BOUNDS ON ZECKENDORF GAMES

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Abstract. Zeckendorf proved that every positive integer n can be written uniquely as the sum of non-adjacent Fibonacci numbers. We use this decomposition to construct a two-player game. Given a fixed integer n and an initial decomposition of $n = nF_1$, the two players alternate by using moves related to the recurrence relation $F_{n+1} = F_n + F_{n-1}$, and whoever moves last wins. The game always terminates in the Zeckendorf decomposition; depending on the choice of moves the length of the game and the winner can vary, though for $n \geq 2$ there is a non-constructive proof that Player 2 has a winning strategy.

Initially the lower bound of the length of a game was order n (and known to be sharp) while the upper bound was of size $n \log n$. Recent work decreased the upper bound to of size n, but with a larger constant than was conjectured. We improve the upper bound and obtain the sharp bound of $\frac{\sqrt{5}+3}{2}\ n - IZ(n) - \frac{1+\sqrt{5}}{2} Z(n)$, which is of order n as $Z(n)$ is the number of terms in the Zeckendorf decomposition of n and $IZ(n)$ is the sum of indices in the Zeckendorf decomposition of n (which are at most of sizes $\log n$ and $\log^2 n$ respectively). We also introduce a greedy algorithm that realizes the upper bound, and show that the longest game on any n is achieved by applying splitting moves whenever possible.

Contents

1. Introduction

The Fibonacci numbers are one the most interesting and famous sequences. Among their fascinating properties, the Fibonacci numbers lend themselves to a beautiful theorem by Edouard Zeckendorf [Ze] which states that each positive integer n can be written uniquely as the sum of distinct, non-consecutive Fibonacci numbers. This sum is called the *Zeckendorf decomposition* of n and requires that we define the Fibonacci numbers by $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5...$ instead of the usual $1, 1, 2, 3, 5...$ to create uniqueness. Baird-Smith, Epstein, Flint and Miller [BEFM1, BEFM2] create a game based on the Zeckendorf decomposition. We quote from [BEFM2] to describe the game.

We introduce some notation. By $\{1^n\}$ or $\{{F_1}^n\}$ we mean n copies of 1, the first Fibonacci number. If we have 3 copies of F_1, 2 copies of F_2, and 7 copies of F_4, we write either $\{{F_1}^3 \wedge {F_2}^2 \wedge {F_4}^7\}$ or $\{1^3 \wedge 2^2 \wedge 5^7\}$.

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Definition 1.1 (The Two Player Zeckendorf Game)**.** *At the beginning of the game, there is an unordered list of n 1's. Let $F_1 = 1, F_2 = 2$, and $F_{i+1} = F_i + F_{i-1}$; therefore the initial list is $\{F_1{}^n\}$. On each turn, a player can do one of the following moves.*

(1) *If the list contains two consecutive Fibonacci numbers, F_{i-1}, F_i, then a player can change these to F_{i+1}. We denote this move $\{F_{i-1} \wedge F_i \to F_{i+1}\}$.*
(2) *If the list has two of the same Fibonacci number, F_i, F_i, then*
   (a) *if $i = 1$, a player can change F_1, F_1 to F_2, denoted by $\{F_1 \wedge F_1 \to F_2\}$,*
   (b) *if $i = 2$, a player can change F_2, F_2 to F_1, F_3, denoted by $\{F_2 \wedge F_2 \to F_1 \wedge F_3\}$, and*
   (c) *if $i \geq 3$, a player can change F_i, F_i to F_{i-2}, F_{i+1}, denoted by $\{F_i \wedge F_i \to F_{i-2} \wedge F_{i+1}\}$.*

The players alternate moving. The game ends when one player moves to create the Zeckendorf decomposition.

The moves of the game are derived from the Fibonacci recurrence, either combining terms to make the next in the sequence or splitting terms with multiple copies. A proof that this game is well defined and ends at the Zeckendorf decomposition can be found in [BEFM2].

We introduce some further notation and state some simple results.

- Let $i_{\max}(n)$ be the largest index of terms in the Zeckendorf decomposition of n. The order of $i_{\max}(n)$ is at most $\log n$; this follows immediately from the exponential growth of the Fibonacci numbers, as we can never use a summand larger than the original number n.

- Let δ_i denote the number of F_i's in the Zeckendorf decomposition of n. Then $n = \sum_{i=1}^{i_{\max}(n)} \delta_i F_i$.

- Let $Z(n)$ denote the number of terms in the Zeckendorf decomposition of n, and $Z(n) = \sum_{i=1}^{i_{\max}(n)} \delta_i$. The order of $Z(n)$ is at most $\log n$ since $Z(n) \leq i_{\max}(n)$.

- Let $IZ(n)$ denote the sum of indices in the Zeckendorf decomposition of n, and $IZ(n) = \sum_{i=1}^{i_{\max}(n)} i\, \delta_i$. The order of $IZ(n)$ is at most $\log^2 n$; this follows trivially from summing the indices and recalling the largest index used is of order $\log n$.

- The original upper bound for the game was of order $n \log n$, and the lower bound was found to be sharp at $n - Z(n)$ in [BEFM2]. The upper bound on the number of moves was improved to $3n - 3Z(n) - IZ(n) + 1$ in [LLMMSXZ]. Since the order of $Z(n)$ and $IZ(n)$ are both less than n, we observe that the upper and lower bounds are both of order n.

