CONSTRUCTIVE WINNING STRATEGIES FOR SOME BLACK HOLE ZECKENDORF GAMES

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1. The Zeckendorf Game

The beauty of the Fibonacci numbers is undeniable: a simple sequence, recursively defined by the sum of the two previous numbers, that has the tendency to show up in both natural and surprising places. Indexing so that $F_1 = 1$, $F_2 = 2$ and $F_{k+1} = F_k + F_{k-1}$, Zeckendorf proved a particularly interesting fact about the Fibonacci numbers, namely that any positive integer n can be written as the sum of non-adjacent Fibonacci numbers, known as the number's Zeckendorf decomposition [Ze]. Baird-Smith, Epstein, Flint, and Miller [BEFM1, BEFM2] created a game from the process of converting a positive integer into its Zeckendorf decomposition using the moves of $F_i + F_{i-1} = F_{i+1}$ and $2F_i = F_{i+1} + F_{i-2}$, where F_i is the i^{th} Fibonacci number. We outline the rules to the original Zeckendorf game as follows.

(1) **Setup:** The game is played on a board with columns corresponding to each of the Fibonacci numbers, indexing so that the 1^{st} column corresponds with $F_1 = 1$, the 2^{nd} column corresponds with $F_2 = 2$ and the m^{th} column corresponds with F_m, the m^{th} Fibonacci number. All n pieces begin in the 1^{st} column.

(2) **Gameplay:** Players alternate, selecting their moves from the following.
   (a) Adding consecutive terms: If the board contains pieces in both F_i and F_{i-1} columns, players can remove one piece from each column to add as one piece in the F_{i+1} column.
   (b) Merging 1's: If the board contains more than one piece in the F_1 column, players can remove two pieces from the F_1 column to merge as one piece in the F_2 column.
   (c) Splitting: If the board contains more than one piece in the F_2 column, players can split two pieces from the F_2 column to place one piece in each F_1 and F_3. For $i \geq 3$, players can split two pieces in the F_i column to place one in each F_{i-2} and F_{i+1}.

(3) **Winning:** The last player to move wins.

They proved that the game is playable, meaning it always ends in finite time, and that the final board placed down will be equal to the Zeckendorf decomposition of n. Moreover, they showed that for all $n \neq 2$, Player 2 has a winning strategy. Notably, this is not a constructive winning strategy, and instead relies on a parity stealing argument. If one assumes that Player 1 has a winning strategy, Player 2 later has the opportunity to steal it, therefore Player 2 must have a winning strategy. With increasing n, the number of possible game positions grows exponentially, making the construction of a winning solution for Player 2 challenging. Even for n as small as 14, as in Figure 1, it is not obvious how Player 2 wins without use of brute force.

Date: March 25, 2025.

Key words and phrases. Zeckendorf decompositions, Fibonacci numbers, game theory, recreational mathematics.

The authors were supported by Williams College, The Finnerty Fund, The College of William & Mary Charles Center, and NSF Grant DMS2241623, and thank the organizations for their generous support. Special thanks to Paul Baird-Smith, whose code we edited to play through the Zeckendorf Black Hole game.

