

# On Zeckendorf-Niven Numbers and Arithmetic Progressions

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## Abstract

A positive integer is Zeckendorf-Niven (respectively, Lucas-Niven) if it is divisible by the number of summands in its Zeckendorf decomposition (respectively, Lucas decomposition). We show that there exist infinitely many Zeckendorf-Niven numbers and Lucas-Niven numbers in every arithmetic progression. Furthermore, we provide bounds on the maximum number of consecutive Zeckendorf-Niven terms in certain arithmetic progressions.

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# 1 Introduction

A *Niven number* is a positive integer that is divisible by the sum of its digits in base 10 representation. This definition was first introduced by Niven during a lecture in 1977. Since then, Niven numbers have been studied in the literature in various context [2–5, 12, 13, 15]. Moreover, the definition of Niven numbers has also been generalized and modified in many different ways. For example, Grundman [8] defined  $b$ -Niven numbers as positive integers that are divisible by their digit sums in base  $b$ , while Gohn et al. [7] studied  $b$ -prodigious numbers, i.e., positive integers that are divisible by the product of their nonzero digits in base  $b$ .

One of the variations that we study are the Zeckendorf-Niven numbers, introduced by Ray and Cooper in 2006 [16]. To understand the definition, we begin with the following theorem by Zeckendorf [20]. Throughout this paper, we use  $\mathbb{N}$  to denote the set of positive integers, and for all nonnegative integers  $i$ ,  $F_i$  denotes the  $i$ -th Fibonacci number, i.e.,  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_i = F_{i-1} + F_{i-2}$  for all integers  $i \geq 2$ .

**Theorem 1.1.** *For every  $n \in \mathbb{N}$ , there exists a unique sequence  $(\zeta_i(n))_{i=2}^{\infty}$  such that  $\zeta_i(n) \in \{0, 1\}$  and  $\zeta_i(n) + \zeta_{i+1}(n) \leq 1$  for all  $i \geq 2$ , and*

$$n = \sum_{i=2}^{\infty} \zeta_i(n) F_i. \quad (1)$$

Note that the sum in equation (1) starts at  $i = 2$ , or else we clearly lose uniqueness (either from adding  $F_0$  as a summand or from switching between  $F_1$  and  $F_2$ .) This sum is called the *Zeckendorf decomposition* of  $n$ . Although the Zeckendorf decomposition is presented as an infinite sum in equation (1), there are only finitely many nonzero summands since all Fibonacci numbers are positive. For every  $n \in \mathbb{N}$ , the number of nonzero summands in the Zeckendorf decomposition of  $n$  is defined as  $s_Z(n)$ , i.e.,

$$s_Z(n) := \sum_{i=2}^{\infty} \zeta_i(n).$$

We say that  $n$  is *Zeckendorf-Niven* if  $s_Z(n)$  divides  $n$ .

Similarly, we define Lucas-Niven numbers as follows. For all nonnegative integers  $i$ , let  $L_i$  denote the  $i$ -th Lucas number, i.e.,  $L_0 = 2$ ,  $L_1 = 1$ , and  $L_i = L_{i-1} + L_{i-2}$  for all integers  $i \geq 2$ . Brown [1] proved that every positive integer has a unique *Lucas decomposition*.

**Theorem 1.2.** *For every  $n \in \mathbb{N}$ , there exists a unique sequence  $(\lambda_i(n))_{i=0}^{\infty}$  such that  $\lambda_i(n) \in \{0, 1\}$  and  $\lambda_i(n) + \lambda_{i+1}(n) \leq 1$  for all  $i \geq 0$ ,  $\lambda_0(n) + \lambda_2(n) \leq 1$ , and*

$$n = \sum_{i=0}^{\infty} \lambda_i(n) L_i.$$

We say that  $n$  is *Lucas-Niven* if  $s_L(n)$  divides  $n$ , where

$$s_L(n) := \sum_{i=0}^{\infty} \lambda_i(n).$$

A common direction of study related to  $b$ -Niven numbers is their distribution in arithmetic progressions. An *arithmetic progression* is an infinite sequence of positive integers with the same common difference between consecutive terms. If the common difference is  $d$ , then we call this arithmetic progression a  $d$ -AP. Cooper and Kennedy [3] proved that the maximum number of consecutive Niven terms in a 1-AP is 20, while Grundman [8] and Wilson [19] generalized this result and proved that the maximum number of consecutive  $b$ -Niven terms in a 1-AP is  $2b$ . Later, Grundman et al. [10] studied the maximum number of consecutive  $b$ -Niven terms in a  $d$ -AP when  $d > 1$ . Harrington et al. [11] also showed that every arithmetic progression contains infinitely many  $b$ -Niven numbers.

One of the key results on Zeckendorf-Niven numbers in arithmetic progressions is provided by Grundman [9], and she proved that any sequence of consecutive Zeckendorf-Niven numbers greater than 6 has a maximum length of 4. We show that there are infinitely many Zeckendorf-Niven numbers as well as infinitely many Lucas-Niven numbers in any given arithmetic progression, with the proofs presented in Sections 2 and 3. We also show in Section 2 that every arithmetic progression contains an arbitrary number of consecutive terms that are all non-Zeckendorf-Niven. In Section 4, we give upper and lower bounds on the maximum number of consecutive Zeckendorf-Niven terms in a 2-AP. In Section 5, we prove that the maximum number of consecutive terms in an  $F_d$ -AP that share the same  $s_Z$ -value is three, and show that there are infinitely many of such occurrences where all three terms are also Zeckendorf-Niven.

## 2 Infinitely many Zeckendorf-Niven numbers in every arithmetic progression

We begin this section with several preliminary results on Fibonacci numbers.