- Finally, several deterministic games have been introduced in [LLMMSXZ]. These are defined in terms of the priority of moves; that is, each move in a strategy will follow whichever move is available and comes first in the ordering of moves.
  - Combine Largest: adding consecutive indices from largest to smallest, adding 1's, splitting from largest to smallest.
  - Split Largest: splitting from largest to smallest, adding consecutive indices from largest to smallest, adding 1's.

– Split Smallest: splitting from smallest to largest, adding 1's, adding consecutive indices from smallest to largest.

It was shown in the same paper that the Combine Largest and Split Largest games both realize the shortest game.

Since the lower bound of the game has been shown to be sharp, we focus on the upper bound of the game. One of our main result is a proof of a conjecture from [BEFM2] that the longest game on any n is achieved by applying splitting moves whenever possible.

Theorem 1.2. *The longest game on any n is achieved by applying split moves or combine 1's (in any order) whenever possible, and, if there is no split or combine 1 move available, combine consecutive indices from smallest to largest.*

This algorithm is not deterministic. Thus, there are many game paths that follow this algorithm. For instance, it can be easily shown that the Split Smallest game described in [LLMMSXZ] is a deterministic example of this algorithm, and therefore realizes the longest game.

Now that we have an algorithm that achieves the longest game, we are interested in the upper bound of the game length. The previous upper bound was already very close to the known lower bound (both of order n). Nevertheless, we are able to further close the gap.

Theorem 1.3. *Let $a_i = F_{i+2} - i - 2$ ($i > 0$, $i \in \mathbb{N}$). The upper bound of the game is given by $\sum_{i=1}^{i_{\max}(n)} a_i \, \delta_i$ which is at most $\frac{\sqrt{5}+3}{2}\, n \; - \; IZ(n) \; - \; \frac{1+\sqrt{5}}{2} Z(n)$.*

It was originally conjectured in [BEFM2] that the number of moves in the Split Smallest game grows linearly with n, with a constant of the golden mean squared, which is equivalent to $\frac{\sqrt{5}+3}{2}$. We observe that this conjecture has been shown here, since the order of $Z(n)$ and $IZ(n)$ are less than n.

Though this bound is very close to the actual longest game, it is not a strict upper bound for most n. In fact, we observed during the construction of this upper bound that this bound is sharp if and only if the game on n can be played with only split and combine 1 moves, and identified all such n.

Theorem 1.4. *A game can be played with only splitting and combine 1 moves, if and only if $n \; = \; F_k - 1$ ($k \geq 2$).*

In Section 2 we prove Theorem 1.2. Then in Section 3 we prove Theorems 1.3 and 1.4. Finally in Section 4 we give some possible directions for future research.

2. Strategy to achieve the longest game

We start by introducing some notation that we use in our proofs.

Following the notation introduced in [LLMMSXZ], we let MC_i denote the number of combining moves at the index i with $i \geq 2$, with MC_1 the number of combine 1 moves. Similarly the number of splitting moves at i is denoted MS_i for $i \geq 2$. We refer to combining moves at i by C_i, and splitting moves at i by S_i.

The move of adding 1's is usually considered a combining move as the case in [BEFM1, BEFM2, LLMMSXZ], but for the sake of our proofs, we consider combine 1 (C_1) to be a splitting move, and also refer to it as S_1 in this section.

We begin with the proof of Theorem 1.2, which is a greedy algorithm that achieves the longest game path, starting with two lemmas.

As a reminder, the algorithm requires moves to be in the following order: choose any split or combine 1 move whenever possible, then combine consecutive terms with smallest indices.

Lemma 2.1. *If the aforementioned strategy gives us choice of moves at some game state G, then starting from this game state, no matter which move we choose at this step, there exists a path that follows our strategy and has the same length as our initial path.*

Proof. In paths following our strategy, the only game states that allow choice of moves are game states with at least two splitting (including S_1) moves available.

Let P be a game path starting from any game state that follows our strategy, and let G be any game state that P visits that allows a choice. Let P' be another path that starts from the same game state as P and follows our strategy such that P' follows the same moves as P until they differ in choice of move for the first time at G.

Let P choose S_i and P' choose S_j at G. We want to construct the steps for P' such that it has same length as P.

Since P' could have chosen S_i, we must have at least two F_i at G. Since S_j does not decrease the number of F_i, we still have at least two F_i after this step. Thus for the next step, P' can still choose S_i. By similar reasons, P' can always imitate the moves that P takes after S_i until P takes a S_j. We know that P must take S_j at some step because we had at least two F_j at G, and the only moves that decrease the number of F_j's are S_j, C_j, and C_{j+1}. Since our strategy prioritizes split moves over combine moves, we must take S_j at some point in P.

Thus the moves in P' after G are as follows: perform S_j and S_i in the first two steps, and then imitate the moves of P after S_i until P takes S_j. Since P follows our strategy of prioritizing split moves, P' does also. In this way, P and P' take the same set of steps but in different order, so they reach the same game state with the same number of steps. After that, P and P' follow exactly same steps until game terminates.

Thus, we prove that no matter which move we choose at some game state G, there exists a path that follows our strategy and has same length as our initial path. □

Lemma 2.2. *Starting from any game state, all paths that follow our strategy have the same length.*

Proof. We show that an arbitrary game path P that follows our strategy is no longer or shorter than any other path that follow our strategy, given that they start from same game state.

Suppose for the sake of contradiction that P and P' both follow our strategy but differ in length. Then P and P' must differ by at least 1 move. Let G be the first game state where P and P' differ in choice of moves. By Lemma 2.1, there exists a path P_1 that chooses the same move as P' at G, has the same length as P, and follows our strategy.

