

EXTENDING SUPPORT FOR THE CENTERED MOMENTS OF THE LOW LYING ZEROES OF CUSPIDAL NEWFORMS

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ABSTRACT. We study low-lying zeroes of L -functions and their n -level density, which relies on a smooth test function ϕ whose Fourier transform $\hat{\phi}$ has compact support. Assuming the generalized Riemann hypothesis, we compute the n^{th} centered moments of the 1-level density of low-lying zeroes of L -functions associated with weight k , prime level N cuspidal newforms as $N \rightarrow \infty$, where $\text{supp}(\hat{\phi}) \subset (-2/n, 2/n)$. The Katz-Sarnak density conjecture predicts that the n -level density of certain families of L -functions is the same as the distribution of eigenvalues of corresponding families of orthogonal random matrices. We prove that the Katz-Sarnak density conjecture holds for the n^{th} centered moment of the 1-level density for test functions with $\hat{\phi}$ supported in $(-2/n, 2/n)$, for families of cuspidal newforms split by the sign of their functional equations. Our work provides better bounds on the percent of forms vanishing to a certain order at the central point. Previous work handled the 1-level for support up to 2 and the n -level up to $\min(2/n, 1/(n-1))$; we are able to remove the second restriction on the support and extend the result to what one would expect, based on the 1-level, by finding a tractable vantage to evaluate the combinatorial zoo of terms which emerge.

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1. INTRODUCTION

Since Montgomery and Dyson's discovery that the two point correlation of the zeros of the Riemann zeta function agree with the pair correlation function for eigenvalues of the Gaussian

Date: October 23, 2022.

2010 Mathematics Subject Classification. 11M26 (primary), 11M41, 15A52 (secondary).

Key words and phrases. Low-lying zeros, optimal test functions, n -level densities, Katz-Sarnak Conjectures.

This work was done during the 2011, 2012, 2015, 2017, and 2021 SMALL REU program in Number Theory and Probability at Williams College supported by NSF grants DMS1561945 and DMS1659037. We also thank Roger Heath-Brown for his assistance on exponential sums.

Unitary Ensemble (see [Mon73]), the connection between the zeros of L -functions and the zeros of random matrices has been a major area of study. It is now widely believed that the statistical behavior of families of L -functions can be modeled by ensembles of random matrices. Based on the observation that the spacing statistics of high zeros associated with cuspidal L -functions agree with the corresponding statistics for eigenvalues of random unitary matrices under Haar measure (see [RS96], for example), it was originally believed that only the unitary ensemble was important to number theory. However, Katz and Sarnak [KS99a, KS99b] showed that these statistics are the same for all classical compact groups. These statistics, the n -level correlations, are unaffected by finite numbers of zeros. In particular, they fail to identify differences in behavior near $s = 1/2$.

The n -level density statistic was introduced to distinguish the behavior of families of L -functions close to this central point. Based partially on an analogy with the function field setting, Katz and Sarnak conjectured that the low-lying zeros of families of L -functions behave like the eigenvalues near 1 of classical compact groups (unitary, symplectic, and orthogonal). The behavior of the eigenvalues near 1 is different for each matrix group. A growing body of evidence has shown that this conjecture holds for test functions with suitably restricted support for a wide range of families of L -functions. For a non-exhaustive list, see [AM13, AAI⁺15, DM05, DM06, ERGR12, FM11, Gao13, Gul05, HM07, ILS99, Mil04, MP10, OS93, OS06, RR07, Roy01, Rub01, ST12, You04]. Much of the previous work is focused on the $n = 1$ case. We study the n^{th} centered moment of the 1-level density; see Remark 1.5 for comments on how our result may be generalized to the n -level density.

Here, we consider the family of L -functions associated with holomorphic automorphic forms, and split this family according to the sign of the functional equation of the L -function. We prove that the Katz-Sarnak conjecture holds for the n^{th} centered moment of the 1-level density for test functions ϕ with Fourier transform $\hat{\phi}$ supported in $(-\frac{2}{n}, \frac{2}{n})$. Hughes and Miller [HM07] looked at the same family, and proved that the conjecture holds for $\hat{\phi}$ supported in $(-\frac{1}{n-1}, \frac{1}{n-1})$ for $n \geq 2$. However, new terms emerge on both the number theory and random matrix theory side which obstructed calculations beyond this support, though based on the 1-level result it was thought that one should be able to go up to $2/n$ for the n -level. Our main result is to develop an approach to handle all the combinatorial terms which emerge for support up to $2/n$, thus extending the support to what the 1-level suggests. Thus, though our calculations are long and technical, the final result matches expectations.

We first introduce some standard notation and definitions. Let $H_k^*(N)$ be the set of holomorphic cusp forms of weight k which are newforms of level N . Every $f \in H_k^*(N)$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n. \quad (1.1)$$

where $q = e^{2\pi iz}$. Set $\lambda_f(n) = a_f(n) n^{-(k-1)/2}$. The L -function associated to f is

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}. \quad (1.2)$$

The completed L -function is

$$\Lambda(s, f) := \left(\frac{\sqrt{N}}{2\pi} \right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(s, f), \quad (1.3)$$

and it satisfies the functional equation $\Lambda(s, f) = \epsilon_f \Lambda(1-s, f)$ with $\epsilon_f = \pm 1$. Therefore, $H_k^*(N)$ splits into two disjoint subsets, $H_k^+(N) := \{f \in H_k^*(N) : \epsilon_f = +1\}$ and $H_k^-(N) := \{f \in H_k^*(N) : \epsilon_f = -1\}$. Each L -function has a set of non-trivial zeros $\rho_f = \frac{1}{2} + i\gamma_f$. The Generalized Riemann Hypothesis is the statement that all $\gamma_f \in \mathbb{R}$ for all f .

Let ϕ be an even Schwartz function such that its Fourier transform has compact support. We are interested in moments of the smooth counting function (also called the one-level density or linear statistic)

$$D(f; \phi) := \sum_{\gamma_f} \phi \left(\frac{\log R}{2\pi} \gamma_f \right) \quad (1.4)$$

when averaged over $H_k^+(N)$ or $H_k^-(N)$ (the split cases) as $N \rightarrow \infty$ through the primes, with k held fixed. Here γ_f runs through the non-trivial zeros of $L(s, f)$, and R is its analytic conductor ($R = k^2 N$ for these families). We rescale the zeros by $\log R$ as this is the order of the number of zeros with imaginary part less than a large absolute constant.¹ Because of the rapid decay of ϕ , most of the contribution in (1.4) is from zeros near the central point. We use the uniform average over $f \in H_k^\sigma(N)$ (for σ one of $+$ or $-$), in the sense that if Q is a function defined on $f \in H_k^\sigma(N)$, then the average of Q over $H_k^\sigma(N)$ is

$$\langle Q(f) \rangle_\sigma := \frac{1}{|H_k^\sigma(N)|} \sum_{f \in H_k^\sigma(N)} Q(f). \quad (1.5)$$

The corresponding statistic to $D(f; \phi)$ in random matrix theory, denoted $Z_\phi(U)$, is defined as follows. First, for a Schwartz function ϕ on the real line, define

$$F_M(\theta) := \sum_{j=-\infty}^{\infty} \phi \left(\frac{M}{2\pi} (\theta + 2\pi j) \right), \quad (1.6)$$

which is 2π -periodic and localized on a scale of $1/M$. For U an $M \times M$ unitary matrix with eigenvalues $e^{i\theta_n}$, set

$$Z_\phi(U) := \sum_{n=1}^M F_M(\theta_n). \quad (1.7)$$

Note that going from $e^{i\theta_n}$ to θ_n is well defined, since $F_M(\theta)$ is 2π -periodic. We often consider U to be a special orthogonal matrix when the eigenvalues occur in complex-conjugate pairs, and thus are doubly counted.

We show that the random matrix moments of Z_ϕ correctly model the moments of $D(f; \phi)$, in the sense that the n^{th} centered moment of $D(f; \phi)$ averaged over $H_k^+(N)$ equals the n^{th} centered moment of Z_ϕ averaged over $\text{SO}(\text{even})$, and $H_k^-(N)$ similarly corresponds to $\text{SO}(\text{odd})$. Our main result is the following.

Theorem 1.1. *Assume GRH for $L(s, f)$ and Dirichlet L -functions. Let $n \geq 2, k \geq 2$ and $\text{supp}(\hat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$. Then the n^{th} centered moment of $D(f; \phi)$ averaged over $H_k^\pm(N)$ converges as $N \rightarrow \infty$ through the primes to the n^{th} centered moment of $Z_\phi(U)$ averaged over $\text{SO}(\text{even/odd})$.*

Theorem 1.1 follows immediately from Theorems 1.2 and 1.3 below. It is conjectured that the n^{th} centered moments from number theory agree with random matrix theory for any Schwartz test function; our results above may be interpreted as providing additional evidence for this conjecture.

We find closed form expressions for the n^{th} centered moments of both $D(f, \phi)$ and $Z_\phi(U)$ for test functions supported in $\text{supp}(\hat{\phi}) \subset \left(-\frac{1}{n-a}, \frac{1}{n-a}\right)$ for some fixed positive integer a , with the additional condition that $\text{supp}(\hat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right)$. To this end, we define σ_ϕ^2 , $R(m, i; \phi)$ and $S(n, a; \phi)$

¹Our analysis is greatly simplified by all forms in the family having the same analytic conductor. Varying conductors are easily handled in 1-level calculations, but cause technical difficulties through cross terms once $n \geq 2$, see [Mil04].

for an even Schwartz function ϕ and integers m and $i \leq m$:

$$\sigma_\phi^2 := 2 \int_{-\infty}^{\infty} |y| \widehat{\phi}(y)^2 dy \quad (1.8)$$

and

$$\begin{aligned} R(m, i; \phi) &:= 2^{m-1} (-1)^{m+1} \sum_{\ell=0}^{i-1} (-1)^\ell \binom{m}{\ell} \left(-\frac{1}{2} \phi^m(0) \right. \\ &\quad \left. + \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_2) \cdots \widehat{\phi}(x_{\ell+1}) \int_{-\infty}^{\infty} \phi^{m-\ell}(x_1) \frac{\sin(2\pi x_1(1 + |x_2| + \cdots + |x_{\ell+1}|))}{2\pi x_1} dx_1 \cdots dx_{\ell+1} \right) \end{aligned} \quad (1.9)$$

and

$$S(n, a; \phi) := \sum_{\ell=0}^{\lfloor \frac{a-1}{2} \rfloor} \frac{n!}{(n-2\ell)! \ell!} R(n-2\ell, a-2\ell; \phi) \left(\frac{\sigma_\phi^2}{2} \right)^\ell. \quad (1.10)$$

Our main result is the following.

Theorem 1.2. *Let $n \geq 2, k \geq 2$, $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-a}, \frac{1}{n-a}\right)$, $\text{supp}(\widehat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right)$, $D(f; \phi)$ be as in (1.4), σ_ϕ^2 be as in (1.8), $S(n, a; \phi)$ be as in (1.10), and $\mathbb{1}_{\{n \text{ even}\}}$ be the indicator function supported at the even integers. Assume GRH for $L(s, f)$ and for all Dirichlet L -functions. As $N \rightarrow \infty$ through the primes,*

$$\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \langle (D(f; \phi) - \langle D(f; \phi) \rangle_\pm)^n \rangle_\pm = \mathbb{1}_{\{n \text{ even}\}} \cdot (n-1)!! \sigma_\phi^n \pm S(n, a; \phi). \quad (1.11)$$

To prove Theorem 1.1, we show that the n^{th} centered moment of $Z_\phi(U)$ is the same as the above for the corresponding families. Our main random matrix theory result is as follows.

Theorem 1.3. *When $\text{supp}(\widehat{\phi}) \subseteq [-1, 1]$, the means of $Z_\phi(U)$ when averaged with respect to Haar measure over $\text{SO}(\text{even})$ or $\text{SO}(\text{odd})$ are*

$$\mu_\pm := \lim_{\substack{M \rightarrow \infty \\ M \text{ even/odd}}} \mathbb{E}_{\text{SO}(M)} [Z_\phi(u)] = \widehat{\phi}(0) + \frac{1}{2} \int_{-1}^1 \widehat{\phi}(y) dy. \quad (1.12)$$

Let ϕ be a Schwartz class test function such that $\text{supp}(\widehat{\phi}) \subseteq \left[-\frac{1}{n-a}, \frac{1}{n-a}\right]$ and $\text{supp}(\widehat{\phi}) \subseteq \left[-\frac{2}{n}, \frac{2}{n}\right]$. Let σ_ϕ^2 be as in (1.8), $S(n, a; \phi)$ be as in (1.10), and $\mathbb{1}_{\{n \text{ even}\}}$ be the indicator function supported at the even integers. Then the n^{th} centered moment of $Z_\phi(U)$ averaged over $\text{SO}(\text{even/odd})$ is

$$\lim_{\substack{M \rightarrow \infty \\ M \text{ even/odd}}} \mathbb{E}_{\text{SO}(M)} [(Z_\phi(U) - \mu_\pm)^n] = \mathbb{1}_{\{n \text{ even}\}} \cdot (n-1)!! \sigma_\phi^n \pm S(n, a; \phi). \quad (1.13)$$

When not splitting by sign, Hughes and Miller [HM07] study the n^{th} centered moment of $D(f; \phi)$ averaged over $H_k^*(N)$, and prove the following.

Theorem 1.4 (Theorem E.1 of [HM07]). *Assume GRH for $L(s, f)$ and Dirichlet L -functions. For $n \geq 1$ an integer and $2k \geq n$, if $\text{supp}(\widehat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right)$ then the n^{th} centered moment of $D(f; \phi)$ averaged over $H_k^*(N)$ converges as $N \rightarrow \infty$ through the primes to the n^{th} centered moment of $Z_\phi(U)$ averaged over the mean² of $\text{SO}(\text{even})$ and $\text{SO}(\text{odd})$.*

²By the mean of $\text{SO}(\text{even})$ and $\text{SO}(\text{odd})$ we mean the ensemble where half the matrices are $\text{SO}(\text{even})$ and the other half $\text{SO}(\text{odd})$.

The condition $2k \geq n$ arises from controlling error terms, in Appendix C we show how to remove this condition so that Theorem 1.4 holds for all k .

Remark 1.5. Since the families of L -functions we study here are of constant sign, we have a better understanding of the number of zeros at the central point than for many other families. Hence, our arguments can be easily translated to give the n -level density, defined as

$$\frac{1}{|H_k^\pm(N)|} \sum_{f \in H_k^\pm(N)} \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} \phi_1 \left(\frac{\log R}{2\pi} \gamma_f^{(j_1)} \right) \cdots \phi_n \left(\frac{\log R}{2\pi} \gamma_f^{(j_n)} \right), \quad (1.14)$$

where the ϕ_i are even Schwartz functions whose Fourier transforms have compact support. We study the n^{th} centered moments to make it easier to compare our number theory and random matrix theory results.

As noted in [HM07] and [Mil09], another application of centered moments is in bounding the order of vanishing of L -functions at the central point. In Appendix D, we show how to use Theorem 1.2 to bound the probability that a newform with negative sign will have order of vanishing exceeding some r at the central point. Similar calculations may be done for the positive sign family. Our results provide the best known bounds (conditional on GRH) for order of vanishing at the central point when $r \geq 5$, surpassing [ILS99, HM07, BCD⁺20, LM22]. The test function used in these calculations is not optimal for bounding order of vanishing; see [DM22] for a more developed analysis with optimized test functions which give sharper bounds.

The primary obstacle which prevents the extension of support past the $\frac{1}{n-1}$ proven in [HM07] is the emergence of more complicated terms as the support of $\hat{\phi}$ increases. The main insight which allows us to extend support is the observation that many of these terms vanish in the limit (see Proposition 3.8). This allows us to mostly ignore their difficult combinatorics. The terms which do contribute in the limit exhibit nicer symmetries, which allows us to handle their delicate combinatorics and find closed form integrals for them (see Proposition 3.9). Our work also involves many combinatorial simplifications so that our results may be compared with those from random matrix theory. As with number theory, the key result which allows us to obtain greater support in random matrix theory is the vanishing of many of the complicated terms which emerge at larger supports (see Lemma 5.19).

The ability to extend the support of $\hat{\phi}$ up to $2/n$ for the n^{th} centered moment of $H_K^\pm(N)$ is expected from the work of [ILS99], who showed that the density conjecture holds for the first moment of $H_K^\pm(N)$ with $\text{supp}(\hat{\phi}) \subset (-2, 2)$ under GRH for Dirichlet L -functions, though the later work of [HM07] showed that actually reaching this would require handling challenging combinatorics. Note $2/n$ is a natural barrier, appearing in (2.35) and Lemmas 3.6 and 5.1. Proving the density conjecture past this support likely requires new ideas or stronger hypotheses such as the ‘‘Hypothesis S’’ discussed in Section 10 of [ILS99] (or more likely its generalizations).

The structure of this paper is as follows. In Section 2, we review notation and state some needed estimates. In Section 3, we state the main lemmas that assist in calculating the relevant number theory quantities, concentrating on the new terms that did not arise in the computations of [ILS99] and [HM07]. Using these, we prove Theorem 1.2 in Section 3.3. In Section 4, we evaluate the terms which arise in the number theory calculations, proving the two key propositions in Section 3. In Section 5 we go over random matrix theory preliminaries and prove Theorem 1.3. In Appendix A, we prove the lemmas stated in Section 3 and Section 4. In Appendix B, we prove some of the remnant lemmas from Section 5. In Appendix C, we show how to use the tools developed in Section 3 to remove the condition $2k \geq n$ from Theorem 1.4. In Appendix D we use Theorem 1.2 in order to bound the percent of newforms vanishing to a certain order at the central point.

2. PRELIMINARIES

As we are extending the results of [HM07], we have the same preliminaries as they do, which we reproduce with permission.

2.1. Notation.

Definition 2.1 (Gauss Sums). *For χ a character modulo q and $e(x) = e^{2\pi i x}$,*

$$G_\chi(n) := \sum_{a \bmod q} \chi(a) e(an/q), \quad (2.1)$$

and $|G_\chi(n)| \leq \sqrt{q}$.

Definition 2.2 (Ramanujan Sums). *If $\chi = \chi_0$ (the principal character modulo q) in (2.1), then $G_{\chi_0}(n)$ becomes the Ramanujan sum*

$$R(n, q) := \sum_{a \bmod q}^* e(an/q) = \sum_{d|(n, q)} \mu(q/d) d, \quad (2.2)$$

where $*$ restricts the summation to be over all a relatively prime to q .

The Ramanujan sum satisfies the following identity:

$$R(n, q) = \mu\left(\frac{q}{(q, n)}\right) \frac{\varphi(q)}{\varphi\left(\frac{q}{(q, n)}\right)}. \quad (2.3)$$

Definition 2.3 (Kloosterman Sums). *For integers m and n ,*

$$S(m, n; q) := \sum_{d \bmod q}^* e\left(\frac{md}{q} + \frac{n\bar{d}}{q}\right), \quad (2.4)$$

where $d\bar{d} \equiv 1 \bmod q$. We have

$$|S(m, n; q)| \leq (m, n, q) \sqrt{\min\left\{\frac{q}{(m, q)}, \frac{q}{(n, q)}\right\}} \tau(q), \quad (2.5)$$

where $\tau(q)$ is the number of divisors of q ; see Equation 2.13 of [ILS99].

Definition 2.4 (Fourier Transform). *We use the following normalization:*

$$\widehat{\phi}(y) := \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i xy} dx, \quad \phi(x) := \int_{-\infty}^{\infty} \widehat{\phi}(y) e^{2\pi i xy} dy. \quad (2.6)$$

Definition 2.5 (Characteristic Function). *For $A \subset \mathbb{R}$, let*

$$\mathbb{1}_{\{x \in A\}} := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases} \quad (2.7)$$

Throughout, we suppress the argument of a characteristic function when it is clear from context.