**Proposition 2.1** ([14], Theorem 16.1). *For all  $i, j \in \mathbb{N}$  such that  $i \mid j$ , we have  $F_i \mid F_j$ .*

**Proposition 2.2** ([18]). *For all  $d \in \mathbb{N}$ , the sequence  $(F_i)_{i=0}^{\infty}$  is periodic modulo  $d$ .*

The period of the sequence  $(F_i)_{i=0}^{\infty}$  modulo  $d$  is called the  $d$ -th *Pisano period* and is denoted by  $\pi(d)$ . Some of the Pisano periods are known.

**Proposition 2.3** ([6]). *For all  $j \in \mathbb{N}$ , we have  $\pi(F_{2j}) = 4j$  and  $\pi(F_{2j+1}) = 8j + 4$ .*

The following lemma allows us to focus on arithmetic progressions that have a Fibonacci number as the common difference.

**Lemma 2.4.** *For every  $a, d \in \mathbb{N}$ , there exists  $j \in \mathbb{N}$  such that  $F_{2j} \geq a$  and  $(a + kF_{2j})_{k=0}^{\infty}$  is a subsequence of  $(a + kd)_{k=0}^{\infty}$ .*

77 *Proof.* Let  $j = a\pi(d)$ . Then  $F_{2j} \geq F_{2a} \geq a$ . The conclusion follows since  $F_{2j} = F_{2a\pi(d)} - F_0 \equiv$   
78  $0 \pmod{d}$  by the definition of Pisano period.  $\square$

79 The next lemma shows the existence of an appropriate term in the arithmetic progression  
80  $(a + kF_{2j})_{k=0}^{\infty}$  that achieves a prescribed number of nonzero summands in its Zeckendorf  
81 decomposition.

82 **Lemma 2.5.** *Let  $(a + kF_{2j})_{k=0}^{\infty}$  be a given arithmetic progression. Then for every integer*  
83  *$m > s_Z(a)$ , there exists a term  $N$  in the arithmetic progression  $(a + kF_{2j})_{k=0}^{\infty}$  such that*  
84  *$s_Z(N) = m$ .*

85 *Proof.* Let  $i_0 \in \mathbb{N}$  such that  $F_{2i_0j} \geq a$ , and let

$$86 \quad N = a + \sum_{i=i_0+1}^{i_0+m-s_Z(a)} F_{2ij}.$$

87 Then  $N$  is a term in the arithmetic progression  $(a + kF_{2j})_{k=0}^{\infty}$  by Proposition 2.1. Furthermore,  
88 there are no consecutive Fibonacci numbers inside the summation in the definition of  $N$ , so  
89  $s_Z(N) = s_Z(a) + m - s_Z(a) = m$  by the uniqueness of Zeckendorf decomposition.  $\square$

90 Now, we are ready to prove the main theorem of this section.

91 **Theorem 2.6.** *Every arithmetic progression contains infinitely many Zeckendorf-Niven num-*  
92 *bers.*

93 *Proof.* It suffices to show that for every  $a, d \in \mathbb{N}$ , there exists a Zeckendorf-Niven number  
94 in the arithmetic progression  $(a + kd)_{k=0}^{\infty}$ , since the original statement can then be proved  
95 inductively by replacing the value of  $a$  with a larger value at each step. By Lemma 2.4,  
96 we can restrict our attention to arithmetic progressions of the form  $(a + kF_{2j})_{k=0}^{\infty}$  where  
97  $F_{2j} \geq a$ . Since  $F_{2j+1} > F_{2j} \geq a \geq s_Z(a)$ , there exists a term  $N$  in the arithmetic progression  
98  $(a + kF_{2j})_{k=0}^{\infty}$  such that  $s_Z(N) = F_{2j+1}$  by Lemma 2.5. To complete the proof, we are going  
99 to transform the Zeckendorf decomposition of  $N$  term-by-term so that each new summand  
100 is divisible by  $s_Z(N)$  and yet maintains the same residue modulo the common difference  $F_{2j}$   
101 of the arithmetic progression.

102 Let

$$103 \quad N = \sum_{i=1}^{F_{2j+1}} F_{r_i}$$

104 be the Zeckendorf decomposition of  $N$ , where  $r_1 \geq 2$  and  $r_{i+1} - r_i \geq 2$  for all  $1 \leq i \leq F_{2j+1} - 1$ .  
105 Since  $2j + 1$  and  $4j$  are coprime, for each  $r_i$ , there exists an integer  $0 \leq m_i < 2j + 1$  such  
106 that  $m_i \equiv -r_i(4j)^{-1} \pmod{2j + 1}$ , i.e.,  $2j + 1 \mid r_i + 4jm_i$ . Hence,  $F_{2j+1} \mid F_{r_i+4jm_i}$  by  
107 Proposition 2.1. Furthermore, let  $t_i = r_i + 4jm_i + (2j + 1)4ij$  for each  $1 \leq i \leq F_{2j+1}$ , and  
108 define

$$109 \quad N' = \sum_{i=1}^{F_{2j+1}} F_{t_i}.$$

Note that the summation provided is the Zeckendorf decomposition of  $N'$  since  $t_1 \geq 2$  and

$$t_{i+1} - t_i = r_{i+1} - r_i + 4j(m_{i+1} - m_i + 2j + 1) > r_{i+1} - r_i \geq 2$$

for all  $1 \leq i \leq F_{2j+1} - 1$ , thus  $s_Z(N') = F_{2j+1}$ . By Proposition 2.3, we have  $\pi(F_{2j+1}) = 8j + 4$ , which divides  $t_i - (r_i + 4jm_i)$  for all  $1 \leq i \leq F_{2j+1}$ , so  $F_{t_i} \equiv F_{r_i + 4jm_i} \equiv 0 \pmod{F_{2j+1}}$ . Therefore,  $F_{2j+1} \mid N'$  and  $N'$  is Zeckendorf-Niven.

