

# ZECKENDORF'S THEOREM USING INDICES IN AN ARITHMETIC PROGRESSION

AMELIA GILSON, HADLEY KILLEN, TAMÁS LENGYEL, STEVEN J. MILLER,  
NADIA RAZEK, JOSHUA M. SIKTAR, AND LIZA SULKIN

ABSTRACT. Zeckendorf's Theorem states that any positive integer can be uniquely decomposed into a sum of distinct, non-adjacent Fibonacci numbers. There are many generalizations, including results on existence of decompositions using only even indexed Fibonacci numbers. We extend these further and prove that similar results hold when only using indices in a given arithmetic progression. As part of our proofs, we generate a range of new recurrences for the Fibonacci numbers that are of interest in their own right.

## CONTENTS

1.	Introduction	1
2.	Linear Recurrences with Fibonacci Numbers	4
3.	Decomposition Results	8
4.	Conclusion and Future Work	9
	Appendix A. Alternative Proofs of Lemmas 2.2 and 2.4	9
	References	9

## 1. INTRODUCTION

The Fibonacci sequence is defined via the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \tag{1.1}$$

for  $n \geq 2$ , where we need two initial conditions; often these are  $F_0 = 0$  and  $F_1 = 1$ . We can use Binet's Formula to jump to the  $n^{\text{th}}$  term:

$$F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}, \tag{1.2}$$

where  $\phi$  is the Golden ratio  $\frac{1+\sqrt{5}}{2}$ .

There are many interesting properties of the Fibonacci numbers; see for example [Kos]. We focus on Zeckendorf's Theorem; it turns out that if we change the initial conditions, the Fibonacci numbers are equivalent to a decomposition property of the integers.<sup>1</sup>

---

This work was supported in part by NSF Grant DMS1561945. We thank the referee for a careful reading.

<sup>1</sup>If we began with  $F_0 = 0, F_1 = 1$  then  $F_2 = 1$  and we lose uniqueness of decomposition, both because we can add an  $F_0$  as well as we have two ways to represent 1.

**Theorem 1.1.** (*Zeckendorf's Theorem*) Consider the Fibonacci recurrence with initial conditions  $F_0 = 1, F_1 = 2$ . Any positive integer  $n$  can be expressed uniquely as a sum of non-adjacent Fibonacci numbers:

$$N = \sum_{k=0}^{\infty} b_k F_k, \text{ where } b_k \in \{0, 1\} \text{ and } b_k \cdot b_{k+1} = 0. \quad (1.3)$$

Further, the Fibonacci numbers are the unique sequence of positive numbers such that every integer can be expressed uniquely as a sum of non-adjacent terms. Note we could also choose initial conditions  $F_0 = 0, F_1 = 1$  if we only use indices  $k \geq 2$ .

The classical proof is by induction on  $N$ , but other proofs have been developed as well; see [Br, KKMW, Len1, Ost, Ze]. There is an extensive literature on generalizations and variations of Theorem 1.1; see for instance [Al, BM, Br, CHHMPV, DDKMMV, Fr, Ho, Ke, Luo, ML, MW1, MW2]. Zeckendorf decompositions have also been studied in a combinatorial framework in numerous places, including [BCCFLMX, CCGJMSY, FJLLLMSS, KKMW, Len1]. The combinatorial approach initiated in [KKMW] is very useful for studying related problems, such as the distribution of the number of summands and the gaps between them in decompositions.

Previous work derived decomposition results when we can only use Fibonacci numbers whose indices have the same parity. For example, there is the *even Fibonacci representation of  $N$*  (see [CG1, CG2]): every positive integer has a unique decomposition of the form

$$\sum_{k=1}^{\infty} b_k F_{2k} \text{ with } b_k \in \{0, 1, 2\} \text{ and if } b_i = b_j = 2 \text{ then } \exists k \text{ with } i < k < j \text{ and } b_k = 0, \quad (1.4)$$

where we use the initial conditions  $F_2 = 1, F_3 = 2$  to ensure that the decompositions are unique<sup>2</sup>. The Fibonacci recurrence decomposes a summand into two terms: one whose index has the same parity as the original summand, and one whose index has the opposite parity. Thus the existence of decomposition (1.4) is to be expected.

**Example 1.2.** As an example, here are the Zeckendorf and even Fibonacci representations of 83 respectively:

$$\begin{aligned} 83 &= 55 + 21 + 5 + 2 = F_{10} + F_8 + F_5 + F_3 \\ 83 &= 1 \cdot 55 + 1 \cdot 21 + 2 \cdot 3 + 1 \cdot 1 = 1 \cdot F_{10} + 1 \cdot F_8 + 0 \cdot F_6 + 2 \cdot F_4 + 1 \cdot F_2. \end{aligned} \quad (1.5)$$

As stated in Theorem 1.1, the initial conditions for Zeckendorf Decompositions are  $F_0 = 0, F_1 = 1$ . On the other hand, the even Fibonacci Representation (1.4) uses the initial conditions  $F_2 = 1$  and  $F_3 = 2$  to maintain uniqueness of decompositions. For precisely this reason, unlike the decomposition (1.3), we begin summing terms in (1.4) when  $k = 1$ .

