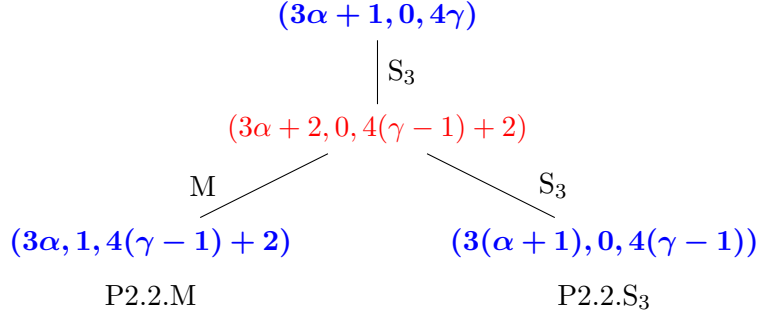
We first show that Player 2 wins $(3\alpha, 0, 4\gamma)$ for all $\alpha \geq \gamma$, noting that when $\alpha = \gamma = 0$ Player 2 trivially wins $(0, 0, 0)$. By Corollary 5.9 Part 1, $(3\alpha, 0, 4\gamma)$ wins for all $\alpha \geq \gamma + 1$, so it is only necessary to consider the game when $\gamma = \alpha$, as we do below.



P2.2.S₃ wins by Corollary 5.9 Part 1 so Player 2 wins $(3\alpha, 0, 4\gamma)$ wins for all $\alpha \geq \gamma$.

Proof of Part 2.

To conclude this lemma, we show that Player 2 wins $(3\alpha + 1, 0, 4\gamma)$ for all α, γ , noting that when $\alpha = \gamma = 0$ Player 2 wins trivially at $(1, 0, 0)$. From Corollary 5.9 Part 1, we already know it wins for all $\alpha \geq \gamma + 1$. Then, consider the tree below.



Player 2 wins at P2.2.M for all $\alpha \leq \gamma - 3$ by Theorem 5.13 Part 3 and wins at P2.2.S₃ for all $\alpha \geq \gamma - 2$ as established by the proof of Part 1. Therefore, Player 2 wins $(3\alpha + 1, 0, 4\gamma)$ for all $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$. \square

Lemma 5.14 is sufficient information for us to now construct a winning strategy for any state $(a, 0, c)$.

Theorem 5.15. *Let $(a, 0, c) = (3\alpha + k_1, 0, 4\gamma + k_3)$ be a board setup for an F_4 Black Hole Zeckendorf game. For $\alpha, \gamma, k_1, k_3 \in \mathbb{Z}^{\geq 0}$, $0 \leq k_1 \leq 2$, and $0 \leq k_3 \leq 3$, either Player 2 or Player 1 has a constructive winning solution, which we outline as follows.*

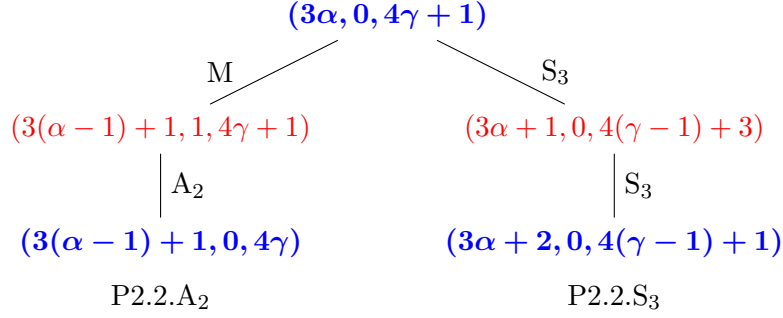
- (1) $c \equiv 1 \pmod{4}$:
 - (a) Player 2 wins $(3\alpha, 0, 4\gamma + 1)$ for all $\alpha \geq \gamma - 1$.
 - (b) Player 1 wins $(3\alpha, 0, 4\gamma + 1)$ for all $\alpha \leq \gamma - 2$.
 - (c) Player 2 wins $(3\alpha + 1, 0, 4\gamma + 1)$ for all α, γ .
 - (d) Player 2 wins $(3\alpha + 2, 0, 4\gamma + 1)$ for all $\alpha \geq \gamma$.
 - (e) Player 1 wins $(3\alpha + 2, 0, 4\gamma + 1)$ for all $\alpha \leq \gamma - 1$.
- (2) $c \equiv 2 \pmod{4}$:
 - (a) Player 1 wins $(3\alpha, 0, 4\gamma + 2)$ for all α, γ .
 - (b) Player 1 wins $(3\alpha + 1, 0, 4\gamma + 2)$ for all $\alpha \geq \gamma + 1$.
 - (c) Player 2 wins $(3\alpha + 1, 0, 4\gamma + 2)$ for all $\alpha \leq \gamma$.
 - (d) Player 1 wins $(3\alpha + 2, 0, 4\gamma + 2)$ for all α, γ .
- (3) $c \equiv 3 \pmod{4}$:
 - (a) Player 1 wins $(3\alpha, 0, 4\gamma + 3)$ for all α, γ .
 - (b) Player 1 wins $(3\alpha + 1, 0, 4\gamma + 3)$ for all $\alpha \geq \gamma + 1$.
 - (c) Player 2 wins $(3\alpha + 1, 0, 4\gamma + 3)$ for all $\alpha \leq \gamma$.
 - (d) Player 1 wins $(3\alpha + 2, 0, 4\gamma + 3)$ for all α, γ .
- (4) $c \equiv 0 \pmod{4}$:
 - (a) Player 2 wins $(3\alpha, 0, 4\gamma)$ for all $\alpha \geq \gamma$.
 - (b) Player 1 wins $(3\alpha, 0, 4\gamma)$ for all $\alpha \leq \gamma - 1$.
 - (c) Player 2 wins $(3\alpha + 1, 0, 4\gamma)$ for all α, γ .
 - (d) Player 2 wins $(3\alpha + 2, 0, 4\gamma)$ for all $\alpha \geq \gamma + 1$.
 - (e) Player 1 wins $(3\alpha + 2, 0, 4\gamma)$ for all $\alpha \leq \gamma$.

Proof. We work through each section of the theorem, referencing a base case as an example, and then demonstrating the winning strategy. Since we aim to create a constructive solution, we assume that both players are unfamiliar with Lemma 5.5, and play out the game when players choose to set the board as $(3\alpha + 1, 1, 4\gamma + k_3)$ and $(3\alpha + 2, 1, 4\gamma + k_3)$, although we already know

that game state loses.