Since P_1 and P' differ in length, they must differ by at least one move. Thus there exists a game state G_1 after G where P_1 and P' differ in their choice for the first time. Again, by Lemma 2.1, there exists a path P_2 that chooses the same move as P' at G_1, has the same length as P_1 (which is equal to length of P), and follows our strategy.

Since the number of steps in any game path is finite, we can repeat this process until we find a P_k with the same length as P, but there no longer exists G_k where P' and P_k can differ in choice of moves. Thus, the rest of steps are deterministic, which means P_k and P' must be exactly the same path, and therefore both have the same length as P, a contradiction.

In conclusion, all paths that start from same game state and follow our strategy have same length. □

Lemma 2.3. *Starting from any game state, if a path does not follow our strategy, then this game path is either not the longest path or there exists a path that has the same length as this path and follows our strategy.*

Proof. Suppose a game path P contains at least one step that is not chosen by our strategy. We want to either find a path P^* that is longer than P, or construct a P^* that follows our strategy and is as long as P.

We look at the last step in P that is not chosen by our strategy. Suppose the step is taken at game state G. Since we consider the last step that does not follow our strategy, all moves after this step must follow our strategy.

There are two situations when a step is not chosen by our strategy: either the combining move taken is not the one with smallest index when no splitting move is available, or a combining move is taken when there is splitting move available. Note that in splitting move we include combine 1's moves, and in combining move we exclude combine 1's. We look at these two cases. In both cases we want to find a path P' that is either longer than P or follows our strategy at and after G and have same length as P.

First, suppose P takes a combining move that is not the smallest when no splitting move is available at G. In this case, there is at most one F_k for any k. Let C_i $(i \geq 2)$ be the smallest combining move at G and let C_j $(j > i)$ be the combining move chosen by P at G. We study the following sub-cases based on whether G contains a F_{i-2} term.

- *Case 1.1.* At G, there is at least one F_{j-2}.

  In this case, we find a path P' that is longer than P. Let P' take the same moves as P before reaching G. At G, path P takes C_j at G and reaches G'. Compared to G, G' contains one less F_{j-1}, one less F_j, and one more F_{j+1}. Let path P' take C_{j-1} and S_j. Then the game state it reaches also contains one less F_{j-1}, one less F_j, and one more F_{j+1} compared to G. Thus P' also reaches G' with one more step than P. After that, P' can imitate the moves P takes until game terminates. In the end, P' is one move longer than P.

- *Case 1.2.* At G, there are no F_{j-2}.

  Again, let P' follow same steps as P and reach the game state G. At G, let P take C_j and P' take C_i. After that, P follows our strategy.

  First, we look at the steps in P. C_j increases number of F_{j+1} by one, so if there is one F_{j+1} at G, we apply S_{j+1}. This is the only possible splitting move at this game state since we assumed there are no F_{j-2}. Similarly, S_{j+1} increases F_{j+2} by one, so if there is one F_{j+2} at G, we apply S_{j+2}. Since we have used up all F_{j-1}'s with C_j, there are no F_{j-1}'s before taking S_{j+1}, so we cannot apply S_{j-1}. Thus S_{j+2} should be the only possible splitting move at this game state.

  We repeat the process of taking the only splitting move until we have to do a combining move. Note that the number of such splitting moves that can be taken by P after taking C_j and before taking another combining move depends on the number of consecutive F_k's $(k > j)$ starting from F_{j+1}. Let $\alpha \geq 0$ be the number of such moves. The move taken in P after these α steps is C_i since there is no splitting move available and C_i is the smallest combining move.

  Now P has taken $1 + \alpha + 1$ steps and P' has taken 1 step. The current game state for P and P' has the same number of F_k's for all $k \leq j - 3$ since S_j and the α splitting moves in P do not affect the number of F_k's $(k \leq j - 3)$, and P and P' both take a C_i move. Thus, they can follow the same steps until either we get a F_{j-2} before we have to take C_j in P' or we never get a F_{j-2} and the next step in P' is C_j. Suppose we took β steps before we stop. We look at these two cases.

- *Case 1.2.1.* We get a F_{j-2} before we have to take C_j in P'.
  This case is similar to Case 1.1 where we find a path longer than P. In P' we take C_{j-1}, S_j, and then follow the α steps described in P. After that, P and P' reach the same game state. Notice that after G, path P took $1+\alpha+1+\beta$ steps, and P' took $1+\beta+2+\alpha$ steps. Thus, by the end of the game, P' is one move longer than P.
- *Case 1.2.2.* We never get a F_{j-2} before the next step in P' is C_j.
  In this case P' takes the C_j move and then follows the α steps described in P. Then P and P' reach the same game state, and share the same steps after that. Notice that the sub-paths of P and P' after G follow our strategy, and P' has the same length as P.

Second, suppose P chooses a combining move when there is a splitting move available. Let C_i ($i \geq 2$) be the combining move that P chooses. Then there must be at least one F_{i-1} and one F_i at G. We consider the following cases based on the number of F_{i-1}'s and F_i's.

