ON A ROLL AGAIN: ANALYSIS OF A DICE REMOVAL GAME

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ABSTRACT. Suppose we have n dice, each with s faces (assume $s \geq n$). On the first turn, roll all of them, and remove from play those that rolled an n. Roll all of the remaining dice. In general, if at a certain turn you are left with k dice, roll all of them and remove from play those that rolled a k. The game ends when you are left with no dice to roll. For $n, s \in \mathbb{N} \setminus \{0\}$ such that $s \geq n$, let Y_n^s be the random variable for the number of turns to finish the game rolling n dice with s faces. We find recursive and non-recursive solutions for $\mathbb{E}(Y_n^s)$ and $\mathrm{Var}(Y_n^s)$, and bounds for both values. We conclude that the expected value grows linearly in both the number of dice and the number of faces, while the standard deviation grows linearly in the number of faces. Moreover, we show that Y_n^s can also be modeled as the maximum of a sequence of i.i.d. geometrically distributed random variables. Although, as far as we know, this game hasn't been studied before, similar problems have. The fixed strategy approach to the TENZI dice game analyzed in [Vea21] has a similar solution to our problem, as the number of turns to finish the game is modeled as the maximum of i.i.d geometric variables. Other problems involving dice have been explored extensively, such as the Coupon Collector's Problem, discussed in [Isa96].

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1. Introduction

As explained in [BB18], dice have been around since around 3000 BCE. Similarly, there are records of dice games dating as far back as the Roman era, when soldiers had the right to play dice in camp. One of the most popular games involved throwing three dice and summing the numbers shown. This is thought to be the first game studied using mathematical calculations, although the text that illustrates them, called "De Vetula", dates back to the 13th century. Of course, the aim wasn't scientific per se, but moral: the author wanted to describe the game, and thus inform the reader on how he could be ruined. In the following centuries, dice games began to be used to introduce or study the theory of probability. For example, Jakob Bernoulli in [Ber13] uses a convolution formula to calculate the chances of the aforementioned game with an arbitrary number of dice, although he only presents the results and not the formula itself, which is supposed to be derived by the reader. Nowadays, the primary aim of studying these games is educational: they are taught as a fun way to present mathematical concepts to students. However, they can still be very valuable in mathematical research. For example, in this paper, using a concrete example, such as the die game, as a starting point helps to get a sense of the equivalence between two formulas to calculate the expected value and variance of the maximum of i.i.d. geometric random variables, which is very complicated to demonstrate mathematically.

The game we study works as follows; Suppose we have n dice, each with s faces (assume $s \geq n$). On the first turn, roll all of them, and remove from play those that rolled an n. Roll all of the remaining dice. In general, if at a certain turn you are left with k dice, roll all of them and remove from play those that rolled a k. The game ends when you are left with no dice to roll.

As an example, let's simulate a simple game.

Example 1.1. *Suppose we have 4 dice, each with 6 faces. We roll all of them on the first turn:* 3224*. We remove from the game the fourth die. On the second turn, roll all three remaining dice:* 363*. Now we remove from the game the first and third dice. On the third turn, roll the remaining die:* 4*. We roll that die again on the fourth turn:* 1*. The game ends in four turns.*

Although, as far as we know, this game hasn't been studied before, similar problems have. The fixed strategy approach to TENZI analyzed in [Vea21] has a similar solution to our problem, as the number of turns to finish the game is modeled as the maximum of i.i.d geometric variables. Other problems involving dice have been explored extensively, such as the Coupon Collector's Problem discussed in [Isa96].

For $n, s \in \mathbb{N} \setminus \{0\}$ such that $s \geq n$, let Y_n^s be the random variable for the number of turns to finish the game rolling n dice with s faces. We model our problem in two different ways. First, we imagine rolling each die separately until we achieve a success and we remove it from the game. This method results in the number of turns to finish the game being modeled as the maximum of a sequence of i.i.d. geometric random variables, which has been studied before in [SR90; Eis08]. Although these papers include some of our results (namely Theorems 3.5, 3.6, 4.3, 4.4, they find them by using advanced methods, while we only resort to the direct formula for calculating the expected value and variance for discrete random variables. Then, we imagine rolling all the dice simultaneously, leading to some successes (i.e., dice that roll the correct number and can be removed from the game) and some failures. This interpretation naturally leads to a recursive solution. For the sake of completeness, we also prove that the formulas found are mathematically equivalent. Furthermore, we find that $n \leq s \leq \mathbb{E}(Y_n^s) \leq ns$ and $s^2 - s \leq \mathrm{Var}(Y_n^s) \leq ns(s-1)$. We conclude that the expected value grows linearly in both the number of dice and the number of faces, while the variance grows quadratically in the number of faces and sublinearly in the number of dice.

In the second section, we write some initial definitions and derive the finite number of possible endings. In the third and fourth sections, we find two different formulations for the expected value and variance of the number of turns needed to finish the game. In particular, in the third section, we imagine that each die is rolled individually until a successful roll is achieved (which would remove it from the game). Thus, the problem is equivalent to studying the maximum of a sequence of i.i.d. geometric random variables, and we derive the non-recursive formulas in Theorems 3.5 and 3.6. In the fourth section, we imagine that, at each turn, all of the dice are rolled simultaneously, leading to the recursive formulas in Theorems 4.3 and 4.4. In the fifth section, we confirm our findings numerically by writing Monte Carlo simulations of the game. In the sixth section, we prove some bounds for both the expected value and variance of the game, and we conclude that the expected value grows linearly in the number of dice and number of faces (Proposition 6.1), while the variance grows quadratically in the number of faces (Proposition 6.3).

We conclude by noting applications of the formulas and bounds we found. Obviously, these can be applied to any order statistics scenario involving a maximum of i.i.d. geometric random variables, such as networks ([Lar+17; ARW23]) or probabilistic data structures in informatics ([BC24]).

2. Initial Considerations

To study the game, we first define formally what we mean by the term "turn".

Definition 2.1. *Given k dice, each with s faces, a turn is formally defined as a finite sequence of k numbers, each between 1 and s.*

We will, somewhat improperly, also refer to the act of throwing the k dice as "turn". On each turn, we will colloquially define a die (or a roll) as successful if it shows k (and is therefore removed from the game). Now we define a game.

Definition 2.2. *Given n dice with s faces, a game is formally defined as a possibly infinite sequence of turns, each with $k \leq n$ dice with s faces, where if we roll k dice on a turn we remove all showing a k, and the game ends when there are no dice to roll.*

We must consider the case where there's an infinite number of turns because we can show that the game doesn't have to finish, though with probability 1 it will.

Example 2.3. *Let us consider a game with 4 dice, each with 6 faces. In this case, we could have 4 dice forever, if at each throw we never get a 4. This shows that the game is not guaranteed to end.*

Definition 2.4. *Given a game with n dice, each with s faces, it is said to be finite if it's a finite sequence of turns. In that case, we define the signature of a game as a sequence of n numbers, one for every different die, each being the last number rolled when the die was removed from the game.*

Example 2.5. *Let us consider Example 1.1. The signature of that game is* 4331.

Example 2.6. *Let us consider the previous problem again. We observe that the only possible signatures are:*

$$4444, 4441, 4422, 4421, 4333, 4331, 4322, 4321.$$

Proposition 2.7. *Given a finite game with n dice, each with s faces, there are 2^{n-1} possible signatures.*

Proof. We proceed by induction. With a single die, the game ends only if a 1 is rolled. Thus, there is exactly one possible outcome and the base case holds.

Assume that for all integers k such that $1 < k < n$, the number of possible outcomes with k dice is 2^{k-1}. Now consider the case with n dice. Each outcome falls into one of the following categories.

- All n dice show n. This accounts for one outcome.
- For each $1 \leq i \leq n-1$, the first i dice show n, followed by an outcome of a game with $n-i$ dice.