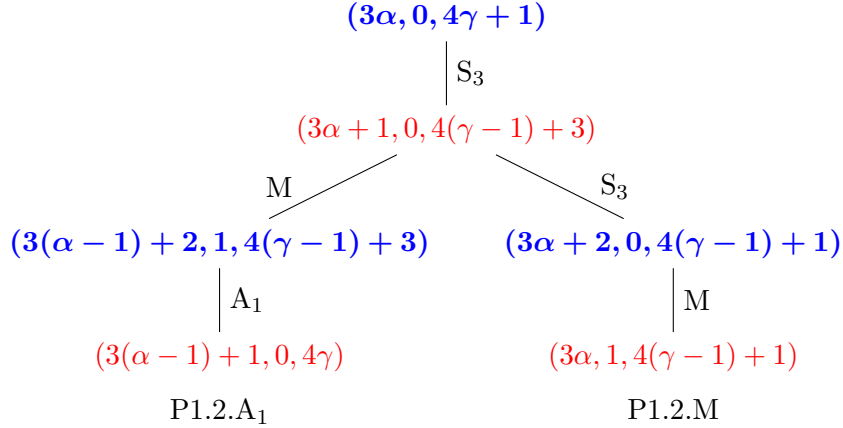
Proof for $c \equiv 1 \pmod{4}$.

Proof of Part 1a: We first consider the game on $(3\alpha, 0, 4\gamma + 1)$ where $\alpha \geq \gamma - 1$. As base cases, we see that Player 2 wins the setup $(0, 0, 1)$ trivially and wins the setup $(0, 0, 5)$ by Theorem 5.3. Then consider the game tree below.



P2.2.A₂ wins by Lemma 5.14 Part 2 and P2.2.S₃ wins by Corollary 5.9 Part 2, since $\alpha \geq \gamma - 1$, so Player 2 wins $(3\alpha, 0, 4\gamma + 1)$ for all $\alpha \geq \gamma - 1$.

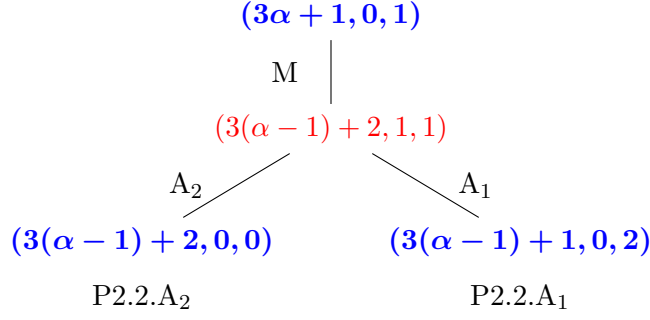
Proof of Part 1b: Consider the game on $(3\alpha, 0, 4\gamma + 1)$ where $\alpha \leq \gamma - 2$. As a base case, note that Player 1 wins $(0, 0, 9)$ by Theorem 5.3. Then, consider the tree below.



P1.2.A₁ wins by Lemma 5.14 Part 2 and P1.2.M wins by Theorem 5.13 Part 2.

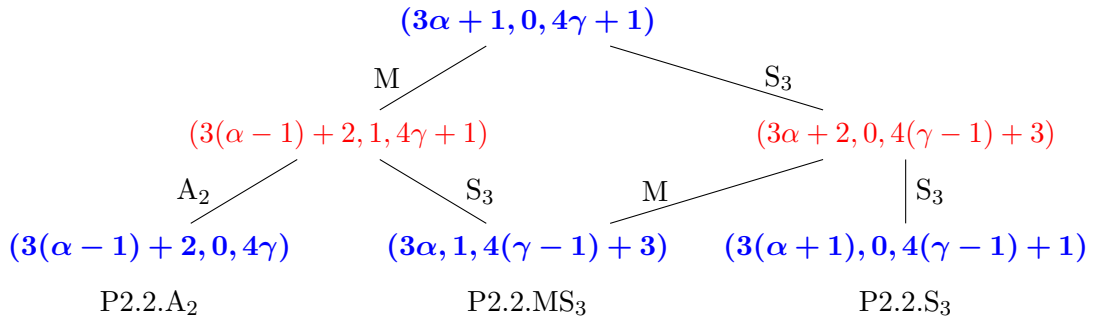
Proof of Part 1c: Next, we consider the game on $(3\alpha + 1, 0, 4\gamma + 1)$, noting that as a base case, Player 2 trivially wins $(1, 0, 1)$.

When $\gamma = 0$, the game tree is as follows.



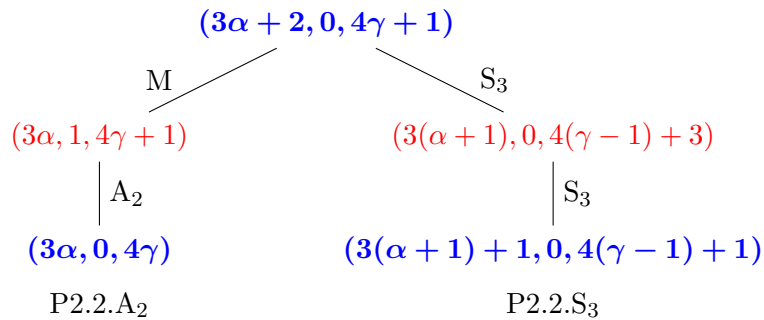
When $\alpha > 1$, Player 2 wins at P2.2.A₂ by Theorem 5.1. When $\alpha = 1$, Player 2 wins at P2.2.A₁ by Corollary 5.4.

We show in the game tree below that Player 2 always wins $(3\alpha + 1, 0, 4\gamma + 1)$ for $\gamma \geq 1$.



By Corollary 5.9 Part 1, P2.2.A₂ wins for all $\alpha \geq \gamma + 2$ and by Corollary 5.9 Part 2, P2.2.S₃ wins for all $\alpha \geq \gamma - 2$. By Theorem 5.13, P2.2.MS₃ wins for all $\alpha \leq \gamma + 1$. Thus, Player 2 wins $(3\alpha + 1, 0, 4\gamma + 1)$ for all $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$.

Proof of Part 1d: Next, we consider the game on $(3\alpha + 2, 0, 4\gamma + 1)$, with $\alpha \geq \gamma$, noting that as a base case Player 2 wins $(2, 0, 1)$ by Corollary 5.2. Now, consider the game on an arbitrary $(3\alpha + 2, 0, 4\gamma + 1)$ with $\alpha \geq \gamma$.



Since, $\alpha \geq \gamma$, P2.2.A₂ wins as proven in Lemma 5.14 Part 1 and P2.2.S₃ wins by Corollary 5.9 Part 2.

Proof of Part 1e: To conclude this section, we consider the game on $(3\alpha + 2, 0, 4\gamma + 1)$ with $\alpha \leq \gamma - 1$. As a base case, note that Player 1 wins $(2, 0, 5)$ by Corollary 5.4. Consider the game below.

$$(3\alpha + 2, 0, 4\gamma + 1)$$

| M

$$(3\alpha, 1, 4\gamma + 1)$$

P1.1.M

Since $\alpha \leq \gamma - 1$, P1.1.M wins by Theorem 5.13 Part 2. Therefore, Player 1 has a winning strategy for $(3\alpha + 2, 0, 4\gamma + 1)$ when $\alpha \leq \gamma - 1$.