- *Case 2.1.* There are more than one F_{i-1}'s at G (i.e., P' can take the move S_{i-1}).
  - *Case 2.1.1.* $i = 2$ (i.e., path P takes C_2).
    At G, path P takes C_2, and path P' takes S_1 and S_2 to reach the same game state. After that P' imitates the steps P takes. In the end, P' is one move longer than P.
  - *Case 2.1.2.* $i = 3$ (i.e., path P takes C_3).
    At G, path P takes C_3, and path P' takes S_2, S_3, S_1 to reach the same game state. After that P' imitates the steps P takes. In the end, P' is two moves longer than P.
  - *Case 2.1.3.* $i > 3$.
    At G, path P takes C_i, and path P' takes S_{i-1}, S_i, C_{i-2} to reach the same game state. After that P' imitates the steps P takes. In the end, P' is two moves longer than P.
- *Case 2.2.* There is exactly one F_{i-1} and more than one F_i at G (i.e., P' can take S_i).
  - *Case 2.2.1.* $i = 2$ (i.e., path P takes C_2).
    At G, path P takes C_2, and path P' takes S_2 and S_1 to reach the same game state. After that P' imitates the steps P takes. In the end, P' is one move longer than P.
  - *Case 2.2.2.* $i > 2$.
    At G, path P takes C_i, and path P' takes S_i and C_{i-1} to reach the same game state. After that P' imitates the steps P takes. In the end, P' is one move longer than P.
- *Case 2.3.* There is exactly one F_{i-1} and one F_i at G.
  Let G' be the game state P reaches after taking C_i at G. Since G is the last game state where our strategy is violated, all steps in P after G follows our strategy. We proved in Lemma 2.2 that starting from any game state, any game path that follows our strategy has same length. Thus, there exists a path that starts from G', performs S_{i+1} only when no other splitting move is available, and has the same length as the

sub-path of P starting from G'. We extend this path so that it starts from the same game state as P and follows the same steps as P until G'. Call the extended path P''.

Note that the only differences between G and G' are that G' has one more F_{i+1} term, one less F_i term and one less F_{i-1} term. If we let P' follow the same moves after G that P does after G', then the game states in P' and P always differ in only these three terms. Thus, P' may be unable to imitate P if P performs S_{i+1}.

We avoid this problem to the maximum extent by letting P' follow the same steps in P'' (instead of P) until either P'' is forced to take S_{i+1} and P' cannot choose the same step, or P' performs all splitting moves before P'' takes a combining move. In both situations, let G'' be the game state reached by P'' here and consider the next step in P'. The next step is either a splitting move or a combining move (there must exist a next step since we can always take C_i). Notice that if there is a splitting move possible, it is either S_{i-1} or S_i since F_{i-1} and F_i are the only two terms which game states in P' have more of than game states in P''.

– *Case 2.3.1.* The next step in P' can be S_{i-1} or S_i.
  Suppose the next step taken in P' is C_i. If the step in P'' before reaching G'' is S_{i+1} and P' failed to follow this move, P' takes S_{i+1} after the C_i move. Then P' reaches G'' with same number of steps as P''. However, according to *Case 2.1* and *Case 2.2*, we can find a path that is longer than P'. Thus there exist a path longer than P.

– *Case 2.3.2.* The next step in P can only be a combining move.
  Again, suppose the next step taken in P' is C_i (and S_{i+1} if P' failed to follow the S_{i+1} move in P''). Then, it reaches G'' with same number of steps as P''. Let P' follow the same moves in P'' after G''.
  If C_i was not the smallest possible combining move P' could have taken, then by our discussion about the case where the smallest combining move is not chosen, we know that there exists either a path longer than P' or a path that has same length as P' and its sub-path (after G) follows our strategy.
  If C_i was the smallest possible combining move, then this step followed our strategy. Thus P' is a path that has the same length as P and whose sub-path (after G) follows our strategy.

In both cases, we can either find a path that is longer than P or a path whose sub-path after G follows our strategy and have same length as P.

Now we can start to construct a path P^* that is either longer than P or has same length as P and follows our strategy.

Since the game takes finitely many steps, there are finitely many game states in P that do not follow our strategy in choosing moves. We denote all such game states in P in reverse chronological order as $G_1, G_2, \ldots, G_k$, where G_1 is the last game state where a move is not chosen in accordance with our strategy and G_k is the first.

We start by looking at the path after G_1. Since our strategy is not followed at G_1, it must fall into either of the two cases discussed above. In both cases, we can either find a path that is longer than P or find a path whose sub-path after G_1 always follow our strategy and is as long as P.

If we find a path that is longer than P, then this is the P^* that we are looking for. Otherwise, we denote the path whose sub-path after G_1 follows our strategy as P_1. Notice that P and P_1 follow same moves before reaching G_1, and P_1 follows our strategy at G_1, so the last game state in P_1 where our strategy is violated is G_2.

By the same argument, we can either find a path that is longer than P_1 or find a path whose sub-path after G_2 always follow our strategy and is as long as P_1. If we find a path that is longer than P_1, then this is the P^* that we are looking for. Otherwise, we find a P_2 whose sub-path after G_2 follows our strategy and has same length as P_1.

We repeat this process until either we find a P^* that is longer than P, or we find a P_k whose sub-path after G_k follows our strategy and has the same length as P. Since G_k is the first game state in P where our strategy is not followed, P_k is a path that follows our strategy from the starting state. Thus P_k is the P^* we want.

In conclusion, starting from any game state, if there is a path P that does not follow our strategy, then it is either not the longest game or there exists a path P' that is as long as P and follows our strategy. □

Proof of Theorem 1.2. By Lemma 2.3, we proved that a path that does not follow our strategy is either not the longest game, or there exists a path that follows our strategy and is as long as the original path. By Lemma 2.2, we know that all paths that start from same game state and follow our strategy have the same length. Thus our strategy gives the longest game. □

3. Upper Bound on the Game Length

In this section, we construct and analyze the order of an upper bound on the game length.