Therefore, the total number of outcomes is

$$1 + \sum_{i=1}^{n-1} 2^{i-1} \; = \; 2^{n-1}.$$

□

3. Subsequent Throws

3.1. **Formalization.** Now we imagine that each die is rolled singly until achieving a successful roll (which would remove it from the game). The probability that a die with s faces, considered individually, is eliminated at each turn is constant and equal to $p = 1/s$. This probability is independent of the outcomes of the other dice or the number of dice being rolled (since the faces are equiprobable). Thus, if we define Z_i^s for $i = 1, 2, \ldots, n$ the random variable for the turn in which die i with s faces is removed from the game, we observe that each Z_i^s follows a geometric distribution with parameter p. Therefore, if, as before, we define Y_n^s as the random variable for the number of turns to finish the game rolling n dice with s faces, we note that Y_n^s corresponds to the turn at which the last die is eliminated, and thus

$$Y_n^s \; = \; \max_i Z_i^s.$$

Definition 3.1. *Let $X : \Omega \to \mathbb{R}$ be a random variable, $p \in \mathbb{R}$ such that $0 \leq p \leq 1$, and $\mathbb{P} : \Omega \to [0, 1]$ a probability measure. X is geometrically distributed with parameter p ($X \sim \mathrm{Geom}(p)$) if $\mathbb{P}(X = k) = (1-p)^{k-1}p$ for all $k \in \mathbb{N} \setminus \{0\}$.*

Definition 3.2. *Let $X : \Omega \to \mathbb{R}$ be a random variable, and $\mathbb{P} : \Omega \to [0, 1]$ a probability measure. $F_X : \mathbb{R} \to [0, 1]$, $F_X(x) = \mathbb{P}(X \leq x)$ is the cumulative distribution function of X.*

Proposition 3.3. *Assume $s \geq n > 0$, $Z_i^s \overset{\text{iid}}{\sim} \mathrm{Geom}(1/s)$, $Y_n^s = \max_{i \leq n} Z_i^s$, and $q = 1 - p$. Then*

$$\mathbb{P}(Y_n^s = y) \; = \; \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} q^{yk} \left(\frac{1}{q^k} - 1 \right). \tag{3.1}$$

Proof. For $y = 1, 2, \ldots$, we calculate the cumulative distribution function using some well-known results in order statistics [CB02, p.229]:

$$F_{Y_n^s}(y) \; = \; F_{\max_i Z_i^s}(y) \; = \; F_{Z_i^s}(y)^n \; = \; \begin{cases} 0 & y < 1; \\ (1 - (1-p)^y)^n & y \geq 1. \end{cases} \tag{3.2}$$

From this, we can also calculate the probability density,

$$\begin{aligned} \mathbb{P}(Y_n^s = y) \; &= \; F_{Y_n^s}(y) - F_{Y_n^s}(y-1) \\ &= \; \left[(1 - (1-p)^y)^n - \left(1 - (1-p)^{y-1}\right)^n \right] \\ &= \; \left[\sum_{k=0}^{n} \binom{n}{k} (-1)^k (1-p)^{yk} - \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1-p)^{(y-1)k} \right] \\ &= \; \sum_{k=0}^{n} \binom{n}{k} (-1)^k ((1-p)^{yk} - (1-p)^{(y-1)k}). \end{aligned}$$

Then, if we let $q = 1 - p$ the above becomes

$$\begin{aligned}\mathbb{P}(Y_n^s = y) &= \sum_{k=1}^{n} \binom{n}{k} (-1)^k q^{yk} \left(1 - \frac{1}{q^k}\right) \\ &= \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} q^{yk} \left(\frac{1}{q^k} - 1\right).\end{aligned}$$

□

In Table 1 we evaluate the cumulative distribution function in (3.2) at some values of y and n, with $s = 6$.

TABLE 1

y	$F_{Y_1^6}(y)$	$F_{Y_2^6}(y)$	$F_{Y_3^6}(y)$	$F_{Y_4^6}(y)$	$F_{Y_5^6}(y)$	$F_{Y_6^6}(y)$
2	0.305556	0.093364	0.028528	0.008717	0.002663	0.000814
4	0.517747	0.268062	0.138788	0.071857	0.037204	0.019262
6	0.665102	0.442361	0.294215	0.195683	0.130149	0.086562
8	0.767432	0.588952	0.451980	0.346864	0.266195	0.204286
10	0.838494	0.703073	0.589523	0.494311	0.414477	0.347537
12	0.887843	0.788266	0.699857	0.621363	0.551673	0.489799
14	0.922113	0.850293	0.784067	0.722999	0.666687	0.614761
16	0.945912	0.894750	0.846355	0.800577	0.757276	0.716316
18	0.962439	0.926289	0.891496	0.858011	0.825783	0.794766
20	0.973916	0.948512	0.923771	0.899676	0.876208	0.853353
30	0.995787	0.991592	0.987415	0.983255	0.979113	0.974988
50	0.999890	0.999780	0.999670	0.999561	0.999451	0.999341

3.2. **Expected Value.** Now we can use Equation (3.1) to find a new equation for $\mathbb{E}(Y_n^s)$, but first we recall (and give a proof of) a well-known result.

Lemma 3.4. *Let $a, b \in \mathbb{R}$, $|a^b| < 1$. Then*

$$\sum_{i=1}^{\infty} i a^{bi} = \frac{a^b}{(1 - a^b)^2}, \qquad \sum_{i=1}^{\infty} i^2 a^{bi} = \frac{a^b(1 + a^b)}{(1 - a^b)^3}. \tag{3.3}$$

Proof. Let $x = a^b \in (-1, 1)$. By the geometric series formula,

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1 - x}.$$

For all x in $[-r, r] \subset (-1, 1)$ the series $\sum_{i=0}^{\infty} i x^{i-1}$ converges uniformly and so we may differentiate both sides with respect to x:

$$\frac{d}{dx}\left(\sum_{i=0}^{\infty} x^i\right) = \sum_{i=0}^{\infty} i x^{i-1} = \frac{1}{(1 - x)^2}.$$

Then, multiplying both terms by x yields:

$$\sum_{i=0}^{\infty} i x^i = \frac{x}{(1 - x)^2}, \qquad |x| < 1.$$

Since the $i = 0$ term contributes 0, we may drop it. So, substituting back $a^b = x$, we have the desired result.

Now lets prove the second result. We know that

$$\sum_{i=0}^{\infty} ix^i \;=\; \frac{x}{(1-x)^2}.$$

For all x in $[-r, r] \subset (-1, 1)$ the series $\sum_{i=0}^{\infty} i^2 x^{i-1}$ converges uniformly and so we may differentiate both sides with respect to x

$$\frac{d}{dx}\left(\sum_{i=0}^{\infty} ix^i\right) \;=\; \sum_{i=0}^{\infty} i^2 x^{i-1} \;=\; \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4} \;=\; \frac{1-x^2}{(1-x)^4} \;=\; \frac{1+x}{(1-x)^3}.$$

Then, multiplying both terms by x yields

$$\sum_{i=0}^{\infty} i^2 x^i \;=\; \frac{x(1+x)}{(1-x)^2}, \qquad |x| < 1.$$

Since the $i = 0$ term contributes 0, we may drop it. So, substituting back $a^b = x$, we have the desired result. □

Theorem 3.5. *Assume* $s \geq n > 0$, $Z_i^s \overset{\text{iid}}{\sim} \text{Geom}(1/s)$, $Y_n^s = \max_{i \leq n} Z_i^s$, *and* $q = 1 - p$. *Then*

$$\mathbb{E}(Y_n^s) \;=\; \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \frac{1}{1-q^k}. \tag{3.4}$$

Proof. Using Lemma 3.4 and $q < 1$, the expected value is

$$\begin{aligned}
\mathbb{E}(Y_n^s) \;&=\; \sum_{y=1}^{\infty} y \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} q^{yk} \left(\frac{1}{q^k} - 1\right) \\
&=\; \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \left(\frac{1}{q^k} - 1\right) \sum_{y=1}^{\infty} y \cdot q^{yk} \\
&=\; \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \left(\frac{1}{q^k} - 1\right) \frac{q^k}{(1-q^k)^2} \\
&=\; \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \frac{1}{1-q^k}.
\end{aligned}$$

□

Note that, as a direct consequence of Theorem 3.5, we have $\mathbb{E}(Y_n^s) < \infty$.

3.3. **Variance.** Now we can use Equation (3.1) to find a new equation for $\text{Var}(Y_n^s)$.

Theorem 3.6. *Assume* $s \geq n > 0$, $Z_i^s \overset{\text{iid}}{\sim} \text{Geom}(1/s)$, $Y_n^s = \max_{i \leq n} Z_i^s$, *and* $q = 1 - p$. *Then*

$$\text{Var}(Y_n^s) \;=\; \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \frac{1+q^k}{(1-q^k)^2} - \mathbb{E}(Y_n^s)^2. \tag{3.5}$$

Proof. Using Lemma 3.4 and $q < 1$, the second moment is

$$\begin{aligned}
\mathbb{E}\left((Y_n^s)^2\right) &= \sum_{y=1}^{\infty} y^2 \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} q^{yk} \left(\frac{1}{q^k} - 1\right) \\
&= \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \left(\frac{1}{q^k} - 1\right) \sum_{y=1}^{\infty} y^2 \cdot q^{yk} \\
&= \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \left(\frac{1}{q^k} - 1\right) \frac{q^k(1+q^k)}{(1-q^k)^3} \\
&= \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \frac{1+q^k}{(1-q^k)^2}.
\end{aligned} \tag{3.6}$$

The variance can then be found using the well-known formula $\mathrm{Var}(Y_n^s) = \mathbb{E}\left((Y_n^s)^2\right) - \mathbb{E}(Y_n^s)^2$. ☐

Note that, as a direct consequence of Theorem 3.6, we have $\mathrm{Var}(Y_n^s) < \infty$.