Proof for $c \equiv 2 \pmod{4}$.

Proof of Part 2a: We begin by showing that Player 1 wins $(3\alpha, 0, 4\gamma + 2)$ for all α, γ , noting that as a base case Player 1 wins $(0, 0, 2)$ by Theorem 5.3. We consider the general case below.

$$(3\alpha, 0, 4\gamma + 2)$$

| S₃

$$(3\alpha + 1, 0, 4\gamma)$$

P1.1.S₃

We showed that $(3\alpha + 1, 0, 4\gamma)$ wins for all α, γ in Lemma 5.14 Part 2, so therefore Player 1 has a winning strategy for $(3\alpha, 0, 4\gamma + 2)$ for all α, γ .

Proof of Part 2b: The fact that Player 1 wins $(3\alpha + 1, 0, 4\gamma + 2)$ for $\alpha \geq \gamma + 1$ follows directly from Corollary 5.9 Part 3.

Proof of Part 2c: Next, we show Player 2 wins $(3\alpha + 1, 0, 4\gamma + 2)$ for all $\alpha \leq \gamma$, noting that as a base case Player 2 wins $(1, 0, 2)$ by Corollary 5.4. We consider the general case below.

$$(3\alpha + 1, 0, 4\gamma + 2)$$

M

S₃

$$(3(\alpha - 1) + 2, 1, 4\gamma + 2)$$

$$(3\alpha + 2, 0, 4\gamma)$$

S₃

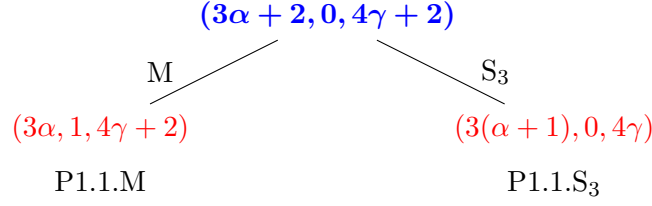
M

$$(3\alpha, 1, 4\gamma)$$

P2.2.S₃

We showed that $(3\alpha, 1, 4\gamma)$ wins for all $\alpha \leq \gamma$ in Theorem 5.13 Part 1, so therefore Player 2 has a winning strategy for $(3\alpha + 1, 0, 4\gamma + 2)$ for all $\alpha \leq \gamma$.

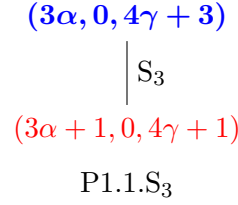
Proof of Part 2d: We conclude this section by showing that Player 1 wins $(3\alpha + 2, 0, 4\gamma + 2)$ for all α, γ , noting that as a base case Player 1 wins $(2, 0, 2)$ by Corollary 5.4. We consider the general case below.



By Theorem 5.13 Part 3, placing P1.1.M wins for all $\alpha \leq \gamma - 2$ and by Lemma 5.14 Part 1 placing P1.1. wins for all $\alpha \geq \gamma - 1$. Thus, Player 1 has a winning strategy for $(3\alpha + 2, 0, 4\gamma)$ for all α, γ .

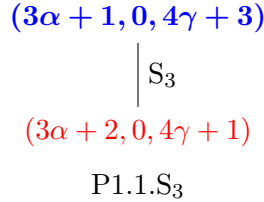
Proof for $c \equiv 3 \pmod{4}$.

Proof of Part 3a: We begin this section by showing that Player 1 wins $(3\alpha, 0, 4\gamma + 3)$ for all α, γ , noting that as a base case Player 1 wins $(0, 0, 3)$ by Theorem 5.3. We consider the general case below.



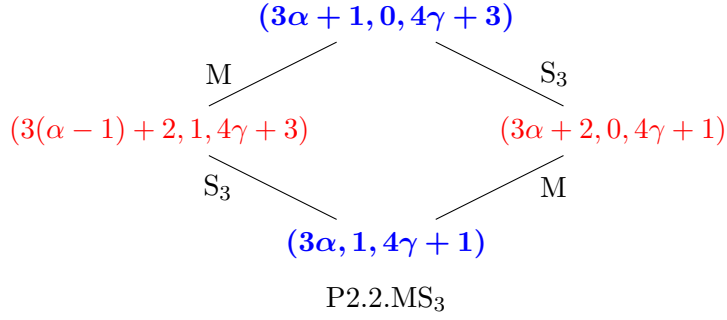
We showed that $(3\alpha + 1, 0, 4\gamma + 1)$ wins for all α, γ earlier in the proof of Part 1c, thus Player 1 has a winning strategy for $(3\alpha, 0, 4\gamma + 3)$ for all α, γ .

Proof of Part 3b: We follow by showing that Player 1 wins $(3\alpha + 1, 0, 4\gamma + 3)$ for all $\alpha \geq \gamma$, noting that as a base case Player 1 wins $(1, 0, 3)$ by Corollary 5.4. We consider the general case below.



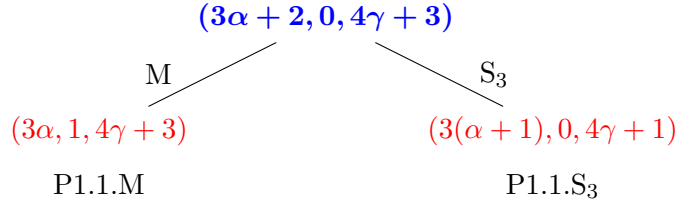
We showed that $(3\alpha + 2, 0, 4\gamma + 1)$ wins for all $\alpha \geq \gamma$ in the proof of 1d, so therefore Player 1 has a winning strategy for $(3\alpha + 1, 0, 4\gamma + 3)$ for all $\alpha \geq \gamma$.

Proof of Part 3c: We then show that Player 2 wins $(3\alpha + 1, 0, 4\gamma + 3)$ for all $\alpha \leq \gamma - 1$, noting that as a base case Player 2 wins $(1, 0, 7)$ by Corollary 5.4. We consider the general case below.



We showed that $(3\alpha, 1, 4\gamma + 1)$ wins for all $\alpha \leq \gamma - 1$ in Theorem 5.13 Part 2, so Player 2 has a winning strategy for $(3\alpha + 1, 0, 4\gamma + 3)$ for all $\alpha \leq \gamma - 1$.

Proof of Part 3d: We conclude this section by showing that Player 1 wins $(3\alpha + 2, 0, 4\gamma + 3)$ for all α, γ , noting that as a base case Player 1 wins $(2, 0, 3)$ by Corollary 5.4. We consider the general case below.