Proof of Theorem 1.3. The first step is to construct an upper bound on the game length. To do this, we first look at changes in the amount of F_2. We start the game with no F_2, and end with δ_2 of F_2. Every time we combine two F_1's, we get a 2 ($F_1 \wedge F_1 \to F_2$) and increase the number of F_2's by one. Every time we split two F_4's, we get a 2 ($F_4 \wedge F_4 \to F_2 \wedge F_5$) and increase the number of F_2's by one. These are the only moves that increase the number of F_2's. Each combining move of F_1 and F_2 ($F_1 \wedge F_2 \to F_3$) and combining move of F_2 and F_3 ($F_2 \wedge F_3 \to F_4$) decreases the number of F_2's by one. Finally splitting two F_2's ($F_2 \wedge F_2 \to F_1 \wedge F_3$) decreases the number of F_2's by two. These are the only moves that decrease the number of F_2's.

Thus we can construct the following equation:

$$MC_1 - 2MS_2 + MS_4 - MC_2 - MC_3 = \delta_2. \tag{3.1}$$

Similarly, for each $3 \le i \le i_{\max}(n)$, we start the game with no F_i, and end with δ_i of the F_i. Every time we combine F_{i-2} and F_{i-1}, we increase the number of F_i's by one. Every time we split two F_{i-1}'s, we increase the number of F_i's by one. Every time we split two F_{i+2}'s, we increase the number of F_i's by one. These are the only moves that increase the number of F_i's. Each combining move of F_{i-1} and F_i and combining move of F_i and F_{i+1} decreases the number of F_i's by one. Finally splitting two F_i's decreases the number of F_i's by two. These are the only moves that decrease the number of F_i's.

Thus we have for $3 \le i \le i_{\max}(n)$

$$MS_{i-1} - 2MS_i + MS_{i+2} + MC_{i-1} - MC_i - MC_{i+1} \ = \ \delta_i. \tag{3.2}$$

Since $i_{\max}(n)$ is the largest index in the final decomposition, we know that for all $i \ge i_{\max}(n)$, $MS_i = MC_i = 0$. Thus for $i \ge i_{\max}(n) - 2$, we can get rid of a few terms in the equation above:

$$MS_{i_{\max}(n)-3} - 2MS_{i_{\max}(n)-2} + MC_{i_{\max}(n)-3} - MC_{i_{\max}(n)-2} - MC_{i_{\max}(n)-1} = \delta_{i_{\max}(n)-2}, \tag{3.3}$$

$$MS_{i_{\max}(n)-2} - 2MS_{i_{\max}(n)-1} + MC_{i_{\max}(n)-2} - MC_{i_{\max}(n)-1} \ = \ \delta_{i_{\max}(n)-1}, \tag{3.4}$$

$$MS_{i_{\max}(n)-1} + MC_{i_{\max}(n)-1} \ = \ \delta_{i_{\max}(n)}. \tag{3.5}$$

Now we have $i_{\max}(n)-1$ linear equations with $2\cdot i_{\max}(n)-3$ variables, we write the system of equations in matrix form.

$$\begin{bmatrix} 1 & -2 & 0 & 1 & \cdots & 0 & 0 & -1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 0 & \cdots & 0 & 0 & 1 & -1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & & & & \cdots & \vdots & \vdots & & & & \cdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} MC_1 \\ MS_2 \\ MS_3 \\ \vdots \\ MS_{i_{\max}(n)-1} \\ MC_2 \\ \vdots \\ MC_{i_{\max}(n)-1} \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \\ \vdots \\ \delta_{i_{\max}(n)-1} \\ \delta_{i_{\max}(n)} \end{bmatrix} \tag{3.6}$$

Let M be the $(i_{\max}(n)-1)\times(2\cdot i_{\max}(n)-3)$ matrix shown in (3.6). We express each entry $m_{i,j}$ in M explicitly:

$$m_{i,j} = \begin{cases} 1 & \text{if } j=i \\ -2 & \text{if } j=i+1 \text{ and } j\le i_{\max}(n)-1 \\ 1 & \text{if } j=i+3 \text{ and } j\le i_{\max}(n)-1 \\ 1 & \text{if } j=i+i_{\max}(n)-2 \text{ and } j\ge 1 \\ -1 & \text{if } j=i+i_{\max}(n)-1 \text{ and } j\le 2\cdot i_{\max}(n)-3 \\ -1 & \text{if } j=i+i_{\max}(n) \text{ and } j\le 2\cdot i_{\max}(n)-3 \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Let A be the left $(i_{\max}(n)-1)\times(i_{\max}(n)-1)$ sub-matrix of M. Notice that A is upper triangular and has 1's in the diagonal, so it is invertible.