Given the decomposition (1.4) result, it is natural to ask whether other subsequences of the Fibonacci numbers also yield unique decompositions, and if so what they are. We prove there are such decompositions when we restrict our indices to be in an arithmetic progression. Before stating our results we first establish some notation.

**Definition 1.3.** ( *$n$ -gap Fibonacci numbers*) For  $n, m \in \mathbb{N}^+$  with  $0 \leq m < n+1$ , let  $\mathcal{F}(k; n, m) = F_{k(n+1)+m}$  equal the Fibonacci numbers whose indices are congruent to  $m$  modulo  $n+1$ . We call  $m$  the offset, and call  $\mathcal{F}(k; n, m)$  an  $n$ -gap subsequence. Note the Fibonacci numbers are a 0-gap subsequence, and the even and odd index results concern 1-gap subsequences.

---

<sup>2</sup>If we were to allow  $k = 0$ , then the  $F_0 = 0$  term from the Fibonacci Sequence would be allowed in our decompositions and we would lose the uniqueness property of Zeckendorf Decompositions.

As we will see, the construction of  $n$ -gap Fibonacci subsequences is based on the theory of **Positive Linear Recurrence Sequences** (PLRS).

**Definition 1.4.** (PLRS) A Positive Linear Recurrence Sequence is a sequence of integers  $\{H_n\}_{n=1}^{\infty}$  with the following properties.

- (1) There are non-negative integers  $L, c_1, \dots, c_L$  such that

$$H_n = c_1 H_{n-1} + c_2 H_{n-2} + \dots + c_L H_{n-L} \quad (1.6)$$

where  $L, c_1, c_L > 0$ .

- (2)  $H_1 = 1$  and for  $1 \leq n < L$ , we have

$$H_n = c_1 H_{n-1} + c_2 H_{n-2} + \dots + c_{n-1} H_1 + 1. \quad (1.7)$$

The study of PLRS is foundational in many papers relating to Zeckendorf Decompositions; see for instance [BCCFLMX, BM, CFHMN, CFHMNPX, DFFHMPP, Ha, ML]. There is an extensive literature on when there is a unique decomposition arising from a given recurrence relation, as well as a host of other properties (such as the distribution of the number of summands in a decomposition, gaps between summands, and digital expansions of these sequences). In particular, if the recurrence relation is a PLRS, then Miller and Wang [MW1, MW2] proved that there exists a unique legal decomposition; for more on these sequences see [BM, Br, Day, DG, Fr, GTNP, Ha, HW, Ho, Ke, KKMW, LT, Len1, PT, Ste1, Ste2], and for other types of decompositions see [Al, CFHMN, CFHMNPX, CCGJMSY, DDKMMV, DDKMV, DFFHMPP].

**Theorem 1.5.** ( $n$ -gap Fibonacci as PLRS) If  $n = 2$  or  $n \geq 3$  is odd, then the  $n$ -gap Fibonacci sequence  $\{\mathcal{F}(k; n, m)\}_{k=1}^{\infty}$  is a PLRS for any  $0 \leq m < n + 1$ . If  $n = 2$  then there is a unique decomposition of every positive integer taking the form (1.4), with initial conditions  $F_2 = 1, F_3 = 2$ . On the other hand, if  $n \geq 3$  is odd then the decomposition is still unique for every positive integer, but it takes the form

$$\sum_{k=0}^{\infty} b_k F_{k(n+1)+m} \quad \text{with} \quad |b_k| \leq a_n \quad \text{for all} \quad k \geq 1, \quad (1.8)$$

again with initial conditions  $F_2 = 1, F_3 = 2$ . Here  $a_n$  refers to  $\phi^n$  rounded to the nearest integer (one of the Lucas numbers).

Just like the even Fibonacci decomposition, we use the initial conditions  $F_2 = 1, F_3 = 2$  for the odd  $n$ -gap Fibonacci decomposition because we want to ensure all decompositions are unique. Here is an example of what these decompositions look like for specific values of  $n$  and  $m$ .

**Example 1.6.** This is the 2-gap decomposition of 143 when  $n = 2, m = 1$ :

$$143 = 2 \cdot 55 + 2 \cdot 13 + 2 \cdot 3 + 1 \cdot 1 = 2 \cdot F_{10} + 2 \cdot F_7 + 2 \cdot F_4 + 1 \cdot F_1. \quad (1.9)$$

Similarly, this is the 2-gap decomposition of 143 when  $n = 2, m = 2$ :

$$143 = 1 \cdot 89 + 2 \cdot 21 + 2 \cdot 5 + 2 \cdot 1 = 1 \cdot F_{11} + 2 \cdot F_8 + 2 \cdot F_5 + 2 \cdot F_2. \quad (1.10)$$

We can also list 3-gap decompositions for 143. Here is the decomposition when  $n = 3, m = 1$ :

$$143 = 4 \cdot 34 + 1 \cdot 5 + 2 \cdot 1 = 4 \cdot F_9 + 1 \cdot F_5 + 2 \cdot F_1. \quad (1.11)$$