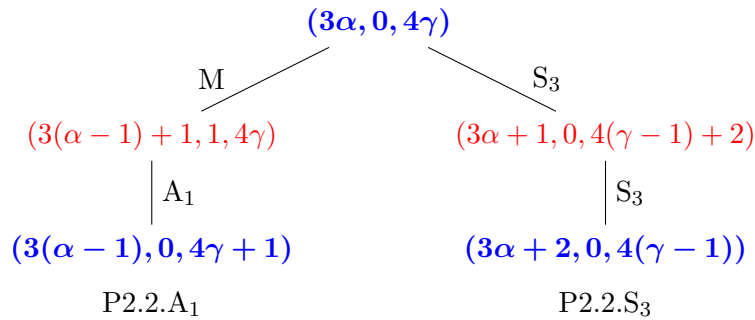


By Theorem 5.13 Part 4, placing P1.1.M wins for all $\alpha \leq \gamma + 2$ and by Corollary 5.9 Part 2, placing P1.1.S₃ wins for all $\alpha \geq \gamma - 1$. Thus, Player 1 has a winning strategy for $(3\alpha + 2, 0, 4\gamma + 3)$ for all α, γ .

Proof for $c \equiv 0 \pmod{4}$.

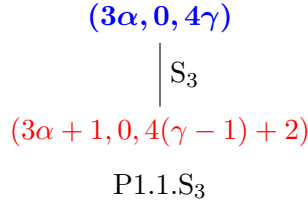
We already proved that a significant portion of Part 4 is true in Lemma 5.14. However, these proofs did not consider the possibility of Player 1 placing $(3\alpha + 1, 1, 4\alpha)$ and $(3\alpha + 2, 1, 4\alpha)$ so were not fully constructive. Here, we provide a constructive strategy.

Proof of Part 4a: We first consider the game on $(3\alpha, 0, 4\gamma)$ where $\alpha \geq \gamma$. As a base case, see that Player 2 wins $(0, 0, 0)$ trivially. Then consider the gameplay below.



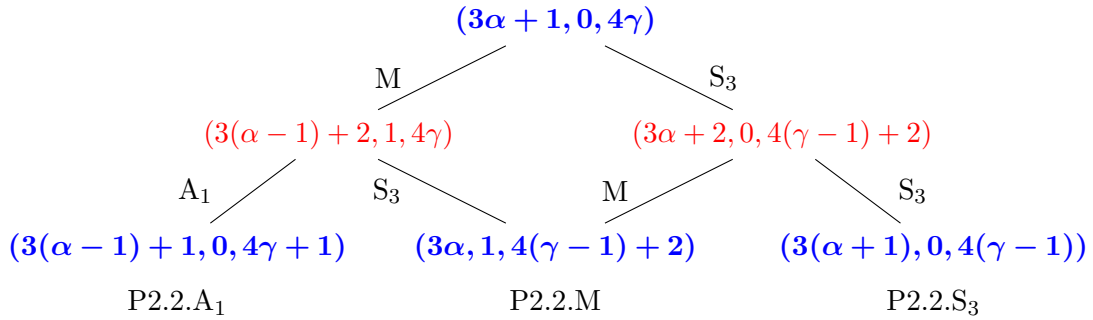
P2.2.A₁ wins by the proof of Part 1a since $\alpha \geq \gamma$. P2.2.S₃ wins by Corollary 5.9 Part 1 since $\alpha \geq \gamma$. Thus, Player 2 wins $(3\alpha, 0, 4\gamma)$ for all $\alpha \geq \gamma$.

Proof of Part 4b: Then, consider the game on $(3\alpha, 0, 4\gamma)$ where $\alpha \leq \gamma - 1$. As a base case, note that Player 1 wins $(0, 0, 4)$ by Theorem 5.3. Then, consider the tree below.



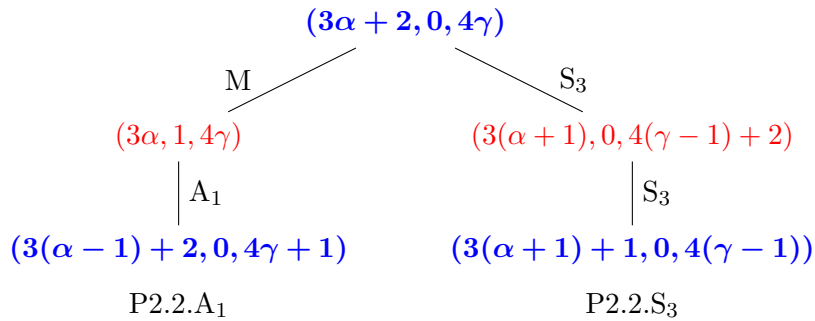
Since $\alpha \leq \gamma - 1$, then P1.1.S₃ wins by the proof of Part 2c.

Proof of Part 4c: Next, we consider the game on $(3\alpha + 1, 0, 4\gamma)$, noting that as a base case, Player 2 trivially wins $(1, 0, 0)$. We show below that $(3\alpha + 1, 0, 4\gamma)$ wins for all α, γ .



We showed P2.2.A₁ wins for all α, γ by the proof of Part 1c. By the proof of Part 4a, P2.2.S₃ wins for all $\alpha \geq \gamma - 2$ and by Theorem 5.13 Part 3, P2.2.M wins for all $\alpha \leq \gamma - 3$. Thus, Player 2 wins $(3\alpha + 1, 0, 4\gamma)$ for all $\alpha, \gamma \in \mathbb{Z}^{\geq 0}$.

Proof of Part 4d: Next, we consider the game on $(3\alpha + 2, 0, 4\gamma)$, with $\alpha \geq \gamma + 1$, noting that as a base case Player 2 wins $(5, 0, 0)$ by Theorem 5.1. Now, consider the game on an arbitrary $(3\alpha + 2, 0, 4\gamma)$ with $\alpha \geq \gamma + 1$.



Since $\alpha \geq \gamma + 1$, P2.2.A₁ wins by the proof of Part 1d and P2.2.S₃ wins as shown in the proof of Part 4c. Then, Player 2 wins $(3\alpha + 2, 0, 4\gamma)$ for all $\alpha \geq \gamma + 1$.

Proof of Part 4e: To conclude the proof of this theorem, we consider the game on $(3\alpha + 2, 0, 4\gamma)$ with $\alpha \leq \gamma$. As a base case, see that Player 1 wins $(2, 0, 0)$ by Theorem 5.1. Consider the game below.

$$(3\alpha + 2, 0, 4\gamma)$$

| M

$$(3\alpha, 1, 4\gamma)$$

P1.1.M

When $\alpha \leq \gamma$, P1.1.M wins by Theorem 5.13 Part 1. Therefore, Player 1 wins $(3\alpha + 2, 0, 4\gamma)$ for all $\alpha \leq \gamma$.

This concludes our proof of Theorem 5.15. □

5.3. Empty Board Game

We now continue to the Empty Board game, to determine which player has a winning strategy for any given $n \in \mathbb{Z}^{>0}$. We define the game so that players can still only place pieces in the outermost columns, meaning that no player can move to the F_2 column until all pieces have been placed on the board. Then, the initial setup is guaranteed to be in the form of some $(a, 0, c)$ above. Players are able to force certain setups, as there are only 2 options for placement, so mirroring remains an option as in the F_3 Black Hole game.