To solve the equation in (3.6), we write M in reduced row-echelon form:

$$\left[\begin{array}{ccccccc|cccccc} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & & & & \cdots & & \vdots & \vdots & & & \cdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right]. \tag{3.8}$$

Thus the solutions for the system of equations are in the form:

$$\left[\begin{array}{c} MC_1 \\ MS_2 \\ MS_3 \\ \vdots \\ MS_{i_{\max}(n)-1} \\ \hline MC_2 \\ MC_3 \\ \vdots \\ MC_{i_{\max}(n)-1} \end{array}\right] = \left[\begin{array}{c} A^{-1} \begin{bmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \\ \vdots \\ \delta_{i_{\max}(n)} \end{bmatrix} \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{array}\right] + MC_2 \left[\begin{array}{c} -1 \\ -1 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] + MC_3 \left[\begin{array}{c} -1 \\ -1 \\ -1 \\ \vdots \\ 0 \\ \hline 0 \\ 1 \\ \vdots \\ 0 \end{array}\right] + \cdots + MC_{i_{\max}(n)-1} \left[\begin{array}{c} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \\ \hline 0 \\ 0 \\ \vdots \\ 1 \end{array}\right]. \tag{3.9}$$

The length of game path is the sum of all MC and MS terms, thus can be written as

$$\begin{bmatrix}1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1\end{bmatrix}\begin{bmatrix} MC_1 \\ MS_2 \\ MS_3 \\ \vdots \\ MS_{i_{\max}(n)-1} \\ MC_2 \\ MC_3 \\ \vdots \\ MC_{i_{\max}(n)-1}\end{bmatrix} \tag{3.10}$$

which is equal to

$$\begin{bmatrix}1 & 1 & 1 & \cdots & 1\end{bmatrix} A^{-1}\begin{bmatrix}\delta_2 \\ \delta_3 \\ \delta_4 \\ \vdots \\ \delta_{i_{\max}(n)}\end{bmatrix} - MC_2 - 2MC_3 - \cdots - (i_{\max}(n)-2)MC_{i_{\max}(n)-1}. \tag{3.11}$$

We calculate A^{-1} with Gauss-Jordan elimination, and let $a_{i,j}$ be the i,j-th entry of A^{-1}:

$$A^{-1} = \begin{bmatrix} 1 & 2 & 4 & \cdots & a_{1,i_{\max}(n)-2} & a_{1,i_{\max}(n)-1} \\ 0 & 1 & 2 & \cdots & a_{2,i_{\max}(n)-2} & a_{2,i_{\max}(n)-1} \\ 0 & 0 & 1 & \cdots & a_{3,i_{\max}(n)-2} & a_{3,i_{\max}(n)-1} \\ \vdots & & & \cdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{3.12}$$

where $a_{i,j} = 2a_{i,j-1} - a_{i,j-3}$ for all $1 \le i \le i_{\max}(n) - 1,\ 4 \le j \le i_{\max}(n) - 1$. Also observe that $a_{i+1,j+1} = a_{i,j}$.

We claim that $a_{1,j} = F_{j+1} - 1$, and prove this with induction. First, we see that $a_{1,1} = F_2 - 1 = 1,\ a_{1,2} = F_3 - 1 = 2,\ a_{1,3} = F_4 - 1 - 4$, so this claim holds for $j = 1, 2, 3$. Then suppose this claim holds for all $j' < j$, consider $a_{1,j}$.

$$a_{1,j} = 2a_{1,j-1} - a_{1,j-3} = 2F_j - F_{j-2} - 1 = F_j + F_{j-1} + F_{j-2} - F_{j-2} - 1 = F_{j+1} - 1. \tag{3.13}$$

By induction, our claim holds for all j.

Then we calculate $\begin{bmatrix}1 & 1 & 1 & \cdots & 1\end{bmatrix} A^{-1}$:

$$\begin{aligned}
&\begin{bmatrix}1 & 1 & 1 & \cdots & 1\end{bmatrix} A^{-1} \\
&= \begin{bmatrix}1 & 1 & 1 & \cdots & 1\end{bmatrix}\begin{bmatrix} F_2 - 1 & F_3 - 1 & \cdots & F_{i_{\max}(n)-1} & F_{i_{\max}(n)} \\ 0 & F_2 - 1 & \cdots & F_{i_{\max}(n)-2} & F_{i_{\max}(n)-1} \\ \vdots & & \cdots & & \vdots \\ 0 & 0 & \cdots & 0 & F_2 - 1 \end{bmatrix} \\
&= \begin{bmatrix}1 & 3 & 7 & \cdots & \sum_{i=2}^{i_{\max}(n)-1} (F_i - 1) & \sum_{i=2}^{i_{\max}(n)} (F_i - 1)\end{bmatrix}.
\end{aligned} \tag{3.14}$$

We add an extra 0 before the sequence of $1,\ 3,\ 7,\ldots$ and express it explicitly as $a_1 = 0,\ a_j = \sum_{i=2}^{j}(F_i - 1)$ for $j \geq 2$. Then we find a formula for $j \geq 2$:

$$a_j = \sum_{i=2}^{j}(F_i - 1) = \sum_{i=1}^{j} F_i - j = (F_j - 1) + (F_{j+1} - 1) - j = F_{j+2} - j - 2. \tag{3.15}$$

Since $a_1 = 0 = F_3 - 1 - 2$ is consistent with this formula, we conclude that $a_j = F_{j+2} - j - 2$ for all j.

Now the game length given in (3.11) becomes

$$\sum_{j=1}^{i_{\max}(n)} a_j\ \delta_j - MC_2 - 2MC_3 - \cdots - (i_{\max}(n) - 2)MC_{i_{\max}(n)-1}. \tag{3.16}$$

Since all MC terms are non-negative, we ignore them for the upper bound. Thus the upper bound on game length is

$$\sum_{j=1}^{i_{\max}(n)} a_j\ \delta_j = a_1\ \delta_1 + a_2\ \delta_2 + a_3\ \delta_3 + \cdots + a_{i_{\max}(n)}\ \delta_{i_{\max}(n)}. \tag{3.17}$$

This is the bound we claimed in Theorem 1.3.

We now analyze the order of this bound.