Definition 2.6 (Delta Function). *For $x, y \in \mathbb{R}$, let*

$$\delta(x, y) := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases} \quad (2.8)$$

Definition 2.7 ((Infinite) GCD). *For $x, y \in \mathbb{Z}$, let (x, y) denote the greatest common divisor of x and y . Set $(x, y^\infty) = \max_{n \in \mathbb{N}} (x, y^n)$ and $(x^\infty, y) = \max_{n \in \mathbb{N}} (x^n, y)$.*

The Bessel function of the first kind occurs frequently in this paper, and so we collect here some standard bounds for it (see, for example, [GR65, Wat66]).

Lemma 2.8. *Let $k \geq 2$ be an integer. The Bessel function satisfies*

- (1) $J_{k-1}(x) \ll 1$,
- (2) $J_{k-1}(x) \ll x$,
- (3) $J_{k-1}(x) \ll x^{k-1}$,
- (4) $J_{k-1}(x) \ll x^{-\frac{1}{2}}$.

We also utilize the Mellin transform of $J_{k-1}(x)$, which we denote by $G_{k-1}(s)$. By (6.561.14) of [GR65] it is

$$\begin{aligned} G_{k-1}(s) &= \int_0^\infty J_{k-1}(x) x^{s-1} dx \\ &= 2^{s-1} \Gamma\left(\frac{k-1+s}{2}\right) / \Gamma\left(\frac{k+1-s}{2}\right) \end{aligned} \quad (2.9)$$

where $1-k < \Re(s) < \frac{3}{2}$. We take $k \geq 2$ so that we may take $\Re(s) \in (-1, 3/2)$. The inverse transform is

$$J_{k-1}(x) = \frac{1}{2\pi i} \int_{\Re(s)=c} G_{k-1}(s) x^{-s} ds \quad (2.10)$$

where $1-k < c < \frac{3}{2}$.

2.2. Fourier coefficients. Let k and N be positive integers with k even and N prime. We denote by $S_k(N)$ the space of all cusp forms of weight k for the Hecke congruence subgroup $\Gamma_0(N)$ of level N . That is, f belongs to $S_k(N)$ if and only if f is holomorphic in the upper half-plane, satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad (2.11)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \gamma \equiv 0 \pmod{N} \right\}$, and vanishes at each cusp of $\Gamma_0(N)$.

Let $f \in S_k(N)$ be a cuspidal newform of weight k and level N ; in our case this means f is a cusp form of level N but not of level 1. It has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz), \quad (2.12)$$

with f normalized so that $a_f(1) = 1$. We normalize the coefficients by defining

$$\lambda_f(n) := a_f(n) n^{-(k-1)/2}. \quad (2.13)$$

$H_k^*(N)$ is the set of all $f \in S_k(N)$ which are newforms of level N . We split this set into two subsets, $H_k^+(N)$ and $H_k^-(N)$, depending on whether the sign of the functional equation of the associated L -function (see Section 1 for details) is 1 or -1. From Equation (2.73) of [ILS99] we have for $N > 1$ that

$$|H_k^\pm(N)| = \frac{k-1}{24} N + O\left((kN)^{5/6}\right). \quad (2.14)$$

This combined with Equation (1.16) of [ILS99], we have that for $N \neq 1$

$$|H_k^*(N)| = \frac{k-1}{12} N + O\left((kN)^{5/6}\right). \quad (2.15)$$

For simplicity we shall deal only with the case when N is prime, a fact which we will occasionally remind the reader of (though, as in [ILS99], similar arguments work for N square-free). For a newform of level N , $\lambda_f(N)$ is related to the sign of the form ([ILS99], Equation 3.5).

Lemma 2.9. *If $f \in H_k^*(N)$ and N is prime, then*

$$\epsilon_f = -i^k \lambda_f(N) \sqrt{N}. \quad (2.16)$$

As $\epsilon_f = \pm 1$, (2.16) implies $|\lambda_f(N)| = 1/\sqrt{N}$. Essential in our investigations are the multiplicative properties of the Fourier coefficients.

Lemma 2.10. *Let $f \in H_k^*(N)$. Then*

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right). \quad (2.17)$$

In particular, if $(m,n) = 1$ then

$$\lambda_f(m)\lambda_f(n) = \lambda_f(mn). \quad (2.18)$$

From [Guy00] we have the following expansion for $\lambda_f(p)^n$:

$$\lambda_f(p)^n = \sum_{\alpha=0}^{n/2} \left[\binom{n}{\alpha} - \binom{n}{\alpha-1} \right] \lambda_f(p^{n-2\alpha}). \quad (2.19)$$

Note that for a prime $p \nmid N$,

$$\lambda_f(p)^2 = \lambda_f(p^2) + 1. \quad (2.20)$$

Now, consider

$$\Delta_{k,N}^\sigma(n) := \sum_{f \in H_k^\sigma(N)} \lambda_f(n), \quad \sigma \in \{+, -, *\}. \quad (2.21)$$

Note we are *not* dividing by the cardinality of the family, which is of order N . Splitting by sign and using Lemma 2.9 we have that if N is prime and $(N,n) = 1$,

$$\begin{aligned} \Delta_{k,N}^\pm(n) &= \sum_{f \in H_k^*(N)} \frac{1}{2} (1 \pm \epsilon_f) \lambda_f(n) \\ &= \frac{1}{2} \Delta_{k,N}^*(n) \mp \frac{i^k \sqrt{N}}{2} \Delta_{k,N}^*(nN). \end{aligned} \quad (2.22)$$

Thus, to execute sums over $f \in H_k^\pm(N)$, it suffices to understand sums over all $f \in H_k^*(N)$. Propositions 2.1, 2.11 and 2.15 of [ILS99] yield a useful form of the Petersson formula.

Lemma 2.11 ([ILS99]). *If N is prime and $(n, N^2) | N$ then*

$$\Delta_{k,N}^*(n) = \Delta'_{k,N}(n) + \Delta_{k,N}^\infty(n), \quad (2.23)$$

where

$$\begin{aligned} \Delta'_{k,N}(n) &= \frac{(k-1)N}{12\sqrt{n}} \delta_{n,\square_Y} \\ &\quad + \frac{(k-1)N}{12} \sum_{\substack{(m,N)=1 \\ m \leq Y}} \frac{2\pi i^k}{m} \sum_{\substack{c \equiv 0 \pmod{N} \\ c \geq N}} \frac{S(m^2, n; c)}{c} J_{k-1} \left(4\pi \frac{\sqrt{m^2 n}}{c} \right), \end{aligned} \quad (2.24)$$

where $\delta_{n,\square_Y} = 1$ only if $n = m^2$ with $m \leq Y$ and 0 otherwise. The remaining piece, $\Delta_{k,N}^\infty(n)$, is called the complementary sum. By Lemma A.1 of [HM07], the complementary sum does not contribute in all cases appearing in this paper.

In the applications we take $Y = N^\epsilon$ and write $c = bN$ for $c \equiv 0 \pmod{N}$. Using the estimate on Kloosterman sums, (2.5), the bound on the Bessel function $J_{k-1}(x) \ll x$ from Lemma 2.8, and (2.14), we can trivially estimate $\Delta'_{k,N}(n)/|H_k^*(N)|$. We obtain the following lemma.

Lemma 2.12. *Assume $(n, N) = 1$. Then*

$$\frac{1}{|H_k^*(N)|} \Delta'_{k,N}(Nn) \ll \sqrt{n} N^{-\frac{3}{2}+\epsilon}. \quad (2.25)$$

2.3. Density and moment sums. Let $f \in H_k^*(N)$, and let $\Lambda(s, f)$ be its associated completed L -function, (1.3). The Generalized Riemann Hypothesis states that all the zeros of $\Lambda(s, f)$ (i.e., the non-trivial zeros of $L(s, f)$) are of the form $\rho_f = \frac{1}{2} + i\gamma_f$ with $\gamma_f \in \mathbb{R}$. The analytic conductor of $\Lambda(s, f)$ is $R = k^2 N$, and its smooth counting function (also called the 1-level density) is

$$D(f; \phi) = \sum_{\gamma_f} \phi\left(\frac{\log R}{2\pi} \gamma_f\right), \quad (2.26)$$

where ϕ is an even Schwartz function whose Fourier transform has compact support and the sum is over all zeros of $\Lambda(s, f)$. Because ϕ decays rapidly, the main contribution to (2.26) is from zeros near the central point. The explicit formula applied to $D(f; \phi)$ gives (see Equation (4.25) of [ILS99])

$$D(f; \phi) = \widehat{\phi}(0) + \frac{1}{2}\phi(0) - P(f; \phi) + O\left(\frac{\log \log R}{\log R}\right), \quad (2.27)$$

where

$$P(f; \phi) = \sum_{p \nmid N} \lambda_f(p) \widehat{\phi}\left(\frac{\log p}{\log R}\right) \frac{2 \log p}{\sqrt{p} \log R}. \quad (2.28)$$

While the derivation of (2.27) in [ILS99] uses GRH for $L(s, \text{sym}^2 f)$, as they remark this formula can be established on average over f by an analysis of the Petersson formula or from properties of $L(s, \text{sym}^2 f \otimes \text{sym}^2 f)$ (see page 88 of [ILS99]). As in [HM07] we shall assume GRH for $L(s, f)$ below for ease of exposition. If $\text{supp}(\widehat{\phi}) \subset (-1, 1)$, [ILS99] show the $P(f, \phi)$ term does not contribute, and hence $\lim_{N \rightarrow \infty} \langle D(f; \phi) \rangle_\sigma = \widehat{\phi}(0) + \frac{1}{2}\phi(0)$ for any $\sigma \in \{+, -, *\}$. Thus it is enough to evaluate:

$$\begin{aligned} \langle (D(f; \phi) - \langle D(f; \phi) \rangle_\sigma)^n \rangle_\sigma &= \left\langle \left(-P(f; \phi) + O\left(\frac{\log \log R}{\log R}\right) \right)^n \right\rangle_\sigma \\ &= (-1)^n \langle P(f; \phi)^n \rangle_\sigma + O\left(\frac{\log \log R}{\log R}\right). \end{aligned} \quad (2.29)$$

As in [HM07], we split by sign and use Lemma 2.9 to obtain

$$\begin{aligned} \sum_{f \in H_k^\pm(N)} P(f; \phi)^n &= \sum_{f \in H_k^*(N)} \frac{1 \pm \epsilon_f}{2} P(f; \phi)^n \\ &= \frac{1}{2} \sum_{f \in H_k^*(N)} P(f; \phi)^n \mp \frac{1}{2} \sum_{f \in H_k^*(N)} i^k \sqrt{N} \lambda_f(N) P(f; \phi)^n. \end{aligned} \quad (2.30)$$

Since $|H_k^+(N)| \sim |H_k^-(N)| \sim \frac{1}{2}|H_k^*(N)|$, as $N \rightarrow \infty$ by (2.14) we have

$$\langle P(f; \phi)^n \rangle_\pm \sim \langle P(f; \phi)^n \rangle_* \mp i^k \sqrt{N} \langle \lambda_f(N) P(f; \phi)^n \rangle_*. \quad (2.31)$$

In conclusion, if $\text{supp}(\widehat{\phi}) \subset (-1, 1)$, we have

$$\lim_{N \rightarrow \infty} \langle (D(f; \phi) - \langle D(f; \phi) \rangle_\pm)^n \rangle_\pm = (-1)^n \lim_{N \rightarrow \infty} S_1^{(n)} \pm (-1)^{n+1} \lim_{N \rightarrow \infty} S_2^{(n)} \quad (2.32)$$

(assuming all limits exist), where

$$S_1^{(n)} := \sum_{p_1 \nmid N, \dots, p_n \nmid N} \prod_{j=1}^n \left(\widehat{\phi}\left(\frac{\log p_j}{\log R}\right) \left(\frac{2 \log p_j}{\sqrt{p_j} \log R} \right) \right) \left\langle \prod_{j=1}^n \lambda_f(p_j) \right\rangle_* \quad (2.33)$$

and

$$S_2^{(n)} := i^k \sqrt{N} \sum_{p_1 \nmid N, \dots, p_n \nmid N} \prod_{j=1}^n \left(\widehat{\phi}\left(\frac{\log p_j}{\log R}\right) \left(\frac{2 \log p_j}{\sqrt{p_j} \log R} \right) \right) \left\langle \lambda_f(N) \prod_{j=1}^n \lambda_f(p_j) \right\rangle_*. \quad (2.34)$$

Now, Lemma 3.1 and Theorem E.1 of [HM07] prove under GRH for $L(s, f)$ that for $\text{supp}(\hat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$ with $2k \geq n$,

$$\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} S_1^{(n)} = \begin{cases} (2m-1)!! \sigma_\phi^{2m} & \text{if } n = 2m \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}. \quad (2.35)$$

where

$$\sigma_\phi^2 = 2 \int_{-\infty}^{\infty} |y| \hat{\phi}(y)^2 dy. \quad (2.36)$$

In Appendix C we show how to remove the condition $2k \geq n$ so (2.35) holds for all k . Hence, to find the n^{th} moment of $D(f, \phi)$, for ϕ with $\text{supp} \hat{\phi} \subset (-\frac{2}{n}, \frac{2}{n})$, it suffices to compute the value of $S_2^{(n)}$. We begin this calculation in the following section.

3. EXTENDING SUPPORT FOR THE MOMENTS OF THE 1-LEVEL DENSITY

In this section, we calculate the n^{th} centered moment of $D(f; \phi)$ with $\text{supp}(\hat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$. Note that (2.32) expresses this value in terms of $S_2^{(n)}$ and $S_1^{(n)}$. We break $S_2^{(n)}$ into subterms and first show that certain subterms vanish in the limit. Then we apply various lemmas to the remaining subterms before calculating them exactly in Section 4. The proofs of these lemmas are mostly standard and can be found in Appendix A.

3.1. Eliminating subterms of $S_2^{(n)}$. As in [HM07], we evaluate

$$S_2^{(n)} := i^k \sqrt{N} \sum_{p_1 \nmid N, \dots, p_n \nmid N} \prod_{j=1}^n \left(\hat{\phi} \left(\frac{\log p_j}{\log R} \right) \left(\frac{2 \log p_j}{\sqrt{p_j} \log R} \right) \right) \left\langle \lambda_f(N) \prod_{j=1}^n \lambda_f(p_j) \right\rangle_*. \quad (3.1)$$

We rewrite this sum over primes as a sum over powers of distinct primes. Suppose $p_1 \cdots p_n = q_1^{n_1} \cdots q_\ell^{n_\ell}$, where q_1, \dots, q_ℓ are distinct. We now have a way to write $S_2^{(n)}$ as a sum over some ℓ -tuple of distinct primes (q_1, \dots, q_ℓ) , given a fixed ℓ -tuple of multiplicities (n_1, \dots, n_ℓ) . Hence, $S_2^{(n)}$ can be written as a sum of terms of the form

$$E'(\vec{n}, \vec{m}) := i^k \sqrt{N} \sum_{\substack{q_1 \nmid N, \dots, q_\ell \nmid N \\ q_j \text{ distinct}}} \prod_{j=1}^{\ell} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \right) \langle \lambda_f(N q_1^{m_1} \cdots q_\ell^{m_\ell}) \rangle_*. \quad (3.2)$$

Here, the m_j 's arise from the multiplicative properties of the Fourier coefficients given by (2.19). We have that $m_j \leq n_j$ and $m_j \equiv n_j \pmod{2}$. Observe that in the sum, there is a distinctness condition attached. It is advantageous to remove this distinctness condition, and we likewise define terms of the form

$$E(\vec{n}, \vec{m}) := i^k \sqrt{N} \sum_{q_1 \nmid N, \dots, q_\ell \nmid N} \prod_{j=1}^{\ell} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \right) \langle \lambda_f(N q_1^{m_1} \cdots q_\ell^{m_\ell}) \rangle_*. \quad (3.3)$$

In the expansion of $S_2^{(n)}$, the terms of the form (3.3) have an attached combinatorial coefficient arising from choosing the indices of the primes, expanding the Fourier coefficients using Lemma 2.10, and inclusion-exclusion from removing the distinctness condition in the sum. In general these combinatorial coefficients are very difficult to calculate, however we are able to determine them in the particular cases where they contribute in the limit. First we have the following property.

Lemma 3.1. *In a term of the form $E(\vec{n}, \vec{m})$ suppose there is some j for which $n_j = m_j$ and $n_j > 1$. Then this term has combinatorial coefficient 0 in the expansion of $S_2^{(n)}$, and does not contribute.*

Since terms in the expansion of $S_2^{(n)}$ must have $m_j \leq n_j$, it suffices in the expansion of $S_2^{(n)}$ to only consider terms $E(\vec{n}, \vec{m})$ which satisfy $n_j > m_j$ or $n_j = m_j = 1$ for each $1 \leq j \leq \ell$. For each term $E(\vec{n}, \vec{m})$, we henceforth may reindex the sums so that $n_j > m_j$ for $1 \leq j \leq \omega$ and $n_j = m_j = 1$ for $\omega + 1 \leq j \leq \ell$ for some $0 \leq \omega \leq \ell$. Noting that $\ell - \omega \leq n$, we define $n' := \ell - \omega$ (so $0 \leq n' \leq n$) and can write $q_{\omega+1} = p_1, q_{\omega+2} = p_2, \dots, q_\ell = p_{n-n'}$. Thus each $E(\vec{n}, \vec{m})$ appearing in $S_2^{(n)}$ has the form

$$E(\vec{n}, \vec{m}) = i^k \sqrt{N} \sum_{q_1 \nmid N, \dots, q_\omega \nmid N} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \right) \\ \times \sum_{p_1 \nmid N, \dots, p_{n-n'} \nmid N} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \left(\frac{2 \log p_i}{\sqrt{p_i} \log R} \right) \langle \lambda_f(N p_1 \cdots p_{n-n'} q_1^{m_1} \cdots q_\omega^{m_\omega}) \rangle_* \quad (3.4)$$

with each $n_j > m_j$ and $n_j > 1$. We next have the following result.

Lemma 3.2. *Let $E(\vec{n}, \vec{m})$ be as in (3.4). If $\text{supp}(\hat{\phi}) \subset \left(-\frac{1}{n-a}, \frac{1}{n-a}\right)$ and $n' \geq a$ then $E(\vec{n}, \vec{m})$ is $O(N^{-\epsilon})$.*

This result states that $E(\vec{n}, \vec{m})$ contributes in the limit only if “most” of the indices satisfy $n_j = m_j$. The next section of our work involves explicitly calculates $E(\vec{n}, \vec{m})$ as expressed in (3.4).

3.2. Simplifying the subterms. Next we apply the Petersson formula to (3.4). Since $|H_k^*(N)| \sim N(k-1)/12$ from (2.15), applying (2.24) to (3.4) gives

$$E(\vec{n}, \vec{m}) = \frac{2^{n+1}\pi}{\sqrt{N}} \sum_{q_1 \nmid N, \dots, q_\omega \nmid N} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \frac{\log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \sum_{p_1 \nmid N, \dots, p_{n-n'} \nmid N} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i}{\sqrt{p_i} \log R} \\ \times \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{b=1}^{\infty} \frac{S(m^2, NQ; Nb)}{b} J_{k-1} \left(\frac{4\pi m \sqrt{Q}}{b\sqrt{N}} \right) + O(N^{-\epsilon}). \quad (3.5)$$

where $Q = p_1 \cdots p_{n-n'} q_1^{m_1} \cdots q_\omega^{m_\omega}$. Note that since $R = k^2 N$ and $\text{supp}(\hat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right) \subseteq (-1, 1)$, for sufficiently large N the condition that $q_j \nmid N$ and $p_i \nmid N$ in (3.5) is automatically satisfied so it can be removed. We restrict the sum over b in (3.5) with the following two lemmas.

Lemma 3.3. *Suppose $\text{supp}(\hat{\phi}) \subseteq \left(-\frac{5}{2(n-n')}, \frac{5}{2(n-n')}\right)$. Then the subterms of $E(\vec{n}, \vec{m})$ in (3.5) for which $(b, N) > 1$ are $O(N^{-\epsilon})$.*

Lemma 3.4. *Suppose $\text{supp}(\hat{\phi}) \subset \left(-\frac{1000}{n-n'}, \frac{1000}{n-n'}\right)$. Then, the subterms of $E(\vec{n}, \vec{m})$ in (3.5) for which $b \geq N^{2022}$ are $O(N^{-12})$.*

Applying these to (3.5) gives

$$E(\vec{n}, \vec{m}) = \frac{2^{n+1}\pi}{\sqrt{N}} \sum_{q_1, \dots, q_\omega} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \frac{\log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \sum_{p_1, \dots, p_{n-n'}} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i}{\sqrt{p_i} \log R} \\ \times \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{\substack{(b, N)=1 \\ b < N^{2022}}} \frac{S(m^2, NQ; Nb)}{b} J_{k-1} \left(\frac{4\pi m \sqrt{Q}}{b\sqrt{N}} \right) + O(N^{-\epsilon}). \quad (3.6)$$

Next, we convert the Kloosterman sums in (3.6) to Gauss sums with the following lemma.