Here is the decomposition when  $n = 3, m = 2$ :

$$143 = 2 \cdot 55 + 4 \cdot 8 + 1 \cdot 1 = 2 \cdot F_{10} + 4 \cdot F_6 + 1 \cdot F_2. \quad (1.12)$$

Finally, here is the decomposition when  $n = 3, m = 3$ :

$$143 = 1 \cdot 89 + 4 \cdot 13 + 1 \cdot 2 = 1 \cdot F_{11} + 4 \cdot F_7 + 1 \cdot F_3. \quad (1.13)$$

In Section 2 we examine some recurrences for  $n$ -gap Fibonacci numbers and discuss how these relate to the more general theory of PLRS. Then in Section 3 we prove Theorem 1.5. Finally, in Section 4 we give some concluding remarks and possible directions for future research.

## 2. LINEAR RECURRENCES WITH FIBONACCI NUMBERS

We began by looking at decompositions using only every third Fibonacci number; in our notation this would be a 2-gap Fibonacci sequence with an offset of 2. We choose this offset so that our first term is  $F_2 = 1$ , consistent with the initial conditions in (1.4). This allows us to begin finding patterns for the general  $n$ -gap Fibonacci sequence. In this case, we define

$$\{\mathcal{F}(k; 2, 2)\}_{k=0}^{\infty} = \{1, 5, 21, 89, 377, 1597, \dots\}. \quad (2.1)$$

The sequence in (2.1) can itself be defined recursively.

**Lemma 2.1.** *For  $k \geq 2$ ,*

$$F_{3k+2} = 4 \cdot F_{3(k-1)+2} + F_{3(k-2)+2}. \quad (2.2)$$

*Proof.* We repeatedly use the recursion  $F_j = F_{j-1} + F_{j-2}$  to calculate

$$\begin{aligned} F_{3k+2} &= F_{3k+1} + F_{3k} \\ &= F_{3k} + F_{3k-1} + F_{3k-1} + F_{3k-2} \\ &= F_{3k-1} + F_{3k-2} + F_{3k-1} + F_{3k-1} + F_{3k-3} + F_{3k-4} \\ &= 3 \cdot F_{3k-1} + F_{3k-2} + F_{3k-3} + F_{3k-4} \\ &= 4 \cdot F_{3k-1} + F_{3k-4} \\ &= 4 \cdot F_{3(k-1)+2} + F_{3(k-2)+2}. \end{aligned} \quad (2.3)$$

□

By a procedure analogous to (2.3) we can also generate the following identities, which hold for all  $k \geq 2$ :

$$\begin{aligned} F_{4k+2} &= 7 \cdot F_{4(k-1)+2} - F_{4(k-2)+2} \\ F_{5k+2} &= 11 \cdot F_{5(k-1)+2} + F_{5(k-2)+2} \\ F_{6k+2} &= 18 \cdot F_{6(k-1)+2} - F_{6(k-2)+2} \\ F_{7k+2} &= 29 \cdot F_{7(k-1)+2} + F_{7(k-2)+2}. \end{aligned} \quad (2.4)$$

Notice that 3, 4, 7, 11, 18, 29, ... are the Lucas numbers, which have the closed form  $\phi^k + (-\phi)^{-k}$ . Since the golden ratio is defined as  $\phi = (1 + \sqrt{5})/2$ , the Lucas numbers are the closest integer to  $\phi^k$  for each  $k > 1$ , because  $|(-\phi)^{-k}| < 1/2$  for  $k > 1$ . This motivates the question of whether every  $n$ -gap Fibonacci subsequence can be defined recursively, and then what decomposition properties they have. Using Binet's formula (1.2) for Fibonacci numbers, we can generalize the formulas (2.2) and (2.4).

**Lemma 2.2.** *For any  $n \geq 2$  we have the following generalization of (2.2):*

$$\mathcal{F}(k; n, m) = a_n \cdot \mathcal{F}(k-1; n, m) + (-1)^{n-1} \cdot \mathcal{F}(k-1; n, m), \quad (2.5)$$

where  $a_n$  henceforth will denote  $\phi^n$  rounded to the nearest integer (the Lucas numbers).

*Proof.* We take advantage of the Lucas numbers and Binet's formula (1.2) to rewrite each term on the right hand side of (2.5):

$$\begin{aligned} a_n \cdot \mathcal{F}(k-1; n, m) &= \left( (\phi^n + (-\phi)^{-n}) \cdot \frac{1}{\sqrt{5}} (\phi^{nk-n+m} - (-\phi)^{n-nk-m}) \right) \\ (-1)^{n-1} \cdot \mathcal{F}(k-2; n, m) &= (-1)^{n-1} \cdot \frac{1}{\sqrt{5}} (\phi^{nk-2n+m} - (-\phi)^{-nk+2n-m}). \end{aligned} \quad (2.6)$$