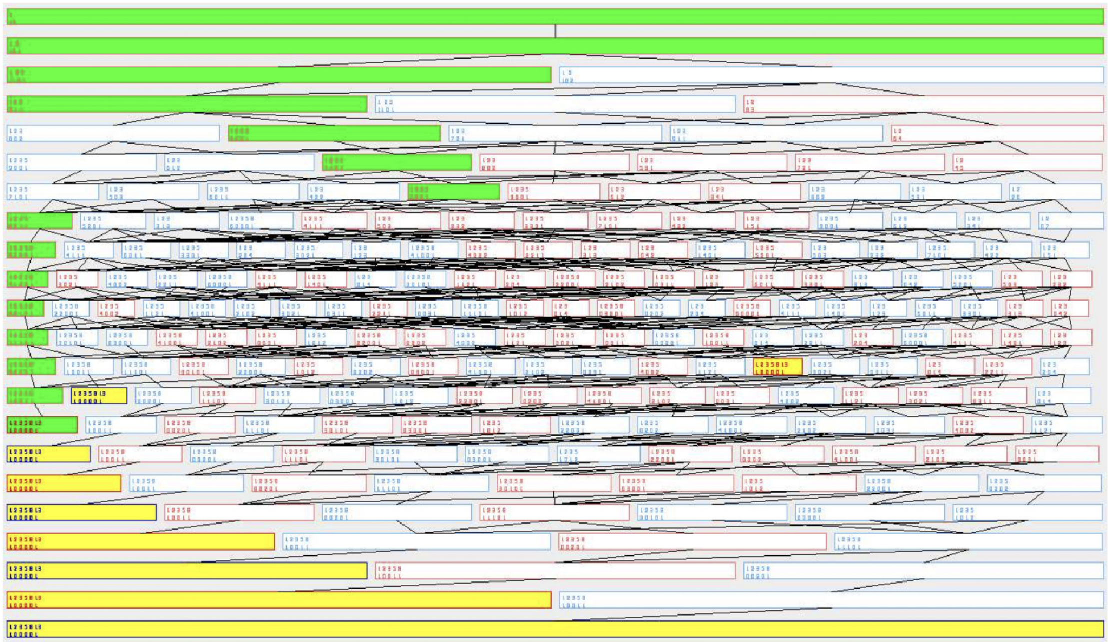


FIGURE 1. Game tree for $n = 14$, with a winning path in green, as included in [BEFM1].

As to date, all attempts at determining a constructive winning strategy for Player 2 have been unsuccessful, numerous projects have explored questions about the game (such as the average length, the longest and shortest games, random games) as well as generalizations; see [BJMNSYY, BCDD+15, CMJDMMN, CDHK+6i, CDHK+6ii, GMRVY, LLMMSXZ, MSY]. Below we report on another new variant explored by the authors (see [CMSS]), where in some situations we are able to constructively provide winning strategies. We describe the new game in §2, discuss winning strategies in §3, and end with some problems for further study in §4.

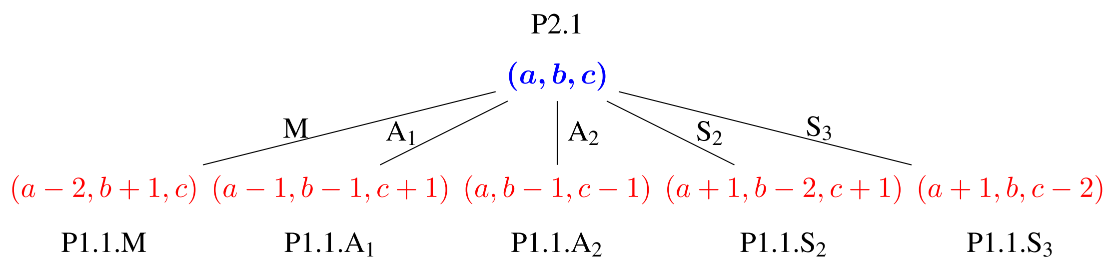
2. THE BLACK HOLE ZECKENDORF GAME

In an attempt to better understand the original Zeckendorf game, and hopefully gain some insight towards constructing a winning strategy for Player 2, in [CMSS] we consider a variation of the game occurring on a smaller board. We describe this new game and state some of our results below, and give a quick discussion of the key ideas in the proofs.

We call this the "F_m Black Hole" variation of the Zeckendorf game, where once a piece is placed on some F_i for $i \geq m$, it falls into the "Zeckendorf Black Hole" and is permanently removed from game play. This variant greatly reduces the number of possible moves a player has, making the game easier to analyze. Here, a game played with n pieces ends in the Zeckendorf decomposition of $n \pmod{F_m}$. As we play through games, we let bold blue denote Player 2 and red denote Player 1. We focus on the case with a black hole on $F_4 = 5$, which allows for the following moves (in the tree below, moves with an "M" are merge moves, with an "A" are adding moves, and an "S" are splitting moves):

	$a \equiv 0 \pmod 3$	$a \equiv 1 \pmod 3$	$a \equiv 2 \pmod 3$
$c \equiv 0 \pmod 4$	$\boldsymbol{\alpha \geq \gamma}$ $\alpha \leq \gamma - 1$	$\boldsymbol{\forall \alpha, \gamma}$	$\boldsymbol{\alpha \geq \gamma + 1}$ $\alpha \leq \gamma$
$c \equiv 1 \pmod 4$	$\boldsymbol{\alpha \geq \gamma - 1}$ $\alpha \leq \gamma - 2$	$\boldsymbol{\forall \alpha, \gamma}$	$\boldsymbol{\alpha \geq \gamma}$ $\alpha \leq \gamma - 1$
$c \equiv 2 \pmod 4$	$\forall \alpha, \gamma$	$\alpha \geq \gamma + 1$ $\boldsymbol{\alpha \leq \gamma}$	$\forall \alpha, \gamma$
$c \equiv 3 \pmod 4$	$\forall \alpha, \gamma$	$\alpha \geq \gamma$ $\boldsymbol{\alpha \leq \gamma - 1}$	$\forall \alpha, \gamma$