Theorem 5.16. *Let $(0, 0, 0)$ be the beginning board for an Empty Board F_4 Black Hole Zeckendorf game with $n \in \mathbb{Z}^{>0}$ pieces. Players can force certain game setups as outlined below.*

- (1) *For any $n \equiv 8, 12 \pmod{16}$, Player 1 can force the game into a setup $(n/4, 0, n/4)$.*
- (2) *For any $n \equiv 0, 4 \pmod{16}$, Player 2 can force the game into a setup $(n/4, 0, n/4)$.*
- (3) *For any $n \equiv 1, 5 \pmod{16}$, Player 1 can force the game into a setup $((n-1)/4 + 1, 0, (n-1)/4)$.*
- (4) *For any $n \equiv 9, 13 \pmod{16}$, Player 2 can force the game into a setup $((n-1)/4 + 1, 0, (n-1)/4)$.*
- (5) *For any $n \equiv 10, 14 \pmod{16}$, Player 1 can force the game into a setup $((n-2)/4 + 2, 0, (n-2)/4)$.*
- (6) *For any $n \equiv 2, 6 \pmod{16}$, Player 2 can force the game into a setup $((n-2)/4 + 2, 0, (n-2)/4)$.*
- (7) *For any $n \equiv 3, 7 \pmod{16}$, Player 1 can force the game into a setup $((n-3)/4 + 3, 0, (n-3)/4)$.*
- (8) *For any $n \equiv 15 \pmod{16}$, Player 1 can force the game into a setup $((n-3)/4, 0, (n-3)/4 + 1)$.*
- (9) *For any $n \equiv 11 \pmod{16}$, Player 2 can force the game into a setup that is either $((n-3)/4 + 3, 0, (n-3)/4)$ or $((n-3)/4, 0, (n-3)/4 + 1)$.*

Proof. First, consider $n \equiv 8, 12 \pmod{16}$. If Player 1 places their first piece in the F_3 column, and then places opposite Player 2 for all other moves, it will be Player 2's turn when there is 1 piece left, so they will be forced to place down $(n/4, 0, n/4)$.

Then, consider $n \equiv 0, 4 \pmod{16}$. Here, Player 2 can always mirror Player 1, until they eventually place down $(n/4, 0, n/4)$.

Next, consider $n \equiv 1, 5 \pmod{16}$. If Player 1 places their first piece in the F_1 column, and then places opposite Player 2 for all other moves, then every board Player 1 places will be of

the form $(m + 1, 0, m)$ for some non-negative integer m , until all n have been placed, resulting in the setup $((n - 1)/4 + 1, 0, (n - 1)/4)$. Here, Player 1 assumes the role of Player 2 in the decomposition phase of the game.

For $n \equiv 9, 13 \pmod{16}$, Player 2 can also force the setup $((n - 1)/4 + 1, 0, (n - 1)/4)$ by mirroring Player 1, until there is one piece left, forcing Player 1 to set down $((n - 1)/4 + 1, 0, (n - 1)/4)$. Again, Player 1 assumes the role of Player 2 in the decomposition phase of the game.

Then, consider $n \equiv 10, 14 \pmod{16}$. If Player 1 places their first piece in the F_1 column, and then places opposite Player 2 for all other moves, then every board Player 1 places will be of the form $(m + 1, 0, m)$ until there is only 1 piece left, which Player 2 is forced to set down in the F_1 column, placing down $((n - 2)/4 + 2, 0, (n - 2)/4)$.

For $n \equiv 2, 6 \pmod{16}$, Player 2 can force the setup $((n - 2)/4 + 2, 0, (n - 2)/4)$ by move mirroring until there are only 2 pieces left. Since pieces can only be placed in the outermost columns, Player 1 is forced to place $((n - 2)/4 + 1, 0, (n - 2)/4)$, allowing Player 2 to place $((n - 2)/4 + 2, 0, (n - 2)/4)$.

Next, consider $n \equiv 3, 7 \pmod{16}$. If Player 1 places their first piece in the F_1 column, and then places opposite Player 2 for all other moves, then every board Player 1 places will be of the form $(m + 1, 0, m)$ until there are only 2 pieces left; Player 2 is forced to set down one in the F_1 column, and Player 1 sets the other, placing down $((n - 3)/4 + 3, 0, (n - 3)/4)$. For this setup, Player 1 assumes the role of Player 2 in the decomposition phase of the game.

Then, consider $n \equiv 15 \pmod{16}$. Here, Player 1 should place their first piece in the F_3 column, and then place opposite Player 2 for all other moves. Then, every board Player 1 places will be of the form $(m, 0, m + 1)$ until they finally place $((n - 3)/4, 0, (n - 3)/4 + 1)$, assuming the role of Player 2 for the decomposition phase of the game.

Lastly, consider $n \equiv 11 \pmod{16}$. By move mirroring, Player 2 can set down the board $((n - 3)/4, 0, (n - 3)/4)$ when there are 3 pieces left. From here, Player 1 either sets $((n - 3)/4, 0, (n - 3)/4 + 1)$ in the next move or $((n - 3)/4 + 3, 0, (n - 3)/4)$ in the following moves. In both situations, Player 1 assumes the role of Player 2 in the decomposition phase of the game. \square

Theorem 5.17. *Player 2 has a constructive strategy for winning an Empty Board F_4 Black Hole Zeckendorf game for any $n \equiv 0, 2, 4, 6, 9, 11, 13 \pmod{16}$ such that $n \neq 2, 32$ in which case Player 1 has the winning strategy. Player 1 has a constructive strategy for winning an Empty Board F_4 Black Hole Zeckendorf game for any $n \equiv 1, 3, 5, 7, 8, 10, 12, 14, 15 \pmod{16}$, such that $n \neq 17, 47$, in which case Player 2 has the winning strategy.*

Proof. We split this proof into sections based on the value of n modulo 4.

Proof for $n \equiv 0 \pmod{4}$.

Player 2 sets down the board as $(n/4, 0, n/4)$, by Theorem 5.16. For $n \equiv 0, 4 \pmod{16}$, we have $n/4 \equiv 0, 1 \pmod{4}$ while $n \equiv 8, 12 \pmod{16}$ implies $n/4 \equiv 2, 3 \pmod{4}$.

For all $n/4 \geq 9$, with $0 \leq k_1 \leq 2$, $0 \leq k_3 \leq 3$ there is no $\alpha \leq \gamma$ such that $3\alpha + k_1 = n/4 = 4\gamma + k_3$. Thus, when $n/4 \equiv 0, 1 \pmod{4}$, Player 2 wins for all $n/4 \geq 9$ by Theorem 5.15 Part 1 and 4 respectively. When $n/4 \equiv 2, 3 \pmod{4}$, Player 1 wins for all $n/4 \geq 9$ by Theorem 5.15 Part 2 and 3 respectively.