We simplify each component algebraically:

$$\begin{aligned} a_n \cdot \mathcal{F}(k-1; n, m) &= \frac{1}{\sqrt{5}} \left( (\phi^n + (-\phi)^{-n}) (\phi^{nk-n+m} - (-\phi)^{n-nk-m}) \right) \\ &= \frac{1}{\sqrt{5}} \left( \phi^n \cdot \phi^{nk-n+m} - \phi^n \cdot (-\phi)^{n-nk-m} \right. \\ &\quad \left. + (-\phi)^{-n} \cdot \phi^{nk-n+m} - (-\phi)^{-n} \cdot (-\phi)^{n-nk-m} \right) \\ &= \frac{1}{\sqrt{5}} \left( \phi^{nk+m} - \phi^n \cdot (-1)^{n-nk-m} \cdot \phi^{n-nk-m} \right. \\ &\quad \left. + (-1)^{-n} \cdot (\phi)^{-n} \cdot \phi^{nk-n+m} - (-\phi)^{-nk-m} \right) \\ &= \frac{1}{\sqrt{5}} \left( \phi^{nk+m} + (-1)^{n-nk} \cdot \phi^{2n-nk-m} \right. \\ &\quad \left. + (-1)^{-n} \cdot \phi^{nk-2n+m} - (-\phi)^{-nk-m} \right), \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} (-1)^{n-1} \cdot \mathcal{F}(k-2; n, m) &= \frac{1}{\sqrt{5}} \left( (-1)^{n-1} (\phi^{nk-2n+m} - (-\phi)^{2n-nk-m}) \right) \\ &= \frac{1}{\sqrt{5}} \left( (-1)^{n-1} \cdot \phi^{nk-2n+m} - (-1)^{n-1} \cdot (-1)^{2n-nk-m} \cdot \phi^{2n-nk-m} \right) \\ &= \frac{1}{\sqrt{5}} \left( (-1)^{n-1} \cdot \phi^{nk-2n+m} - (-1)^{1-n} \cdot (-1)^{2n-nk-m} \cdot \phi^{2n-nk-m} \right) \\ &= \frac{1}{\sqrt{5}} \left( (-1)^{n-1} \cdot \phi^{nk-2n+m} - (-1)^{n-nk} \cdot \phi^{2n-nk-m} \right). \end{aligned} \quad (2.8)$$

We now sum and simplify the above, and obtain

$$\begin{aligned}
& a_n \cdot \mathcal{F}(k-1; n, m) + (-1)^{n-1} \cdot \mathcal{F}(k-1; n, m) \\
&= \frac{1}{\sqrt{5}} \left( \phi^{nk+m} + (-1)^{n-nk} \cdot \phi^{2n-nk-m} + (-1)^{-n} \cdot \phi^{nk-2n+m} - (-\phi)^{-nk-m} \right. \\
&\quad \left. + (-1)^{n-1} \cdot \phi^{nk-2n+m} - (-1)^{n-nk} \cdot \phi^{2n-nk-m} \right) \\
&= \frac{1}{\sqrt{5}} \left( \phi^{nk+m} + \cancel{(-1)^{n-nk} \cdot \phi^{2n-nk-m}} + (-1)^{-n} \cdot \phi^{nk-2n+m} - (-\phi)^{-nk-m} \right. \\
&\quad \left. + (-1)^{n-1} \cdot \phi^{nk-2n+m} - \cancel{(-1)^{n-nk} \cdot \phi^{2n-nk-m}} \right) \\
&= \frac{1}{\sqrt{5}} \left( \phi^{nk+m} + (-1)^{-n} \cdot \phi^{nk-2n+m} - (-\phi)^{-nk-m} + (-1)^{n-1} \cdot \phi^{nk-2n+m} \right) \\
&= \frac{1}{\sqrt{5}} \left( \phi^{nk+m} + \cancel{(-1)^{-n} \cdot \phi^{nk-2n+m}} - (-\phi)^{-nk-1} + \cancel{(-1)^{n-1} \cdot \phi^{nk-2n+m}} \right) \\
&= \frac{1}{\sqrt{5}} \left( \phi^{nk+m} - (-\phi)^{-nk-m} \right) \\
&= \mathcal{F}(k; n, m),
\end{aligned} \tag{2.9}$$

which is the desired result.  $\square$

We can generalize Lemma 2.2 to all  $n$ -gap subsequences of the recurrence relation

$$G_n = G_{n-1} + G_{n-2}, \tag{2.10}$$

where  $G_1$  and  $G_2$  are positive integers; to do this we first prove a recurrence relating  $\{F_\ell\}_{\ell=1}^\infty$  to  $\{G_\ell\}_{\ell=1}^\infty$ .