Lemma 3.5. *Let N be a prime not dividing b, Q, m . Then*

$$S(m^2, NQ; Nb) = -\frac{1}{\varphi(b)} \sum_{\chi(b)} G_\chi(m^2) G_\chi((Q, b^\infty)) \bar{\chi} \left(\frac{Q}{(Q, b^\infty)} \right) \chi(N). \quad (3.7)$$

Applying Lemma 3.5 to (3.6) gives

$$\begin{aligned} E(\vec{n}, \vec{m}) &= -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{q_1, \dots, q_\omega} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \frac{\log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \sum_{p_1, \dots, p_{n-n'}} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i}{\sqrt{p_i} \log R} \\ &\times \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{\substack{(b, N)=1 \\ b < N^{2022}}} \frac{1}{b\varphi(b)} \sum_{\chi(b)} G_\chi(m^2) G_\chi((Q, b^\infty)) \bar{\chi} \left(\frac{Q}{(Q, b^\infty)} \right) \chi(N) J_{k-1} \left(\frac{4\pi m \sqrt{Q}}{b\sqrt{N}} \right) + O(N^{-\epsilon}). \end{aligned} \quad (3.8)$$

The subterms involving non-principal characters in (3.8) are negligible in the limit. This leaves only subterms involving $\chi_0 = \bar{\chi}_0$ modulo b for each b , and $G_{\chi_0}(x) = R(x, b)$, a Ramanujan sum, definition. Additionally, note that $\chi_0(N) = 1$ since $(b, N) = 1$.

Lemma 3.6. *Assume GRH for Dirichlet L -functions and suppose that $\text{supp}(\hat{\phi}) \subset \left(-\frac{2}{n-n'}, \frac{2}{n-n'}\right)$. Then the sum over all non-principal characters in (3.8) is $O(N^{-\epsilon})$.*

Remark 3.7. Lemma 3.6 is the only place in our calculation of $S_2^{(n)}$ where GRH for Dirichlet L -functions or the restriction $\text{supp}(\hat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right)$ is necessary. Additionally, Lemma 3.6 corrects an error made in Lemma 4.7 of [HM07], thus making their work unconditional on this result.

Applying Lemma 3.6 to (3.8) gives

$$\begin{aligned} E(\vec{n}, \vec{m}) &= -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{q_1, \dots, q_\omega} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \frac{\log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \sum_{p_1, \dots, p_{n-n'}} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i}{\sqrt{p_i} \log R} \\ &\times \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{\substack{(b, N)=1 \\ b < N^{2022}}} \frac{R(m^2, b) R((Q, b^\infty), b)}{b\varphi(b)} \chi_0 \left(\frac{Q}{(Q, b^\infty)} \right) J_{k-1} \left(\frac{4\pi m \sqrt{Q}}{b\sqrt{N}} \right) + O(N^{-\epsilon}). \end{aligned} \quad (3.9)$$

In Section 4, we explicitly calculate $E(\vec{n}, \vec{m})$ in two different cases starting from (3.9). Our first result characterizes when $E(\vec{n}, \vec{m})$ vanishes.

Proposition 3.8. *Let $E(\vec{n}, \vec{m})$ be defined as in (3.3) with $n_j + m_j > 2$ for some $1 \leq j \leq \ell$. Under GRH for $L(s, f)$ and all Dirichlet L -functions, if $\text{supp}(\hat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right)$, then $E(\vec{n}, \vec{m}) = O(\log^{-1} N)$ and vanishes in the limit.*

The second result is the explicit calculation of the “main term,” when $n_j = m_j$ for all j where $1 \leq j \leq n$. In this case, we arrive at the following result.

Proposition 3.9. *Let $E(\vec{n}, \vec{m})$ be defined as in (3.3) with $\ell = n$ and $n_j = m_j = 1$ for $1 \leq j \leq n$. Under GRH for $L(s, f)$ and all Dirichlet L -functions, if $\text{supp}(\hat{\phi}) \subset \left(-\frac{1}{n-a}, \frac{1}{n-a}\right)$ and $\text{supp}(\hat{\phi}) \subset$*

$(-\frac{2}{n}, \frac{2}{n})$, then

$$\begin{aligned} E(\vec{n}, \vec{m}) &= i^k \sqrt{N} \sum_{q_1 \nmid N, \dots, q_n \nmid N} \prod_{j=1}^n \hat{\phi} \left(\frac{\log q_j}{\log R} \right) \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right) \langle \lambda_f(N q_1 \cdots q_n) \rangle_* \\ &= (-1)^{n+1} R(n, a; \phi) + O \left(\frac{\log \log N}{\log N} \right), \end{aligned} \quad (3.10)$$

where $R(n, a; \phi)$ is defined as in (1.9).

Next we show how to use Propositions 3.8 and 3.9 to complete the calculation of $S_2^{(n)}$ and prove Theorem 1.2.

3.3. Evaluating $S_2^{(n)}$. In this section we evaluate $S_2^{(n)}$ by proving the following lemma. By doing so, we complete the proof of Theorem 1.2.

Lemma 3.10. *Let $S(n, a; \phi)$ be as in (1.10). Under GRH for $L(s, f)$ and all Dirichlet L -functions, if $\text{supp}(\hat{\phi}) \subset (-\frac{1}{n-a}, \frac{1}{n-a})$ and $\text{supp}(\hat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$, then*

$$S_2^{(n)} = (-1)^{n+1} S(n, a; \phi) + O \left(\frac{\log \log N}{\log N} \right). \quad (3.11)$$

Proof. By Proposition 3.8, $S_2^{(n)}$ may be written as the sum of terms of the form (3.3) with $n_j + m_j \leq 2$ for each j with an error of $O(\log^{-1} N)$. Since $n_j \equiv m_j \pmod{2}$, then $n_j = m_j = 1$ or $n_j = 2$ and $m_j = 0$ for each j . Let E_ℓ denote the term $E(\vec{n}, \vec{m})$ in which $n_j = 2$ and $m_j = 0$ for exactly ℓ values of j . By Lemma 3.2, if $\ell \geq a/2$, then E_ℓ will vanish in the limit. Thus we have that

$$S_2^{(n)} = \sum_{\ell=0}^{\lfloor \frac{a-1}{2} \rfloor} \frac{n!}{2^\ell (n-2\ell)! \ell!} E_\ell + O(\log^{-1} N). \quad (3.12)$$

The combinatorial factor $\frac{n!}{2^\ell (n-2\ell)! \ell!}$ arises from choosing the indices of the primes for which $n_j = m_j = 1$, or $n_j = 2$ and $m_j = 0$. We choose the primes for which $n_j = m_j = 1$ in $\binom{n}{2\ell}$ ways, and put the remaining primes into pairs in $(2\ell - 1)!! = (2\ell)! / (\ell! 2^\ell)$ ways. Multiplying and simplifying gives the desired combinatorial coefficient. Now, to evaluate E_ℓ , we write

$$\begin{aligned} E_\ell &= \sum_{q_1 \nmid N, \dots, q_\ell \nmid N} \prod_{j=1}^{\ell} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right)^2 \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right)^2 \right) \\ &\quad \times i^k \sqrt{N} \sum_{p_1 \nmid N, \dots, p_{n-2\ell} \nmid N} \prod_{i=1}^{n-2\ell} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \left(\frac{2 \log p_i}{\sqrt{p_i} \log R} \right) \langle \lambda_f(N p_1 \cdots p_{n-2\ell}) \rangle_* \end{aligned} \quad (3.13)$$

as in (3.4). By Proposition 3.9, the second line of the product in (3.13) equals $(-1)^{n+1} R(n-2\ell, a-2\ell; \phi) + O \left(\frac{\log \log N}{\log N} \right)$ as defined in (1.9). We factor the remaining prime sums to get

$$E_\ell = (-1)^{n+1} \left[R(n-2\ell, a-2\ell; \phi) + O \left(\frac{\log \log N}{\log N} \right) \right] \left[\sum_{q \nmid N} \hat{\phi} \left(\frac{\log q}{\log R} \right)^2 \frac{4 \log^2 q}{q \log^2 R} \right]^\ell. \quad (3.14)$$

Similar to Lemma B.4 of [Mil02], we have by Riemann–Stieltjes integration that

$$\sum_{q \nmid N} \hat{\phi} \left(\frac{\log q}{\log R} \right)^2 \frac{4 \log^2 q}{q \log^2 R} = 2 \int_{-\infty}^{\infty} |y| \hat{\phi}(y)^2 dy = \sigma_\phi^2, \quad (3.15)$$

where σ_ϕ^2 is given by (1.8). Applying (3.15) to (3.14) we have that

$$E_\ell = (-1)^{n+1} (\sigma_\phi^2)^\ell R(n - 2\ell, a - 2\ell; \phi) + O\left(\frac{\log \log N}{\log N}\right). \quad (3.16)$$

Applying this to (3.12) and comparing with (1.10) completes the proof of the lemma. \square

Combining Lemma 3.10 with (2.32) and (2.35) completes the proof of Theorem 1.2. \square

4. EXPLICIT CALCULATIONS OF $E(\vec{n}, \vec{m})$

We continue the calculation of the terms $E(\vec{n}, \vec{m})$ starting from (3.9) in order to prove Propositions 3.8 and 3.9. In Section 4.1 we convert our sums over primes into integrals by applying the argument principle to $\zeta(s)$. We use these results to complete the proof of Proposition 3.8 in Section 4.2 by breaking up $E(\vec{n}, \vec{m})$ into subterms which we show vanish. Then, we prove Proposition 3.9 by finding a closed form integral for $E(\vec{n}, \vec{m})$ in Section 4.3 and doing combinatorial simplification in Section 4.4. Throughout this section, we will assume that $\text{supp}(\hat{\phi}) \subset \left(-\frac{1}{n-a}, \frac{1}{n-a}\right)$ for some nonnegative integer a . Additionally, we assume $\text{supp}(\hat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right)$, so we may without loss of generality take $a \leq \lceil n/2 \rceil$.

We begin by casing on the value of $r := (Q, b^\infty)$ in (3.9). Recall that $Q = p_1 \cdots p_{n-n'} q_1^{m_1} \cdots q_\omega^{m_\omega}$ and reindex the prime sums so that $r = (Q, b^\infty) = p_1 \cdots p_\alpha q_1^{m_1} \cdots q_\theta^{m_\theta}$ for some $\alpha \leq n - n'$, $\theta \leq \omega$. Thus $E(\vec{n}, \vec{m})$ can be written as a sum of terms of the form

$$\begin{aligned} & -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{q_1, \dots, q_\omega} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \frac{\log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \sum_{p_1, \dots, p_{n-n'}} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i}{\sqrt{p_i} \log R} \\ & \times \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{\substack{(b, N)=1 \\ b < N^{2022} \\ p_1, \dots, p_\alpha, q_1, \dots, q_\theta | b}} \frac{R(m^2, b) R(r, b)}{b \varphi(b)} \chi_0(Q/r) J_{k-1} \left(\frac{4\pi m \sqrt{Q}}{b \sqrt{N}} \right) + O(N^{-\epsilon}). \end{aligned} \quad (4.1)$$

In the expansion of $E(\vec{n}, \vec{m})$, the terms in (4.1) have a combinatorial coefficient attached from choosing the indices of the primes which divide b . In the “main term,” where $\omega = 0$ and so $n_j = m_j = 1$ for all $1 \leq j \leq n$, this coefficient is $\binom{n}{\alpha}$ and we simplify to get the explicit formula

$$\begin{aligned} A &:= E(\vec{n}, \vec{m}) \\ &= -\sum_{\alpha=0}^{a-1} \binom{n}{\alpha} 2^{n+1}\pi \sum_{p_1, \dots, p_\alpha} \prod_{j=1}^{\alpha} \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j}{\sqrt{p_j} \log R} \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{\substack{(b, N)=1 \\ b < N^{2022} \\ p_1, \dots, p_\alpha | b}} \frac{1}{b \varphi(b)} R(m^2, b) R(p_1 \cdots p_\alpha, b) \\ & \times N^{-1/2} \sum_{p_{\alpha+1}, \dots, p_n} J_{k-1} \left(\frac{4\pi m \sqrt{p_1 \cdots p_n}}{b \sqrt{N}} \right) \prod_{j=\alpha+1}^n \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\chi_0(p_j) \log p_j}{\sqrt{p_j} \log R} + O(N^{-\epsilon}). \end{aligned} \quad (4.2)$$

In the sum over α in (4.2), the value of α ranges from 0 to $a - 1$ as if $\alpha > a - 1$ the term is $O(N^{-\epsilon})$, which follows by applying $J_{k-1}(x) \ll x$, $R(m^2, b) \leq m^4$ and $R(p_1 \cdots p_\alpha, b) \leq \varphi(b)$. We denote the term above by A for the remainder of the section.

4.1. Converting from sums to integrals. Now we focus on the inner sum on the second line of (4.2), which we denote by $B(\alpha)$:

$$B(\alpha) := N^{-1/2} \sum_{p_{\alpha+1}, \dots, p_n} J_{k-1} \left(\frac{4\pi m \sqrt{p_{\alpha+1} \cdots p_n}}{b \sqrt{N}} \right) \times \prod_{j=\alpha+1}^n \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\chi_0(p_j) \log p_j}{p_j^{1/2} \log R}, \quad (4.3)$$

where c is some fixed constant. In the context of (4.2), we take $c = p_1 \cdots p_\alpha$. We evaluate $B(\alpha)$ by first converting from sums over primes to sums over integers through inclusion-exclusion. Then we apply Lemma 4.4 to convert the sums over integers into integrals, and finish by doing combinatorial simplification. Our main result is the following.

Lemma 4.1. *Let $B(\alpha)$ be as in (4.3), set $\Phi_{n-\alpha-\delta}(x) = \phi(x)^{n-\alpha-\delta}$, and suppose $\text{supp } \hat{\phi} \subset \left(-\frac{1}{n-a}, \frac{1}{n-a}\right)$. Under the Riemann Hypothesis for $\zeta(s)$, as N tends to infinity,*

$$B(\alpha) = \sum_{\delta=0}^{a-\alpha-1} \binom{n-\alpha}{\delta} \sum_{i=0}^{a-\alpha-\delta-1} (-1)^i \binom{n-\alpha-\delta}{i} \frac{b}{2\pi m \sqrt{c}} \sum_{p_1, \dots, p_\delta} \prod_{j=1}^{\delta} \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\chi_0(p_j) \log p_j}{p_j \log R} \\ \times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\alpha-\delta}} \left(\frac{2 \log(bx \sqrt{N/(cp_1 \cdots p_\delta)})/4\pi m}{\log R} \right) \frac{dx}{\log R} + O(N^{-\epsilon}). \quad (4.4)$$

First, we want to convert the sums over primes in (4.3) into sums over integers. Define

$$C'(\alpha, \beta) := N^{-1/2} \sum_{p_{\alpha+1}, \dots, p_n} \sum_{t_{\alpha+1}, \dots, t_{\alpha+\beta}=2}^{\infty} J_{k-1} \left(\frac{4\pi m \sqrt{cp_{\alpha+\beta+1} \cdots p_n}}{b \sqrt{N/(p_{\alpha+1}^{t_{\alpha+1}} \cdots p_{\alpha+\beta}^{t_{\alpha+\beta}})}} \right) \\ \times \prod_{j=\alpha+1}^{\alpha+\beta} \hat{\phi} \left(\frac{t_j \log p_j}{\log R} \right) \frac{\chi_0(p_j) \log p_j}{p_j^{t_j/2} \log R} \times \prod_{j=\alpha+\beta+1}^n \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\chi_0(p_j) \log p_j}{p_j^{1/2} \log R}. \quad (4.5)$$

This is a remainder term when we convert β of the sums over primes in $B(\alpha)$ to sums over integers. Note that $B(\alpha) = C'(\alpha, 0)$. Next, define

$$C(\alpha, \beta) := N^{-1/2} \sum_{p_{\alpha+1}, \dots, p_{\alpha+\beta}} \sum_{t_{\alpha+1}, \dots, t_{\alpha+\beta}=2}^{\infty} \sum_{v_{\alpha+\beta+1}, \dots, v_n=1}^{\infty} J_{k-1} \left(\frac{4\pi m \sqrt{cv_{\alpha+\beta+1} \cdots v_n}}{b \sqrt{N/(p_{\alpha+1}^{t_{\alpha+1}} \cdots p_{\alpha+\beta}^{t_{\alpha+\beta}})}} \right) \\ \times \prod_{j=\alpha+1}^{\alpha+\beta} \hat{\phi} \left(\frac{t_j \log p_j}{\log R} \right) \frac{\chi_0(p_j) \log p_j}{p_j^{t_j/2} \log R} \times \prod_{j=\alpha+\beta+1}^n \hat{\phi} \left(\frac{\log v_j}{\log R} \right) \frac{\chi_0(v_j) \Lambda(v_j)}{v_j^{1/2} \log R}. \quad (4.6)$$

This is the result of converting the inner prime sums in $C'(\alpha, \beta)$ to sums over integers (which are expressed as $v_{\alpha+\beta}, \dots, v_n$). The following relation between C and C' holds via a partitioning argument.

Property 4.2. *For any $\epsilon > 0$,*

$$C'(\alpha, \beta) = C(\alpha, \beta) - \sum_{i=1}^{a-1-\alpha-\beta} \binom{n-\alpha-\beta}{i} C'(\alpha, \beta+i) + O(N^{-\epsilon}). \quad (4.7)$$

We are able to restrict the sum over i in (4.7) up to $a-1-\alpha-\beta$ as when $i > a-1-\alpha-\beta$ the term is $O(N^{-\epsilon})$ (which be seen by taking $J_{k-1}(x) \ll 1$ in (4.5)). We repeatedly apply Property 4.2 to $B(\alpha)$ to get the following relation between B and C .

Property 4.3. *For any $\epsilon > 0$,*

$$B(\alpha) = \sum_{i=0}^{a-1-\alpha} \binom{n-\alpha}{i} C(\alpha, i) (-1)^i + O(N^{-\epsilon}). \quad (4.8)$$

Proof of Property 4.3. We proceed by induction. Define the sum

$$B'(\alpha, \eta) := \sum_{i=0}^{\eta} \binom{n-\alpha}{i} C(\alpha, i) (-1)^i - \sum_{i=\eta+1}^{a-1-\alpha} \binom{n-\alpha}{i} C'(\alpha, i) \sum_{j=0}^{\eta} (-1)^j \binom{i}{j} + O(N^{-\epsilon}). \quad (4.9)$$

Our inductive hypothesis is that $B'(\alpha, \eta) = B(\alpha)$. The base case $\eta = 0$ holds by Property 4.2 and the fact that $B(\alpha) = C'(\alpha, 0)$. For the inductive step, we assume that $B'(\alpha, k) = B(\alpha)$ for some non-negative integer k and show that under this assumption, $B'(\alpha, k+1) = B(\alpha)$. By the inductive hypothesis we have

$$B(\alpha) = \sum_{i=0}^k \binom{n-\alpha}{i} C(\alpha, i) (-1)^i - \sum_{i=k+1}^{a-1-\alpha} \binom{n-\alpha}{i} C'(\alpha, i) \sum_{j=0}^k (-1)^j \binom{i}{j} + O(N^{-\epsilon}). \quad (4.10)$$

We examine the $i = k+1$ term in the second sum and simplify it using the fact (from the binomial theorem) that $\sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} = 0$:

$$- \binom{n-\alpha}{k+1} C'(\alpha, k+1) \sum_{j=0}^k (-1)^j \binom{k+1}{j} = \binom{n-\alpha}{k+1} (-1)^{k+1} C'(\alpha, k+1). \quad (4.11)$$

Applying Property 4.2 (and reindexing the sum using a change of variables $\ell = k+1+j$) gives

$$\begin{aligned} & \binom{n-\alpha}{k+1} (-1)^{k+1} C'(\alpha, k+1) \\ &= \binom{n-\alpha}{k+1} (-1)^{k+1} \left[C(\alpha, \eta+1) - \sum_{j=1}^{a-\alpha-k} \binom{n-\alpha-k-1}{j} C'(\alpha, k+1+j) \right] + O(N^{-\epsilon}) \\ &= \binom{n-\alpha}{k+1} (-1)^{k+1} C(\alpha, k+1) - \sum_{j=1}^{a-\alpha-k} \binom{n-\alpha}{k+1+j} \binom{k+1+j}{k+1} (-1)^{k+1} C'(\alpha, k+1+j) + O(N^{-\epsilon}) \\ &= \binom{n-\alpha}{k+1} (-1)^{k+1} C(\alpha, k+1) - \sum_{\ell=k+2}^{a-\alpha-1} \binom{n-\alpha}{\ell} \binom{\ell}{k+1} (-1)^{k+1} C'(\alpha, \ell) + O(N^{-\epsilon}). \end{aligned} \quad (4.12)$$

Substituting this term back into (4.10) gives $B(\alpha) = B'(\alpha, k+1)$, proving the inductive hypothesis and completing the proof of the property. \square

Property 4.3 allows us to convert the prime sums in $B(\alpha)$ to sums over integers in $C(\alpha, \beta)$, with β of the sums being over higher powers of primes which are “left over” from this conversion. We convert from sums to integrals using the following lemma, which generalizes Lemma 4.9 of [HM07] and is proven in Appendix A.7.