4. Simultaneous Throws

4.1. **Formalization.** Let's imagine that all of the dice are rolled simultaneously. Since each throw is independent and we are only considering successes and failures, at each turn the number of dice that achieved a successful roll can be modeled as a binomial random variable. The probability of success for each roll is constant at $p = 1/s$, where s is the number of faces of each die.

Definition 4.1. *Let* $X : \Omega \to \mathbb{R}$ *be a random variable,* $n \in \mathbb{N} \setminus \{0\}$*,* $p \in \mathbb{R}$ *such that* $0 \le p \le 1$*, and* $\mathbb{P} : \Omega \to [0,1]$ *a probability measure.* X *is binomially distributed with parameters* n, p *(*$X \sim \mathrm{Bin}(n,p)$*) if* $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ *for all* $k \in \mathbb{N}$.

We now define some useful random variables related to the game. Let Y_n^s be the random variable for the number of turns to finish the game rolling n dice with s faces, and X_n^s the random variable for the number of successful throws when rolling n dice with s faces. It's easy to see that

$$\begin{aligned}
Y_n^s &\in \{1, 2, \dots\}, \\
X_n^s &\sim \mathrm{Bin}(n, 1/s).
\end{aligned}$$

Now we can define $\mathbb{P}(Y_n^s = y)$ recursively. First, let's explore a simpler example, with 4 dice and 6 faces.

Example 4.2. *In the case where* $n = 4$ *and* $s = 6$*, for the game to end in one turn, necessarily each die must roll a* 4 *in the first turn. Thus,*

$$\mathbb{P}(Y_4^6 = 1) = \mathbb{P}(X_4^6 = 4).$$

For the game to end in two turns, the calculation is more complicated. Each possible roll at the first turn has a non-zero probability of leading to the game finishing in two turns, except for the throw that achieves 4 successes, which would end the game and should therefore be discarded. On the following turn, the game should end, thus all remaining dice should be successful. By independence of each turn,

$$\begin{aligned}
\mathbb{P}(Y_4^6 = 2) &= \mathbb{P}(X_4^6 = 0)\mathbb{P}(Y_4^6 = 1) + \mathbb{P}(X_4^6 = 1)\mathbb{P}(Y_3^6 = 1) \\
&\quad + \mathbb{P}(X_4^6 = 2)\mathbb{P}(Y_2^6 = 1) + \mathbb{P}(X_4^6 = 3)\mathbb{P}(Y_1^6 = 1).
\end{aligned}$$

The cases where $y > 2$ *behave exactly as the latter. On the first of the* y *turns, we can roll any combination of numbers as long as it doesn't end the game (i.e., at least one die should be unsuccessful). On the following turn, we are left with the unsuccessful dice, and the game should now end in* $y - 1$ *turns.*

$$\begin{aligned}\mathbb{P}(Y_4^6 = y) &= \mathbb{P}(X_4^6 = 0)\mathbb{P}(Y_4^6 = y-1) + \mathbb{P}(X_4^6 = 1)\mathbb{P}(Y_3^6 = y-1) \\ &\quad + \mathbb{P}(X_4^6 = 2)\mathbb{P}(Y_2^6 = y-1) + \mathbb{P}(X_4^6 = 3)\mathbb{P}(Y_1^6 = y-1).\end{aligned}$$

Generalizing for n dice, each with s faces and any number of turns, gives

$$\mathbb{P}(Y_n^s = y) = \begin{cases} \mathbb{P}(X_n^s = n) & \text{if } y = 1; \\ \sum_{i=0}^{n-1} \mathbb{P}(X_n^s = i)\mathbb{P}(Y_{n-i}^s = y-1) & \text{if } y > 1. \end{cases} \tag{4.1}$$

Note that in this section we will use some of the results found in Section 3, namely the fact that $\mathbb{E}\left((Y_n^s)^2\right) < \infty$, which is a direct consequence of Theorem 3.6. As both the interpretation given in this section and the one provided in Section 3 are valid ways to model this game, it is admissible to reuse these proofs.

4.2. **Expected Value.**

Theorem 4.3. *Assume $s \geq n > 0$, Y_n^s and $p = 1/s$. Then,*

$$\mathbb{E}(Y_n^s) = \begin{cases} s & n = 1; \\ \frac{1+\sum_{i=1}^{n-1}\binom{n}{i}p^i(1-p)^{n-i}\mathbb{E}(Y_{n-i}^s)}{1-(1-p)^n} & n > 1. \end{cases} \tag{4.2}$$

Proof. Note that $Y_1^s \sim \mathrm{Geom}(1/s)$, thus $\mathbb{E}(Y_1^s) = s$. For $n > 1$, by Equation (4.1):

$$\begin{aligned}\mathbb{E}(Y_n^s) &= \mathbb{P}(X_n^s = n)\cdot 1 + \sum_{y=2}^{\infty} y \sum_{i=0}^{n-1} \mathbb{P}(X_n^s = i)\mathbb{P}(Y_{n-i}^s = y-1) \\ &= \mathbb{P}(X_n^s = n) + \sum_{i=0}^{n-1}\mathbb{P}(X_n^s = i)\sum_{y=2}^{\infty} y\mathbb{P}(Y_{n-i}^s = y-1) \\ &= \mathbb{P}(X_n^s = n) + \sum_{i=0}^{n-1}\mathbb{P}(X_n^s = i)\left[\sum_{y=2}^{\infty}(y-1)\mathbb{P}(Y_{n-i}^s = y-1) + \sum_{y=2}^{\infty}\mathbb{P}(Y_{n-i}^s = y-1)\right].\end{aligned}$$

Moreover, $\sum_{y=2}^{\infty}\mathbb{P}(Y_{n-i}^s = y-1) = 1$. Therefore,

$$\begin{aligned}\mathbb{E}(Y_n^s) &= \mathbb{P}(X_n^s = n) + \sum_{i=0}^{n-1}\mathbb{P}(X_n^s = i)\left[\mathbb{E}(Y_{n-i}^s) + 1\right] \\ &= \sum_{i=0}^{n-1}\mathbb{P}(X_n^s = i)\mathbb{E}(Y_{n-i}^s) + \sum_{i=0}^{n}\mathbb{P}(X_n^s = i) \\ &= \sum_{i=1}^{n-1}\mathbb{P}(X_n^s = i)\mathbb{E}(Y_{n-i}^s) + \mathbb{P}(X_n^s = 0)\mathbb{E}(Y_n^s) + 1.\end{aligned} \tag{4.3}$$

From this, we obtain the following formula, which can be solved for $\mathbb{E}(Y_n^s)$ because $\mathbb{E}(Y_n^s) \leq \mathbb{E}\left((Y_n^s)^2\right) < \infty$ due to Theorem 3.6, and gives

$$\mathbb{E}(Y_n^s) = \frac{\sum_{i=1}^{n-1}\mathbb{P}(X_n^s = i)\mathbb{E}(Y_{n-i}^s) + 1}{1 - \mathbb{P}(X_n^s = 0)}.$$

Since X_n^s is binomial and $p = 1/s$ is the probability of success for each dice, we have the desired result of

$$\mathbb{E}({Y_n}^s) = \frac{\sum_{i=1}^{n-1}\binom{n}{i}p^i(1-p)^{n-i}\mathbb{E}(Y_{n-i}^s) + 1}{1-(1-p)^n}.$$

□

4.3. **Variance.** We calculate $\mathrm{Var}(Y_n^s)$ using a strategy similar to the one used in Theorem 4.3.