Finally, by Proposition 2.3,  $\pi(F_{2j}) = 4j$ , which divides  $t_i - r_i$  for all  $1 \leq i \leq F_{2j+1}$ , implying that  $F_{2j} \mid F_{t_i} - F_{r_i}$ . Hence,  $F_{2j} \mid N' - N$ , i.e.,  $N'$  is a term in the arithmetic progression  $(a + kF_{2j})_{k=0}^{\infty}$ .  $\square$

Before ending this section, we provide a result concerning the distribution of non-Zeckendorf-Niven numbers in arithmetic progressions.

**Theorem 2.7.** *Every arithmetic progression contains infinitely many subsequences of  $\ell$  consecutive non-Zeckendorf-Niven terms, where  $\ell$  is an arbitrary positive integer.*

*Proof.* Let  $(a + kd)_{k=0}^{\infty}$  be a given arithmetic progression. Let  $g \in \mathbb{N}$  such that  $g > a + (\ell - 1)d$ , and let

$$j = \text{lcm}(\{\pi(s_Z(a + kd) + g) : 0 \leq k \leq \ell - 1\} \cup \{\pi(d)\}).$$

Note that  $j \geq 2$  since  $\pi(n) \geq 2$  for all integers  $n \geq 2$ . Further let  $m \in \mathbb{N}$  such that  $jm \geq r + 2$ , where  $F_r$  is the largest summand in the Zeckendorf decomposition of  $a + (\ell - 1)d$ .

For each integer  $0 \leq k \leq \ell - 1$ , let

$$a_k = a + kd + \sum_{i=1}^g F_{j(m+i)}.$$

If we replace  $a + kd$  by its Zeckendorf decomposition, then the above summation is the Zeckendorf decomposition of  $a_k$  due to the construction of  $jm$  and the fact that  $j \geq 2$ . Hence,  $s_Z(a_k) = s_Z(a + kd) + g$ , implying that  $s_Z(a_k) \mid \sum_{i=1}^g F_{j(m+i)}$  by the definition of  $j$ . However,  $s_Z(a_k) > g \geq a + kd > 0$ , so  $s_Z(a_k) \nmid a_k$ , i.e.,  $a_k$  is non-Zeckendorf-Niven. The conclusion follows since  $a_k - a = kd + \sum_{i=1}^g F_{j(m+i)} \equiv 0 \pmod{d}$ .  $\square$

### 3 Infinitely many Lucas-Niven numbers in every arithmetic progression

To prove a result analogous to Theorem 2.6 for the Lucas sequence, we need to make small modifications to our technique due to the lack of divisibility properties in the Lucas sequence such as those given by Proposition 2.1 and Lemma 2.4.

Similar to the Fibonacci sequence, it is known that the Lucas sequence is also periodic modulo  $d$  for all  $d \in \mathbb{N}$  [18]. Denote this period by  $\pi_L(d)$ . In the following proposition, we are going to state an interesting relation between  $\pi(d)$  and  $\pi_L(d)$  as well as some property of  $\pi(d)$ , which directly implies a subsequent lemma on  $\pi_L(d)$ .

**Proposition 3.1** ([18]).

(i) For every  $d \in \mathbb{N}$  such that  $d \not\equiv 0 \pmod{5}$ , we have  $\pi(d) = \pi_L(d)$ .

(ii) For every prime  $p$  such that  $p \equiv -1 \pmod{10}$ , we have  $\pi(p) \mid p - 1$ .

**Lemma 3.2.** For every prime  $p$  such that  $p \equiv -1 \pmod{10}$ , we have  $\pi_L(p) \mid p - 1$ .

The next lemma is an analogue of Lemma 2.5.

**Lemma 3.3.** Let  $(a + kd)_{k=0}^{\infty}$  be a given arithmetic progression. Then for every integer  $m > s_L(a)$ , there exists a term  $N$  in the arithmetic progression  $(a + kd)_{k=0}^{\infty}$  such that  $s_L(N) = m$ .

*Proof.* It suffices to show that there exists a term  $N$  in the arithmetic progression  $(a + kd)_{k=0}^{\infty}$  such that  $s_L(N) = s_L(a) + 1$ , since the original statement can then be proved inductively by replacing the value of  $a$  with  $N$  at each step.

Let

$$a = \sum_{i=1}^{s_L(a)} L_{r_i}$$

be the Lucas decomposition of  $a$ . To simplify the notation, let  $r = r_{s_L(a)}$ . Define

$$N = L_{r-1+3\pi_L(d)} + L_{r-2+2\pi_L(d)} + \sum_{i=1}^{s_L(a)-1} L_{r_i}.$$

Note that the summation provided is the Lucas decomposition of  $N$  since  $(r - 1 + 3\pi_L(d)) - (r - 2 + 2\pi_L(d)) \geq 2$  and  $(r - 2 + 2\pi_L(d)) - r_{s_L(a)-1} \geq (r - 2 + 2\pi_L(d)) - (r - 2) \geq 2$ , thus  $s_L(N) = s_L(a) + 1$ . Furthermore,

$$N - a = L_{r-1+3\pi_L(d)} + L_{r-2+2\pi_L(d)} - L_r = (L_{r-1+3\pi_L(d)} - L_{r-1}) + (L_{r-2+2\pi_L(d)} - L_{r-2}),$$

which is divisible by  $d$ . Therefore,  $N$  is a term in the arithmetic progression  $(a + kd)_{k=0}^{\infty}$ .  $\square$

This leads to the main theorem of this section.