Lemma 4.4. *Set $\Phi_{n-\eta-\gamma}(x) = \phi(x)^{n-\eta-\gamma}$. Under the Riemann Hypothesis for $\zeta(s)$, if $\text{supp}(\hat{\phi}) \subset (-\frac{1}{n-a}, \frac{1}{n-a})$, then, as N tends to infinity,*

$$\begin{aligned} & \sum_{v_1, \dots, v_{n-\eta}} \left[\prod_{i=1}^{n-\eta} \hat{\phi} \left(\frac{\log v_i}{\log R} \right) \left(\frac{\chi_0(v_i) \Lambda(v_i)}{\sqrt{v_i} \log R} \right) \right] J_{k-1} \left(\frac{4\pi m \sqrt{cv_1 \cdots v_{n-\eta}}}{b\sqrt{N}} \right) \\ &= \sum_{\gamma=0}^{a-\eta-1} \sum_{j=\gamma}^{a-\eta-1} (-1)^{j-\gamma} \binom{n-\eta}{j} \binom{j}{\gamma} \frac{b\sqrt{N}}{2\pi m \sqrt{c}} \sum_{v_1, \dots, v_\gamma=1}^{\infty} \left[\prod_{i=1}^{\gamma} \hat{\phi} \left(\frac{\log v_i}{\log R} \right) \frac{\chi_0(v_i) \Lambda(v_i)}{v_i \log R} \right] \\ & \quad \times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\eta-\gamma}} \left(\frac{2 \log(bx \sqrt{N/(cv_1 \cdots v_\gamma)/4\pi m})}{\log R} \right) \frac{dx}{\log R} + O(N^{1/2-\epsilon}). \end{aligned} \quad (4.13)$$

We apply Lemma 4.4 in order to convert the sums in $C(\alpha, \beta)$ into integrals (see (4.6) for the definition of $C(\alpha, \beta)$). To do so, we first define the following term which emerges from applying Lemma 4.4:

$$D(\alpha, \beta, \gamma) := \frac{b}{2\pi m \sqrt{c}} \sum_{p_{\alpha+1}, \dots, p_{\alpha+\beta+\gamma}} \sum_{t_{\alpha+1}, \dots, t_{\alpha+\beta}=2}^{\infty} \sum_{t_{\alpha+\beta+1}, \dots, t_{\alpha+\beta+\gamma}=1}^{\infty} \quad (4.14)$$

$$\times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\alpha-\beta-\gamma}} \left(\frac{2 \log(bx \sqrt{N''}/4\pi m)}{\log R} \right) \frac{dx}{\log R} \prod_{j=\alpha+\beta+1}^{\alpha+\beta+\gamma} \hat{\phi} \left(\frac{t_j \log p_j}{\log R} \right) \frac{\chi_0(p_j) \log p_j}{p_j^{t_j} \log R}. \quad (4.15)$$

Here, $N'' = N/(cp_{\alpha+1}^{t_{\alpha+1}} \cdots p_{\alpha+\beta+\gamma}^{t_{\alpha+\beta+\gamma}})$. The three sums in (4.15) can be interpreted as follows: the first α primes are those that divide b ; the next two sums involving β powers of primes are those left over from converting to sums over integers; and the last sums over γ integers are those left over from applying Lemma 4.4. We have the following relation between $C(\alpha, \beta)$ and $D(\alpha, \beta, \gamma)$.

Property 4.5. *We have*

$$\binom{n-\alpha}{\beta} C(\alpha, \beta) = \sum_{\gamma=0}^{a-\alpha-\beta-1} D(\alpha, \beta, \gamma) \left[\sum_{i=0}^{a-\alpha-\beta-\gamma-1} (-1)^i \binom{n-\alpha}{\gamma+\beta} \binom{n-\alpha-\beta-\gamma}{i} \binom{\gamma+\beta}{\gamma} \right] + O(N^{-\epsilon}). \quad (4.16)$$

Proof. Carefully applying Lemma 4.4 to $C(\alpha, i)$ using the definition in (4.6) gives

$$\binom{n-\alpha}{\beta} C(\alpha, \beta) = \binom{n-\alpha}{\beta} \sum_{\gamma=0}^{a-\alpha-\beta-1} D(\alpha, \beta, \gamma) \left[\sum_{j=\gamma}^{a-\alpha-\beta-1} (-1)^{j-\gamma} \binom{n-\alpha-\beta}{j} \binom{j}{\gamma} \right] + O(N^{-\epsilon}). \quad (4.17)$$

The property follows from reindexing the sum by setting $i = j - \gamma$ and simplifying. \square

We apply Property 4.5 to our formula for $B(\alpha)$ given by Property 4.3.

Property 4.6. *We have*

$$B(\alpha) = \sum_{\delta=0}^{a-\alpha-1} \binom{n-\alpha}{\delta} \sum_{i=0}^{a-\alpha-\delta-1} (-1)^i \binom{n-\alpha-\delta}{i} \sum_{\gamma=0}^{\delta} (-1)^{\delta-\gamma} \binom{\delta}{\gamma} D(\alpha, \delta-\gamma, \gamma) + O(N^{-\epsilon}). \quad (4.18)$$

Proof. The proof follows from applying Property 4.5 to (4.8) and collecting terms with $\delta = \beta + \gamma$. \square

We want to eliminate the remaining sums over powers of primes in D by recombining terms. First we define the following term.

$$G(\alpha, \delta) := \frac{b}{2\pi m \sqrt{c}} \sum_{p_1, \dots, p_{\delta}} \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\alpha-\delta}} \left(\frac{2 \log(bx \sqrt{N/(cp_1 \cdots p_{\delta})}/4\pi m)}{\log R} \right) \frac{dx}{\log R} \\ \times \prod_{j=\alpha+1}^{\alpha+\delta} \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\chi_0(p_j) \log p_j}{p_j \log R}. \quad (4.19)$$

$D(\alpha, \beta, \gamma)$ and $G(\alpha, \delta)$ (where $\delta = \beta + \gamma$, as above) satisfy the following relation:

Property 4.7. *We have*

$$G(\alpha, \delta) = \sum_{\gamma=0}^{\delta} (-1)^{\delta-\gamma} \binom{\delta}{\gamma} D(\alpha, \delta-\gamma, \gamma). \quad (4.20)$$

To prove this, we first prove an intermediate result.

Lemma 4.8. *Let $f(t_1, \dots, t_n)$ be a symmetric function which takes as an input a finite sequence t_1, \dots, t_n of arbitrary length and define the following transform $\mathcal{T}(i, j)$ on f :*

$$\mathcal{T}(i, j)(f) := \sum_{t_1, \dots, t_i=2}^{\infty} \sum_{s_1, \dots, s_j=1}^{\infty} f(t_1, \dots, t_i, s_1, \dots, s_j). \quad (4.21)$$

Then

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \mathcal{T}(i, n-i)(f) = f([1]^n) \quad (4.22)$$

where $[1]^n$ is a sequence of n 1's.

Proof. We proceed by induction on n . The base case $n = 1$ holds immediately. Assume the result holds up to n and define a new function $g(t_1, \dots, t_n) = f(t_1, \dots, t_n, 1)$. Then

$$\begin{aligned} f([1]^{n+1}) &= g([1]^n) \\ &= \sum_{i=0}^n (-1)^i \binom{n}{i} \sum_{t_1, \dots, t_i=2}^{\infty} \sum_{s_1, \dots, s_{n-i}=1}^{\infty} f(t_1, \dots, t_i, s_1, \dots, s_{n-i}, 1) \\ &= \sum_{i=0}^n \binom{n}{i} [(-1)^i \mathcal{T}(i, n+1-i)(f) + (-1)^{i+1} \mathcal{T}(i+1, n+1-(i+1))(f)] \\ &= \sum_{i=0}^{n+1} (-1)^i \mathcal{T}(i, n+1-i)(f) \binom{n+1}{i}, \end{aligned} \quad (4.23)$$

proving the inductive hypothesis and the result. \square

Proof of Property 4.7. This is just a special case of Lemma 4.8. It follows from setting $f([1]^\delta) = G(\alpha, \delta)$ and $\mathcal{T}(i, j) = D(\alpha, i, j)$. \square

Applying Property 4.7 to (4.18) gives the following relation between $B(\alpha)$ and $G(\alpha, \delta)$.

Property 4.9. *We have*

$$B(\alpha) = \sum_{\delta=0}^{a-\alpha-1} \binom{n-\alpha}{\delta} \sum_{i=0}^{a-\alpha-\delta-1} (-1)^i \binom{n-\alpha-\delta}{i} G(\alpha, \delta). \quad (4.24)$$

Applying the definition of $G(\alpha, \delta)$ from (4.19) to (4.24) completes the proof of Lemma 4.1. \square

4.2. Vanishing off the diagonal. In this section we complete the proof of Proposition 3.8 using Lemma 4.1. First we apply Lemma 4.1 to (4.1) with $c = p_1 \cdots p_\alpha q_1^{m_1} \cdots q_\omega^{m_\omega}$. Since $E(\vec{n}, \vec{m})$ is a sum of a fixed number of terms of the form (4.1), in order to prove Proposition 3.8 it suffices to show that these terms vanish when $n_j + m_j > 2$ for some j . We do so in Lemma 4.10, which relies on eliminating the sum over b using Lemma 4.11 followed by careful bounding of the result. Since we eventually show that these terms vanish, we omit the combinatorial coefficients. Recall that $Q = p_1 \cdots p_{n-n'} q_1^{m_1} \cdots q_\omega^{m_\omega}$ and $r = (Q, b^\infty) = p_1 \cdots p_\alpha q_1^{m_1} \cdots q_\theta^{m_\theta}$. Then, $E(\vec{n}, \vec{m})$ can be written as a sum of terms of the form

$$\begin{aligned}
 & \sum_{q_1, \dots, q_\omega} \prod_{j=1}^{\omega} \hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \frac{\log^{n_j} q_j}{q_j^{(n_j+m_j)/2} \log^{n_j} R} \sum_{p_1, \dots, p_{\alpha+\delta}} \prod_{i=1}^{\alpha+\delta} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_j}{p_j \log R} \\
 & \times \sum_{m \leq N^\epsilon} \frac{1}{m^2} \sum_{\substack{(b, N)=1 \\ b < N^{2022} \\ p_1, \dots, p_\alpha, q_1, \dots, q_\theta | b}} \frac{R(m^2, b) R(p_1 \cdots p_\alpha q_1^{m_1} \cdots q_\theta^{m_\theta}, b)}{\varphi(b)} \chi_0(p_{\alpha+1} \cdots p_{\alpha+\delta} q_{\theta+1}^{m_{\theta+1}} \cdots q_\omega^{m_\omega}) \\
 & \times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\alpha-\delta}} \left(\frac{2 \log(bx \sqrt{N/(p_1 \cdots p_{\alpha+\delta} q_1^{m_1} \cdots q_\omega^{m_\omega})/4\pi m})}{\log R} \right) \frac{dx}{\log R} + O(N^{-\epsilon}).
 \end{aligned} \tag{4.25}$$

We simplify this term in a few ways. First, we extend the b sum by removing the condition $b < N^{2022}$, as the integral decays rapidly with respect to b . Next, the principal character modulo b equals 1 when $(b, p_{\alpha+1} \cdots p_{\alpha+\delta} q_{\theta+1} \cdots q_\omega) = 1$ and 0 otherwise, so we may add this condition to the sum over b and remove the character. Thus (4.25) equals

$$\begin{aligned}
 & \sum_{q_1, \dots, q_\omega} \prod_{j=1}^{\omega} \hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \frac{\log^{n_j} q_j}{q_j^{(n_j+m_j)/2} \log^{n_j} R} \sum_{p_1, \dots, p_{\alpha+\delta}} \prod_{i=1}^{\alpha+\delta} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_j}{p_j \log R} \\
 & \times \sum_{m \leq N^\epsilon} \frac{1}{m^2} \sum_{\substack{(b, N p_{\alpha+1} \cdots p_{\alpha+\delta} q_{\theta+1} \cdots q_\omega)=1 \\ p_1, \dots, p_\alpha, q_1, \dots, q_\theta | b}} \frac{R(m^2, b) R(p_1 \cdots p_\alpha q_1^{m_1} \cdots q_\theta^{m_\theta}, b)}{\varphi(b)} \\
 & \times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\alpha-\delta}} \left(\frac{2 \log(bx \sqrt{N/(p_1 \cdots p_{\alpha+\delta} q_1^{m_1} \cdots q_\omega^{m_\omega})/4\pi m})}{\log R} \right) \frac{dx}{\log R} + O(N^{-\epsilon}).
 \end{aligned} \tag{4.26}$$

We break up the term (4.26) in two ways. First, we convert the sums over q_j and p_i into sums over distinct primes, requiring us to case on when some of the primes in the sum are equal. Next, we case on the multiplicity of the primes dividing b . Thus we see that $E(\vec{n}, \vec{m})$ can be written as a sum of terms of the form

$$\begin{aligned}
 F(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}) &:= \sum_{\substack{q_1, \dots, q_\ell \\ q_j \text{ distinct}}} \prod_{j=1}^{\ell} \hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{a_j} \frac{\log^{a_j} q_j}{q_j^{b_j} \log^{a_j} R} \sum_{m \leq N^\epsilon} \frac{1}{m^2} \sum_{\substack{(b', N q_1 \cdots q_\ell)=1 \\ b=b' q_1^{c_1} \cdots q_\ell^{c_\ell}}} \frac{R(m^2, b) R(q_1^{d_1} \cdots q_\ell^{d_\ell}, b)}{\varphi(b)} \\
 & \times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\nu}} \left(\frac{2 \log(b' x \sqrt{N q_1^{e_1} \cdots q_\ell^{e_\ell}}/4\pi m)}{\log R} \right) \frac{dx}{\log R}
 \end{aligned} \tag{4.27}$$

with an error term of $O(N^{-\epsilon})$, where a_i, b_i, c_i , and d_i are positive integers, and the e_i 's are integers. Additionally, we have that $\sum a_j = \nu$ and $b_i > 1$ for some i since $n_j + m_j > 2$ for some j . Lastly, we have that $b_j \geq d_j$ for all j , since $n_j \geq m_j$ in (4.26). Since $E(\vec{n}, \vec{m})$ is the sum of terms of the form (4.27) with some $b_i > 1$, proving the following lemma completes the proof of Proposition 3.8. The next lemma is also useful in Section 4.3.

Lemma 4.10. *Let $F(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e})$ be defined as in (4.27) with a_j, b_j, c_j, d_j positive integers, e_j integers, $b_j \geq d_j$ for all $1 \leq j \leq \kappa$. If $b_i > 1$ or $d_i < c_i$ for some i , then $F(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}) \ll O(\log^{-1} N)$.*

In order to prove this lemma, we need the following integral identity, which generalizes Section 7 of [ILS99] and is proven in Appendix A.8.

Lemma 4.11. *Let ϕ be an even Schwartz class function such that the Fourier transform $\hat{\phi}$ has compact support. Then*

$$\begin{aligned} & \sum_{(b,M)=1} \frac{R(1,b)R(m^2,b)}{\varphi(b)} \int_0^\infty J_{k-1}(y) \hat{\phi}\left(\frac{2\log(by\sqrt{Q}/4\pi m)}{\log R}\right) \frac{dy}{\log R} \\ &= \delta\left(\frac{m}{(m,M^\infty)}, 1\right) \frac{\varphi(M)}{M} \left(-\frac{1}{2} \int_{-\infty}^\infty \phi(x) \sin\left(2\pi x \frac{\log(k^2 Q/16\pi^2 m^2)}{\log R}\right) \frac{dx}{2\pi x} + \frac{1}{4}\phi(0)\right) \\ &+ O_{\epsilon'}\left(m^{\epsilon'} \frac{\log \log M}{\log R}\right), \end{aligned} \quad (4.28)$$

where the implied constant depends on ϵ' .

While the error term limits the effectiveness of this bound in the m aspect, in practice we only need to take $\epsilon' \ll 1/n$. To prove Lemma 4.10, we only need the above lemma to show that the sum over b is bounded by m^ϵ . We use the full result in Section 4.3 to prove Proposition 3.9.

Proof of Lemma 4.10. We begin by expanding the fraction in the sum over b in (4.27) using the multiplicative properties of Ramanujan sums and φ and the fact that $b = b' q_1^{c_1} \cdots q_\kappa^{c_\kappa}$ with $(b', N q_1 \cdots q_\ell) = 1$. Doing so gives

$$\frac{R(m^2, b) R(q_1^{d_1} \cdots q_\kappa^{d_\kappa}, b)}{\varphi(b)} = X \frac{R(m^2, b') R(m^2, q_1^{c_1} \cdots q_\kappa^{c_\kappa}) R(1, b')}{\varphi(b')}, \quad (4.29)$$

where $X = R(q_1^{d_1} \cdots q_\kappa^{d_\kappa}, q_1^{c_1} \cdots q_\kappa^{c_\kappa}) / \varphi(q_1^{c_1} \cdots q_\kappa^{c_\kappa})$. Applying (4.29) to (4.27) gives

$$\begin{aligned} F(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}) &= \sum_{\substack{q_1, \dots, q_\ell \\ q_j \text{ distinct}}} \prod_{j=1}^\ell \hat{\phi}\left(\frac{\log q_j}{\log R}\right)^{a_j} \frac{\log^{a_j} q_j}{q_j^{b_j} \log^{a_j} R} \sum_{m \leq N^\epsilon} X \frac{R(m^2, q_1^{c_1} \cdots q_\kappa^{c_\kappa})}{m^2} \\ &\times \sum_{(b', N q_1 \cdots q_\ell)=1} \frac{R(m^2, b') R(1, b')}{\varphi(b')} \int_{x=0}^\infty J_{k-1}(x) \widehat{\Phi_{n-\nu}}\left(\frac{2\log(b'x\sqrt{Q}/4\pi m)}{\log R}\right) \frac{dx}{\log R}, \end{aligned} \quad (4.30)$$

where $Q = N q_1^{e_1} \cdots q_\ell^{e_\ell}$. By Lemma 4.11, we have that the sum over b' in (4.30) is $\ll m^\epsilon$. Applying this bound to (4.30) gives

$$F(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}) \ll \sum_{\substack{q_1, \dots, q_\ell \\ q_j \text{ distinct}}} \prod_{j=1}^\ell \hat{\phi}\left(\frac{\log q_j}{\log R}\right)^{a_j} \frac{\log^{a_j} q_j}{q_j^{b_j} \log^{a_j} R} \sum_{m \leq N^\epsilon} X \frac{|R(m^2, q_1^{c_1} \cdots q_\kappa^{c_\kappa})|}{m^{2-\epsilon}}. \quad (4.31)$$

We bound the sum over m using the multiplicative properties of Ramanujan sums as

$$\sum_{m \leq N^\epsilon} X \frac{|R(m^2, q_1^{c_1} \cdots q_\kappa^{c_\kappa})|}{m^{2-\epsilon}} \ll \left[\sum_{(m', q_1 \cdots p_\kappa)=1} \frac{1}{(m')^{2-\epsilon}} \right] \left[\prod_{i=1}^\kappa \sum_{t \geq 0} \frac{|R(q_i^{2t}, q_i^{c_i})| |R(q_i^{d_i}, q_i^{c_i})|}{q_i^{(2-\epsilon)t} \varphi(q_i^{c_i})} \right]. \quad (4.32)$$

The sum over m' converges absolutely. Now we analyze the sum over t , primarily relying on (2.3) to bound the Ramanujan sums. When $2t < c_i - 1$, then $R(q_i^{2t}, q_i^{c_i}) = 0$. When $2t = c_i - 1$, then $R(q_i^{2t}, q_i^{c_i}) = q_i^{2t}$. When $2t \geq c_i$, we have that $R(q_i^{2t}, q_i^{c_i}) = \varphi(q_i^{c_i}) \leq q_i^{c_i}$.