Then, it is only necessary to explicitly consider the cases when $n/4 \leq 8$. We outline the outline the value of n , the cases, the winners, and the part of Theorem 5.15 that determines

the winner below.

Value of n	Board Setup	Winner	Part of Theorem 5.15
$n = 4$	$(1, 0, 1) = (3(0) + 1, 0, 4(0) + 1)$	Player 2	1c
$n = 8$	$(2, 0, 2) = (3(0) + 2, 0, 4(0) + 2)$	Player 1	2d
$n = 12$	$(3, 0, 3) = (3(1) + 0, 0, 4(0) + 3)$	Player 1	3a
$n = 16$	$(4, 0, 4) = (3(1) + 1, 0, 4(1) + 0)$	Player 2	4c
$n = 20$	$(5, 0, 5) = (3(1) + 2, 0, 4(1) + 1)$	Player 2	1d
$n = 24$	$(6, 0, 6) = (3(2) + 0, 0, 4(1) + 2)$	Player 1	2a
$n = 28$	$(7, 0, 7) = (3(2) + 1, 0, 4(1) + 3)$	Player 1	3b
$n = 32$	$(8, 0, 8) = (3(2) + 2, 0, 4(2) + 0)$	Player 1	4e

Thus, Player 2 wins for all $n \equiv 0, 4 \pmod{16}$ such that $n \neq 32$ and Player 1 wins for all $n \equiv 8, 12 \pmod{16}$, as well as $n = 32$.

Proof for $n \equiv 1 \pmod{4}$.

Player 1 sets down the board as $((n-1)/4 + 1, 0, (n-1)/4)$ by Theorem 5.16, therefore assuming the role of Player 2. For $n \equiv 1, 5 \pmod{16}$, we have $(n-1)/4 \equiv 0, 1 \pmod{4}$ while $n \equiv 9, 13 \pmod{16}$ implies $(n-1)/4 \equiv 2, 3 \pmod{4}$.

For all $(n-1)/4 \geq 5$, such that $0 \leq k_1 \leq 2, 0 \leq k_3 \leq 3$ there is no $\alpha \leq \gamma$ such that $3\alpha + k_1 = (n-1)/4 + 1$ and $4\gamma + k_3 = (n-1)/4$. When $(n-1)/4 \geq 5$, Player 1 wins for all $(n-1)/4 \equiv 0, 1 \pmod{4}$, by Theorem 5.15 Parts 1 and 4. When $(n-1)/4 \equiv 2, 3 \pmod{4}$, Player 2 wins for all $(n-1)/4$ by Theorem 5.15 Parts 2 and 3.

Then, it is only necessary to explicitly consider the cases when $(n-1)/4 \leq 4$. We outline the value of n , the cases, the winners, and the part of 5.15 that determines the winner below.

Value of n	Board Setup	Winner	Part of Theorem 5.15
$n = 1$	$(1, 0, 0) = (3(0) + 1, 0, 4(0) + 0)$	Player 1	4c
$n = 5$	$(2, 0, 1) = (3(0) + 2, 0, 4(0) + 1)$	Player 1	1d
$n = 9$	$(3, 0, 2) = (3(1) + 0, 0, 4(0) + 2)$	Player 2	2a
$n = 13$	$(4, 0, 3) = (3(1) + 1, 0, 4(0) + 3)$	Player 2	3b
$n = 17$	$(5, 0, 4) = (3(1) + 2, 0, 4(1) + 0)$	Player 2	4e

Therefore, Player 1 wins for all $n \equiv 1, 5 \pmod{16}$ such that $n \neq 17$ and Player 2 wins for all $n \equiv 9, 13 \pmod{16}$, as well as $n = 17$.

Proof for $n \equiv 2 \pmod{4}$.

Player 2 sets down the board as $((n-2)/4 + 2, 0, (n-2)/4)$ by Theorem 5.16. For $n \equiv 2, 6 \pmod{16}$, we have $(n-2)/4 \equiv 0, 1 \pmod{4}$ while $n \equiv 10, 14 \pmod{16}$ implies $(n-2)/4 \equiv 2, 3 \pmod{4}$. For all $(n-2)/4 \geq 1$, and $0 \leq k_1 \leq 2, 0 \leq k_3 \leq 3$ there is no $\alpha \leq \gamma$ such that $3\alpha + k_1 = (n-2)/4 + 2$ and $4\gamma + k_3 = (n-2)/4$. When $(n-2)/4 \equiv 0, 1 \pmod{4}$, Player 2 wins for all $(n-2)/4 \geq 1$ by Theorem 5.15 Parts 1 and 4. When $(n-2)/4 \equiv 2, 3 \pmod{4}$, Player 1 wins for all $(n-2)/4 \geq 1$ by Theorem 5.15 Parts 2 and 3.

Then, it is only necessary to explicitly consider the case $(2, 0, 0)$, which Player 1 wins by Theorem 5.1. Hence, Player 2 wins for all $n \equiv 2, 6 \pmod{16}$ such that $n \neq 2$ and Player 1 wins for all $n \equiv 10, 14 \pmod{16}$, as well as $n = 2$.

Proof for $n \equiv 3 \pmod{4}$.

By Theorem 5.16, Player 1 either sets down $((n-3)/4+3, 0, (n-3)/4)$ or $((n-3)/4, 0, (n-3)/4+1)$. In both situations, they assume the role of Player 2 for the decomposition phase of the game.

For $n \equiv 3, 7 \pmod{16}$, we have $(n-3)/4 \equiv 0, 1 \pmod{4}$. Here, Player 1 should set down $((n-3)/4+3, 0, (n-3)/4)$. Then, for all $(n-3)/4 \geq 0$, and $0 \leq k_1 \leq 2, 0 \leq k_3 \leq 3$ there is no $\alpha \leq \gamma$ such that $3\alpha + k_1 = (n-3)/4+3$ and $4\gamma + k_3 = (n-3)/4$. So, for all $n \equiv 3, 7 \pmod{16}$, Player 1 wins by Theorem 5.15 Parts 1 and 4.

Next, for $n \equiv 15 \pmod{16}$, we have $(n-3)/4+1 \equiv 0 \pmod{4}$. Player 1 should set down $((n-3)/4, 0, (n-3)/4+1)$; then, for all $(n-3)/4 > 11$, and $0 \leq k_1 \leq 2, 0 \leq k_3 \leq 3$ there is no $\alpha \leq \gamma$ such that $3\alpha + k_1 = (n-3)/4$ and $4\gamma + k_3 = (n-3)/4+1$. Then, Player 1 wins by Theorem 5.15 Part 4. We explicitly consider $n \equiv 15 \pmod{16}$ such that $(n-3)/4 \leq 11$ below.