**Theorem 3.4.** Every arithmetic progression contains infinitely many Lucas-Niven numbers.

*Proof.* With the same logic presented in the proof of Theorem 2.6, it suffices to show that for every  $a, d \in \mathbb{N}$ , there exists a Lucas-Niven number in the arithmetic progression  $(a + kd)_{k=0}^{\infty}$ . By Dirichlet's Theorem, there are infinitely many primes  $p$  such that  $p \equiv -1 \pmod{10\pi_L(d)}$ . Let  $p > s_L(a)$  be such a prime. Note that  $\gcd(p - 1, \pi_L(d)) \leq 2$ , which implies that  $\gcd(\pi_L(p), \pi_L(d)) \leq 2$  by Lemma 3.2. Since  $p > s_L(a)$ , there exists a term  $N$  in the arithmetic progression  $(a + kd)_{k=0}^{\infty}$  such that  $s_L(N) = p$  by Lemma 3.3. Let

$$N = \sum_{i=1}^p L_{r_i}$$

be the Lucas decomposition of  $N$ . We are going to complete this proof by considering the following two cases.

*Case 1:*  $\gcd(\pi_L(p), \pi_L(d)) = 1$ .

For each  $1 \leq i \leq p$ , there exists an integer  $0 \leq m_i < \pi_L(p)$  such that

$$m_i \equiv (-r_i + 1)\pi_L(d)^{-1} \pmod{\pi_L(p)},$$

i.e.,  $\pi_L(p) \mid r_i - 1 + \pi_L(d)m_i$ . Let  $t_i = r_i + \pi_L(d)(m_i + i\pi_L(p))$  and define

$$N' = \sum_{i=1}^p L_{t_i}.$$

Note that  $\pi_L(p) \geq 2$ , so the summation provided is the Lucas decomposition of  $N'$  since  $t_1 \geq 2$  and  $t_{i+1} - t_i = r_{i+1} - r_i + \pi_L(d)\pi_L(p) \geq 2$  for all  $1 \leq i \leq p-1$ . Consequently,  $s_L(N') = p$ . Moreover,  $t_i \equiv 1 \pmod{\pi_L(p)}$  for all  $1 \leq i \leq p$ , so  $N' \equiv \sum_{i=1}^p L_1 \equiv 0 \pmod{p}$ , implying that  $N'$  is Lucas-Niven. Furthermore, since  $\pi_L(d) \mid t_i - r_i$  for all  $1 \leq i \leq p$ , we have  $d \mid N' - N$ . Therefore,  $N'$  is a term in the arithmetic progression  $(a + kd)_{k=0}^\infty$ .

*Case 2:*  $\gcd(\pi_L(p), \pi_L(d)) = 2$ .

Let  $E = \{1 \leq i \leq p : r_i \text{ is even}\}$  and  $O = \{1 \leq i \leq p : r_i \text{ is odd}\}$ . We consider the following subcases.

*Case 2.1:*  $|E| \geq |O|$ .

Partition  $E$  into  $E_1$  and  $E_2$  such that  $|E_1| = |E| - |O|$  and  $|E_2| = |O|$ . For each  $1 \leq i \leq p$ , there exists an integer  $0 \leq m_i < \pi_L(p)/2$  such that

- $m_i \equiv -(r_i/2)(\pi_L(d)/2)^{-1} \pmod{\pi_L(p)/2}$  if  $i \in E_1$ , i.e.,  $\pi_L(p) \mid r_i + \pi_L(d)m_i$ ;
- $m_i \equiv -((r_i - 2)/2)(\pi_L(d)/2)^{-1} \pmod{\pi_L(p)/2}$  if  $i \in E_2$ , i.e.,  $\pi_L(p) \mid r_i - 2 + \pi_L(d)m_i$ ;
- and
- $m_i \equiv -((r_i - 1)/2)(\pi_L(d)/2)^{-1} \pmod{\pi_L(p)/2}$  if  $i \in O$ , i.e.,  $\pi_L(p) \mid r_i - 1 + \pi_L(d)m_i$ .

Similar to Case 1, let  $t_i = r_i + \pi_L(d)(m_i + i\pi_L(p))$  and define  $N' = \sum_{i=1}^p L_{t_i}$ . With the same reasoning as in Case 1, we have  $s_L(N') = p$ , and we also know that  $N'$  is a term in the arithmetic progression  $(a + kd)_{k=0}^\infty$ . Moreover,  $t_i \equiv 0 \pmod{\pi_L(p)}$  if  $i \in E_1$ ,  $t_i \equiv 2 \pmod{\pi_L(p)}$  if  $i \in E_2$ , and  $t_i \equiv 1 \pmod{\pi_L(p)}$  if  $i \in O$ , so

$$N' \equiv \sum_{i \in E_1} L_0 + \sum_{i \in E_2} L_2 + \sum_{i \in O} L_1 \equiv 2(|E| - |O|) + 3|O| + |O| \equiv 2p \equiv 0 \pmod{p},$$

implying that  $N'$  is Lucas-Niven.

*Case 2.2:*  $|E| < |O|$ .

Partition  $O$  into  $O_1$  and  $O_2$  such that  $|O_1| = |O| - |E|$  and  $|O_2| = |E|$ . For each  $1 \leq i \leq p$ , there exists an integer  $0 \leq m_i < \pi_L(p)/2$  such that

- $m_i \equiv -((r_i - 2)/2)(\pi_L(d)/2)^{-1} \pmod{\pi_L(p)/2}$  if  $i \in E$ , i.e.,  $\pi_L(p) \mid r_i - 2 + \pi_L(d)m_i$ ;
- $m_i \equiv -((r_i - 1)/2)(\pi_L(d)/2)^{-1} \pmod{\pi_L(p)/2}$  if  $i \in O_1$ , i.e.,  $\pi_L(p) \mid r_i - 1 + \pi_L(d)m_i$ ;  
and
- $m_i \equiv -((r_i + 1)/2)(\pi_L(d)/2)^{-1} \pmod{\pi_L(p)/2}$  if  $i \in O_2$ , i.e.,  $\pi_L(p) \mid r_i + 1 + \pi_L(d)m_i$ .