When $d_i \geq c_i$, we may apply the bound $|R(q_i^{d_i}, q_i^{c_i})| \leq \varphi(q_i^{c_i})$ to find that the sum over t is $O(q_i^{\epsilon \lfloor c_i/2 \rfloor})$. When $d_i < c_i$, we use the bound $|R(q_i^{d_i}, q_i^{c_i})| \leq q_i^{d_i}$ to find that the sum over t is $O(q_i^{-1+\epsilon \lfloor c_i/2 \rfloor})$. Applying these bounds to (4.31) gives

$$F(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}) \ll \sum_{\substack{q_1, \dots, q_\ell \\ q_j \text{ distinct}}} \prod_{j=1}^{\ell} \hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{a_j} \frac{\log^{a_j} q_j}{q_j^{b_j + \eta_j - \epsilon \lfloor c_j/2 \rfloor} \log^{a_j} R}, \quad (4.33)$$

where $\eta_j = 1$ if $d_j < c_j$ and 0 otherwise. Removing distinctness from (4.33) and factoring gives

$$F(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}) \ll \prod_{j=1}^{\ell} \left[\sum_p \hat{\phi} \left(\frac{\log p}{\log R} \right)^{a_j} \frac{\log^{a_j} p}{p^{b_j + \eta_j - \epsilon \lfloor c_j/2 \rfloor} \log^{a_j} R} \right]. \quad (4.34)$$

Set $x_j = b_j + \eta_j - \epsilon \lfloor c_j/2 \rfloor$. If $x_j > 1$, the sum over p in (4.34) is $O(\log^{-a_j} R)$. If $x_j = 1$, then the sum is $O(1)$. First suppose that $d_j \geq c_j$. By assumption we have that $b_j \geq d_j$ so $b_j \geq c_j$ so if $b_j = 1$ then $x_j = 1$ and if $b_j > 1$ then $x_j > 1$. If $d_j < c_j$, then we have $x_j > 1$. We have shown that each $x_j \geq 1$, so each term in the product in (4.34) is at most $O(1)$. By assumption there exists some i for which either $d_i < c_i$ or $b_i > 1$. By the above arguments, we have that $x_i > 1$, so the i th factor in (4.34) is then $O(\log^{-a_i} R) \ll O(\log^{-1} N)$, since $a_i > 0$ by assumption. Taking the product over all j , we then have that $F(\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}) \ll O(\log^{-1} N)$, completing the proof of the lemma. \square

By the above lemma and the arguments preceding (4.27), $E(\vec{n}, \vec{m})$ can be written as the sum of finitely many terms, the number of which is independent of N and each of which is $O(\log^{-1} N)$ if $n_j + m_j > 2$ for some j . This completes the proof of Proposition 3.8.

4.3. Converting to closed form integrals. Now we resume proving Proposition 3.9 by first applying Lemma 4.1 to (4.2). Then we apply Lemma 4.11 to the result before simplifying by carefully bounding error terms. Applying Lemma 4.1 to (4.2) and simplifying gives

$$\begin{aligned} A &= - \sum_{\alpha=0}^{a-1} \binom{n}{\alpha} \sum_{\delta=0}^{a-\alpha-1} \binom{n-\alpha}{\delta} \sum_{i=0}^{a-\alpha-\delta-1} (-1)^i \binom{n-\alpha-\delta}{i} 2^n \sum_{p_1, \dots, p_{\alpha+\delta}} \sum_{m \leq N^\epsilon} \frac{1}{m^2} \\ &\quad \times \sum_{\substack{(b, N p_{\alpha+1} \cdots p_{\alpha+\delta})=1 \\ p_1, \dots, p_\alpha | b}} \frac{R(m^2, b) R(p_1 \cdots p_\alpha, b)}{\varphi(b)} \prod_{j=1}^{\alpha+\delta} \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j}{p_j \log R} \\ &\quad \times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\alpha-\delta}} \left(\frac{2 \log(bx \sqrt{N/(p_1 \cdots p_{\alpha+\delta})}/4\pi m)}{\log R} \right) \frac{dx}{\log R} + O(N^{-\epsilon}), \end{aligned} \quad (4.35)$$

where we eliminate the character and modify the sum over b as in the beginning of Section 4.2. Let $H(\alpha, \delta)$ denote the sum over primes in (4.35) for some fixed α, δ so that

$$\begin{aligned} H(\alpha, \delta) &:= \sum_{p_1, \dots, p_{\alpha+\delta}} \prod_{j=1}^{\alpha+\delta} \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j}{p_j \log R} \sum_{m \leq N^\epsilon} \frac{1}{m^2} \sum_{\substack{(b, N p_{\alpha+1} \cdots p_{\alpha+\delta})=1 \\ p_1, \dots, p_\alpha | b}} \frac{R(m^2, b) R(p_1 \cdots p_\alpha, b)}{\varphi(b)} \\ &\quad \times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\alpha-\delta}} \left(\frac{2 \log(bx \sqrt{N}/(4\pi m \sqrt{p_1 \cdots p_{\alpha+\delta}}))}{\log R} \right) \frac{dx}{\log R}. \end{aligned} \quad (4.36)$$

Our main result is the following.

Lemma 4.12. *Let $H(\alpha, \delta)$ be defined as above. Then*

$$\begin{aligned} H(\alpha, \delta) = & -2^{-1-\alpha-\delta}(-1)^\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_2) \cdots \widehat{\phi}(x_{\alpha+\delta+1}) \\ & \times \left[\int_{-\infty}^{\infty} \phi^{n-\alpha-\delta}(x_1) \frac{\sin(2\pi x_1(1 + |x_2| + \cdots + |x_{\alpha+1}| - |x_{\alpha+2}| - \cdots - |x_{\alpha+\delta+1}|))}{2\pi x_1} dx_1 \right. \\ & \left. - \frac{1}{2} \phi^{n-\alpha-\delta}(0) \right] dx_2 \cdots dx_{\alpha+\delta+1} + O\left(\frac{\log \log N}{\log N}\right). \end{aligned} \quad (4.37)$$

First, we transform the sum over the primes $p_1, \dots, p_{\alpha+\delta}$ in $H(\alpha, \delta)$ to a sum over distinct primes.

Property 4.13. *A distinctness condition may be added to (4.36), introducing an error of $O(\log^{-1} N)$.*

Proof. To introduce the distinctness condition, we apply inclusion-exclusion by casing on which primes are equal. If $p_i = p_j$ for some $1 \leq i \leq \alpha$ and $\alpha + 1 \leq j \leq \alpha + \delta$, then the corresponding term of $H(\alpha, \delta)$ is zero due to the condition on the sum over b . Thus, without loss of generality, we let $p_1 \cdots p_\alpha = q_1^{u_1} \cdots q_{\alpha'}^{u_{\alpha'}}$ and $p_{\alpha+1} \cdots p_{\alpha+\delta} = q_{\alpha'+1}^{u_{\alpha'+1}} \cdots q_{\alpha'+\delta'}^{u_{\alpha'+\delta'}}$, where the primes q_i are distinct and at least one $u_i > 1$. Thus, when adding a distinctness condition to $H(\alpha, \delta)$, we add additional terms of the form

$$\begin{aligned} & \sum_{\substack{q_1, \dots, q_{\alpha'+\delta'} \\ q_i \text{ distinct}}} \prod_{j=1}^{\alpha'+\delta'} \widehat{\phi}\left(\frac{\log q_j}{\log R}\right)^{u_j} \frac{\log^{u_j} q_j}{q_j^{u_j} \log^{u_j} R} \sum_{m \leq N^\epsilon} \frac{1}{m^2} \sum_{\substack{(b, Nq_{\alpha'+1} \cdots q_{\alpha'+\delta'})=1 \\ q_1^{u_1}, \dots, q_{\alpha'}^{u_{\alpha'}} | b}} \frac{R(m^2, b) R(q_1^{u_1} \cdots q_{\alpha'}^{u_{\alpha'}}, b)}{\varphi(b)} \\ & \times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\alpha-\delta}} \left(\frac{2 \log(bx\sqrt{N}/(4\pi m \sqrt{q_1^{u_1} \cdots q_{\alpha'+\delta'}^{u_{\alpha'+\delta'}}}))}{\log R} \right) \frac{dx}{\log R}. \end{aligned} \quad (4.38)$$

After breaking up (4.38) based on the multiplicities of the primes dividing b , we may appeal to Lemma 4.10 to find that these terms are $O(\log^{-1} N)$ since at least one $u_i > 1$, completing the proof of the property. \square

Now that we have introduced a distinctness condition, we case on the multiplicities of the primes dividing b . Suppose we take the b sum in (4.36) over all b satisfying $b = b' p_1^{c_1} \cdots p_\alpha^{c_\alpha}$ with $(b', p_1 \cdots p_\alpha) = 1$. Then, again utilizing Lemma 4.10, we find that if any $c_i > 1$ then the corresponding term is $O(\log^{-1} N)$. Thus, the only case which contributes is when each $c_i = 1$, so we find that

$$\begin{aligned} H(\alpha, \delta) = & \sum_{\substack{p_1, \dots, p_{\alpha+\delta} \\ p_i \text{ distinct}}} \prod_{j=1}^{\alpha+\delta} \widehat{\phi}\left(\frac{\log p_j}{\log R}\right) \frac{\log p_j}{p_j \log R} \sum_{m \leq N^\epsilon} \frac{1}{m^2} \sum_{\substack{b=b' p_1 \cdots p_\alpha \\ (b', Np_1 \cdots p_{\alpha+\delta})=1}} \frac{R(m^2, b) R(p_1 \cdots p_\alpha, b)}{\varphi(b)} \\ & \times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\alpha-\delta}} \left(\frac{2 \log(bx\sqrt{N}/(4\pi m \sqrt{p_1 \cdots p_{\alpha+\delta}}))}{\log R} \right) \frac{dx}{\log R} + O(\log^{-1} N). \end{aligned} \quad (4.39)$$

We simplify the fraction in the b sum using the multiplicative properties of Ramanujan sums and φ and the fact that $b = b' p_1 \cdots p_\alpha$ with $(b', Np_1 \cdots p_{\alpha+\delta}) = 1$. Doing so gives

$$\frac{R(m^2, b) R(p_1 \cdots p_\alpha, b)}{\varphi(b)} = \frac{R(m^2, b') R(m^2, p_1 \cdots p_\alpha) R(1, b')}{\varphi(b')}, \quad (4.40)$$

where we use the fact that $R(x, x) = \varphi(x)$ from (2.3). Applying this to (4.39) and setting $Q = Np_1 \cdots p_\alpha / (p_{\alpha+1} \cdots p_{\alpha+\delta})$ gives

$$\begin{aligned} H(\alpha, \delta) &= \sum_{\substack{p_1, \dots, p_{\alpha+\delta} \\ p_i \text{ distinct}}} \prod_{j=1}^{\alpha+\delta} \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j}{p_j \log R} \sum_{m \leq N^\epsilon} \frac{1}{m^2} R(m^2, p_1 \cdots p_\alpha) \\ &\quad \times \sum_{(b', Np_1 \cdots p_{\alpha+\delta})=1} \frac{R(m^2, b') R(1, b')}{\varphi(b')} \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\alpha-\delta}} \left(\frac{2 \log(b' x \sqrt{Q} / (4\pi m))}{\log R} \right) \frac{dx}{\log R} \\ &\quad + O(\log^{-1} N). \end{aligned} \quad (4.41)$$

Now we are ready to apply Lemma 4.11 to the sum over b' in (4.41). First we show that the resulting error term, which arises from the error term in (4.28), is $O\left(\frac{\log \log N}{\log N}\right)$. Using the multiplicativity of Ramanujan sums, we find that the sum over m in the error term is

$$\sum_{m \leq N^\epsilon} \frac{R(m^2, p_1 \cdots p_\alpha)}{m^2} O\left(m^\epsilon \frac{\log \log N}{\log N}\right) \ll \frac{\log \log N}{\log N} \sum_{(m', p_1 \cdots p_\alpha)=1} \frac{1}{(m')^{2-\epsilon}} \prod_{i=1}^{\alpha} \left[\sum_{t \geq 0} \frac{R(p_i^{2t}, p_i)}{p_i^{(2-\epsilon)t}} \right]. \quad (4.42)$$

The sum over m' converges absolutely. When $t = 0$, $R(p_i^{2t}, p_i) = R(1, p_i) = 1$. When $t > 0$, $R(p_i^{2t}, p_i) = \varphi(p_i) < p_i$. From this it is clear that the sum over t is bounded above by an absolute constant independent of p_i so the sum over m is $O\left(\frac{\log \log N}{\log N}\right)$. Introducing this into the sum over primes, we find that in all the error term is $O\left(\frac{\log \log N}{\log N}\right)$. Thus, after applying Lemma 4.11 to (4.41), we have that

$$\begin{aligned} H(\alpha, \delta) &= \sum_{\substack{p_1, \dots, p_{\alpha+\delta} \\ p_i \text{ distinct}}} \prod_{j=1}^{\alpha+\delta} \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j}{p_j \log R} \sum_{m \leq N^\epsilon} \frac{1}{m^2} R(m^2, p_1 \cdots p_\alpha) \delta \left(\frac{m}{(m, M^\infty)}, 1 \right) \frac{\varphi(M)}{M} \\ &\quad \times \left(-\frac{1}{2} \int_{-\infty}^{\infty} \phi(x)^{n-\alpha-\delta} \sin \left(2\pi x \frac{\log(k^2 Q / 16\pi^2 m^2)}{\log R} \right) \frac{dx}{2\pi x} + \frac{1}{4} \phi(0)^{n-\alpha-\delta} \right) + O\left(\frac{\log \log N}{\log N}\right), \end{aligned} \quad (4.43)$$

where $M = Np_1 \cdots p_{\alpha+\delta}$. We must have that $m = p_1^{t_1} \cdots p_{\alpha+\delta}^{t_{\alpha+\delta}}$, since otherwise $\delta\left(\frac{m}{(m, M^\infty)}, 1\right) = 0$, as N does not divide m . Additionally, we use that $\varphi(p_i)/p_i = 1 - 1/p_i$ and note that as N grows large, $\varphi(N)/N \rightarrow 1$, as N is prime. Lastly, we recall that $Q = Np_1 \cdots p_\alpha / (p_{\alpha+1} \cdots p_{\alpha+\delta})$ allowing us to simplify (4.43) as

$$\begin{aligned} H(\alpha, \delta) &= \sum_{\substack{p_1, \dots, p_{\alpha+\delta} \\ p_i \text{ distinct}}} \prod_{j=1}^{\alpha+\delta} \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j}{p_j \log R} \left(1 - \frac{1}{p_j} \right) \sum_{0 \leq t_1, \dots, t_{\alpha+\delta} \leq \epsilon \log N} \frac{R(p_1^{2t_1} \cdots p_\alpha^{2t_\alpha}, p_1 \cdots p_\alpha)}{p_1^{2t_1} \cdots p_{\alpha+\delta}^{2t_{\alpha+\delta}}} \\ &\quad \times \left(-\frac{1}{2} \int_{-\infty}^{\infty} \phi(x)^{n-\alpha-\delta} \sin \left(2\pi x \left(1 + \frac{\log p_1}{\log R} + \cdots + \frac{\log p_\alpha}{\log R} - \frac{\log p_{\alpha+1}}{\log R} - \cdots - \frac{\log p_{\alpha+\delta}}{\log R} \right) \right) \frac{dx}{2\pi x} \right. \\ &\quad \left. + \frac{1}{4} \phi(0)^{n-\alpha-\delta} \right) + O\left(\frac{\log \log N}{\log N}\right). \end{aligned} \quad (4.44)$$

We show that the only term in (4.44) which contributes is when $t_1 = \cdots = t_{\alpha+\delta} = 0$. We chose some i and sum over $t_i \geq 1$ and $t_\ell \geq 0$ for $\ell \neq i$:

$$\begin{aligned}
& \sum_{\substack{p_1, \dots, p_{\alpha+\delta} \\ p_i \text{ distinct}}} \prod_{j=1}^{\alpha+\delta} \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j}{p_j \log R} \left(1 - \frac{1}{p_j} \right) \sum_{\substack{0 \leq t_1, \dots, t_{\alpha+\delta} \leq \epsilon \log N \\ t_i \neq 0}} \frac{|R(p_1^{2t_1}, p_1)| \cdots |R(p_{\alpha}^{2t_{\alpha}}, p_{\alpha})|}{p_1^{2t_1} \cdots p_{\alpha+\delta}^{2t_{\alpha+\delta}}} \\
& \times \left(-\frac{1}{2} \int_{-\infty}^{\infty} \phi(x)^{n-\alpha-\delta} \sin \left(2\pi x \left(1 + \frac{\log p_1}{\log R} + \cdots + \frac{\log p_{\alpha}}{\log R} - \frac{\log p_{\alpha+1}}{\log R} - \cdots - \frac{\log p_{\alpha+\delta}}{\log R} \right) \right) \frac{dx}{2\pi x} \right. \\
& \left. + \frac{1}{4} \phi(0)^{n-\alpha-\delta} \right) + O \left(\frac{\log \log N}{\log N} \right). \tag{4.45}
\end{aligned}$$

Noting that the integral above is bounded and $\hat{\phi}$ is bounded, we take absolute values and remove distinctness over the prime sum to bound (4.45) by

$$\sum_{\substack{p_1, \dots, p_{\alpha+\delta} \\ p_j < R}} \prod_{j=1}^{\alpha+\delta} \frac{\log p_j}{p_j \log R} \sum_{\substack{0 \leq t_1, \dots, t_{\alpha+\delta} \leq \epsilon \log N \\ t_i \neq 0}} \frac{|R(p_1^{2t_1}, p_1)| \cdots |R(p_{\alpha}^{2t_{\alpha}}, p_{\alpha})|}{p_1^{2t_1} \cdots p_{\alpha+\delta}^{2t_{\alpha+\delta}}}. \tag{4.46}$$

We factor (4.46) into sums over primes as

$$\prod_{j=1}^{\alpha+\delta} \left[\sum_{p_j < R} \frac{\log p_j}{p_j \log R} \sum_{\delta(i,j) \leq t_j \leq \epsilon \log N} \frac{R(p_j^{2t_j}, p_j^{s_j})}{p_j^{2t_j}} \right], \tag{4.47}$$

where $\delta(i, j) = 1$ if $j = i$ and 0 otherwise, and $s_j = 1$ for $1 \leq j \leq \alpha$ and 0 for $\alpha + 1 \leq j \leq \alpha + \delta$. When $t_j = 0$, $|R(1, p^{s_j})| = 1$. When $t_j \geq 1$, we bound the summand by $1/p_j^{2t_j - s_j}$ so the sum over $t_j \geq 1$ is $O(1/p_j)$. Thus when $j \neq i$, the sum over t_j is $O(1)$ and we find that the sum of p_j is $O(1)$ as well. When $j = i$, the sum over t_i is $O(1/p_i)$, as there is no $t_i = 0$ term. Thus, we have that

$$\sum_{p_i < R} \frac{\log p_i}{p_i \log R} \sum_{1 \leq t_i \leq \epsilon \log N} \frac{R(p_i^{2t_i}, p_i^{s_i})}{p_i^{2t_i}} \ll \sum_{p_i < R} \frac{\log p_i}{p_i^2 \log R} \ll O(\log^{-1} N). \tag{4.48}$$

Taking a product over all j in (4.47) we find that the entire term is $O(\log^{-1} N)$. For each choice of i , summing over $t_i \geq 1$ yields similar results, so the only term in (4.44) not absorbed by the error term is the one for which each $t_j = 0$, so $m = 1$. We further simplify (4.44) in two ways. First, since $R(1, p) = -1$, we may eliminate the Ramanujan sums and introduce a factor of $(-1)^\alpha$. Additionally, note that we may multiply out by $(1 - 1/p_j)$ in the product in (4.44) to get a sum involving $\frac{\log p_j}{p_j}$ and a sum involving $\frac{\log p_j}{p_j^2}$. In the latter case, we bound the sum by $O(\log^{-1} N)$ and it is absorbed by the error term. Thus we have from (4.44) that

$$\begin{aligned}
H(\alpha, \delta) &= (-1)^\alpha \sum_{\substack{p_1, \dots, p_{\alpha+\delta} \\ p_i \text{ distinct}}} \prod_{j=1}^{\alpha+\delta} \hat{\phi} \left(\frac{\log p_j}{\log R} \right) \frac{\log p_j}{p_j \log R} \\
&\times \left(-\frac{1}{2} \int_{-\infty}^{\infty} \phi(x)^{n-\alpha-\delta} \sin \left(2\pi x \left(1 + \frac{\log p_1}{\log R} + \cdots + \frac{\log p_{\alpha}}{\log R} - \frac{\log p_{\alpha+1}}{\log R} - \cdots - \frac{\log p_{\alpha+\delta}}{\log R} \right) \right) \frac{dx}{2\pi x} \right. \\
&\left. + \frac{1}{4} \phi(0)^{n-\alpha-\delta} \right) + O \left(\frac{\log \log N}{\log N} \right). \tag{4.49}
\end{aligned}$$

We wish to remove the distinctness condition from (4.49).