Value of n	Board Setup	Winner	Part of Theorem 5.15
$n = 15$	$(3, 0, 4) = (3(1) + 0, 0, 4(1) + 0)$	Player 1	4a
$n = 31$	$(7, 0, 8) = (3(2) + 1, 0, 4(2) + 0)$	Player 1	4c
$n = 47$	$(11, 0, 12) = (3(3) + 2, 0, 4(3) + 0)$	Player 2	4e
$n = 47$	$(14, 0, 11) = (3(4) + 2, 0, 4(2) + 3)$	Player 2	3d

Lastly, for $n \equiv 11 \pmod{16}$, we have $(n-3)/4 \equiv 2 \pmod{4}$ and $(n-3)/4+1 \equiv 3 \pmod{4}$. Therefore, Player 2 can force Player 1 to set the board such that $c \equiv 2, 3 \pmod{4}$. If Player 1 places $((n-3)/4+3, 0, (n-3)/4)$, then since there exists no $\alpha \leq \gamma$ that satisfies $3\alpha + k_1 = (n-3)/4+3$ and $4\gamma + k_3 = (n-3)/4$, Player 2 wins by Theorem 5.15 Part 3.

If Player 1 places $((n-3)/4, 0, (n-3)/4+1)$, Player 2 wins for all $(n-3)/4 \geq 12$ by Theorem 5.15 Part 3. We explicitly consider $n \equiv 11 \pmod{16}$ such that $(n-3)/4 \leq 11$ below.

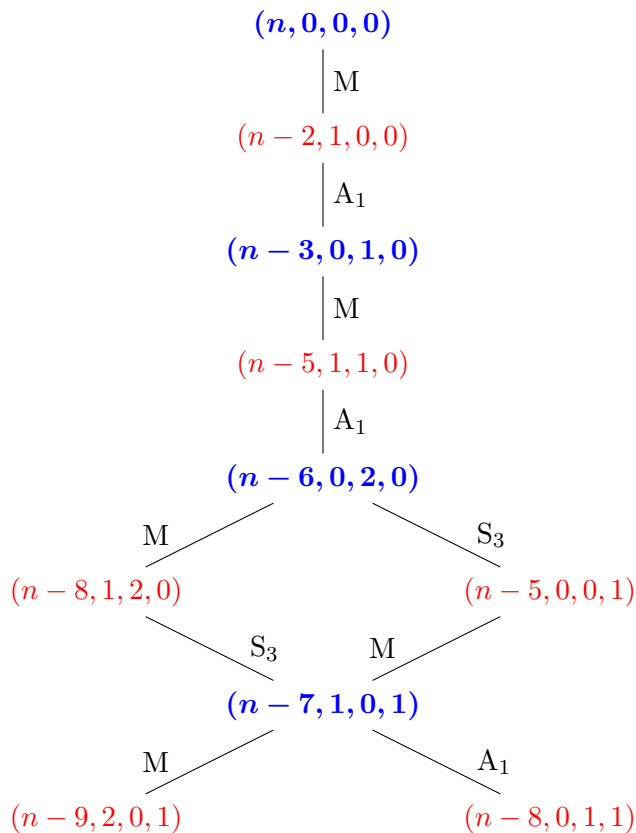
Value of n	Board Setup	Winner	Part of Theorem 5.15
$n = 11$	$(2, 0, 3) = (3(0) + 2, 0, 4(0) + 3)$	Player 2	3d
$n = 27$	$(6, 0, 7) = (3(2) + 0, 0, 4(1) + 3)$	Player 2	3a
$n = 43$	$(10, 0, 11) = (3(3) + 1, 0, 4(2) + 3)$	Player 2	3b

So then, we find that Player 1 has a constructive winning solution for all $n \equiv 1, 3, 5, 7, 8, 10, 12, 14, 15 \pmod{16}$ such that $n \neq 17, 47$, for which Player 2 has a winning solution. Player 2 has a constructive winning solution for all $n \equiv 0, 2, 4, 6, 9, 11, 13 \pmod{16}$ such that $n \neq 2, 32$, for which Player 1 has a winning solution. \square

6. Future Work

To determine a constructive solution for the Empty Board F_4 Black Hole Zeckendorf Game as we defined it, it was only necessary to determine which player wins for any given $(a, 0, c)$. Still, a complete analysis of (a, b, c) for any value of b could give more insight into the original problem, and could be an interesting area for future work. Redefining the Empty Board F_4 Black Hole Zeckendorf Game to allow players to place in the F_2 column would also be interesting, as the strategy of move mirroring is no longer as applicable.

Note also that the strategy of reducing modulo F_m is not as immediately successful when the black hole is at F_m such that $m \geq 5$. See below an example of a game with a black hole on $F_5 = 8$.



In the typical Zeckendorf game with $n > 8$, Player 1 is forced to place $(n - 8, 0, 1, 1)$. Here, Player 1 can now also place $(n - 9, 2, 0, 1)$, making it challenging for Player 2 to place a piece in the black hole without giving Player 1 the opportunity to do so first. Expanding to black holes on higher Fibonacci numbers may very well prove a fruitful step in determining a constructive solution to the original Zeckendorf game.

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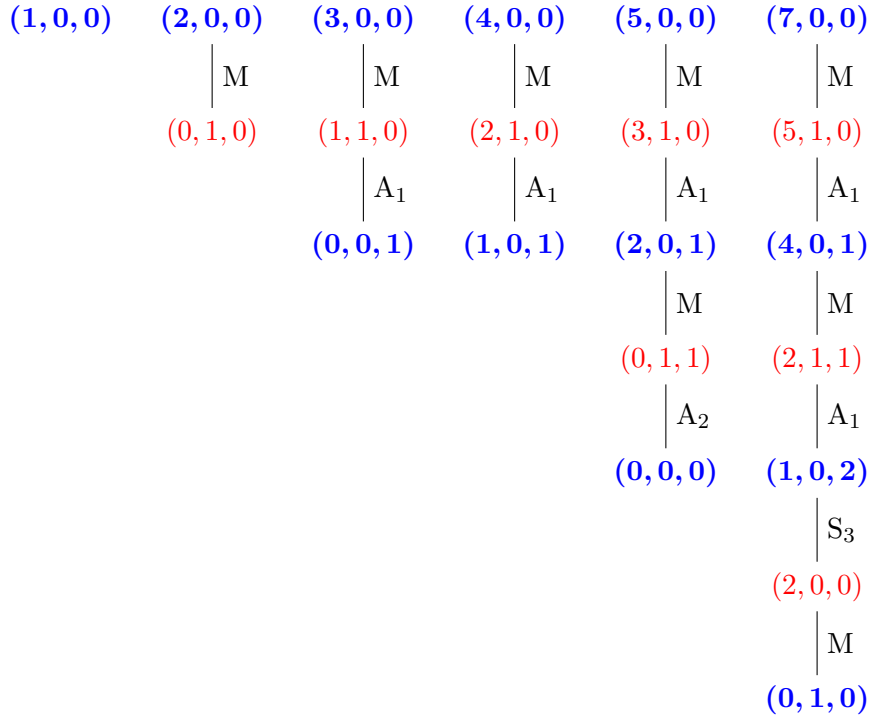
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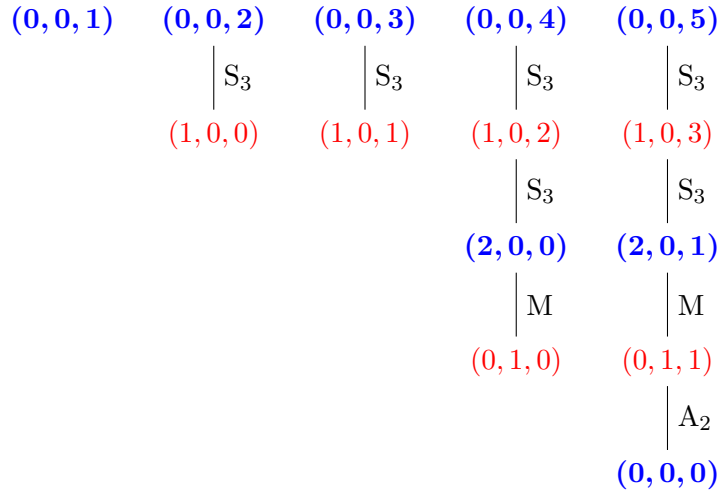
Appendix A. Base Case Games for $(a, 0, 0)$