Once again, letting  $t_i = r_i + \pi_L(d)(m_i + i\pi_L(p))$  and defining  $N' = \sum_{i=1}^p L_{t_i}$  give us  $s_L(N') = p$  and that  $N'$  is a term in the arithmetic progression  $(a + kd)_{k=0}^\infty$ . Finally,  $t_i \equiv 2 \pmod{\pi_L(p)}$  if  $i \in E$ ,  $t_i \equiv 1 \pmod{\pi_L(p)}$  if  $i \in O_1$ , and  $t_i \equiv -1 \pmod{\pi_L(p)}$  if  $i \in O_2$ , so by defining  $L_{-1} = -1$ , we have

$$N' \equiv \sum_{i \in E} L_2 + \sum_{i \in O_1} L_1 + \sum_{i \in O_2} L_{-1} \equiv 3|E| + (|O| - |E|) + (-1)|E| \equiv p \equiv 0 \pmod{p},$$

implying that  $N'$  is Lucas-Niven. □

## 4 Maximum number of consecutive Zeckendorf-Niven terms in a 2-AP

Motivated by Grundman's proof [9] on the maximum number of consecutive Zeckendorf-Niven terms in a 1-AP, we define  $z_6(n) = \sum_{i=2}^6 \zeta_i(n)F_i$  for every  $n \in \mathbb{N}$ . In the following two lemmas, we demonstrate some interesting relation between the values of  $z_6(n)$  and Zeckendorf-Niven numbers.

**Lemma 4.1.** *For every integer  $n \geq 10$ , if  $z_6(n) = 7$  and  $\zeta_7(n) = 0$ ,  $z_6(n) = 4$ , or  $z_6(n) = 9$ , then  $n$  and  $n + 2$  cannot be both Zeckendorf-Niven.*

*Proof.* If  $z_6(n) = 7 = F_3 + F_5$  and  $\zeta_7(n) = 0$ , then  $z_6(n + 2) = 9 = F_2 + F_6$ ; if  $z_6(n) = 4 = F_2 + F_4$ , then  $\zeta_6(n) = 0$ , so  $z_6(n + 2) = 6 = F_2 + F_5$ ; if  $z_6(n) = 9 = F_2 + F_6$ , then  $\zeta_7(n) = 0$ , so  $z_6(n + 2) = 11 = F_4 + F_6$ . Hence,  $s_Z(n) = s_Z(n + 2)$  in all cases. Since  $n \geq 10$  and  $z_6(n) \in \{7, 4, 9\}$ , we have  $s_Z(n) \geq 3$ . As a result,  $s_Z(n)$  does not divide  $(n + 2) - n = 2$ , so  $n$  and  $n + 2$  cannot be both Zeckendorf-Niven. □

**Lemma 4.2.** *For every integer  $n \geq 4$ , if  $z_6(n) = 1$  or  $z_6(n) = 3$  and both  $n$  and  $n + 2$  are Zeckendorf-Niven, then  $s_Z(n) = 2$ .*

235 *Proof.* If  $z_6(n) = 1 = F_2$ , then  $z_6(n+2) = 3 = F_4$  since  $\zeta_5(n) = \zeta_5(n+2) = 0$ ; if  
 236  $z_6(n) = 3 = F_4$ , then  $z_6(n) = 5 = F_5$  since  $\zeta_6(n) = \zeta_6(n+2) = 0$ . Hence,  $s_Z(n) = s_Z(n+2)$ .  
 237 If both  $n$  and  $n+2$  are Zeckendorf-Niven, then  $s_Z(n)$  divides  $(n+2) - n = 2$ . Since  $n \geq 4$   
 238 and  $z_6(n) = \{1, 3\}$ , we have  $s_Z(n) \geq 2$ . Therefore,  $s_Z(n) = 2$ .  $\square$

239 The following two theorems establish an upper bound and a lower bound for the maximum  
 240 number of consecutive Zeckendorf-Niven terms in a 2-AP.

241 **Theorem 4.3.** *The only sequences of eight or more consecutive Zeckendorf-Niven terms in*  
 242 *a 2-AP are subsequences of 2, 4, 6, 8, 10, 12, 14, 16, 18.*

243 *Proof.* It is easy to verify that 2, 4, 6, 8, 10, 12, 14, 16, 18 are Zeckendorf-Niven. Now, sup-  
 244 pose that  $(n+2k)_{k=0}^7$  is not a subsequence of 2, 4, 6, 8, 10, 12, 14, 16, 18 and every term is  
 245 Zeckendorf-Niven. Since 7, 19, 20 are not Zeckendorf-Niven, we have  $n \geq 21$ .

246 By Lemma 4.1,  $z_6(n+2k) \notin \{4, 9\}$  for any integer  $0 \leq k \leq 6$ . We also have  $z_6(n+2k) \neq 2$   
 247 for any integer  $0 \leq k \leq 5$ ; otherwise,  $z_6(n+2k+2) = 4$  since  $\zeta_5(n+2k) = \zeta_5(n+2k+2) = 0$ .  
 248 Similarly, we have  $z_6(n+2k) \neq 0$  for any integer  $0 \leq k \leq 4$ ; otherwise,  $z_6(n+2k+2) = 2$   
 249 since  $\zeta_4(n+2k) = \zeta_4(n+2k+2) = 0$ . If  $z_6(n) = 11 = F_4 + F_6$ , then the Zeckendorf  
 250 decomposition of  $n+2$  is

$$251 \quad F_j + \sum_{i=j+2}^{\infty} \zeta_i(n) F_i,$$

252 where  $j \geq 7$  is the smallest integer such that  $\zeta_j(n) = \zeta_{j+1}(n) = 0$ . This leads to a contradic-  
 253 tion since  $z_6(n+2) = 0$ , so we have  $z_6(n) \neq 11$ .