Property 4.14. *Equation (4.49) holds with the distinctness condition in the prime sum removed.*

Proof. When removing the distinctness condition, we use inclusion-exclusion to eliminate terms where some $p_i = p_j$. Suppose $p_1 \cdots p_{\alpha+\delta} = q_1^{a_1} \cdots q_\ell^{a_\ell}$, where $q_i \neq q_j$ when $i \neq j$ and where some $a_j > 1$. Noting that the integral and $\widehat{\phi}$ in (4.49) are bounded, we bound these terms by

$$\sum_{\substack{q_1, \dots, q_\ell \\ q_i \text{ distinct} \\ q_i < R}} \prod_{j=1}^{\ell} \frac{\log^{a_j} q_j}{q_j^{a_j} \log^{a_j} R} \ll \sum_{\substack{q_1, \dots, q_\ell \\ q_i < R}} \prod_{j=1}^{\ell} \frac{\log^{a_j} q_j}{q_j^{a_j} \log^{a_j} R} \ll \prod_{j=1}^{\ell} \left[\sum_{q_j < R} \frac{\log^{a_j} q_j}{q_j^{a_j} \log^{a_j} R} \right]. \quad (4.50)$$

When $a_j = 1$ the sum is $O(1)$. When $a_j > 1$ the sum is $O(\log^{-a_j} R)$. Since some $a_j > 1$, taking the product over j in (4.50) we find that the term is $O(\log^{-2} R)$, completing the proof. \square

To complete the proof of Lemma 4.12, we apply Riemann–Stieltjes integration to each of the prime sums in (4.49) without the distinctness condition to find that

$$\begin{aligned} H(\alpha, \delta) &= -2^{-1-\alpha-\delta} (-1)^\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_2) \cdots \widehat{\phi}(x_{\alpha+\delta+1}) \\ &\quad \times \left[\int_{-\infty}^{\infty} \phi^{n-\alpha-\delta}(x_1) \frac{\sin(2\pi x_1(1 + |x_2| + \cdots + |x_{\alpha+1}| - |x_{\alpha+2}| - \cdots - |x_{\alpha+\delta+1}|))}{2\pi x_1} dx_1 \right. \\ &\quad \left. - \frac{1}{2} \phi^{n-\alpha-\delta}(0) \right] dx_2 \cdots dx_{\alpha+\delta+1} + O\left(\frac{\log \log N}{\log N}\right) \end{aligned} \quad (4.51)$$

as desired. \square

4.4. Simplifying the main term. In this section we finish the proof of Proposition 3.9 by applying Lemma 4.12 to (4.35) and simplifying. This step is mostly combinatorial, although we need the following lemma.

Lemma 4.15. *We have*

$$\int_{-\infty}^{\infty} \widehat{\phi}(y) (\sin(z + 2\pi x|y|) + \sin(z - 2\pi x|y|)) dy = 2 \sin(z) \phi(xy). \quad (4.52)$$

Proof. Using that $\sin(z + 2\pi x|y|) + \sin(z - 2\pi x|y|) = 2 \sin(z) \cos(2\pi xy)$ we have that

$$\begin{aligned} \int_{-\infty}^{\infty} \widehat{\phi}(y) (\sin(z + 2\pi x|y|) + \sin(z - 2\pi x|y|)) dy &= 2 \sin(z) \int_{-\infty}^{\infty} \widehat{\phi}(y) \cos(2\pi xy) dy \\ &= 2 \sin(z) \int_{-\infty}^{\infty} \widehat{\phi}(y) \Re(\exp(2\pi i xy)) dy \\ &= 2 \sin(z) \phi(xy). \end{aligned} \quad (4.53)$$

\square

Applying Lemma 4.12 to (4.35) gives

$$\begin{aligned} A &= \sum_{\alpha=0}^{a-1} \sum_{\delta=0}^{a-\alpha-1} \binom{n}{\alpha+\delta} \binom{\alpha+\delta}{\alpha}^{a-\alpha-\delta-1} \sum_{i=0}^{a-\alpha-\delta-1} (-1)^i \binom{n-\alpha-\delta}{i} 2^{n-1-\alpha-\delta} (-1)^\alpha \\ &\quad \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_2) \cdots \widehat{\phi}(x_{\alpha+\delta+1}) \\ &\quad \times \left[\int_{-\infty}^{\infty} \phi^{n-\alpha-\delta}(x_1) \frac{\sin(2\pi x_1(1 + |x_2| + \cdots + |x_{\alpha+1}| - |x_{\alpha+2}| - \cdots - |x_{\alpha+\delta+1}|))}{2\pi x_1} dx_1 \right. \\ &\quad \left. - \frac{1}{2} \phi^{n-\alpha-\delta}(0) \right] dx_2 \cdots dx_{\alpha+\delta+1} + O\left(\frac{\log \log N}{\log N}\right). \end{aligned} \quad (4.54)$$

Our first step is to eliminate the integral over $\phi^{n-\alpha-\delta}(0)$ in (4.54) when $\alpha + \delta > 0$. We fix some $\nu \leq a$ and collect the terms of (4.54) for which $\alpha + \delta = \nu$:

$$\begin{aligned} & \left[\sum_{\alpha=0}^{\nu} \binom{\nu}{\alpha} (-1)^{\alpha} \right] \binom{n}{\nu} \sum_{i=0}^{a-\nu-1} (-1)^i \binom{n-\nu}{i} 2^{n-1-\nu} \\ & \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_2) \cdots \widehat{\phi}(x_{\nu+1}) \left[-\frac{1}{2} \phi^{n-\nu}(0) \right] dx_2 \cdots dx_{\nu+1}. \end{aligned} \quad (4.55)$$

By the binomial theorem, $\sum_{\alpha=0}^{\nu} \binom{\nu}{\alpha} (-1)^{\alpha} = (1-1)^{\nu} = 0$ for $\nu > 0$, so the sum over α in (4.55) is 0 unless $\nu = 0$. Thus, the terms where $\alpha + \delta = \nu$ cancel when $\nu > 0$. When $\nu = 0$, we pull out the $-\frac{1}{2}\phi^n(0)$ term and find that

$$\begin{aligned} A &= \sum_{\alpha=0}^{a-1} \sum_{\delta=0}^{a-\alpha-1} \binom{n}{\alpha+\delta} \binom{\alpha+\delta}{\alpha} \sum_{i=0}^{a-\alpha-\delta-1} (-1)^i \binom{n-\alpha-\delta}{i} 2^{n-1-\alpha-\delta} (-1)^{\alpha} I(\alpha, \delta) \\ &\quad - 2^{n-2} \phi^n(0) \sum_{i=0}^{a-1} (-1)^i \binom{n}{i} + O\left(\frac{\log \log N}{\log N}\right) \end{aligned} \quad (4.56)$$

where

$$\begin{aligned} I(\alpha, \delta) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_2) \cdots \widehat{\phi}(x_{\alpha+\delta+1}) \int_{-\infty}^{\infty} \phi^{n-\alpha-\delta}(x_1) \\ &\quad \times \frac{\sin(2\pi x_1(1 + |x_2| + \cdots + |x_{\alpha+1}| - |x_{\alpha+2}| - \cdots - |x_{\alpha+\delta+1}|))}{2\pi x_1} dx_1 \cdots dx_{\alpha+\delta+1}. \end{aligned} \quad (4.57)$$

We want to simplify the first sum over α in (4.56), which we denote by A' . By Lemma 4.15 we have that $I(\alpha, \delta) = 2I(\alpha, \delta-1) - I(\alpha+1, \delta-1)$. We want to express A' in terms of $I(\alpha, 0)$. We do so with the following result:

Lemma 4.16. *Let $I(\alpha, \delta)$ be defined as above. Then*

$$I(\alpha, \delta) = \sum_{j=0}^{\delta} 2^{\delta-j} (-1)^j \binom{\delta}{j} I(\alpha + j, 0). \quad (4.58)$$

Proof. We prove the following claim holds by induction, after which setting $k = \delta$ completes the proof of the lemma:

$$I(\alpha, \delta) = \sum_{j=0}^k 2^{k-j} (-1)^j \binom{k}{j} I(\alpha + j, \delta - k). \quad (4.59)$$

The base case $k = 0$ holds immediately. Suppose the result holds up to k . Then using that $I(\alpha, \delta) = 2I(\alpha, \delta-1) - I(\alpha+1, \delta-1)$ we have that

$$\begin{aligned} I(\alpha, \delta) &= \sum_{j=0}^k 2^{k-j} (-1)^j \binom{k}{j} (2I(\alpha + j, \delta - k - 1) - I(\alpha + j + 1, \delta - k - 1)) \\ &= \sum_{j=0}^{k+1} 2^{k+1-j} (-1)^j \left[\binom{k+1}{j+1} - \binom{k}{j} \right] I(\alpha + j, \delta - k - 1) \\ &= \sum_{j=0}^{k+1} 2^{k+1-j} (-1)^j \binom{k+1}{j} I(\alpha + j, \delta - k - 1) \end{aligned} \quad (4.60)$$

completing the inductive hypothesis and the proof of the lemma. \square

Applying Lemma 4.16 to (4.56) gives

$$A' = \sum_{\alpha=0}^{a-1} \sum_{\delta=0}^{a-\alpha-1} \sum_{j=0}^{\delta} \binom{n}{\alpha+\delta} \binom{\alpha+\delta}{\alpha} \sum_{i=0}^{a-\alpha-\delta-1} (-1)^{\alpha+j+i} \binom{n-\alpha-\delta}{i} 2^{n-1-\alpha-j} \binom{\delta}{j} I(\alpha+j, 0). \quad (4.61)$$

We group terms in the above with fixed $\omega = \alpha + j$. Doing so and simplifying gives

$$A' = 2^{n-1} \sum_{\omega=0}^{a-1} 2^{-\omega} (-1)^{\omega} I(\omega, 0) \sum_{\alpha=0}^{\omega} \sum_{\delta=\omega-\alpha}^{a-1-\alpha} \sum_{i=0}^{a-\delta-\alpha-1} (-1)^i \binom{n}{\delta+\alpha} \binom{\delta+\alpha}{\alpha} \binom{n-\delta-\alpha}{i} \binom{\delta}{\omega-\alpha}. \quad (4.62)$$

We set $\delta = \ell + \omega - \alpha$ to change variables in the sum over δ , giving

$$A' = 2^{n-1} \sum_{\omega=0}^{a-1} 2^{-\omega} (-1)^{\omega} I(\omega, 0) \sum_{\alpha=0}^{\omega} \sum_{\ell=0}^{a-1-\omega} \sum_{i=0}^{a-\ell-\omega-1} (-1)^i \binom{n}{\ell+\omega} \binom{\ell+\omega}{\alpha} \binom{n-\ell-\omega}{i} \binom{\ell+\omega-\alpha}{\omega-\alpha}. \quad (4.63)$$

We rewrite the binomial coefficients in (4.63) as

$$A' = 2^{n-1} \sum_{\omega=0}^{a-1} 2^{-\omega} (-1)^{\omega} I(\omega, 0) \sum_{\alpha=0}^{\omega} \sum_{\ell=0}^{a-1-\omega} \sum_{i=0}^{a-\ell-\omega-1} (-1)^i \binom{n}{\ell+\omega+i} \binom{\ell+\omega+i}{\ell+i} \binom{\omega}{\alpha} \binom{\ell+i}{i}. \quad (4.64)$$

Grouping terms with fixed $m = \ell + i$ and rearranging gives

$$A' = 2^{n-1} \sum_{\omega=0}^{a-1} 2^{-\omega} (-1)^{\omega} I(\omega, 0) \sum_{\alpha=0}^{\omega} \binom{\omega}{\alpha} \sum_{m=0}^{a-1-\omega} \binom{n}{m+\omega} \binom{m+\omega}{m} \sum_{i=0}^m (-1)^i \binom{m}{i}. \quad (4.65)$$

As a consequence of the binomial theorem, the sum over i is zero unless $m = 0$ as it is the binomial expansion of $(1-1)^m$. Thus, summing over α yields $\binom{n}{\omega} 2^{\omega}$ so we simplify the entire term as

$$A' = 2^{n-1} \sum_{\omega=0}^{a-1} \binom{n}{\omega} (-1)^{\omega} I(\omega, 0). \quad (4.66)$$

Applying this to (4.56), we find that $A = (-1)^{n+1} R(n, a; \phi) + O\left(\frac{\log \log N}{\log N}\right)$, where $R(n, a; \phi)$ is defined as in (1.9). This completes the calculation of the main term and the proof of Proposition 3.9.

5. EXTENDING SUPPORT FOR RANDOM MATRIX THEORY

In this section, we compute the n^{th} centered moment of $Z_{\phi}(U)$ for test functions ϕ with $\text{supp}(\widehat{\phi}) \subseteq [-\frac{2}{n}, \frac{2}{n}]$. This computation plays a crucial role in the proof of Theorem 1.3. We focus on the case where $n \geq 3$ as [HM07] have proved the $n = 1, 2$ case.

5.1. Introduction. If U is an $M \times M$ unitary matrix, all of the eigenvalues of U have norm 1, which we denote by $e^{i\theta_1}, \dots, e^{i\theta_M}$. Then for any test function ϕ (ϕ is even, real-valued, and integrable) with suitable decay so that the following sums converge, define the 2π periodic function

$$\begin{aligned} F_M(\theta) &:= \sum_{j=-\infty}^{\infty} \phi\left(\frac{M}{2\pi}(\theta + 2\pi j)\right) \\ &= \frac{1}{M} \sum_{k=-\infty}^{\infty} \widehat{\phi}\left(\frac{k}{M}\right) e^{ik\theta}. \end{aligned} \quad (5.1)$$

Note that this function corresponds to g_M in [HR03]. Since ϕ decays rapidly, this function measures the closeness of the input point to the angle 0 in some sense. Define

$$Z_\phi(U) := \sum_{n=1}^M F_M(\theta_n). \quad (5.2)$$

The function Z_ϕ is well-defined since F_M is 2π -periodic. Moreover, since $F_M(\theta)$ measures how close θ is to the “central point” 0, the function Z_ϕ measures how close the eigenvalues of U are to the “central point” 1 on the unit circle.

In this section, we focus on studying the centered moments of $Z_\phi(U)$ averaging over the $\text{SO}(\text{even})$ and $\text{SO}(\text{odd})$ groups as they correspond to $H_k^+(N)$ and $H_k^-(N)$ respectively. Weyl’s explicit representation of Haar measure would allow us to compute the higher moments explicitly. However, to facilitate the comparison with number theory, we first compute the cumulants as in [HR03] and [HM07]. The cumulants $C_\ell^{\text{SO}(\text{odd})}$ and $C_\ell^{\text{SO}(\text{even})}$ are defined to satisfy the following equality of formal power series:

$$\sum_{\ell=1}^{\infty} C_\ell^{\text{SO}(\text{even})}(\phi) \frac{\lambda^\ell}{\ell!} = \lim_{\substack{M \text{ even} \\ M \rightarrow \infty}} \log \mathbb{E}_{\text{SO}(M)}[\exp(\lambda Z_\phi(U))], \quad (5.3)$$

$$\sum_{\ell=1}^{\infty} C_\ell^{\text{SO}(\text{odd})}(\phi) \frac{\lambda^\ell}{\ell!} = \lim_{\substack{M \text{ odd} \\ M \rightarrow \infty}} \log \mathbb{E}_{\text{SO}(M)}[\exp(\lambda Z_\phi(U))]. \quad (5.4)$$

Given the first n cumulants, one can compute the first n moments and vice-versa, as we now explain. For $n > 1$, if μ'_n is the n^{th} centered moment then

$$\mu'_n = \sum_{\substack{2k_2+3k_3+\dots+nk_n=n \\ k_j \geq 0}} \left(\frac{C_2}{2!}\right)^{k_2} \cdots \left(\frac{C_n}{n!}\right)^{k_n} \frac{n!}{k_2! \cdots k_n!}. \quad (5.5)$$

A similar formula recovers the C_n from the μ'_n .

Set $S(x) = \frac{\sin(\pi x)}{\pi x}$ and define

$$\begin{aligned} Q_n(\phi) &:= 2^{n-1} \sum_{m=1}^n \sum_{\substack{\lambda_1+\dots+\lambda_m=n \\ \lambda_j \geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \cdots \lambda_m!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(x_1)^{\lambda_1} \cdots \phi(x_m)^{\lambda_m} \\ &\quad \times S(x_1 - x_2) S(x_2 - x_3) \cdots S(x_{m-1} - x_m) S(x_m + x_1) dx_1 \cdots dx_m. \end{aligned} \quad (5.6)$$

We have the following result due to [HR03].

Lemma 5.1. *Let ϕ be a Schwartz test function such that $\text{supp}(\hat{\phi}) \subseteq [-\frac{2}{n}, \frac{2}{n}]$. For $n \geq 3$,*

$$\begin{aligned} C_n^{\text{SO}(\text{even})}(\phi) &= Q_n(\phi) \\ C_n^{\text{SO}(\text{odd})}(\phi) &= -Q_n(\phi). \end{aligned} \quad (5.7)$$

Moreover, for $n \geq 4$,

$$C_2^{\text{SO}(\text{even})} = C_2^{\text{SO}(\text{odd})} = 2 \int_{-\infty}^{\infty} |y| \hat{\phi}(y)^2 dy = \sigma_\phi^2 \quad (5.8)$$

where σ_ϕ^2 is defined as in (1.8).

Thus, in order to prove Theorem 1.3, it suffices to calculate $Q_n(\phi)$. The main result of this section is Proposition 5.2.

Proposition 5.2. *Let ϕ be a Schwartz test function such that $\text{supp}(\hat{\phi}) \subseteq \left[-\frac{1}{n-a}, \frac{1}{n-a}\right]$ for some nonnegative integer a and $\text{supp}(\hat{\phi}) \subseteq \left[-\frac{2}{n}, \frac{2}{n}\right]$. Let $R(n, a; \phi)$ be as in (1.9). Then*

$$Q_n(\phi) = R(n, a; \phi). \quad (5.9)$$

Assuming Proposition 5.2, we now prove Theorem 1.3.