**Lemma 2.3.** *If  $\{G_\ell\}_{\ell=1}^\infty$  satisfies (2.10) then for  $n \geq 3$ ,*

$$G_n = F_{n-2} \cdot G_1 + F_{n-1} \cdot G_2. \tag{2.11}$$

*Proof.* We proceed by strong induction. The base case will be  $n = 3$ , which is verified as follows:

$$G_3 = G_2 + G_1 = G_2 \cdot 1 + G_1 \cdot 1 = G_2 \cdot F_2 + G_1 \cdot F_1. \tag{2.12}$$

As for the inductive step, we assume for all  $3 \leq j \leq k$  that

$$G_j = F_{j-2} \cdot G_1 + F_{j-1} \cdot G_2, \tag{2.13}$$

and we can finish the proof by demonstrating that

$$G_{k+1} = F_{k-1} \cdot G_1 + F_k \cdot G_2. \tag{2.14}$$

We can show (2.14) by using (2.13) for  $j = k$  and  $j = k-1$ , along with the recurrence (1.1):

$$\begin{aligned}
G_{k+1} &= G_k + G_{k-1} \\
&= F_{k-2} \cdot G_1 + F_{k-1} \cdot G_2 + F_{k-3} \cdot G_1 + F_{k-2} \cdot G_2 \\
&= (F_{k-2} + F_{k-3}) \cdot G_1 + (F_{k-1} + F_{k-2}) \cdot G_2 \\
&= F_{k-1} \cdot G_1 + F_k \cdot G_2,
\end{aligned} \tag{2.15}$$

as desired.  $\square$

Now we can prove our generalization of Lemma 2.2.

**Lemma 2.4.** *For  $k \geq 2$ , if  $\{G_\ell\}_{\ell=1}^\infty$  satisfies (2.10) then*

$$G_{nk+m} = a_n \cdot G_{n(k-1)+m} + (-1)^{n-1} \cdot G_{n(k-2)+m}, \quad (2.16)$$

where  $n, k, m$ , and  $a_n$  are defined as before, regardless of the initial conditions.

*Proof.* We can use (2.5) in conjunction with (2.11) to compute

$$\begin{aligned} a_n \cdot G_{n(k-1)+m} + (-1)^{n-1} \cdot G_{n(k-2)+m} &= a_n \cdot (F_{n(k-1)+m-2} \cdot G_1 + F_{n(k-1)+m-1} \cdot G_2) \\ &\quad + (-1)^{n-1} (F_{n(k-2)+m-2} \cdot G_1 + F_{n(k-2)+m-1} \cdot G_2) \\ &= (a_n \cdot F_{n(k-1)+m-2} + (-1)^{n-1} \cdot F_{n(k-2)+m-2}) \cdot G_1 \\ &\quad + (a_n \cdot F_{n(k-1)+m-1} + (-1)^{n-1} \cdot F_{n(k-2)+m-1}) \cdot G_2 \\ &= F_{nk+m-2} \cdot G_1 + F_{nk+m-1} \cdot G_2 \\ &= G_{nk+m}. \end{aligned} \quad (2.17)$$

□

**Remark 2.5.** *It turns out there are alternative proofs to Lemmas 2.2 and 2.4 that rely on the established generating function theory of multisections. We provide the details in Appendix A.*

For convenience, we restate the definition of Positive Linear Recurrence Sequences (Definition 1.4) so that we can solidify the framework to be used in Section 3.

**Definition 1.4.** *A Positive Linear Recurrence Sequence is a sequence of integers  $\{H_n\}_{n=1}^\infty$  with the following properties.*

- (1) *There are non-negative integers  $L, c_1, \dots, c_L$  such that*

$$H_n = c_1 H_{n-1} + c_2 H_{n-2} + \dots + c_L H_{n-L} \quad (2.18)$$

*where  $L, c_1, c_L > 0$ .*

- (2)  *$H_1 = 1$  and for  $1 \leq n < L$ , we have:*

$$H_n = c_1 H_{n-1} + c_2 H_{n-2} + \dots + c_{n-1} H_1 + 1. \quad (2.19)$$

Here is a simple example demonstrating how to prove that a sequence is a PLRS, particularly the 2-gap sequence.

**Example 2.6.** *We can show directly that*

$$\{\mathcal{F}(k; 2, 2)\}_{k=0}^\infty = \{1, 5, 21, 89, 377, \dots\} \quad (2.20)$$

*is a PLRS. Define*

$$\{G_k\}_{k=1}^\infty = \{1, 5, 21, 89, 377, \dots\}, \quad (2.21)$$

*and we check each condition in Definition 1.4.*

- (1) *The first condition is true, because we can take  $L = 2, c_1 = 4$ , and  $c_2 = 1$ ; then our recurrence is  $G_k = 4G_{k-1} + G_{k-2}$ .*  
 (2) *The second condition also holds; since  $5 = 4 \cdot 1 + 1$ , we conclude  $G_1 = 1$ , and  $G_2 = 4G_1 + 1$ .*

In the next section we generalize Example 2.6.

### 3. DECOMPOSITION RESULTS

In this section we restate and prove the main result of the paper, building on the intuition developed in Section 2. We quote a primary result from [MW1] that yields the uniqueness of decompositions for the sequences in Section 2.