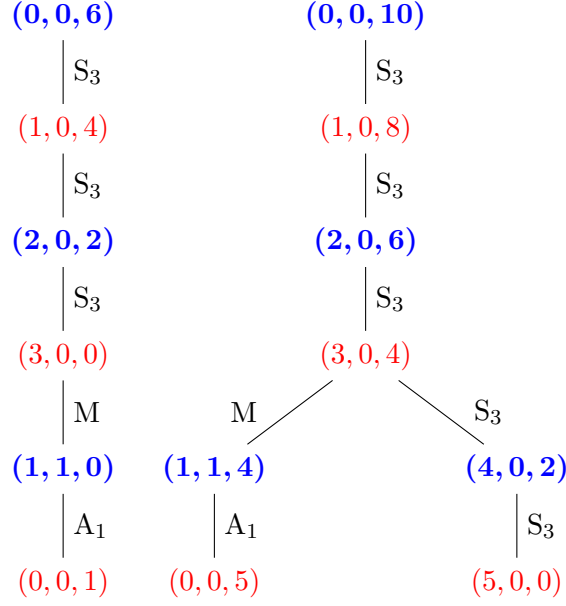
For the base cases of Theorem 5.1, we have the following games, showing that Player 2 wins for $a = 1, 3, 4, 5, 7$ and Player 1 wins for $(2, 0, 0)$.



Appendix B. Base Case Games for $(0, 0, c)$

For the base cases of Theorem 5.3, we have the following games, showing that Player 1 wins for $c = 2, 3, 4, 6, 10$ and Player 2 wins for $c = 1, 5$.

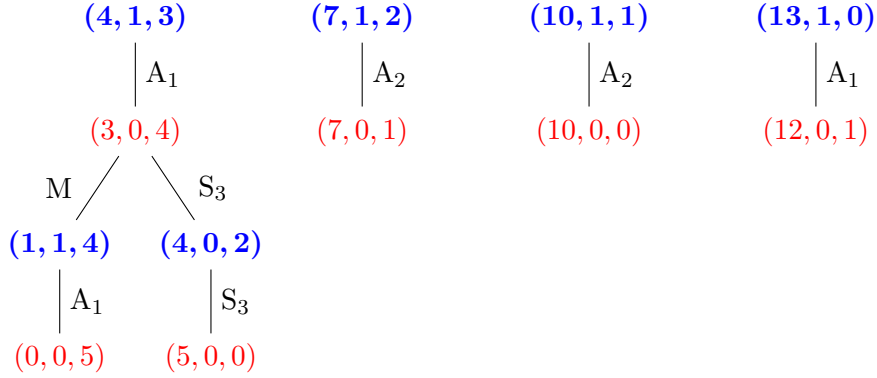




Appendix C. Base Case Games for Lemma 5.5:

We showed in Lemma 5.5 that Player 1 wins $(3\alpha + 1, 1, 4\gamma + k_3)$ for all $4\gamma + \alpha + k_3 - 1 \neq 3$ and wins $(3\alpha + 2, 1, 4\gamma + k_3)$ for all $4\gamma + \alpha + k_3 + 1 \neq 3$. We show here that Player 1 also wins for these exception cases. For both, $\gamma = 0$ since $\alpha, \gamma, k_3 \in \mathbb{Z}^{\geq 0}$.

First, consider $(3\alpha + 1, 1, 4\gamma + k_3)$, which Player 1 must also win for $\alpha + k_3 = 4$. The possible solutions (α, k_3) such that $\alpha, k_3 \in \mathbb{Z}^{\geq 0}$ and $0 \leq k_3 \leq 3$ are $(1, 3)$, $(2, 2)$, $(3, 1)$, and $(4, 0)$. We show that Player 1 has a winning move in the corresponding games $(4, 1, 3)$, $(7, 1, 2)$, $(10, 1, 1)$, and $(12, 0, 1)$.



$(5, 0, 0)$ and $(10, 0, 0)$ win by Theorem 5.1. $(0, 0, 5)$ wins by Theorem 5.3. $(7, 0, 1)$ and $(12, 0, 1)$ win by Corollary 5.2. Therefore, Player 1 has a winning strategy for all $(3\alpha + 2, 1, 4\gamma + k_3)$.

Next consider $(3\alpha + 2, 1, 4\gamma + k_3)$ which must win for all $\alpha + k_3 = 2$ when $\alpha, k_3 \in \mathbb{Z}^{\geq 0}$. The possible solutions (α, k_3) to $\alpha + k_3 = 2$ such that $\alpha, k_3 \in \mathbb{Z}^{\geq 0}$ are $(0, 2)$, $(1, 1)$ and $(2, 0)$. We show that Player 1 has a winning move in the corresponding games $(2, 1, 2)$, $(5, 1, 1)$, and $(8, 1, 0)$

$$\begin{array}{ccc}
(2, 1, 2) & (5, 1, 1) & (8, 1, 0) \\
\left| A_2 \right. & \left| A_2 \right. & \left| A_1 \right. \\
(2, 0, 1) & (5, 0, 0) & (7, 0, 1)
\end{array}$$

$(2, 0, 1)$ and $(7, 0, 1)$ win by Corollary 5.2 and $(5, 0, 0)$ wins by Theorem 5.1. Therefore, Player 1 has a winning strategy for all $(3\alpha + 2, 1, 4\gamma + k_3)$.

Appendix D. Code

In this project, we modified code by Paul Baird-Smith for [?] to function for the Black Hole Zeckendorf game. Our code can be found here: while Baird-Smith's code is available at <https://github.com/paulbsmith1996/ZeckendorfGame/>.