We show that $a_j \leq \frac{3+\sqrt{5}}{2}\ F_j - j - \frac{1+\sqrt{5}}{2}$ using Binet's formula

$$F_j = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{j+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{j+1}\right) \tag{3.18}$$

and the formula $a_j = F_{j+2} - j - 2$. Thus,

$$\begin{aligned} a_j &= \frac{F_{j+2}}{F_j}\ F_j - j - 2 \\ &= \frac{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{j+3} - \left(\frac{1-\sqrt{5}}{2}\right)^{j+3}\right)}{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{j+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{j+1}\right)}\ F_j - j - 2 \\ &= \frac{3+\sqrt{5}}{2}\ F_j + \left(\frac{1-\sqrt{5}}{2}\right)^{j+1} - j - 2. \end{aligned} \tag{3.19}$$

Since $-1 < \frac{1-\sqrt{5}}{2} < 0$, we have $\left(\frac{1-\sqrt{5}}{2}\right)^{j+1} \leq \left(\frac{1-\sqrt{5}}{2}\right)^{2} = \frac{3-\sqrt{5}}{2}$, so

$$a_j \leq \frac{3+\sqrt{5}}{2}\ F_j + \frac{3-\sqrt{5}}{2} - j - 2 = \frac{3+\sqrt{5}}{2}\ F_j - j - \frac{1+\sqrt{5}}{2}. \tag{3.20}$$

Therefore we have

$$\begin{aligned}\sum_{j=1}^{i_{\max}(n)} a_j\ \delta_j &\leq \sum_{j=1}^{i_{\max}(n)} \left(\frac{\sqrt{5}+3}{2}\ F_j - j - \frac{1+\sqrt{5}}{2}\right)\delta_j \\ &= \frac{\sqrt{5}+3}{2}\sum_{j=1}^{i_{\max}(n)} F_j\ \delta_j - \sum_{j=1}^{i_{\max}(n)} j\ \delta_j - \frac{1+\sqrt{5}}{2}\sum_{j=1}^{i_{\max}(n)} \delta_j \\ &= \frac{\sqrt{5}+3}{2}\ n\ -\ IZ(n)\ -\ \frac{1+\sqrt{5}}{2}Z(n). \qquad (3.21)\end{aligned}$$

In conclusion, the game length is at most $\frac{\sqrt{5}+3}{2}\ n\ -\ IZ(n)\ -\ \frac{1+\sqrt{5}}{2}Z(n)$. □

In this proof, equation (3.16) is the exact game length and (3.17) is the bound we gave in Theorem 1.3. From these two equations, we observe that our upper bound is strict if and only if the game on n can be played with only splitting and combine 1 moves.

Remark 3.1. *The same method can be used to calculate game length even if the game does not start from all 1's by replacing δ_i's with the difference in number of F_i's between starting state and final state at each position. Also if we take the MS_i terms as free variables instead of pivot variables, we can use this method to calculate the lower bound of the game.*

We move on to the proof of Theorem 1.4. The reason we are interested in games that can be played with only splitting moves is to identify for which n is our upper bound in Theorem 1.3 strict.

Lemma 3.2. *If $n = F_k - 1$ $(k \geq 2)$, then we can play the game with only split and combine 1 moves (starting from any game state).*

Proof. To prove this, we first prove that any game state (except the final one) in the game on $n = F_k - 1$ $(k \geq 2)$ has at least two F_i's for some i.

Suppose for the sake of contradiction that there is a game state that has at most one of any F_i and is not the final game state. Since this game state is not the final state, there are moves that we can apply. Since this game state has at most one of any F_i, we cannot apply any splitting moves. Thus, there must be some combining moves available. We apply the combining move with the largest index, say $F_{i-1} \wedge F_i \to F_{i+1}$. Note that we cannot have F_{i+1} in this game state, or we would have chosen to combine F_i and F_{i+1}. Thus after this move, we still have at most one of any F_i. We repeat this process until we reach the final state, and each game state we visit has at most one of any F_i.

Now we consider the Zeckendorf Decomposition of $n = F_k - 1$. It has to be in the form of $F_1 + F_3 + F_5 + \cdots$ or $F_2 + F_4 + F_6 + \cdots$ because if we add 1 to these decompositions, we can get a Fibonacci number. By assumption, we reach the final state with a combining move. Let $F_{j-1} \wedge F_j \to F_{j+1}$ be the last step we took. Since there is a F_{j+1} in the final decomposition of n, there must also be a F_{j-1}. Thus we had two F_{j-1}'s before the last step we took. However, we just showed that each game state we visit has at most one of any F_i, which is a contradiction.

Now that we have proved that any game state (except the final one) in the game on $n = F_k - 1$ $(k \geq 2)$ has at least two F_i's for some i, we know that we can apply a splitting or combine 1 move at any game state until the game terminates. Thus, the game can be played with only splitting and combine 1 moves. □

Lemma 3.3. *If we can play the game with only splitting and combining 1 moves, then n can only be in the form $F_k - 1$ $(k \geq 2)$.*

Proof. We consider all possible n that are not in the form of $F_k - 1$ $(k \geq 2)$ and divide them into three cases based on their final decomposition. In each case, we prove that the game on n cannot be played with only splitting moves.

Case 1: The smallest term in the final decomposition of n is at least F_3.

Suppose that the game on n can be played with only splitting and combine 1 moves. Since the smallest term in n's final decomposition is at least F_3, we have to generate at least one F_3 at some point of the game. Thus there must exist split 2 moves in the game. Let t be the last split 2 move in the game.