Theorem 4.4. *Assume $s \geq n > 0$, Y_n^s and $p = 1/s$. Then*

$$\mathrm{Var}(Y_n^s) \;=\; \begin{cases} s^2\,(2 - 1/s) - s^2 & n = 1; \\ \frac{\sum_{i=1}^{n-1}\binom{n}{i}p^i(1-p)^{n-i}\mathbb{E}\left(\left(Y_{n-i}^s\right)^2\right)-1}{1-(1-p)^n} + 2\mathbb{E}(Y_n^s) - \mathbb{E}(Y_n^s)^2 & n > 1. \end{cases} \tag{4.4}$$

Proof. Note that $Y_1^s \sim \mathrm{Geom}(1/s)$, thus, due to [CB02, p.97] and Theorem 4.3, $\mathrm{Var}(Y_1^s) \;=\; (2 - 1/s) - s^2$. For $n > 1$, similarly to the calculation of the expected value, we can use Equation (4.1) to derive

$$\begin{aligned} \mathbb{E}\left((Y_n^s)^2\right) \;&=\; \mathbb{P}(X_n^s = n)\cdot 1 + \sum_{y=2}^{\infty} y^2 \sum_{i=0}^{n-1}\mathbb{P}(X_n^s = i)\mathbb{P}(Y_{n-i}^s = y-1) \\ &=\; \mathbb{P}(X_n^s = n) + \sum_{i=0}^{n-1}\mathbb{P}(X_n^s = i)\sum_{y=2}^{\infty}\left((y-1)^2 + 2y - 1\right)\mathbb{P}(Y_{n-i}^s = y-1) \\ &=\; \mathbb{P}(X_n^s = n) + \sum_{i=0}^{n-1}\mathbb{P}(X_n^s = i)\left[\mathbb{E}\left(\left(Y_{n-i}^s\right)^2\right) + \sum_{y=2}^{\infty}(2(y-1)+1)\mathbb{P}(Y_{n-i}^s = y-1)\right] \\ &=\; \mathbb{P}(X_n^s = n) + \sum_{i=0}^{n-1}\mathbb{P}(X_n^s = i)\left[\mathbb{E}\left(\left(Y_{n-i}^s\right)^2\right) + 2\mathbb{E}\left(Y_{n-i}^s\right) + 1\right] \\ &=\; \sum_{i=0}^{n-1}\mathbb{P}(X_n^s = i)\left[\mathbb{E}\left(\left(Y_{n-i}^s\right)^2\right) + 2\mathbb{E}(Y_{n-i}^s)\right] + \sum_{i=0}^{n}\mathbb{P}(X_n^s = i) \\ &=\; \mathbb{P}(X_n^s = 0)\mathbb{E}\left((Y_n^s)^2\right) + \sum_{i=1}^{n-1}\mathbb{P}(X_n^s = i)\mathbb{E}\left(\left(Y_{n-i}^s\right)^2\right) + 2\sum_{i=0}^{n-1}\mathbb{P}(X_n^s = i)\mathbb{E}(Y_{n-i}^s) + 1. \end{aligned} \tag{4.5}$$

Using Equation (4.2) and the fact that $\mathbb{E}\left((Y_n^s)^2\right) \;<\; \infty$ because of Theorem 3.6, we can find a recursive formula for $\mathbb{E}\left((Y_n^s)^2\right)$, and thus for the variance. In particular,

$$\begin{aligned} \mathbb{E}\left((Y_n^s)^2\right) &= \frac{\sum_{i=1}^{n-1}\mathbb{P}(X_n^s = i)\mathbb{E}\left(\left(Y_{n-i}^s\right)^2\right) + 2\sum_{i=0}^{n-1}\mathbb{P}(X_n^s = i)\mathbb{E}(Y_{n-i}^s) + 1}{1 - \mathbb{P}(X_n^s = 0)} \\ &= \frac{\sum_{i=1}^{n-1}\mathbb{P}(X_n^s = i)\mathbb{E}\left(\left(Y_{n-i}^s\right)^2\right) - 1}{1 - \mathbb{P}(X_n^s = 0)} + 2\mathbb{E}(Y_n^s). \end{aligned}$$

Writing explicitly $\mathbb{P}(X_n^s = i)$ we have

$$\mathbb{E}\left((Y_n^s)^2\right) \;=\; \frac{\sum_{i=1}^{n-1}\binom{n}{i}p^i(1-p)^{n-i}\mathbb{E}\left(\left(Y_{n-i}^s\right)^2\right) - 1}{1 - (1-p)^n} + 2\mathbb{E}(Y_n^s). \tag{4.6}$$

We can find the variance using the well-known formula $\mathrm{Var}(Y_n^s) \;=\; \mathbb{E}((Y_n^s)^2) - \mathbb{E}\left(Y_n^s\right)^2$ to obtain the desired equation.

□

4.4. **Equivalency.** We have found the moments of Y_n^s using two different methods, which led to a recurrence in Section 4 and an explicit formula in Section 3. These two approaches could be proved to be equivalent, but we limit our study to proving the equivalence of the formulas for the variance and the expected value. In order to do that, we need the convolution identity, a well-known result that can be found in [Mil17], and the Cauchy-Hadamard property of power series, which can be found for example in [Rud10, p.69].

Lemma 4.5 (Convolution identity)**.** *Let $f(z)$ and $g(z)$ be two exponential generating functions:*

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}, \qquad g(z) \;=\; \sum_{n=0}^{\infty} b_n \frac{z^n}{n!}.$$

Their product is given by

$$f(z)g(z) \;=\; \sum_{n=0}^{\infty} \left[\sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j} \right] \frac{z^n}{n!}.$$

Lemma 4.6 (Cachy-Hadamard)**.** *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with real coefficients. If*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0,$$

then the series converges uniformly for any $z \in \mathbb{R}$.

The following result and its proof closely follow the content of [SR90].

Theorem 4.7. *Let $s = 1/p \geq 1$ and $0 < n \leq s$. The expected value defined by Equation* (3.4)

$$\mathbb{E}(Y_n^s) \;=\; \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \frac{1}{1-(1-p)^k}$$

is the solution of the recurrence relation Equation (4.2)

$$\mathbb{E}(Y_n^s) \;=\; \frac{1 + \sum_{i=1}^{n-1} \binom{n}{i} p^i (1-p)^{n-i} \mathbb{E}(Y_{n-i}^s)}{1-(1-p)^n}$$

defined for every $n > 1$ with $\mathbb{E}(Y_1^s) = 1/p$.

Proof. For simpler notation, let $q = (1-p)$ and define $x_n = \mathbb{E}(Y_n^s)$ for $n \geq 1$, with $x_0 := 0$.
For $n = 1$, we have $x_1 = 1/p$, and the theorem holds in this case. This also includes the case $s = 1$.

For $n > 1$ and $s > 1$, we can rewrite the recurrence as

$$x_n \;=\; 1 + \sum_{i=0}^{n} \binom{n}{i} q^{n-i} p^i x_{n-i} \;=\; 1 + \sum_{j=0}^{n} \binom{n}{j} q^j p^{n-j} x_j. \tag{4.7}$$

Therefore, the exponential generating function $F(z) := \sum_{n=0}^{\infty} x_n \frac{z^n}{n!}$ becomes

$$\sum_{n=0}^{\infty} x_n \frac{z^n}{n!} \;=\; \sum_{n=0}^{\infty} \frac{z^n}{n!} \left[1 + \sum_{j=0}^{n} \binom{n}{j} q^j p^{n-j} x_j \right] - 1.$$

Note that we subtracted 1 to correct for the term $n = 0$, since x_0 does not satisfy Equation 4.7. Using Lemma 4.5, with $a_j = q^j x_j$ and $b_{n-j} = p^{n-j}$, we have

$$F(z) \;=\; e^z + F(qz) e^{(1-q)z} - 1.$$

Then, defining $H(z) := F(z) e^{-z}$, we have

$$H(z) \;=\; H(qz) + (1 - e^{-z}).$$

This equation is recursive, since $H(qz)$ can be expressed in terms of $H(q^2z)$, $H(q^3z)$, and so on. In other words,

$$H(z) = H(q^m z) + \sum_{k=0}^{m-1} (1 - e^{-q^k z}).$$

Expanding the first term on the right-hand side:

$$H(q^m z) \;=\; F(q^m z) e^{-q^m z} \;=\; e^{-q^m z} \sum_{n=0}^{\infty} x_n \frac{(q^m z)^n}{n!}.$$

Now assume that $x_n \le n/p$. Under this assumption, the series above is bounded by the power series $\sum_{n=0}^{\infty} \frac{n}{p} \frac{(q^m z)^n}{n!}$, which converges uniformly for all $z \in \mathbb{R}$ by Lemma 4.6, since

$$\lim_{n\to\infty} \sqrt[n]{a_n} \;=\; \lim_{n\to\infty} \sqrt[n]{\frac{n}{p \cdot n!}} \;=\; 0.$$