By [HM07, Theorem 1.4], for $\text{supp}(\hat{\phi}) \subseteq \left[-\frac{1}{j}, \frac{1}{j}\right]$, then the first j moments (resp. cumulants) of $Z_\phi(U)$ averaged with respect to Haar measure are equal to the moments (resp. cumulants) of a Gaussian. In particular, for $j \geq 3$ we have $C_j^{\text{SO(even)}}(\phi) = C_j^{\text{SO(odd)}}(\phi) = 0$. Recall that $\text{supp}(\hat{\phi}) \subseteq \left[-\frac{1}{n-a}, \frac{1}{n-a}\right]$ and $\text{supp}(\hat{\phi}) \subseteq \left[-\frac{2}{n}, \frac{2}{n}\right]$, so we may without loss of generality take $a \leq \lceil n/2 \rceil$. Hence, restricting the sum in (5.5) to those terms with $k_3 = \dots = k_{n-a} = 0$ does not change its value. Moreover, $a \leq \lceil n/2 \rceil$ and $\sum_{\ell=2}^n \ell k_\ell = n$ imply that $k_n, k_{n-1}, \dots, k_{n-a+1} \in \{0, 1\}$ and at most one of $k_n, k_{n-1}, \dots, k_{n-a+1}$ is equal to 1.

Thus, we can rewrite (5.5) as

$$\mu'_n = \mathbb{1}_{\{n \text{ even}\}} \left(\frac{C_2}{2} \right)^{n/2} \frac{n!}{(n/2)!} + \sum_{\substack{2k_2 + (n-\ell) = n \\ 0 \leq \ell \leq a-1}} \left(\frac{C_2}{2!} \right)^{k_2} \left(\frac{C_{n-\ell}}{(n-\ell)!} \right) \frac{n!}{k_2!}. \quad (5.10)$$

Observing that $2k_2 + (n-\ell) = n$ forces $\ell = 2k_2$ and specializing to SO(even) , we have

$$\begin{aligned} \lim_{\substack{M \rightarrow \infty \\ M \text{ even}}} \left[\left(Z_\phi(U) - C_1^{\text{SO(even)}} \right)^n \right] &= n! \sum_{k_2=0}^{\lfloor \frac{a-1}{2} \rfloor} \frac{C_{n-2k_2}^{\text{SO(even)}}}{k_2!(n-2k_2)!} \left(\frac{C_2^{\text{SO(even)}}}{2} \right)^{k_2} \\ &\quad + \mathbb{1}_{\{n \text{ even}\}} \left(C_2^{\text{SO(even)}} \right)^{n/2} \frac{n!}{2^{(n/2)}(n/2)!}. \end{aligned} \quad (5.11)$$

The analogous equation holds for SO(odd) . Now, applying Theorems Lemma 5.1 and Proposition 5.2 to the right hand side of (5.11) and simplifying completes the proof of Theorem 1.3 after comparing with (1.10).

The remainder of the section is devoted to proving Proposition 5.2. The section is structured as follows. In Section 5.2, we prove Lemma 5.11, which allows us to express $Q_n(\phi)$ as the product of a combinatorial term and an integral term. In Section 5.3 we evaluate this combinatorial term, and in Section 5.4 we calculate the integral term.

5.2. Preliminaries. In this section, we work towards Proposition 5.2 by evaluating $Q_n(\phi)$ as defined in (5.6) when $\text{supp}(\hat{\phi}) \subseteq \left[-\frac{1}{n-a}, \frac{1}{n-a}\right]$ and $a \leq \lceil n/2 \rceil$. The main result of this subsection is Lemma 5.11, which splits $Q_n(\phi)$ into a combinatorial term and an integral term which we will then evaluate separately.

Equation (5.27) of [HM07] gives (independent of the choice of support) that

$$Q_n(\phi) = 2^{n-2} \int_0^\infty \dots \int_0^\infty \hat{\phi}(y_1) \dots \hat{\phi}(y_n) K(y_1, \dots, y_n) dy_1 \dots dy_n, \quad (5.12)$$

where

$$K(y_1, \dots, y_n) = \sum_{m=1}^n \sum_{\substack{\lambda_1 + \dots + \lambda_m = n \\ \lambda_j \geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \dots \lambda_m!} \sum_{\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}} \prod_{\ell=1}^m \chi_{\{|\sum_{j=1}^n \eta(\ell, j) \epsilon_j y_j| \leq 1\}} \quad (5.13)$$

and

$$\eta(\ell, j) = \begin{cases} +1 & \text{if } j \leq \sum_{k=1}^\ell \lambda_k \\ -1 & \text{if } j > \sum_{k=1}^\ell \lambda_k. \end{cases} \quad (5.14)$$

Important in our evaluation of $Q_n(\phi)$ will be the following identity given by Soshnikov [Sos00]:

$$z = \log(1 + (e^z - 1)) = \sum_{n=1}^{\infty} z^n \sum_{m=1}^n \sum_{\substack{\lambda_1 + \dots + \lambda_m = n \\ \lambda_j \geq 1}} \frac{(-1)^{m+1}}{m} \frac{1}{\lambda_1! \dots \lambda_m!}. \quad (5.15)$$

5.2.1. *Simplifying $K(y_1, \dots, y_n)$.* To evaluate $Q_n(\phi)$, we first discuss how we will interpret the expression $K(y_1, \dots, y_n)$ for $y_1, \dots, y_n \in [0, \frac{1}{n-a}]$. Throughout this paper, if $I \subseteq \{1, \dots, n\}$, we write

$$\chi_I = \chi_{\{y_1 + \dots + y_n > 1 + 2 \sum_{i \in I} y_i\}}. \quad (5.16)$$

Definition 5.3. A system of parameters (or s.o.p.) is an ordered tuple $(m, \lambda_1, \dots, \lambda_m, \epsilon_1, \dots, \epsilon_n)$ with $1 \leq m \leq n$, $\lambda_1 + \dots + \lambda_m = n$, $\lambda_i \geq 1$ for all $1 \leq i \leq m$, and $\epsilon_j = \pm 1$ for each $1 \leq j \leq n$.

Given a system of parameters S , we may use $\eta_S(\ell, j)$ to denote the function $\eta(\ell, j)$ where the λ_k are taken from S . When it is clear from context that the λ_k are taken from the s.o.p. S , we simply denote this function $\eta(\ell, j)$. Fix $n \geq 2a$ and a s.o.p. $S = (m, \lambda_1, \dots, \lambda_m, \epsilon_1, \dots, \epsilon_n)$. Consider the product

$$\prod_{\ell=1}^m \chi_{\{|\sum_{j=1}^n \eta(\ell, j) \epsilon_j y_j| \leq 1\}} \quad (5.17)$$

from (5.13). Fix $1 \leq \ell_0 \leq m$. In order to study (5.17), we study the complement of the indicator functions in (5.17), given by

$$\chi_{\{|\sum_{j=1}^n \eta(\ell_0, j) \epsilon_j y_j| > 1\}}. \quad (5.18)$$

For $y_1, \dots, y_n \in [0, \frac{1}{n-a}]$, if $\sum_{j=1}^n \eta(\ell_0, j) \epsilon_j y_j > 1$ then we cannot find $y'_1, \dots, y'_n \in [0, \frac{1}{n-a}]$ such that $\sum_{j=1}^n \eta(\ell_0, j) \epsilon_j y_j < -1$ because $a \leq \lceil n/2 \rceil$. Thus the indicator function (5.18) is identical to (5.16) for a particular choice of I . Moreover, there exists $y_i \in [0, \frac{1}{n-a}]$ such that (5.18) is nonzero if and only if one of the following (mutually exclusive) conditions holds:

- (i) $|\{1 \leq j \leq n : \eta(\ell_0, j) \epsilon_j = +1\}| \leq a - 1$, or
- (ii) $|\{1 \leq j \leq n : \eta(\ell_0, j) \epsilon_j = -1\}| \leq a - 1$.

If case (i) holds, we define

$$J_{\ell_0} = \{1 \leq j \leq n : \eta(\ell_0, j) \epsilon_j = +1\} \quad (5.19)$$

and say that J_{ℓ_0} has sign $\zeta_{\ell_0} = +1$.

If case (ii) holds, we define

$$J_{\ell_0} = \{1 \leq j \leq n : \eta(\ell_0, j) \epsilon_j = -1\} \quad (5.20)$$

and say that J_{ℓ_0} has sign $\zeta_{\ell_0} = -1$.

If neither case holds, then J_{ℓ_0} is undefined.

Lemma 5.4. If $S = (m, \lambda_1, \dots, \lambda_m, \epsilon_1, \dots, \epsilon_n)$ is a system of parameters and $J \subseteq [1, n]$ is any subset, then there is at most one $\ell_0 \in [1, m]$ and $\zeta \in \{\pm 1\}$ such that $\eta(\ell_0, i) \epsilon_i = \zeta$ for $i \in J$ and $\eta(\ell_0, j) \epsilon_j = -\zeta$ for $j \notin J$.

Proof. Suppose $\ell_1 > \ell_0$ and that both ℓ_0 and ℓ_1 have this property for some ζ_0 and ζ_1 . Without loss of generality, we assume that $J = \{i : \eta(\ell_0, i) \epsilon_i = -1\}$. It is clear that we cannot also have $I = \{i : \eta(\ell_1, i) \epsilon_i = -1\}$, so we may assume that $I = \{i : \eta(\ell_1, i) \epsilon_i = +1\}$, but then we must have $\eta(\ell_0, j) = -\eta(\ell_1, j)$ for all j , and this is clearly impossible. \square

In particular, if J_{ℓ_0} and J_{ℓ_1} are both defined, then $J_{\ell_0} \neq J_{\ell_1}$.

Definition 5.5. For a s.o.p. $S = (m, \lambda_1, \dots, \lambda_m, \epsilon_1, \dots, \epsilon_n)$, let $\{\ell_1, \dots, \ell_t\} \subseteq \{1, \dots, m\}$ be the set of indices for which I_{ℓ_j} is defined. Define

$$J(S) := \{J_{\ell_1}, \dots, J_{\ell_t}\}. \quad (5.21)$$

Define

$$I(S) := \{I_1, \dots, I_r\} \quad (5.22)$$

to be the subset of elements of $J(S)$ which are minimal with respect to inclusion. That is, $I(S)$ consists of those elements of $J(S)$ which do not strictly contain any other elements of $J(S)$. By Lemma 5.4, for each $i \in [1, r]$ there is a unique ℓ_i such that $I_i = J_{\ell_i}$. Finally, define the function

$$\sigma^S(y_1, \dots, y_n) := \sum_{i=1}^r \sum_{1 \leq j_1 < \dots < j_i \leq r} (-1)^i (\chi_{I_{j_1}} \cdots \chi_{I_{j_i}})(y_1, \dots, y_n), \quad (5.23)$$

and the quantity

$$A(S) := \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \cdots \lambda_m!}. \quad (5.24)$$

The next lemma provides a sort of “Möbius inversion formula” for σ^S .

Lemma 5.6. For any s.o.p. S , we have

$$\sigma^S(y_1, \dots, y_n) = \begin{cases} -1 & \text{if } \chi_I(y_1, \dots, y_n) = 1 \text{ for some } I \in I(S) \\ 0 & \text{otherwise.} \end{cases} \quad (5.25)$$

Proof. Fix (y_1, \dots, y_n) . Suppose there are k elements in $I(S)$ whose support contains (y_1, \dots, y_n) . If $k = 0$ the result is immediate. Now, for $k \geq 1$ and $1 \leq i \leq k$, there are $\binom{k}{i}$ terms in the i th summand of with coefficient $(-1)^i$ and all the other terms vanish. Thus we have

$$\sigma^S(y_1, \dots, y_n) = \sum_{i=1}^k \binom{k}{i} (-1)^i = (1 - 1)^k - 1 = -1. \quad (5.26)$$

□

We now have the following.

Lemma 5.7. For $(y_1, \dots, y_n) \in \left[0, \frac{1}{n-a}\right]^n$,

$$K(y_1, \dots, y_n) = \sum_{t=1}^n (-1)^t \sum_{\substack{(I_1, \dots, I_t) \\ \text{valid}}} (\chi_{I_1} \cdots \chi_{I_t})(y_1, \dots, y_n) \sum_{\substack{\text{s.o.p. } S \text{ with} \\ I_1, \dots, I_t \in I(S)}} A(S). \quad (5.27)$$

Proof. The product (5.17) vanishes at (y_1, \dots, y_n) if and only if there is some $J \in J(S)$ such that χ_J is supported at (y_1, \dots, y_n) if and only if there is some $I \in I(S)$ such that χ_I is supported at (y_1, \dots, y_n) . So, by Lemma 5.6,

$$\prod_{\ell=1}^m \chi_{\{|\sum_{j=1}^n \eta(\ell, j) \epsilon_j y_j| \leq 1\}}(y_1, \dots, y_n) = 1 + \sigma^S(y_1, \dots, y_n). \quad (5.28)$$

Substituting (5.28) into (5.13), we have that

$$K(y_1, \dots, y_n) = \sum_{m=1}^n \sum_{\substack{\lambda_1 + \dots + \lambda_m = n \\ \lambda_j \geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \cdots \lambda_m!} \cdot 2^n + \sum_{\text{s.o.p.'s } S} A(S) \sigma^S(y_1, \dots, y_n). \quad (5.29)$$

Applying (5.15), we find that the first sum is 0. Expanding the second sum using the definition of σ^S from (5.23) and rearranging completes the proof. □

5.2.2. Simplifying $Q_n(\phi)$. In this section we simplify $Q_n(\phi)$ by applying Lemma 5.7 to (5.12). First we define further notation which allows us to express $Q_n(\phi)$ (through Lemma 5.11) in terms of combinatorial quantities which we then compute in Section 5.3.1 and 5.3.2.

The symmetric group S_n acts naturally on sets of (unordered) t -tuples of subsets of $[1, n]$ by permuting the elements in each subset of each tuple. Take such a t -tuple (I_1, \dots, I_t) and some $I_j = \{i_1, \dots, i_k\}$. Given some $\tau \in S_n$, we have that $\tau(I_j) = \{\tau(i_1), \dots, \tau(i_k)\}$. Let $\chi_{I_1} \cdots \chi_{I_t}$ and $\chi_{J_1} \cdots \chi_{J_t}$ be elements of Ω such that there exists a permutation $\tau \in S_n$ so that for each $1 \leq \ell \leq t$, $\tau(I_\ell) = J_\ell$. Then

$$\int_0^\infty \cdots \int_0^\infty \hat{\phi}(y_1) \cdots \hat{\phi}(y_n) (\chi_{I_1} \cdots \chi_{I_t} - \chi_{J_1} \cdots \chi_{J_t})(y_1, \dots, y_n) dy_1 \cdots dy_n = 0. \quad (5.30)$$

This motivates the following definition.

Definition 5.8. *The symmetric group S_n acts naturally on sets of (unordered) t -tuples of subsets of $[1, n]$, as described above. A t -class is an orbit of this action.*

Now, for a t -class C , let

$$\int C dy := \int_0^\infty \cdots \int_0^\infty \hat{\phi}(y_1) \cdots \hat{\phi}(y_n) \chi(y_1, \dots, y_n) dy_1 \cdots dy_n \quad (5.31)$$

where $\chi = \chi_{I_1} \cdots \chi_{I_t}$ with (I_1, \dots, I_t) any element of C . (5.30) shows that $\int C dy$ is well defined.

Definition 5.9. *We call an unordered tuple (I_1, \dots, I_t) of subsets of $\{1, \dots, n\}$ valid if $I_1, \dots, I_t \in I(S)$ for some s.o.p. S and $\chi_{I_1} \cdots \chi_{I_t}$ is supported at some point in $\left[0, \frac{1}{n-a}\right]^n$.*

Definition 5.10. *We call a t -class valid if it contains at least one valid tuple.*

We are now ready to prove the main result of the section.

Lemma 5.11. *For a s.o.p. S and a t -class C , set*

$$T(S, C) := \# \{(I_1, \dots, I_t) \in C : I_1, \dots, I_t \in I(S)\}. \quad (5.32)$$

We have

$$Q_n(\phi) = 2^{n-2} \sum_{t=1}^n (-1)^t \sum_{\text{valid } t\text{-classes } C} \left(\sum_{\text{s.o.p.'s } S} T(S, C) A(S) \right) \int C dy. \quad (5.33)$$

Proof. Given a valid t -class C , there is a valid tuple $(I_1, \dots, I_t) \in C$ for which $\chi_{I_1} \cdots \chi_{I_t}$ is supported at some point $(y_1, \dots, y_n) \in \left[-\frac{1}{n-a}, \frac{1}{n-a}\right]^n$. Therefore, if $\tau \in S_n$, then $\chi_{\tau(I_1)} \cdots \chi_{\tau(I_t)}$ is supported at $(y_{\tau(1)}, \dots, y_{\tau(n)})$. Since S_n acts transitively on C , this means that every tuple in C is valid. Now, applying Lemma 5.7 to (5.12) and grouping tuples into t -classes completes the proof. \square

Lemma 5.11 shows that in order to calculate $Q_n(\phi)$ it suffices to calculate $\sum T(S, C) A(S)$ and $\int C dy$ for t -classes C . In Section 5.3, we calculate $\sum T(S, C) A(S)$ and then in Section 5.4 we calculate $\int C dy$.

5.3. Computing the combinatorial piece. In this section, we calculate $\sum T(S, C) A(S)$ for valid t -classes C , where $T(S, C)$ and $A(S)$ are defined as in (5.32) and (5.24), respectively. In Section 5.3.1 we find a closed form for the case $t = 1$, and then in Section 5.3.2 we show that when $t \geq 2$ the quantity vanishes.

5.3.1. *Computing for valid 1-classes.* In this section, we compute the terms in (5.33) for which $t = 1$. We first classify the valid 1-tuples.

Lemma 5.12. *If I and J are subsets of $[1, n]$ such that $|I \cup J| \geq a$, then $\chi_I \cdot \chi_J$ is identically zero on $\left[0, \frac{1}{n-a}\right]^n$.*

Proof. Let I and J be as in the hypotheses, and assume for contradiction that both $y_1 + \dots + y_n > 1 + 2 \sum_{i \in I} y_i$ and $y_1 + \dots + y_n > 1 + 2 \sum_{j \in J} y_j$ for some $(y_1, \dots, y_n) \in \left[0, \frac{1}{n-a}\right]^n$. Since $y_h \leq \frac{1}{n-a}$ for every h , $\sum_{h \notin I \cup J} y_h \leq 1$, so we must have that $\sum_{i \in I} y_i < \sum_{j \in J \setminus I} y_j$ and similarly $\sum_{j \in J} y_j < \sum_{i \in I \setminus J} y_i$ by our assumptions. Adding these inequalities gives

$$\sum_{i \in I} y_i + \sum_{j \in J} y_j < \sum_{i \in I \setminus J} y_i + \sum_{j \in J \setminus I} y_j, \quad (5.34)$$

which is a contradiction, as all the y_h 's are nonnegative and the terms on the right are a subset of those on the left. \square

Lemma 5.13. *If I is a subset of $[1, n]$, then the 1-tuple (I) is valid if and only if $|I| \leq a - 1$.*

Proof. If $|I| > a - 1$, then (I) is not valid by Lemma 5.12, taking both subsets to be I .

Now suppose $|I| \leq a - 1$. Let $y_j = 1/(n-a)$ for each $j \notin I$ and let $y_i = 0$ for each $i \in I$. It is clear that $\chi_I(y_1, \dots, y_n) = 1$. Now consider the system of parameters $S = (m, \lambda_1, \dots, \lambda_m, \epsilon_1, \dots, \epsilon_n)$, where $m = 1$, $\lambda_1 = n$, and $\epsilon_i = -1$ if and only if $i \in I$. Clearly, $I \in I(S)$. Therefore, (I) is valid. \square

It follows from Lemma 5.13 that the valid 1-classes are exactly the classes

$$C_f := \{(I) : I \subseteq [1, n], |I| = f\} \quad (5.35)$$

with $0 \leq f \leq a - 1$.