254 By Lemma 4.1 again,  $z_6(n+2k) \neq 7$  for any integer  $0 \leq k \leq 6$  if  $\zeta_7(n+2k) = 0$ . If  
 255  $z_6(n+2k) = 3$  for some integer  $0 \leq k \leq 4$ , then  $s_Z(n+2k) = 2$  by Lemma 4.2. As a  
 256 result, we have  $\zeta_7(n+2k) = \zeta_7(n+2k+4) = 0$  since  $n \geq 21$  and  $z_6(n+2k+4) = 7$ ,  
 257 which is a contradiction. Hence,  $z_6(n+2k) \neq 3$  for any integer  $0 \leq k \leq 4$ . From this, we  
 258 deduce that  $z_6(n+2k) \neq 1$  for any integer  $0 \leq k \leq 3$ ; otherwise,  $z_6(n+2k+2) = 3$  since  
 259  $\zeta_5(n+2k) = \zeta_5(n+2k+2) = 0$ .

260 Next, if  $z_6(n+2k) = 12 = F_2 + F_4 + F_6$  for some integer  $0 \leq k \leq 2$ , then the Zeckendorf  
 261 decomposition of  $n+2k+2$  is

$$262 \quad F_2 + F_j + \sum_{i=j+2}^{\infty} \zeta_i(n+2k) F_i,$$

263 where  $j \geq 7$  is the smallest integer such that  $\zeta_j(n+2k) = \zeta_{j+1}(n+2k) = 0$ . This leads to a  
 264 contradiction since  $z_6(n+2k+2) = 1$ , so we have  $z_6(n+2k) \neq 12$  for any integer  $0 \leq k \leq 2$ .  
 265 Consequently,  $z_6(n+2k) \neq 10$  for any integer  $0 \leq k \leq 1$ ; otherwise,  $z_6(n+2k+2) = 12$   
 266 since  $\zeta_7(n+2k) = \zeta_7(n+2k+2) = 0$ . Furthermore,  $z_6(n) \neq 8$ ; otherwise,  $z_6(n+2) = 10$   
 267 since  $\zeta_7(n) = \zeta_7(n+2) = 0$ .

268 If  $z_6(n+2k) = 7$  for some integer  $0 \leq k \leq 2$  and  $\zeta_7(n+2k) = 1$ , then the Zeckendorf  
 269 decomposition of  $n+2k+2$  is

$$270 \quad F_2 + F_j + \sum_{i=j+2}^{\infty} \zeta_i(n+2k) F_i,$$

where  $j \geq 8$  is the smallest integer such that  $\zeta_j(n+2k) = \zeta_{j+1}(n+2k) = 0$ . This again leads to a contradiction since  $z_6(n+2k+2) = 1$ , so we have  $z_6(n+2k) \neq 7$  for any integer  $0 \leq k \leq 2$ . Then we also have  $z_6(n) \neq 5$ ; otherwise,  $z_6(n+2) = 7$  since  $\zeta_6(n) = \zeta_6(n+2) = 0$ . The only remaining case is  $z_6(n) = 6 = F_2 + F_5$ . If  $\zeta_7(n) = 1$ , then the Zeckendorf decomposition of  $n+2$  is

$$F_j + \sum_{i=j+2}^{\infty} \zeta_i(n) F_i,$$

where  $j \geq 8$  is the smallest integer such that  $\zeta_j(n) = \zeta_{j+1}(n) = 0$ . This leads to a contradiction since  $z_6(n+2) = 0$ . If  $\zeta_7(n) = 0$ , then  $z_6(n+4) = 10 = F_3 + F_6$ , so  $s_Z(n) = s_Z(n+4)$ . Since  $n \geq 21$  and  $z_6(n) = 6$ , we have  $s_Z(n) \geq 3$ , thus  $s_Z(n) = 4$ . On the other hand, the Zeckendorf decomposition of  $n+8$  is

$$F_2 + F_j + \sum_{i=j+2}^{\infty} \zeta_i(n) F_i,$$

where  $j \geq 7$  is the smallest integer such that  $\zeta_j(n) = \zeta_{j+1}(n) = 0$ . Hence,  $z_6(n+8) = 1$ , which implies that  $s_Z(n+8) = 2$  by Lemma 4.2. The only possibility is  $n = F_2 + F_5 + F_8 + F_{10} = 82$ . However,  $s_Z(n) = 4 \nmid n$ , contradicting that  $n$  is Zeckendorf-Niven.  $\square$

**Theorem 4.4.** *For every  $j \in \mathbb{N}$ , let  $n = 27 + F_{120j+17}$ . Then  $n, n+2, n+4, n+6, n+8$  is a sequence of five consecutive Zeckendorf-Niven terms in a 2-AP.*

*Proof.* The Zeckendorf decomposition of  $n$  is  $F_2 + F_5 + F_8 + F_{120j+17}$ , and it is easy to verify that  $s_Z(n) = s_Z(n+4) = 4$ ,  $s_Z(n+2) = s_Z(n+8) = 3$ , and  $s_Z(n+6) = 5$ . Note that  $\pi(3) = 8$ ,  $\pi(4) = 6$ , and  $\pi(5) = 20$ , so  $\text{lcm}\{\pi(3), \pi(4), \pi(5)\} = 120$ , implying that  $F_{120j+17} \equiv F_{17} = 1597 \equiv 37 \pmod{3 \cdot 4 \cdot 5}$ . The result follows since  $4 \mid 27 + 37$ ,  $3 \mid 29 + 37$ ,  $4 \mid 31 + 37$ ,  $5 \mid 33 + 37$ , and  $3 \mid 35 + 37$ .  $\square$