The uniform convergence of the series, combined with the continuity of its partial sums, ensures the continuity of the full series. Therefore, taking the limit as $m \to \infty$, and noting that $q \in (0,1)$ implies $q^m z \to 0$, we obtain

$$H(z) \;=\; H(0) + \sum_{k=0}^{\infty} (1 - e^{-q^k z}) \;=\; \sum_{k=0}^{\infty} (1 - e^{-q^k z}) \implies F(z) \;=\; e^z \sum_{k=0}^{\infty} (1 - e^{-q^k z}).$$

The series converges for any $q \in (0,1)$, since $(1 - e^{-q^k z}) \sim q^k z$ as $k \to \infty$ and $z \sum q^k$ is a convergent geometric series. Using the definition of $F(z)$ and the properties of the exponential function, we have

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} x_n \;=\; F(z) \;=\; \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^n}{n!} \left(1 - (1-q^k)^n\right) \;=\; \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{\infty} \left(1 - (1-q^k)^n\right),$$

where Tonelli's Theorem, found for example in [JP12, p.67], justifies swapping the sums. We observe that the resulting expression is solved by

$$x_n \;=\; \sum_{k=0}^{\infty} \left(1 - (1-q^k)^n\right) \;=\; \sum_{k=0}^{\infty} \left[1 - \sum_{j=0}^{n} \binom{n}{j} (-1)^j (q^k)^j\right] \;=\; \sum_{k=0}^{\infty} \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} (q^k)^j.$$

Inverting the order of the sums and using the properties of the geometric series, we have the desired result

$$x_n \;=\; \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} \left(\sum_{k=0}^{\infty} (q^j)^k\right) \;=\; \sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} \frac{1}{1-q^j}.$$

We now need to verify whether x_n, as defined in the previous equation, indeed satisfies $x_n \le n/p$, in accordance with the assumption made earlier. This bound is established in Proposition 6.1, thereby confirming that the solution we derived is valid. □

Theorem 4.8. *Let $s = 1/p \ge 1$ and $0 < n \le s$. The second moment (see Equation* (3.6)*)*

$$\mathbb{E}\left((Y_n^s)^2\right) \;=\; \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \frac{1 + (1-p)^k}{(1 - (1-p)^k)^2}$$

is the solution of the recurrence relation in Equation (4.6)

$$\mathbb{E}\left((Y_n^s)^2\right) \;=\; \frac{\sum_{i=1}^{n-1} \binom{n}{i} p^i (1-p)^{n-i} \mathbb{E}\left(\left(Y_{n-i}^s\right)^2\right) - 1}{1 - (1-p)^n} + 2\mathbb{E}(Y_n^s)$$

defined for every $n > 1$ with $\mathbb{E}(Y_1^{s2}) = (2-p)/p^2$.

Proof. This proof closely follows the approach used in the proof of Theorem 4.7. We adopt the same notation. Let $y_n = \mathbb{E}\left((Y_n^s)^2\right)$ for $n \geq 1$ and define $y_0 := 0$.
For $n = 1$, we have $y_1 = (2-p)/p^2$, and the theorem holds in this case. This also includes the case $s = 1$. For $n > 1$ and $s > 1$, we can rewrite the recurrence as

$$y_n \;=\; 1 + 2\sum_{i=0}^{n}\binom{n}{i}q^i p^{n-i}x_i + \sum_{i=0}^{n}\binom{n}{i}q^i p^{n-i}y_i.$$

Define the exponential generating function $G(z) := \sum_{n=0}^{\infty} y_n \frac{z^n}{n!}$ and its transformed counterpart $T(z) := G(z)e^{-z}$. Following the same steps as in the previous proof and using linearity, we derive the functional equation:

$$T(z) = T(qz) + 2H(qz) + (1 - e^{-z})$$

where $H(z)$ is defined as in the previous proof.
Assume now that $y_n \leq 2n/p^2$. Under this assumption, we can apply Lemma 4.6 to show that the series defining $T(z)$ converges uniformly on $\mathbb{R}$, and hence $T(z)$ is well-defined and continuous for all $z \in \mathbb{R}$. Therefore, since $q \in (0,1)$, it follows that $T(q^m z) \to 0$ as $m \to \infty$. Iterating the previous equation and substituting the explicit form of $H(z)$, we get

$$G(z) \;=\; e^z \sum_{k=0}^{\infty}\left[2H\left(q^{k+1}z\right) + \left(1 - e^{-q^k z}\right)\right] \;=\; e^z \sum_{k=0}^{\infty}\left[2\sum_{l=0}^{\infty}\left(1 - e^{-q^{(l+k+1)}z}\right) + \left(1 - e^{-q^k z}\right)\right].$$

We observe that there exist t combinations of (l,k) such that $(l+k+1) = t$, as we can have $k \in (0, 1, \ldots, t-1)$. Therefore we write

$$G(z) \;=\; \sum_{t=0}^{\infty}(2t+1)(e^z - e^{(1-q^t)z}) \;\Longrightarrow\; y_n \;=\; \sum_{t=0}^{\infty}(2t+1)(1-(1-q^t)^n).$$

From the previous proof we know that

$$y_n \;=\; \sum_{t=0}^{\infty}(2t+1)\sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}(q^t)^j \;=\; \sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\sum_{t=0}^{\infty}(2t+1)(q^j)^t.$$

Using the geometric series identity and Lemma 3.4, we have the desired result:

$$y_n \;=\; \sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\left(\frac{2q^j}{(1-q^j)^2} + \frac{1}{1-q^j}\right) \;=\; \sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}\frac{1+q^j}{(1-q^j)^2}.$$

We now need to verify whether y_n, as defined in the previous equation, indeed satisfies $y_n \leq 2n/p^2$, in accordance with the assumption made earlier. This bound is established in Proposition 6.3, thereby confirming that the solution we derived is valid. □

5. Experimental Results

All formulas are implemented in C++ and tested against Monte Carlo simulations. The necessary input is the number of games to simulate (N), the number of dice (D), and the number of faces of each die (S). The output consists of two numbers: the simulated expected value and the simulated variance.

5.1. **Some Numerical Results.** Before proceeding with the simulations, we calculate the probabilities of a few different games in Table 2.

TABLE 2

Number of turns	$D = 2, S = 3$	$D = 4, S = 6$	$D = 6, S = 10$
1	0.111111	0.000771	0.000001
2	0.197531	0.007945	0.000046
4	0.148758	0.040354	0.001258
8	0.037115	0.076752	0.013965
16	0.001519	0.035998	0.041628
32	0.000002	0.002317	0.019033

5.2. **Generating Random Sequence.** The simulations are conducted considering the rolls as subsequent. As seen in Section 3.1, that interpretation views the turn at which the game ends as the maximum of i.i.d. geometric random variables. First, we generate those variables by simulating playing the game. The vector Y, passed by reference as an argument of the function, is populated with the generated numbers.

```
void generateRandomSequence(long long N, int D, int S, vector<long long> &Y) {
    if(D>S) throw invalid_argument("S must be greater or equal to D");

    int it = N, x, dice; long long maxx = 0, nThrows;
    srand(chrono::high_resolution_clock::now().time_since_epoch().count());
    while(it--) {
        dice = D; maxx = 0;
        while(dice--) {
            nThrows = 0; x = 0;
            while(x!=S) {
                x = rand() % S + 1;
                nThrows++;
            }
            maxx = max(maxx, nThrows);
        }
        Y[it] = maxx;
    }

    return ;
}
```

5.3. **Expected Value.** We approximate the expected value using the sample average estimator.

```
double estimateEV(long long N, vector<long long> &Y) {
    long long summ = 0;
    for(long long i=0; i<N; i++) summ += Y[i];
    return ((double) summ)/N;
}
```

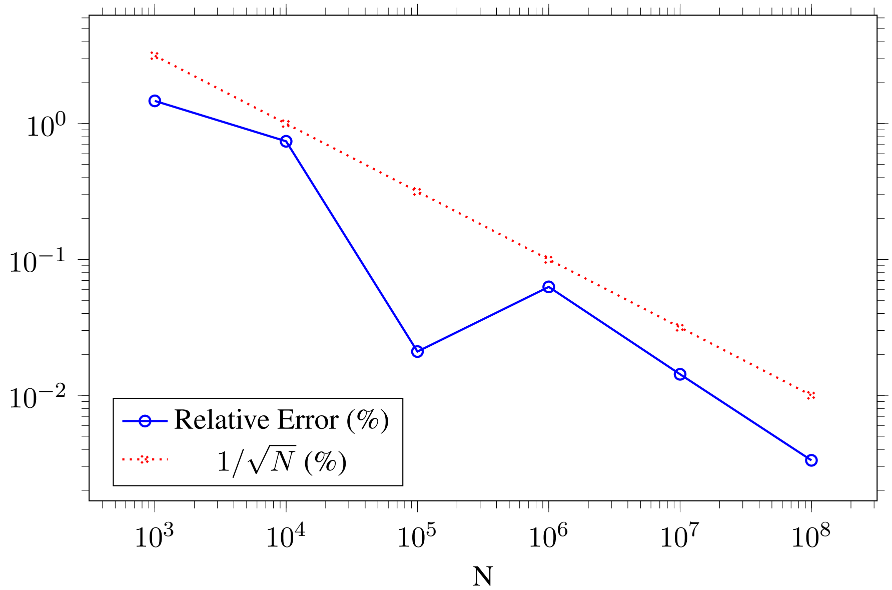
In Table 3 we can see the simulated results and their relative errors against Equation (3.4).