FIGURE 2. Winners for board setups $(a, 0, c) = (3\alpha + k_1, 0, 4\gamma + k_3)$ with $\alpha, \gamma, k_1, k_3 \in \mathbb{Z}^{\geq 0}, 0 \leq k_1 \leq 2, 0 \leq k_3 \leq 3$ in an F_4 Black Hole Zeckendorf Game. Player 2 wins are depicted in bold blue, and Player 1 wins are depicted in red.



We also consider an empty board phase of the game in which players take turns placing pieces in the outermost columns, with the last player to place assuming the role of Player 2 during the decomposition phase of the game. Defining this phase of the game as such allows players to use move mirroring as a strategy. In order to force an advantageous set-up, players can place pieces in the column opposite their opponent. Combining these two variations of the original game, we find the following main result.

Theorem 2.1. *Player 2 has a constructive strategy for winning an Empty Board F_4 Black Hole Zeckendorf game for any $n \equiv 0, 2, 4, 6, 9, 11, 13 \pmod{16}$ such that $n \neq 2, 32$ in which case Player 1 has the winning strategy. Player 1 has a constructive strategy for winning an Empty Board F_4 Black Hole Zeckendorf game for any $n \equiv 1, 3, 5, 7, 8, 10, 12, 14, 15 \pmod{16}$, such that $n \neq 17, 47$, in which case Player 2 has the winning strategy.*

To provide a constructive solution, we determine winning positions for a board setup $(a, 0, c)$ as outlined in Figure 2, and then provide a path from winning position to winning position. Knowing which player wins for a given setup allows us to determine which player move mirroring benefits, giving us our winners as stated in Theorem 2.1. However, it is not immediately clear that the positions in the table are in fact winning positions. In order to show that they are, we show that certain players win key intermediate positions using nonconstructive methods. See, for example Lemma 2.3, which uses the following corollary.

Corollary 2.2. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. Player 2 has a winning strategy at $(1, 0, c)$ for all $c \neq 3 \in \mathbb{Z}^{\geq 0}$ and at $(0, 1, c)$ for all $c \neq 1, 2, 6 \in \mathbb{Z}^{\geq 0}$. Player 1 has a winning strategy at $(2, 0, c)$ for all $c \neq 1 \in \mathbb{Z}^{\geq 0}$ and at $(2, 1, c)$ for all $c \in \mathbb{Z}^{\geq 0}$.*

Lemma 2.3. *Let (a, b, c) be a game state for an F_4 Black Hole Zeckendorf game. For all $\alpha, \gamma, k_1, k_3 \in \mathbb{Z}^{\geq 0}$, such that $1 \leq k_1 \leq 2$, and $0 \leq k_3 \leq 3$, Player 1 has a winning strategy for $(3\alpha + k_1, 1, 4\gamma + k_3)$.*

Proof. We proceed by a non-constructive proof. For contradiction's sake, suppose Player 2 wins $(3\alpha + k_1, 1, 4\gamma + k_3)$ when $k_1 = 1, 2$. Then, consider the game tree in Figure 3, where r is the number of rounds in which Player 1 then the Player 2 plays. We note that Player 1 has other possible moves, but we only consider moves relevant to the proof.