At step t, one F_1 is generated. Since there are no F_1 in the final decomposition, we have to do a combine 1 move to decrease the number of F_1's, producing a F_2 with this move. Since there is no F_2 in the final decomposition, we have to do a split 2 move, which contradicts our assumption that step t is the last split 2 move.

Case 2: The first two terms in the final decomposition of n are F_1 and F_4.

Suppose that the game on n can be played with only splitting and combine 1 moves. Since we have F_4 in the final decomposition, we have to generate a F_4 at some point of the game. Thus there exist split 3 moves in the game. To do the split move, we need a F_3, so there must exist a split 2 move. Let step v be the last split 2 move in the game.

At step v, one F_3 and one F_1 is generated. Since there is no F_3 in the final decomposition, there exists a split 3 move after step v, and we denote it step t. Then in step t, there is one F_1 generated.

Since both step v and t generates F_1, there exists a game state after step v that contains 2 F_1. Therefore, there exists a combine 1 move after step v. With this move, one F_2 is generated. Since there is no F_2 in the final decomposition, we have to do a split 2 move to get rid of this F_2 which contradicts our assumption that v is the last split 2 move.

Case 3: The smallest term of n's decomposition is F_1 or F_2, and the first 2 terms in the decomposition are not F_1 and F_4.

For any such n, suppose the game can be played without any combining moves except combine 1.

In the following proof, we define the gap between two terms as the difference between their indices. Notice that for any n not in the form of $F_k - 1$ $(k \geq 2)$, its decomposition either has a smallest term of F_3 or larger (*Case 1*), or contains two consecutive terms with a gap of at least 3. We make the following claim.

Claim 1. If a game is played with only split or combine 1 moves, then no game state contains two consecutive terms with a gap larger than 3.

Proof of Claim 1. Since the first step of the game is always $F_1 \wedge F_1 \to F_2$, which only generates a gap of 1, the claim is true for the first step. Suppose the claim is true for mth step. For the $(m+1)$th step, if we split 2's, it generates a gap of at most 2, and the gap between F_3 and other terms larger than F_3 shrinks.

If we combine 1's, we generate a gap of 1, and the gap between F_2 and other terms larger than F_2 shrinks.

Every time we split $F_i \wedge F_i \to F_{i+1} \wedge F_{i-2}$ $(i > 2)$, we generate a gap of 3, which is the gap between F_{n+1} and F_{i-2}, but the gaps between F_{i-2} and any terms smaller than F_{n-2} shrink, and the gaps between F_{i+1} with any terms larger than F_{i+1} shrink. In other words, every time we split, we generate a gap of 3 but also make other gaps smaller.

Therefore, for all the situations above, the claim is true for $(m+1)$th step, and by induction Claim 1 is true.

We now use this claim to finish our proof. Note that for any n in Case 3, there will be at least two terms with a gap of at least 3 and no terms in between them. From Claim 1, we also proved that when we only use combine 1 and splitting moves, we can never generate a gap that is more than 3. Therefore, if n's decomposition has a gap of more than 3, then it is out of our consideration. For other n, we can find an i such that F_{i-2} and F_{i+1} are two terms in the final decomposition, and there is neither F_{i-1} nor F_i in the final decomposition.

Since F_{i+1} is in the final decomposition, there must exist a step where S_i is performed. This is because the first time F_{i+1} is generated in the game must come from the S_i. So, let step t be the last S_i.

In order for the S_i happen, there must be at least two F_i's generated prior. Thus, there must exist a step of splitting F_{i-1} moves before step t (this is because the first time F_i is generated in the game must come from the splitting F_{i-1} move). So, let step v be the last S_{i-1} in the game.

Since F_{i-2} is in the final decomposition, F_{i-3} is not in the final decomposition (because the final decomposition does not contain two consecutive Fibonacci numbers). Note that step v has generated one F_{i-3}, so there must be a S_{i-3} after step v to get rid of this F_{i-3} (here, the combine 1 move is also considered as a splitting move), and we can call it step r.

Since step v has generated one F_i and F_i is not in the final decomposition, there must be a S_i after step v in order to get rid of this F_i, and we can call it step s.

Since both step r and step s are after step v and each of them has generated one F_{i-2}, there must exist $2F_{i-2}$ at some point after step v. As a result, there must be a S_{i-2} after step v, and we can call it step w.

Since step w has generated one F_{i-1} and F_{i-1} is not in the final decomposition, there must be a S_{i-1} after step w in order to get rid of this F_{i-1}. In other words, there is a S_{i-1} after step v, which contradicts with our assumption that step v is the last occurrence of S_{i-1}.

Therefore, Lemma 3.3 is proved. □

The proof for Theorem 1.4 follows directly from Lemma 3.2 and 3.3.

4. Future Work

It is worth noting that while our upper bound is sharp for some n, there is still plenty of room for it to be tightened. It was alluded to previously in the proof of 1.3 that this could be done by quantifying the number of each combining moves, but such work is beyond the scope of this paper. In recent work, Cusenza et. al. [CDHKKMMTYZ] investigated winning strategies for alliances of players in a multi-person generalization of the Zeckendorf game; one can similarly investigate the number of moves of various strategies in these settings.

Additionally, there are many ways the Zeckendorf game can be generalized (see for example [BEFM1]). With that in mind, we ask the following questions in regards to how our work relates to other similar games.

- Can our methods for analysis of game bounds be performed on generalized games?
- Can we generalize the strategies suggested in this paper to achieve the longest or shortest game length in generalized games?

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