TABLE 3

N	Simulation	Relative Error (%)	Time (ms)
1000	12.1020	1.469843	0.5680
10000	12.0151	0.741226	4.9243
100000	11.9292	0.020993	46.435
1000000	11.9192	0.062853	319.72
10000000	11.9284	0.014285	3270.2
100000000	11.9263	0.003322	38557

We plot Table 3 in the following Figure 1.

FIGURE 1. Relative Error, $D = 4$, $S = 6$.



5.4. **Variance.** We approximate the variance using the population variance. This estimator requires the exact expected value, which is calculated separately and given as input.

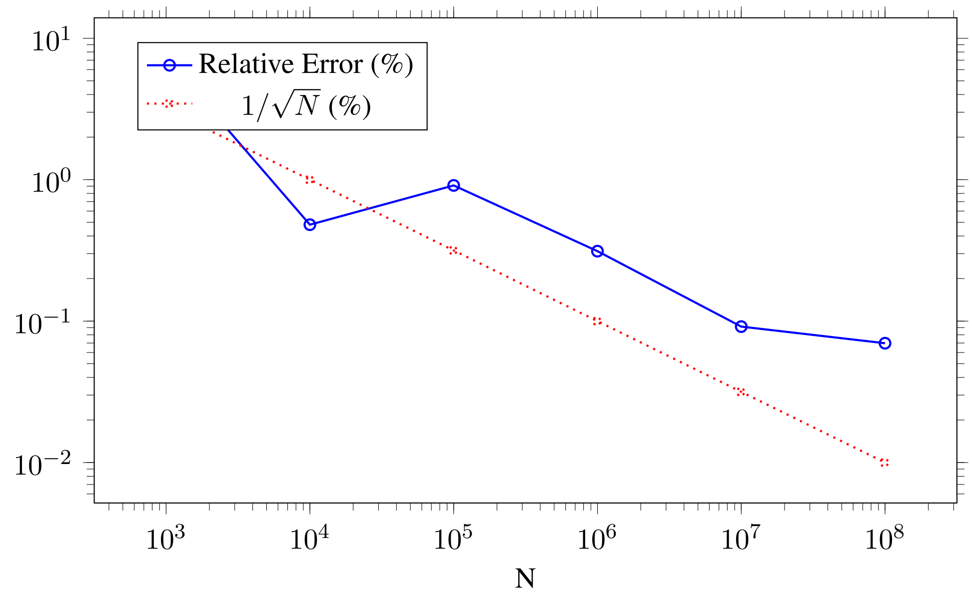
```
double estimateVariance(long long N, double mu, vector<long long> &Y) {
    double num=0;
    for(long long i=0; i<N; i++) {
        num += (Y[i]-mu)*(Y[i]-mu);
    }
    return num/N;
}
```

In Table 4 we can see the simulated results and their relative errors against Equation (3.5).
We plot Table 4 in the following Figure 2.

TABLE 4

N	Simulation	Relative Error (%)	Time (ms)
1000	39.8103	7.224043	0.5714
10000	42.7039	0.480650	4.9772
100000	43.3010	0.910862	46.617
1000000	42.7761	0.312391	321.66
10000000	42.8709	0.091465	3291.5
100000000	42.8802	0.069791	38719

FIGURE 2. Relative Error, $D = 4$, $S = 6$.



6. BOUNDS

We establish a lower and upper bound for both the expected value and the variance. When possible, we ascertain its order in both the number of dice n and the number of faces s.

6.1. **Expected Value.** We approach the problem as we did in Section 3, considering the throws subsequently. We are interested in bounding the value $\mathbb{E}(Y_n^s)$, where $Y_n^s = \max_i Z_i^s$, and Z_i^s are geometrically distributed i.i.d. random variables.

Proposition 6.1. *Assume* $s \geq n > 0$, $Z_i^s \overset{\text{iid}}{\sim} \text{Geom}(1/s)$, $Y_n^s = \max_{i \leq n} Z_i^s$. *Then*

$$n \;\leq\; s \;\leq\; \mathbb{E}(Y_n^s) \;\leq\; ns. \tag{6.1}$$

Proof. We have $n \leq s$ (by assumption). Since $Y_n^s = \max_{i \leq n} Z_i^s \geq Z_1$, it follows that $\mathbb{E}(Y_n^s) \geq \mathbb{E}(Z_1) = s$ by the monotonicity of the expected value. Also, since $\mathbb{E}(Z_i^s) = s > 0$, we have

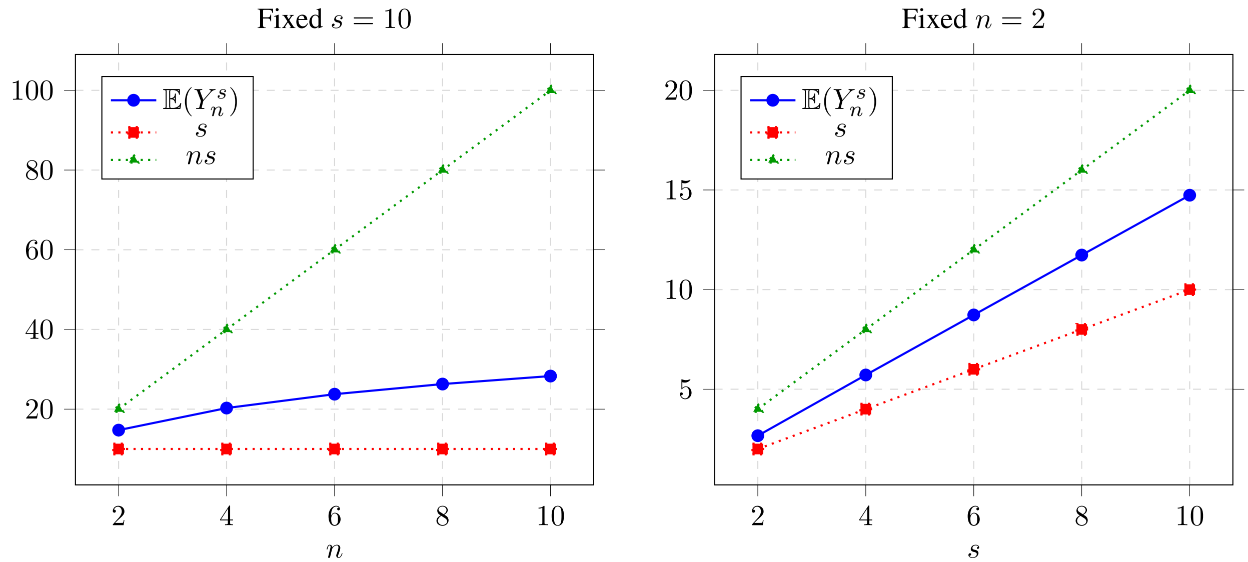
$$\mathbb{E}(Y_n^s) \;=\; \mathbb{E}\left(\max_{i \leq n} Z_i^s\right) \;\leq\; \mathbb{E}\left(\sum_{i=1}^{n} Z_i^s\right) \;=\; \sum_{i=1}^{n} \mathbb{E}(Z_i^s) \;=\; ns$$

by the linearity of the expected value. □

It follows that $\mathbb{E}(Y_n^s) = O(n)$ for fixed values of s and $\mathbb{E}(Y_n^s) = O(s)$ for fixed values of n.

This is verified numerically in Figure 3.

FIGURE 3. Growth rate of the expected value.



This bound can be further improved by using the exact value of some $\mathbb{E}(Y_n^s)$ for small values of n.