**Theorem 3.1.** (*Generalized Zeckendorf's Theorem for PLRS*) *Let  $\{H_j\}_{j=0}^\infty$  be a Positive Linear Recurrence Sequence. Then*

- (1) *there is a unique legal decomposition for each positive integer  $N \geq 0$ , and*
- (2) *there is a bijection between the set  $S_j$  of integers in  $[H_j, H_{j+1})$  and the set  $D_j$  of legal decompositions  $\sum_{i=1}^j b_i \cdot H_{j+1-i}$ .*

We now generalize the result from Example 2.6 to the  $n$ -gap Fibonacci sequences, using Theorem 3.1. Notably, this generalization only extends to odd  $n$  due to the  $(-1)^{n+1}$  factor of the second term in each recurrence relation amongst the list (2.4). Thus the sequences we study in the next theorem are of the form  $\{F_{nk+m}\}_{k=0}^\infty$ , where  $n \geq 3$  is a fixed positive odd integer.

**Theorem 3.2.** ( *$n$ -gap Fibonacci as PLRS*) *If  $n \geq 3$  is odd, then the  $n$ -gap Fibonacci sequence  $\{\mathcal{F}(k; n, m)\}_{k=0}^\infty$  is a PLRS.*

*Proof.* We consider each condition of Definition 1.4 individually:

- (1) The first condition holds via the recurrence relation (2.5). Since  $F_{nk+1} = a_n \cdot F_{n(k-1)+1} + (-1)^{n-1} \cdot F_{n(k-2)+1}$ , we have the following relation for odd  $n$ :

$$F_{nk+1} = a_n \cdot F_{n(k-1)+1} + F_{n(k-2)+1}. \quad (3.1)$$

Since  $a_n$  is a power of a positive number rounded to the nearest integer, we may satisfy the first condition of Definition 1.4 with the following parameters:

$$L = 2n, \quad c_1 = a_n, \quad c_2 = \cdots = c_{2n-1} = 0, \quad c_{2n} = 1. \quad (3.2)$$

- (2) Since  $L = 2$ , we only need to check the condition (2.19) for  $m = 3$ . For this condition to hold, the following needs to be true for all such sequences of odd  $n$ :

$$G_3 = a_n \cdot G_2 + G_1 + 1. \quad (3.3)$$

We can rewrite (3.3) with Fibonacci numbers as follows:

$$\begin{aligned} F_{2n+1} &= a_n \cdot F_{n+1} + F_1 + 1 \Rightarrow \\ F_{2n+1} &= a_n \cdot F_{n+1} + 2. \end{aligned} \quad (3.4)$$

The existence of this representation of  $G_3$  assures that the proof is complete.  $\square$

Ultimately, this result links together two seemingly unrelated properties: the coefficients of certain PLRS and the necessary conditions for uniqueness of decompositions. We see that  $a_n$  acts both as the coefficient of the first term of the  $n$ -gap Fibonacci recurrence and as the highest coefficient necessary for an integer decomposition using the terms generated by the recurrence. The sign of the second term of the recurrence in turn determines whether the integer decompositions are unique, and where a positive term corresponds to uniqueness. This naturally extends to linear combinations of  $n$ -gap Fibonacci, i.e. the recurrences of the form

$$G_k = G_{k-1} + G_{k-2}, \quad (3.5)$$

where  $G_1, G_2 \in \mathbb{Z}^+$ .



## 4. CONCLUSION AND FUTURE WORK

Our method of looking specifically at  $n$ -gap Fibonacci sequences has lead us to several generalizations of Zeckendorf's theorem. We were able to connect these problems to the literature on PLRS by concluding that odd gap Fibonacci sequences are PLRS, and by utilizing results on the number of decompositions of natural numbers that exist using the elements of said sequences. The natural open problem to investigate is to determine whether these results can be extended to even integers  $n \geq 4$ .

Aside from the even integers case, there are also natural enumeration questions to consider. We could study the number of decompositions that arise if we remove the restriction placed by the recursive relationship, but still including the restriction on the number of copies of each summand. Alternatively, we could remove the restriction on the number of copies and investigate how to count the resulting decompositions.

Another possible direction is exploring different types of sequences beyond the Fibonacci numbers, such as Skiponaccis ( $S_k = S_{k-1} + S_{k-3}$ ), Tribonaccis ( $T_k = T_{k-1} + T_{k-2} + T_{k-3}$ ), and so on, and seeing if we can find potential positive linear recursive sequences by changing the values of the coefficients. We could also explore trying to extend our work to cover sequences of the form  $G_n = \alpha G_{n-1} + \beta G_{n-2}$ , where  $\alpha$  and  $\beta$  are arbitrary integers.

## APPENDIX A. ALTERNATIVE PROOFS OF LEMMAS 2.2 AND 2.4

One can use multisection techniques [GL, Len2] for the generating functions to obtain alternative proofs of Lemmas 2.2 and 2.4. The steps are as follows.