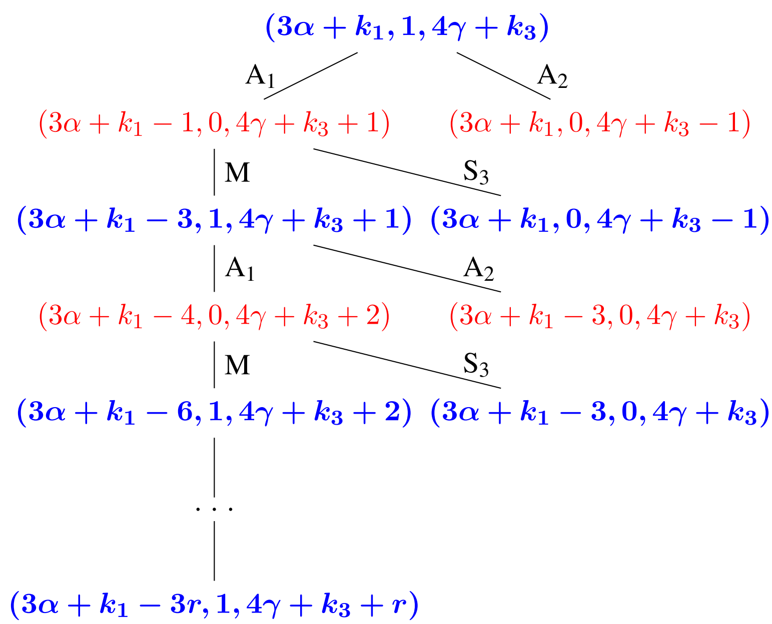


FIGURE 3. Game Tree for the Game State $(3\alpha + k_1, 1, 4\alpha + k_3)$.

By assumption, Player 2 has a winning strategy regardless of what the other player places. Suppose Player 1 places $(3\alpha + k_1 - 1, 0, 4\gamma + 1)$ for their first move. Then, Player 2 can place either $(3\alpha + k_1 - 3, 1, 4\gamma + 1)$ or $(3\alpha + k_1, 0, 4\gamma - 1)$. But as shown in the tree, Player 1 had the opportunity to place $(3\alpha + k_1, 0, 4\gamma - 1)$ in the round before; so by assumption, placing it is a losing move. It follows that in order to win, Player 2 must place $(3\alpha + k_1 - 3, 1, 4\gamma + 1)$. As shown in the tree, Player 1 has the same options as before, so if they place $(3\alpha + k_1 - 4, 0, 4\gamma + k_3 + 2)$, then by assumption, Player 2 must place $(3\alpha + k_1 - 6, 1, 4\gamma + k_3 + 2)$.

Then, the game eventually reduces down to Player 2 placing $(3\alpha + k_1 - 3r, 1, 4\gamma + k_3 + r)$ after r rounds of Player 1 then Player 2 placing. After the α^{th} round, Player 2 will place $(k_1, 1, 4\gamma + k_3 + \alpha)$. If $k_1 = 1$, then Player 1 can add from columns F_2 and F_3 to place $(1, 0, 4\gamma + \alpha + k_3 - 1)$ which wins by Corollary 2.2 for all $4\gamma + \alpha + k_3 - 1 \neq 3$. If $k_1 = 2$, then Player 1 can add from columns F_1 and F_2 to place $(1, 0, 4\gamma + \alpha + k_3 + 1)$ which wins by Corollary 2.2 for all $4\gamma + \alpha + k_3 + 1 \neq 3$. Additionally, we show in the appendix in [CMSS] that Player 1 also has a winning strategy for the cases when $4\gamma + \alpha + k_3 \pm 1 = 3$. This is a contradiction to the assumption that Player 2 has a winning strategy, so then Player 1 must have some winning strategy when $(3\alpha + k_1, 1, 4\gamma + k_3)$ is placed for $k_1 = 1, 2$. □

Through similar proofs that show which players win intermediate positions, we are able to show that Figure 2 accurately describes which player wins a given position. From here, we provide a path from winning position to position, thereby providing a constructive solution to the game.

3. EXAMPLE WINNING STRATEGY

Let us reconsider the case when $n = 14$, but this time when playing the Empty Board Black Hole Zeckendorf Game with a Black Hole on F_4. If Player 1 places their first piece in the first column, and then mirrors Player 2's moves, then they can force Player 2 to set the board down as $(5, 0, 3)$. Player 1 can then use the strategy below to limit Player 2's moves, therefore leading to a Player 1 win.

$$\mathbf{(5,0,3)}$$
$$\big|\ \mathrm{S}_3$$
$$(6,0,1)$$
$$\big|\ \mathrm{M}$$
$$\mathbf{(4,1,1)}$$
$$\big|\ \mathrm{A}_2$$
$$(4,0,0)$$
$$\big|\ \mathrm{M}$$
$$\mathbf{(2,1,0)}$$
$$\big|\ \mathrm{A}_1$$
$$(1,0,1)$$

4. Future Work

For our definition of the empty board game, it was only necessary to analyze which player won for board setups $(a, 0, c)$. Allowing players to place pieces in the center column during the empty board phase could be an interesting area for future work, as the strategy of move mirroring is no longer as applicable and there are more possible setups for the decomposition phase of the game to analyze.

Note also that for any $m \geq 5$, the strategy of reducing the game modulo F_m is not as immediately successful, as it is challenging for either player to place a piece in the black hole without first giving another player the opportunity to do so.

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