Proposition 6.2. *Assume* $s \geq n > 0$, $Z_i^s \overset{\text{iid}}{\sim} \text{Geom}(1/s)$, $Y_n^s = \max_{i \leq n} Z_i^s$. *Then*

(1) if n *is even,* $\mathbb{E}(Y_n^s) \leq \frac{n}{2}\frac{3s^2-2s}{2s-1}$, *and*

(2) if n *is odd,* $\mathbb{E}(Y_n^s) \leq \frac{n}{2}\frac{3s^2-2s}{2s-1} + s$.

Proof. We use Theorem 3.5 to calculate the expected value for $n = 2$.

$$\mathbb{E}(Y_2^s) = \frac{2}{1-(1-1/s)} - \frac{1}{1-(1-1/s)^2} = 2s - \frac{s^2}{2s-1} = \frac{3s^2-2s}{2s-1}.$$

Now, we assume n is even and we rewrite Y_n^s as

$$Y_n^s = \max(Z_1^s, ..., Z_n^s) = \max[\max(Z_1^s, Z_2^s), ..., \max(Z_{n-1}^s, Z_n^s)] \leq \sum_{i=1,3,...,n-1} \max(Z_i^s, Z_{i+1}^s).$$

Note that since $\{Z_n^s\}_n$ are i.i.d., then $\max(Z_i^s, Z_{i+1}^s) \sim Y_2^s$. In particular, they have the same expected value. By the monotonicity and linearity of the expected value,

$$\mathbb{E}(Y_n^s) \leq \sum_{i=1,3,...,n-1} \mathbb{E}[\max(Z_i^s, Z_{i+1}^s)] = \sum_{i=1,3,...,n-1} \mathbb{E}(Y_2^s) = \frac{n}{2}\frac{3s^2-2s}{2s-1}. \tag{6.2}$$

We can extend this formula to an odd number of dice. Let $m = n + 1$, with n even. Then

$$\begin{aligned} Y_m &= \max(Z_1^s, ..., Z_n^s, Z_{n+1}^s) \\ &= \max[\max(Z_1^s, Z_2^s), ..., \max(Z_{n-1}^s, Z_n^s), Z_{n+1}^s] \\ &\leq \sum_{i=1,3,...,n-1} \max(Z_i^s, Z_{i+1}^s) + Z_{n+1}^s. \end{aligned}$$

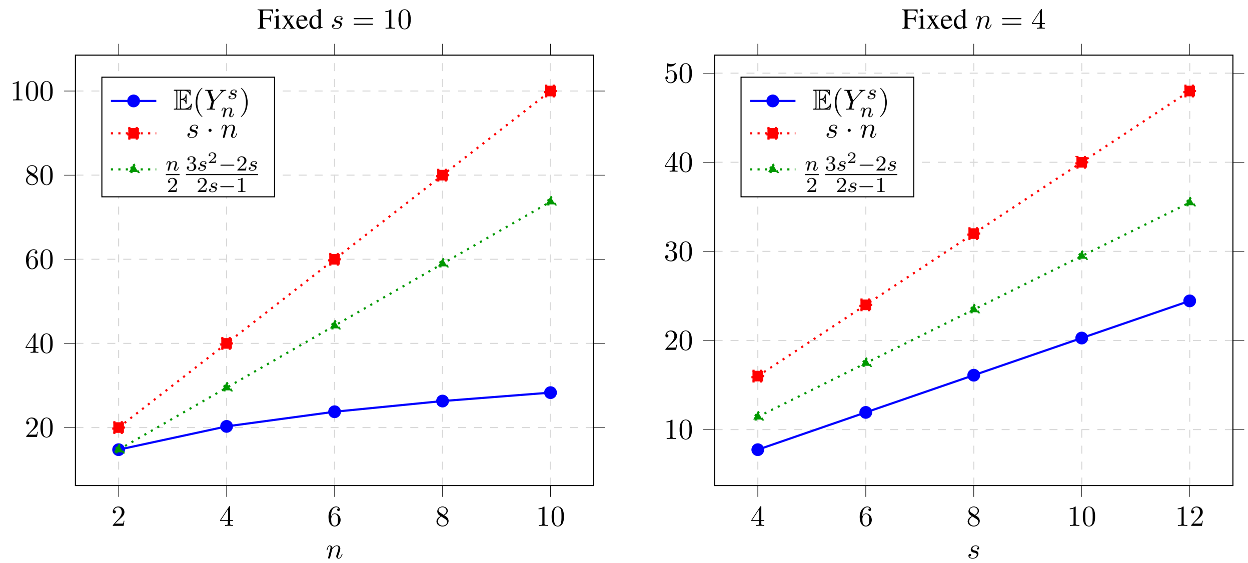
Similarly, by the monotonicity and linearity of the expected value,

$$\mathbb{E}(Y_m^s) \le \sum_{i=1,3,\dots,n-1} \mathbb{E}\left[\max(Z_i^s, Z_{i+1}^s)\right] + \mathbb{E}(Z_{n+1}^s) = \sum_{i=1,3,\dots,n-1} \mathbb{E}(Y_2^s) + \mathbb{E}(Z_{n+1}^s) = \frac{n}{2}\frac{3s^2-2s}{2s-1} + s. \tag{6.3}$$

□

We show numerically how the bound in Proposition 6.2 is an improvement over the bound in Proposition 6.1 in Figure 4.

FIGURE 4. Comparison of bounds in Equations (6.1) vs (6.2).



6.2. **Variance.** Once again, we consider the rolls as subsequent.

Proposition 6.3. *Assume* $s \ge n > 0$, $Z_i^s \overset{\text{iid}}{\sim} \text{Geom}(1/s)$, $Y_n^s = \max_{i \le n} Z_i^s$. *Then*

$$s^2\left(2 - \frac{1}{s}\right) \le \mathbb{E}\left((Y_n^s)^2\right) \le ns^2\left(2 - \frac{1}{s}\right), \tag{6.4}$$

and therefore

$$s^2 - s \le \text{Var}(Y_n^s) \le s^2(2n-1) - ns. \tag{6.5}$$

Proof. Since $Z_i^s > 0$, then $(Y_n^s)^2 = (\max_{i \le n} Z_i^s)^2 = \max_{i \le n}(Z_i^s)^2$.
We observe that

$$(Z_1^s)^2 \le \max_{i \le n}(Z_i^s)^2 \le \sum_{i=1}^{n}(Z_i^s)^2.$$

Since each $Z_i^s \sim \text{Geom}(1/s)$, its second moment can be found in [CB02, p.97] and it is

$$\mathbb{E}\left((Z_i^s)^2\right) = \frac{2 - \frac{1}{s}}{\left(\frac{1}{s}\right)^2} = s^2\left(2 - \frac{1}{s}\right).$$

Therefore, by the monotonicity and linearity of the expected value, it follows that

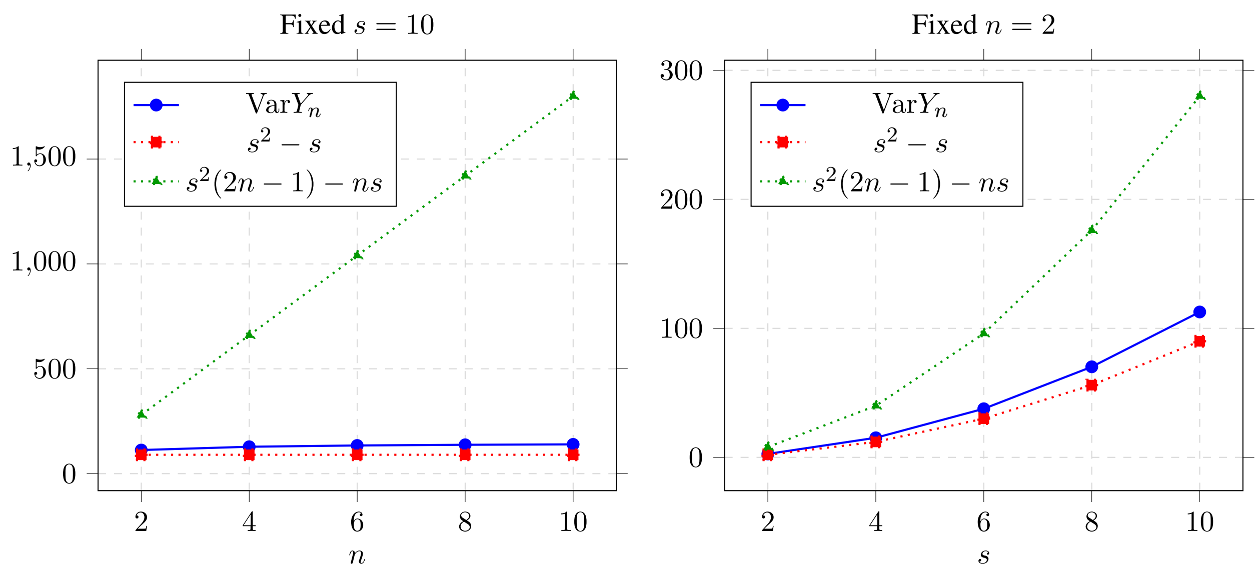
$$s^2\left(2 - \frac{1}{s}\right) = \mathbb{E}\left((Z_1^s)^2\right) \le \mathbb{E}\left(\max_{i \le n}(Z_i^s)^2\right) \le \sum_{i=1}^{n}\mathbb{E}\left((Z_i^s)^2\right) = ns^2\left(2 - \frac{1}{s}\right).$$

□

It follows that $\mathrm{Var}(Y_n^s) = O(s^2)$ (as $s \to \infty$, for fixed values of n).