## 5 Consecutive Zeckendorf-Niven terms in an $F_d$ -AP with the same $s_Z$ -value

Other than 1-APs and 2-APs, the study of consecutive Zeckendorf-Niven numbers in a general  $d$ -AP is difficult. Here, we choose a special case to consider, namely when the common difference is a Fibonacci number, denoted by  $F_d$ . We further restrict our attention to the scenario when the consecutive terms share the same  $s_Z$ -value.

Generalizing the notation from Section 4, we define  $z_{x,y}(n) = \sum_{i=x}^y \zeta_i(n) F_i$  for all  $n, x, y \in \mathbb{N}$  (if  $i < 2$ , then we define  $\zeta_i(n) = 0$  for all  $n \in \mathbb{N}$ .) Now, we are ready to present the two main results in this section.

**Theorem 5.1.** *For every integer  $d \geq 2$ , there exists infinitely many  $a \in \mathbb{N}$  such that  $s_Z(a) = s_Z(a + F_d) = s_Z(a + 2F_d)$ . Furthermore, there does not exist  $a \in \mathbb{N}$  such that  $s_Z(a) = s_Z(a + F_d) = s_Z(a + 2F_d) = s_Z(a + 3F_d)$ .*

304 *Proof.* For every  $a \in \mathbb{N}$  such that  $z_{d-2,d+3}(a) = F_{d-1}$ , since  $z_{d-2,d+3}(a + F_d) = F_{d+1}$  and  
 305  $z_{d-2,d+3}(a + 2F_d) = F_{d+2}$ , we have  $s_Z(a) = s_Z(a + F_d) = s_Z(a + 2F_d)$ .

306 Now, suppose that  $s_Z(a) = s_Z(a + F_d) = s_Z(a + 2F_d) = s_Z(a + 3F_d)$  for some  $a \in \mathbb{N}$ . Note  
 307 that  $z_{d-1,d+1}(a + kF_d) \neq 0$  for all  $k \in \{0, 1, 2\}$ ; otherwise,  $s_Z(a + (k+1)F_d) = s_Z(a + kF_d) + 1$ .  
 308 Consequently, we also have  $z_{d-1,d+1}(a + kF_d) \neq F_{d+1}$  for all  $k \in \{0, 1\}$ ; otherwise,  $z_{d-1,d+1}(a +$   
 309  $(k+1)F_d) = 0$ . Furthermore,  $z_{d-1,d+1}(a) \neq F_{d-1}$ ; otherwise,  $z_{d-1,d+1}(a + F_d) \in \{0, F_{d+1}\}$ . If  
 310  $z_{d-1,d+1}(a + kF_d) = F_{d-1} + F_{d+1}$  for some  $k \in \{0, 1\}$ , then the Zeckendorf decomposition of  
 311  $a + (k+1)F_d$  is

$$312 \quad \sum_{i=2}^{d-3} \zeta_i(a + kF_d)F_i + F_{d-1} + F_j + \sum_{i=j+2}^{\infty} \zeta_i(a + kF_d)F_i,$$

313 where  $j \geq d+2$  is the smallest integer such that  $\zeta_j(a + kF_d) = \zeta_{j+1}(a + kF_d) = 0$ . Since  
 314  $s_Z(a + kF_d) = s_Z(a + (k+1)F_d)$ , we have  $j = d+2$ . As a result, the Zeckendorf decomposition  
 315 of  $a + (k+2)F_d$  is

$$316 \quad \sum_{i=2}^{d-3} \zeta_i(a + kF_d)F_i + F_{j'} + \sum_{i=j'+2}^{\infty} \zeta_i(a + kF_d)F_i,$$

317 where  $j' \geq d+3$  is the smallest integer such that  $\zeta_{j'}(a + kF_d) = \zeta_{j'+1}(a + kF_d) = 0$ . This leads  
 318 to a contradiction since  $s_Z(a + kF_d) > s_Z(a + (k+2)F_d)$ . Thus  $z_{d-1,d+1}(a + kF_d) \neq F_{d-1} + F_{d+1}$   
 319 for all  $k \in \{0, 1\}$ . Therefore,  $z_{d-1,d+1}(a) = F_d$ .

320 It is useful to observe that  $2F_i = F_{i-2} + F_{i+1}$  for all integers  $i \geq 2$ . Next, we have  
 321  $z_{d-3,d+1}(a) \neq F_d$ ; otherwise,  $z_{d-1,d+1}(a + F_d) \in \{0, F_{d+1}\}$ . If  $z_{d-3,d+1}(a) = F_{d-3} + F_d$ , then  
 322 the Zeckendorf decomposition of  $a + F_d$  is

$$323 \quad \sum_{i=2}^{d-5} \zeta_i(a)F_i + F_{d-1} + F_j + \sum_{i=j+2}^{\infty} \zeta_i(a)F_i,$$

324 where  $j \geq d+1$  is the smallest integer such that  $\zeta_j(a) = \zeta_{j+1}(a) = 0$ . Since  $s_Z(a) = s_Z(a + F_d)$ ,  
 325 we have  $j = d+1$ . This leads to a contradiction since  $z_{d-1,d+1}(a + F_d) = F_{d-1} + F_{d+1}$ . Hence,  
 326 it remains to consider the case when  $z_{d-3,d+1}(a) = F_{d-2} + F_d$ .