The following generating function expansions for Fibonacci Numbers and Lucas Numbers are well-known.

$$\begin{aligned} f(x, 1, 0) &= \sum_{k=0}^{\infty} F_k x^k = \frac{x}{1-x-x^2} \\ l(x, 1, 0) &= \sum_{k=0}^{\infty} L_k x^k = \frac{2-x}{1-x-x^2}. \end{aligned} \tag{A.1}$$

Then by invoking the multisection technique on (A.1) we obtain

$$\begin{aligned} f(x, n, m) &= \sum_{k=0}^{\infty} F_{kn+m} x^k = \frac{F_m + (F_{n+m} - F_m L_n)x}{1 - L_n x + (-1)^n x^2} \\ l(x, n, m) &= \sum_{k=0}^{\infty} L_{kn+m} x^k = \frac{L_m + (L_{n+m} - L_m L_n)x}{1 - L_n x + (-1)^n x^2}. \end{aligned} \tag{A.2}$$

For the special case  $n = 1$  and  $m = 0$  [Len2], we have that

$$f(x, n, 0) = \frac{F_n x}{1 - L_n x + (-1)^n x^2} \text{ and } l(x, n, 0) = \frac{2 - L_n x}{1 - L_n x + (-1)^n x^2}. \tag{A.3}$$

The key realization is that the generating extend to negative integer values for the parameters  $m$  and  $n$ .

## REFERENCES

- [Al] H. Alpert, *Differences of multiple Fibonacci numbers*, Integers: Electronic Journal of Combinatorial Number Theory **9** (2009), 745–749.
- [BCCFLMX] N. Borade, D. Cai, D. Z. Chang, B. Fang, A. Liang, S. J. Miller, and W. Xu, *Gaps of Summands of the Zeckendorf Lattice*, to appear in the Fibonacci Quarterly. <https://arxiv.org/pdf/1909.01935.pdf>.
- [BM] I. Ben-Ari and S. J. Miller, *A Probabilistic Approach to Generalized Zeckendorf Decompositions*, SIAM Journal on Discrete Mathematics, **30** (2016), no. 2, 1302–1332.

- [Br] J. L. Brown, Jr., *Zeckendorf's Theorem and Some Applications*, The Fibonacci Quarterly, Vol. 2, No. 3 (Oct. 1964), pages 163–168.
- [CFHMN] M. Catral, P. Ford, P. E. Harris, S. J. Miller, and D. Nelson, *Legal Decompositions Arising from Non-positive Linear Recurrences*, Fibonacci Quarterly **54** (2016), no. 4, 3448–365.
- [CFHMNPX] M. Catral, P. Ford, P. E. Harris, S. J. Miller, D. Nelson, Z. Pan and H. Xu, *New Behavior in Legal Decompositions Arising from Non-positive Linear Recurrences*, Fibonacci Quarterly **55** (2017), no. 3, 252–275. Expanded arXiv version: <http://arxiv.org/pdf/1606.09309>.
- [CCGJMSY] E. Chen, R. Chen, L. Guo, C. Jiang, S. J. Miller, J. M. Siktart and Peter Yu, *Gaussian Behavior in Zeckendorf Decompositions From Lattices*, The Fibonacci Quarterly, **57** (2019), no. 3, 201–212.
- [CG1] F. Chung, R. L. Graham, *On irregularities of distribution, in Finite and Infinite Sets*, Colloq. Math. Soc. János Bolyai **37** (1981), 181–222.
- [CG2] F. Chung and R. L. Graham, *Well dispersed sequences in  $[0, 1]^d$* , Journal of Number Theory **189** (2018), 1–24.
- [CHHMPV] K. Cordwell, M. Hlavacek, C. Huynh, S. J. Miller, C. Peterson, and Y. Vu, *Summand Minimality and Asymptotic Convergence of Generalized Zeckendorf Decompositions*, Research in Number Theory **43** (2018), no. 3, 1–27.
- [Day] D. E. Daykin, *Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers*, J. London Mathematical Society **35** (1960), 143–160.
- [DDKMMV] P. Demontigny, T. Do, A. Kulkarni, S. J. Miller, D. Moon and U. Varma, *Generalizing Zeckendorf's Theorem to  $f$ -decompositions*, Journal of Number Theory **141** (2014), 136–158.
- [DDKMV] P. Demontigny, T. Do, A. Kulkarni, S. J. Miller and U. Varma, *A Generalization of Fibonacci Far-Difference Representations and Gaussian Behavior*, Fibonacci Quarterly **52** (2014), no. 3, 247–273.
- [DFFHMPP] R. Dorward, P. Ford, E. Fourakis, P. E. Harris, S. J. Miller, E. Palsson, and H. Paugh, *New Behavior in Legal Decompositions Arising From Non-Positive Linear Recurrences*, Fibonacci Quarterly, **55** (2017), no. 3, 252–275.
- [DG] M. Drmota and J. Gajdosik, *The distribution of the sum-of-digits function*, J. Théor. Nombrés Bordeaux **10** (1998), no. 1, 17–32.
- [FJLLLMSS] E. Fang, J. Jenkins, Z. Lee, D. Li, E. Lu, S.J. Miller, D. Salgado, and J. Siktart, *Central Limit Theorems for Compound Paths on the Two-Dimensional Lattice*, to appear in Fibonacci Quarterly.
- [Fr] A. S. Fraenkel, *Systems of Numeration*, Amer. Math. Monthly **92** (1985), no. 2, 105–114.
- [GL] I. Gessel and T. Lengyel, *On the order of Stirling numbers and alternating binomial coefficient sums*, *Fibonacci Quart.* 39(2001), 444–454.
- [GTNP] P. J. Grabner, R. F. Tichy, I. Nemes, and A. Pethö, *Generalized Zeckendorf expansions*, Appl. Math. Lett. **7** (1994), no. 2, 25–28.
- [Ha] N. Hamlin, *Representing Positive Integers as a Sum of Linear Recurrence Sequences*, Abstracts of Talks, Fourteenth International Conference on Fibonacci Numbers and Their Applications (2010), pages 2–3.
- [HW] N. Hamlin and W. A. Webb, *Representing positive integers as a sum of linear recurrence sequences*, Fibonacci Quarterly **50** (2012), no. 2, 99–105.
- [Ho] V. E. Hoggatt, *Generalized Zeckendorf theorem*, Fibonacci Quarterly **10** (1972), no. 1 (special issue on representations), pages 89–93.
- [Ke] T. J. Keller, *Generalizations of Zeckendorf's theorem*, Fibonacci Quarterly **10** (1972), no. 1 (special issue on representations), pages 95–102.
- [KKMW] M. Kologlu, G. Kopp, S. J. Miller, and Y. Wang, *On the Number of Summands in Zeckendorf Decompositions*, Journal of Number Theory **49** (2011), no. 2, 116–130.
- [Kos] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, 2001.
- [LT] M. Lamberger and J. M. Thuswaldner, *Distribution properties of digital expansions arising from linear recurrences*, Math. Slovaca **53** (2003), no. 1, 1–20.
- [Len1] T. Lengyel, *A Counting Based Proof of the Generalized Zeckendorf's Theorem*, Fibonacci Quarterly **44** (2006), no. 4, 324–325.
- [Len2] T. Lengyel, *Divisibility properties by multisection*, *Fibonacci Quart.* 41(2003), 72–79.