Once again, we show in Figure 5 how the numerical values are coherent with the theory.

FIGURE 5. Actual variance and the proposed bounds.



Now we need a lemma that expresses a fundamental property of associated random variables. Its proof can be found in [EPW67].

Lemma 6.4. *Let $X = (X_1, \dots, X_n^s)$ be a vector of independent random variables. Then for any real-valued functions f and g that are non-decreasing in each argument,*

$$\mathrm{Cov}(f(X), g(X)) \geq 0.$$

We use this lemma to establish a new upper bound for the variance of the maximum of independent random variables.

Lemma 6.5. *Let $\{Z_i^s\}_{i=1}^n$ be a sequence of independent random variables. Then*

$$\mathrm{Var}\left(\max_{i \leq n}(Z_i^s)\right) \leq \sum_{i=1}^{n} \mathrm{Var}(Z_i^s). \tag{6.6}$$

Proof. Let $\Gamma = \{1, .., n\} \setminus \{\arg\max_{i \leq n} Z_i^s\}$. By the independence of the random variables, we have

$$\begin{aligned}\sum_{i=1}^{n} Var(Z_i^s) &= \text{Var}\left(\sum_{i=1}^{n} Z_i^s\right) \\ &= \text{Var}\left(\max_{i \leq n} Z_i^s + \sum_{i \in \Gamma} Z_i^s\right) \\ &= \text{Var}\left(\max_{i \leq n} Z_i^s\right) + \text{Var}\left(\sum_{i \in \Gamma} Z_i^s\right) + 2\,\text{Cov}\left(\max_{i \leq n} Z_i^s, \sum_{i \in \Gamma} Z_i^s\right) \\ &\geq \text{Var}\left(\max_{i \leq n} Z_i^s\right)\end{aligned}$$

where the inequality follows from the non-negativity of the covariance term due to Lemma 6.4. ☐

Proposition 6.6. *Assume* $s \geq n > 0$, $Z_i^s \overset{\text{iid}}{\sim} \text{Geom}(1/s)$, $Y_n^s = \max_{i \leq n} Z_i^s$. *Then*

$$\text{Var}(Y_n^s) \leq ns(s-1). \tag{6.7}$$

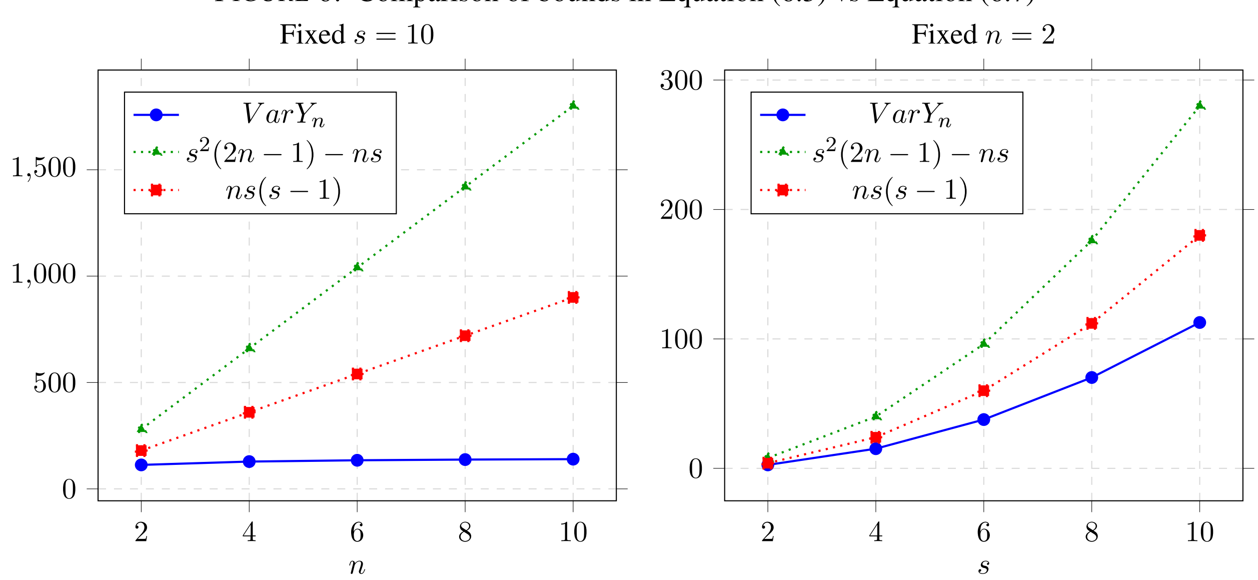
Proof. The proposition is a direct consequence of Lemma 6.5:

$$\text{Var}(Y_n^s) = \text{Var}\left(\max_{i \leq n} Z_i^s\right) \leq \sum_{i=1}^{n} \text{Var}(Z_i^s) = n\frac{1 - 1/s}{1/s^2} = ns(s-1).$$

☐

In Figure 6 we show how this bound is an improvement over Equation (6.5).

FIGURE 6. Comparison of bounds in Equation (6.5) vs Equation (6.7)



References

[Ber13] Jakob Bernoulli. *Ars conjectandi, opus posthumum: accedit tractatus de seriebus infinitis, et epistola Gallice scripta de ludo pilæ reticularis*. Impensis Thurnisiorum Fratrum, 1713.

[EPW67] J. D. Esary, F. Proschan, and D. W. Walkup. "Association of Random Variables, with Applications". In: *The Annals of Mathematical Statistics* 38.5 (1967), pp. 1466–1474.

[SR90] W. Szpankowski and V. Rego. "Yet another application of binomial recurrence: order statistics". In: *Computing* 43 (1990), pp. 401–410.

[Isa96] R. Isaac. *The Pleasures of Probability*. Undergraduate Texts in Mathematics / Readings in Mathematics. Springer New York, 1996, p. 259. ISBN: 9780387944159.

[CB02] G. Casella and R. L. Berger. *Statistical Inference*. Duxbury advanced series in statistics and decision sciences. Thomson Learning, 2002, p. 686. ISBN: 9780534243128.

[Eis08] B. Eisenberg. "On the expectation of the maximum of IID geometric random variables". In: *Statistics & Probability Letters* 78.2 (2008), pp. 135–143. ISSN: 0167-7152.

[Rud10] Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 2010.

[JP12] J. Jacod and P. Protter. *Probability Essentials*. Springer Science & Business Media, 2012, p. 752. ISBN: 9783642514319.

[Lar+17] James Larisch et al. "CRLite: A scalable system for pushing all TLS revocations to all browsers". In: *2017 IEEE Symposium on Security and Privacy (SP)*. IEEE. 2017, pp. 539–556.

[Mil17] S. J. Miller. *The Probability Lifesaver*. Princeton University Press, 2017, pp. 71–74. ISBN: 9780691149547.

[BB18] Marie-France Bru and Bernard Bru. "Dice Games". In: *Statistical Science* 33.2 (2018), pp. 285–297. ISSN: 08834237, 21688745. URL: https://www.jstor.org/stable/26770996 (visited on 10/14/2025).

[Vea21] Michael H Veatch. "An Analysis of TENZI Using Combinatorics and Markov Chains". In: *Mathematics Magazine* 94.3 (2021), pp. 173–185.

[ARW23] Guus Avis, Filip Rozpędek, and Stephanie Wehner. "Analysis of multipartite entanglement distribution using a central quantum-network node". In: *Physical Review A* 107.1 (2023), p. 012609.

[BC24] Ziyad Benomar and Christian Coester. "Learning-augmented priority queues". In: *Advances in Neural Information Processing Systems* 37 (2024), pp. 124163–124197.

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