327 Since  $F_{d-2} + 2F_d = 2F_{d-2} + F_{d+1} = F_{d-4} + F_{d-1} + F_{d+1}$ , we have  $z_{d-1,d+1}(a + F_d) \in$   
 328  $\{F_{d-1}, F_{d-1} + F_{d+1}\}$ . We have shown that  $z_{d-1,d+1}(a + F_d) \neq F_{d-1} + F_{d+1}$ , so  $z_{d-1,d+1}(a + F_d) =$   
 329  $F_{d-1}$ , which implies that  $\zeta_{d+2}(a) = 1$ . Let  $j \geq d+3$  be the smallest integer such that  
 330  $\zeta_j(a) = \zeta_{j+1}(a) = 0$ , and let  $j' \leq d-3$  be the largest integer such that  $\zeta_{j'}(a) = \zeta_{j'-1}(a) = 0$ .  
 331 If  $\zeta_{j'-2}(a) = 1$ , then the Zeckendorf decomposition of  $a$  is

$$332 \quad \sum_{i=2}^{j'-4} \zeta_i(a)F_i + F_{j'-2} + \sum_{i=0}^{\frac{d-2-(j'+1)}{2}} F_{j'+1+2i} + F_d + \sum_{i=0}^{\frac{j-1-(d+2)}{2}} F_{d+2+2i} + \sum_{i=j+2}^{\infty} \zeta_i(a)F_i$$

333 and that of  $a + F_d$  is

$$334 \quad \sum_{i=2}^{j'-4} \zeta_i(a)F_i + F_{j'} + \sum_{i=0}^{\frac{d-2-(j'+1)}{2}} F_{j'+2+2i} + F_j + \sum_{i=j+2}^{\infty} \zeta_i(a)F_i.$$

335 This leads to a contradiction since  $s_Z(a) > s_Z(a + F_d)$ . Thus  $\zeta_{j'-2}(a) = 0$ , the Zeckendorf  
 336 decomposition of  $a$  is

$$337 \quad \sum_{i=2}^{j'-3} \zeta_i(a) F_i + \sum_{i=0}^{\frac{d-2-(j'+1)}{2}} F_{j'+1+2i} + F_d + \sum_{i=0}^{\frac{j-1-(d+2)}{2}} F_{d+2+2i} + \sum_{i=j+2}^{\infty} \zeta_i(a) F_i,$$

338 and that of  $a + F_d$  is

$$339 \quad \sum_{i=2}^{j'-3} \zeta_i(a) F_i + F_{j'-1} + \sum_{i=0}^{\frac{d-2-(j'+1)}{2}} F_{j'+2+2i} + F_j + \sum_{i=j+2}^{\infty} \zeta_i(a) F_i.$$

340 Since  $s_Z(a) = s_Z(a + F_d)$ , we have  $j = d + 3$ . Then the Zeckendorf decomposition of  $a + 2F_d$   
 341 is

$$342 \quad \sum_{i=2}^{j'-3} \zeta_i(a) F_i + F_{j'-1} + \sum_{i=0}^{\frac{d-4-(j'+1)}{2}} F_{j'+2+2i} + F_{d+1} + F_{d+3} + \sum_{i=d+5}^{\infty} \zeta_i(a) F_i$$

343 and that of  $a + 3F_d$  is

$$344 \quad \sum_{i=2}^{j'-3} \zeta_i(a) F_i + F_{j'-1} + \sum_{i=0}^{\frac{d-4-(j'+1)}{2}} F_{j'+2+2i} + F_{j''} + \sum_{i=j''+2}^{\infty} \zeta_i(a) F_i,$$

345 where  $j'' \geq d + 4$  is the smallest integer such that  $\zeta_{j'}(a) = \zeta_{j'+1}(a) = 0$ . This also leads to a  
 346 contradiction since  $s_Z(a) > s_Z(a + 3F_d)$ .  $\square$

347 **Theorem 5.2.** *For every integer  $d \geq 3$ , there exists infinitely many  $a \in \mathbb{N}$  such that  $a$ ,  
 348  $a + F_d$ , and  $a + 2F_d$  are Zeckendorf-Niven and  $s_Z(a) = s_Z(a + F_d) = s_Z(a + 2F_d)$ .*

349 *Proof.* Let  $m \geq 2$  be a factor of  $F_d$ . Then for any nonnegative integer  $j$ , let

$$350 \quad a = F_{d-1} + \sum_{i=1}^{m-1} F_{d+2+\pi(m)(i+j)}.$$

351 Since  $\pi(m) \geq 2$ , this is the Zeckendorf decomposition of  $a$ . Moreover, the Zeckendorf  
 352 decompositions of  $a + F_d$  and  $a + 2F_d$  are

$$353 \quad a + F_d = F_{d+1} + \sum_{i=1}^{m-1} F_{d+2+\pi(m)(i+j)}$$

354 and

$$355 \quad a + 2F_d = F_{d+2} + \sum_{i=1}^{m-1} F_{d+2+\pi(m)(i+j)},$$

respectively. Hence,  $s_Z(a) = s_Z(a + F_d) = s_Z(a + 2F_d) = m$ . Since  $m$  divides  $F_d$ , in order for  $a$ ,  $a + F_d$ , and  $a + 2F_d$  to be Zeckendorf-Niven, it suffices to show that  $m$  divides  $a$ . This follows since

$$a \equiv F_{d-1} + (m-1)F_{d+2} = F_{d-1} + (m-1)(F_{d-1} + 2F_d) = mF_{d-1} + 2(m-1)F_d \equiv 0 \pmod{m},$$

where the first congruence is due to the definition of the Pisano period.  $\square$

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