# ZECKENDORF'S THEOREM USING INDICES IN AN ARITHMETIC PROGRESSION

- [Luo] D.C. Luo, *Nonuniqueness Properties of Zeckendorf Related Decompositions*, <https://arxiv.org/pdf/2004.08316.pdf>.
- [ML] R. Li and S. J. Miller, *A Collection of Central Limit Type Results in Generalized Zeckendorf Decompositions*, The 17th International Fibonacci Conference, Fibonacci Quarterly, **55** (2017), 205–214.
- [MW1] S. J. Miller and Y. Wang, *From Fibonacci Numbers to Central Limit Type Theorems*, Journal of Combinatorial Theory, Series A **119** (2012), no. 7, 1398–1413. Expanded arXiv version: <https://arxiv.org/pdf/1008.3202>.
- [MW2] S. J. Miller and Y. Wang, *Gaussian Behavior in Generalized Zeckendorf Decompositions*, Combinatorial and Additive Number Theory, CANT 2011 and 2012 (Melvyn B. Nathanson, editor), Springer Proceedings in Mathematics & Statistics (2014), 159–173.
- [Ost] A. Ostrowski, *Bemerkungen zur Theorie der Diophantischen Approximationen*, Abh. Math. Sem. Hamburg **1** (1922), 77–98; 250–251.
- [PT] A. Pethő and R. F. Tichy, *On digit expansions with respect to linear recurrences*, J. Number Theory **33** (1989), 243–256.
- [Ste1] W. Steiner, *Parry expansions of polynomial sequences*, Integers **2** (2002), Paper A14.
- [Ste2] W. Steiner, *The Joint Distribution of Greedy and Lazy Fibonacci Expansions*, Fibonacci Quarterly **43** (2005), 60–69.
- [Ze] E. Zeckendorf, *Représentation des nombres naturels par une somme des nombres de Fibonacci ou de nombres de Lucas*, Bulletin de la Société Royale des Sciences de Liège **41** (1972), 179–182.

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213

*Email address:* [ameliasg97@gmail.com](mailto:ameliasg97@gmail.com)

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213

*Email address:* [hadley.killen@gmail.com](mailto:hadley.killen@gmail.com)

DEPARTMENT OF MATHEMATICS, OCCIDENTAL COLLEGE, 1600 CAMPUS ROAD, LOS ANGELES, CA 90041

*Email address:* [lengyel@oxy.edu](mailto:lengyel@oxy.edu)

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, AND DEPARTMENT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267

*Email address:* [sjm1@williams.edu](mailto:sjm1@williams.edu)

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213

*Email address:* [nadia.razek97@gmail.com](mailto:nadia.razek97@gmail.com)

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TENNESSEE-KNOXVILLE, KNOXVILLE, TN 37916

*Email address:* [jsiktar@vols.utk.edu](mailto:jsiktar@vols.utk.edu)

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213

*Email address:* [liza.sulkin@gmail.com](mailto:liza.sulkin@gmail.com)