Lemma 5.14. *Let $1 \leq f \leq a - 1$. Let $S = (m, \lambda_1, \dots, \lambda_m, \epsilon_1, \dots, \epsilon_n)$ be a system of parameters with $m \geq 2$ and suppose $(I) \in C_f$ is such that, for some $1 \leq \ell \leq m$, we have $I = J_\ell \in J(S)$. Define $\Lambda_\ell := \lambda_1 + \dots + \lambda_\ell$. Then $I \in I(S)$ if and only if $[\Lambda_{\ell-1} + 1, \Lambda_\ell] \not\subseteq I$ and $[\Lambda_\ell + 1, \Lambda_{\ell+1}] \not\subseteq I$. If $\ell = m$, then we set $[\Lambda_m + 1, \Lambda_{m+1}]$ to $[1, \Lambda_1] = [1, \lambda_1]$.*

Proof. Assume without loss of generality that J_ℓ has sign $\zeta_\ell = -1$, i.e. $J_\ell = \{j : \eta(\ell, j)\epsilon_j = -1\}$.

For any $\ell' < \ell$, we have

$$\eta(\ell, j)\epsilon_j = \begin{cases} -\eta(\ell', j)\epsilon_j & \text{if } j \in [\Lambda_{\ell'} + 1, \Lambda_\ell], \\ \eta(\ell', j)\epsilon_j & \text{if } j \notin [\Lambda_{\ell'} + 1, \Lambda_\ell]. \end{cases} \quad (5.36)$$

If $[\Lambda_{\ell-1} + 1, \Lambda_\ell] \subseteq J_\ell$, then $J_{\ell-1} = J_\ell \setminus [\Lambda_{\ell-1} + 1, \Lambda_\ell] \subsetneq J_\ell$. In particular, J_ℓ is not minimal, so $J_\ell \notin I(S)$. Similarly, if $[\Lambda_\ell + 1, \Lambda_{\ell+1}] \subseteq J_\ell$ then $J_{\ell+1} = J_\ell \setminus [\Lambda_\ell + 1, \Lambda_{\ell+1}]$ so J_ℓ is not minimal.

Now assume J_ℓ is not minimal, so there exists some $J_{\ell'} \subsetneq J_\ell$.

First, suppose the sign of $J_{\ell'}$ is $\zeta_{\ell'} = -1$. Suppose that $\ell' < \ell$. By (5.36), $J_{\ell'} \setminus [\Lambda_{\ell'} + 1, \Lambda_\ell] = J_\ell \setminus [\Lambda_{\ell'} + 1, \Lambda_\ell]$, while $J_{\ell'} \cap [\Lambda_{\ell'} + 1, \Lambda_\ell]$ and $J_\ell \cap [\Lambda_{\ell'} + 1, \Lambda_\ell]$ are disjoint with union $[\Lambda_{\ell'} + 1, \Lambda_\ell]$. So, so $J_{\ell'} \subsetneq J_\ell$ implies $[\Lambda_{\ell-1} + 1, \Lambda_\ell] \subseteq [\Lambda_{\ell'} + 1, \Lambda_\ell] \subset J_\ell$. Similarly, if $\ell' > \ell$, we have $[\Lambda_\ell + 1, \Lambda_{\ell+1}] \subset J_\ell$.

Next suppose that the sign $\zeta_{\ell'} = 1$. Suppose that $\ell' < \ell$. By (5.36), $J_{\ell'} \cap [\Lambda_{\ell'} + 1, \Lambda_\ell] = J_\ell \cap [\Lambda_{\ell'} + 1, \Lambda_\ell]$, while $J_{\ell'} \setminus [\Lambda_{\ell'} + 1, \Lambda_\ell]$ and $J_\ell \setminus [\Lambda_{\ell'} + 1, \Lambda_\ell]$ are disjoint with union $[1, n] \setminus [\Lambda_{\ell'} + 1, \Lambda_\ell]$. Since J_ℓ is not minimal, we must have $[\Lambda_\ell + 1, \Lambda_{\ell+1}] \subset [1, n] \setminus [\Lambda_{\ell-1} + 1, \Lambda_\ell] \subset J_\ell$. When $\ell' > \ell$, by the same reasoning we have that $[\Lambda_{\ell-1} + 1, \Lambda_\ell] \subseteq J_\ell$. \square

Lemma 5.15. Fix $1 \leq f \leq a - 1$. We have

$$\sum_{\substack{\text{s.o.p.'s } S \\ \text{with } m \geq 2}} T(S, C_f) A(S) = 2n!(-1)^n \sum_{\substack{c+d \leq n \\ c, d \geq 0}} (-1)^{c+d+1} G(n, f, c, d) \frac{1}{(n-c-d)!c!d!}, \quad (5.37)$$

where

$$G(n, f, c, d) = \binom{n}{f} - \binom{n-c}{f-c} - \binom{n-d}{f-d} + \binom{n-c-d}{f-c-d}. \quad (5.38)$$

Proof. Let $S = (m, \lambda_1, \dots, \lambda_m, \epsilon_1, \dots, \epsilon_n)$ denote a variable system of parameters. By Lemma 5.4, we can rewrite $T(S, C_f)$ as

$$T(S, C_f) = \sum_{\ell=1}^m \mathbb{1}_{\{J_\ell \in I(S) \text{ and } \#J_\ell=f\}}. \quad (5.39)$$

We sum over systems of parameters by first summing over all values of m , then summing over all possible values of ℓ , then summing over all possible values of $c = \lambda_\ell$ and $d = \lambda_{\ell+1}$, then summing over all possible values of $\lambda_1, \dots, \lambda_m$ and finally summing over all possible choices of $\epsilon_1, \dots, \epsilon_n$. For fixed $m, \lambda_1, \dots, \lambda_m$, the $\epsilon_1, \dots, \epsilon_n$ and J_ℓ, ζ_ℓ uniquely determine each other, so we may rewrite the innermost sum as

$$\sum_{(\epsilon_j) \in \{\pm 1\}^n} A(S) \mathbb{1}_{\{J_\ell \in I(S) \text{ and } J_\ell: \#J_\ell=f\}} = A(S) \sum_{\zeta_\ell \in \{\pm 1\}} \sum_{\#J_\ell=f} \mathbb{1}_{\{J_\ell \in I(S)\}}. \quad (5.40)$$

By Lemma 5.14, the sum over J_ℓ is $G(n, f, c, d)$, since we can choose a general f element subset in $\binom{n}{f}$ ways, and we need to subtract off when the c element subset $[\lambda_1 + \dots + \lambda_{\ell-1} + 1, \lambda_1 + \dots + \lambda_\ell] \subseteq I$ or when the d element subset $[\lambda_1 + \dots + \lambda_\ell + 1, \lambda_1 + \dots + \lambda_{\ell+1}] \subseteq I$. Then, we add back in the case when both subsets are contained in J_ℓ since we have double counted it. Finally, there are 2 choices for ζ_ℓ . We have

$$\sum_{\substack{\text{s.o.p.'s } S \\ \text{with } m \geq 2}} T(S, C_f) A(S) = \sum_{m=2}^n \sum_{\ell=1}^m \sum_{\substack{c, d \geq 1 \\ c+d \leq n}} \sum_{\substack{\lambda_1 + \dots + \lambda_m = n \\ \lambda_i \geq 1, \lambda_\ell = c, \lambda_{\ell+1} = d}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \dots \lambda_m!} 2 G(n, f, c, d). \quad (5.41)$$

Noting that for each value of ℓ the inner summand is the same, we can set $\ell = m - 1$ and write

$$\sum_{\substack{\text{s.o.p.'s } S \\ \text{with } m \geq 2}} T(S, C_f) A(S) = 2n! \sum_{\substack{c, d \geq 1 \\ c+d \leq n}} \frac{G(n, f, c, d)}{c!d!} \sum_{m=2}^n m \sum_{\lambda_1 + \dots + \lambda_{m-2} = n-c-d} \frac{(-1)^{m+1}}{m} \frac{1}{\lambda_1! \dots \lambda_{m-2}!}. \quad (5.42)$$

The sum over m equals $(-1)^{n+c+d+1}/(n-c-d)!$, which follows from evaluating the coefficient of z^n in

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n = e^{-z} = \frac{1}{1 + (e^z - 1)} = \sum_{n=0}^{\infty} z^n \sum_{m=1}^n \sum_{\substack{\lambda_1 + \dots + \lambda_m = n \\ \lambda_j \geq 1}} \frac{(-1)^m}{\lambda_1! \dots \lambda_m!}. \quad (5.43)$$

Applying this to (5.42) gives

$$\sum_{\substack{\text{s.o.p.'s } S \\ \text{with } m \geq 2}} T(S, C_f) A(S) = 2n!(-1)^n \sum_{\substack{c+d \leq n \\ c, d \geq 1}} (-1)^{c+d+1} G(n, f, c, d) \frac{1}{(n-c-d)!c!d!}. \quad (5.44)$$

Now, we can extend the sum to include when $c = 0$ or $d = 0$ to complete the proof as in this case $G(n, f, c, d) = 0$. \square

We complete our evaluation of the case when $m \geq 2$ with the following lemma, proven in Appendix B.1.

Lemma 5.16. *Fix $1 \leq f \leq a - 1$. We have*

$$\sum_{\substack{\text{s.o.p.'s } S \\ \text{with } m \geq 2}} T(S, C_f) A(S) = 2 \binom{n}{f} \left((-1)^{n+f+1} - 1 \right). \quad (5.45)$$

Now we evaluate the case when $m = 1$.

Lemma 5.17. *Fix $1 \leq f \leq a - 1$. We have*

$$\sum_{\substack{\text{s.o.p.'s } S \\ \text{with } m=1}} T(S, C_f) A(S) = 2 \binom{n}{f}. \quad (5.46)$$

Proof. We let $S = (1, \lambda_1, \epsilon_1, \dots, \epsilon_n)$ denote a variable system of parameters. Since $m = 1$, we have $\lambda_1 = n$ and $A(S) = \frac{(-1)^2}{1} \frac{n!}{n!} = 1$ for all S . Now, as in (5.40), we may rewrite the sum over $\epsilon_1, \dots, \epsilon_n$ as a sum over J_1, ζ_1 . Since $m = 1$, any f -element $J_1 \in J(S)$ will be minimal. So,

$$\sum_{\substack{\text{s.o.p.'s } S \\ \text{with } m \geq 2}} T(S, C_f) A(S) = A(S) \sum_{\zeta_1 \in \{\pm 1\}} \sum_{\#J_1=f} \mathbb{1}_{\{J_1 \in I(S)\}} = \sum_{\zeta_1 \in \{\pm 1\}} \sum_{\#J_1=f} 1 = 2 \binom{n}{f}. \quad (5.47)$$

□

Adding equations (5.45) and (5.46) gives the main result of the section.

Lemma 5.18. *Fix $1 \leq f \leq a - 1$. Then*

$$\sum_{\text{s.o.p.'s } S} T(S, C_f) A(S) = 2(-1)^{n+f+1} \binom{n}{f}. \quad (5.48)$$

5.3.2. The vanishing of valid t -classes for $t \geq 2$. In this section, we show that all terms with $t \geq 2$ in (5.33) vanish. Our main result is the following.

Lemma 5.19. *Let C be a valid t -class with $t \geq 2$. Then*

$$\sum_{\text{s.o.p.'s } S} T(S, C) A(S) = 0. \quad (5.49)$$

Throughout this section, let $S = (m, \lambda_1, \dots, \lambda_m, \epsilon_1, \dots, \epsilon_n)$ be a system of parameters, C a valid class, and $(I_1, \dots, I_t) \in C$ a tuple of subsets of $[1, n]$ such that for each $1 \leq i \leq t$, there is some ℓ_i and $\zeta_{\ell_i} \in \{\pm 1\}$ such that $I_i = \{j : \eta(\ell_i, j)\epsilon_j = \zeta_{\ell_i}\}$. I.e., $I_i = J_{\ell_i}$ with sign ζ_{ℓ_i} . Reorder the I_i so that $\ell_1 < \ell_2 < \dots < \ell_t$ and set $I'_i := I_i - \bigcap_{k=1}^t I_k$ and $j_i = \Lambda_{\ell_i} = \sum_{k=1}^{\ell_i} \lambda_k$. To begin, we prove lemmas which characterize (I_1, \dots, I_t) .

Lemma 5.20. *Set $I_1 = J_{\ell_1}$ with sign ζ_{ℓ_1} and suppose there is some minimal T such that $I_T = J_{\ell_T}$ with sign $\zeta_{\ell_T} = -\zeta_{\ell_1}$. Then, for all $i \geq T$, we have $I_i = \zeta_{\ell_i}$ with sign $\zeta_{\ell_i} = -\zeta_{\ell_1}$.*

Proof. Assume WLOG that $\zeta_{\ell_1} = -1$ so $I_1 = \{j : \eta(\ell_1, j)\epsilon_j = -1\}$ and let T be the smallest value such that $I_T = \{j : \eta(\ell_T, j)\epsilon_j = 1\}$. Suppose there exists some $s > T$ such that $I_s = \{j : \eta(\ell_s, j)\epsilon_j = -1\}$. If $j \leq j_{T-1}$ or $j > j_s$, then $\eta(\ell_T, j) = \eta(\ell_s, j)$, so $j \in I_T \cup I_s$ so $[1, j_{T-1}] \cup [j_s + 1, n] \subseteq I_T \cup I_s$. Similarly, if $j \in [j_{T-1} + 1, j_s]$, then $\eta(\ell_{T-1}, j) = -\eta(\ell_s, j)$, so $j \in I_{T-1} \cup I_s$ so $[j_{T-1} + 1, j_s] \subseteq I_{T-1} \cup I_s$. Since $[1, j_{T-1}] \cup [j_s + 1, n] \cup [j_{T-1} + 1, j_s] = [1, n]$ and $a \leq n/2$, we must have that either $|I_T \cup I_s| \geq a$ or $|I_{T-1} \cup I_s| \geq a$. Then, by Lemma 5.12, C is not valid, a contradiction. Thus such an s cannot exist so $I_i = \{j : \eta(\ell_i, j)\epsilon_j = +1\}$ for all $i \geq T$. □

The above lemma shows that the sign of (I_1, I_2, \dots, I_t) can switch at most once. We call the minimal T such that $\zeta_{\ell_T} = -\zeta_{\ell_1}$ the *transition point* of (I_1, \dots, I_t) . If I_1, I_2, \dots, I_t all have the same sign (so that no such T exists), then we set $T = 1$.

Lemma 5.21. *Let T be the transition point of (I_1, \dots, I_t) . Then*

$$\bigcup_{i=1}^t I'_i = I'_{T-1} \cup I'_T = [1, n] \setminus [j_{T-1} + 1, j_T], \quad (5.50)$$

$$I'_{T-1} \cap I'_T = \emptyset, \quad \text{and} \quad (5.51)$$

$$\bigcap_{i=1}^t I_i = I_{T-1} \cap I_T \subseteq [j_{T-1} + 1, j_T], \quad (5.52)$$

taking indices cyclically in $[1, t]$ and intervals cyclically in $[1, n]$ so that $I'_0 := I'_t$ and $I_0 := I_t$ and $[j_0 + 1, j_1] = [j_t + 1, n] \cup [1, j_1]$. Additionally, if $1 \leq i \leq t$ with $i \neq T - 1$, then

$$(I_i \cap [j_i + 1, j_{i+1}]) \cup (I_{i+1} \cap [j_i + 1, j_{i+1}]) = [j_i + 1, j_{i+1}] \quad \text{and} \quad (5.53)$$

$$(I_i \cap [j_i + 1, j_{i+1}]) \cap (I_{i+1} \cap [j_i + 1, j_{i+1}]) = \emptyset. \quad (5.54)$$

In other words, the restriction of I_i and I_{i+1} to the interval $[j_i + 1, j_{i+1}]$ forms a partition of the interval. If $i = t$ and $T \neq 1$, again taking indices and intervals cyclically, we set $I_{t+1} = I_1$ and $[j_t + 1, j_1] = [j_t + 1, n] \cup [1, j_1]$.

Proof. We consider indices and intervals cyclically in $[1, t]$ and $[1, n]$ respectively, as in Lemma 5.21.

For $j \in [j_{T-1} + 1, j_T]$, the value $\eta(\ell_i, j)\epsilon_j\zeta_{\ell_i}$ is independent of i since for any i, i' either $\zeta_{\ell_i}/\zeta_{\ell_{i'}}$ and $\eta(\ell_i, j)/\eta(\ell_{i'}, j)$ are both 1 or both -1 . So, for any $j \in [j_{T-1} + 1, j_T]$ either $j \in I_i$ for all i of $j \notin I_i$ for all i . So, $\bigcup_{i=1}^t I'_i \subset [1, n] \setminus [j_{T-1} + 1, j_T]$ and $\bigcap_{i=1}^t I_i \cap [j_{T-1} + 1, j_T] = I_{T-1} \cap I_T \cap [j_{T-1} + 1, j_T]$.

For $j \notin [j_{T-1} + 1, j_T]$, we have $\eta(\ell_{T-1}, j) = \eta(\ell_T, j)$ and so $\eta(\ell_{T-1}, j)\epsilon_j, \zeta_{\ell_{T-1}} = -\eta(\ell_T, j)\epsilon_j, \zeta_{\ell_T}$. So, every $j \notin [j_{T-1} + 1, j_T]$ belongs to exactly one of I'_{T-1} and I'_T . We conclude that $\bigcup_{i=1}^t I'_i = I'_{T-1} \cup I'_T \subset [1, n] \setminus [j_{T-1} + 1, j_T]$ and $I_{T-1} \cap I_T \subset [j_{T-1} + 1, j_T]$. So, $I_{T-1} \cap I_T \cap [j_{T-1} + 1, j_T] = I_{T-1} \cap I_T = \bigcap_{i=1}^t I_i$.

For $i \neq T - 1$, we have $\eta(\ell_i, j) = -\eta(\ell_{i+1}, j)$ if and only if $j \in [j_i + 1, j_{i+1}]$. Since $\zeta_{\ell_i} = \zeta_{\ell_{i+1}}$, this means $\eta(\ell_i, j)\epsilon_j\zeta_{\ell_i} = -\eta(\ell_{i+1}, j)\epsilon_j\zeta_{\ell_{i+1}}$ if and only if $j \in [j_i + 1, j_{i+1}]$. Hence, each $j \in [j_i + 1, j_{i+1}]$ is contained in exactly one of I_i and I_{i+1} , as desired. \square

Definition 5.22. *For each $1 \leq i \leq t$, set*

$$r_i = |I_i \cap [j_i + 1, j_{i+1}]| \quad \text{and} \quad (5.55)$$

$$s_i = |I_{i+1} \cap [j_i + 1, j_{i+1}]|. \quad (5.56)$$

We call the ordered tuple $(T, r_1, s_1, \dots, r_t, s_t)$ the *structure* of (I_1, \dots, I_t) in S where T is the transition point of (I_1, \dots, I_t) . If $(T, r_1, s_1, \dots, r_t, s_t)$ is a structure for some $(I_1, \dots, I_t) \in C$, we call it a *valid structure* for C .

By Lemma 5.21, $r_{T-1} = s_{T-1} = |\bigcap_{k=1}^t I_k|$. Lemma 5.21 also shows that when $i \neq T - 1$, $r_i + s_i = |[j_i + 1, j_{i+1}]| = j_{i+1} - j_i = \lambda_{\ell_{i+1}} + \dots + \lambda_{\ell_{i+1}}$. The following lemma shows that the two tuples with the same structure are in the same t -class.

Lemma 5.23. *Let C be a valid t -class and let $(I_1, \dots, I_t) \in C$ such that $(I_1, \dots, I_t) \in I(S)$ for some s.o.p. S . Let (J_1, \dots, J_t) be another tuple such that $(J_1, \dots, J_t) \in I(P)$ for some s.o.p. P . If the structure of (I_1, \dots, I_t) in S is the same as the structure of (J_1, \dots, J_t) in P , then $(J_1, \dots, J_t) \in C$.*

Proof. We first set notation. Set $S = (m, \lambda_1, \dots, \lambda_m, \epsilon_1, \dots, \epsilon_n)$ and $P = (m, \lambda'_1, \dots, \lambda'_m, \epsilon'_1, \dots, \epsilon'_n)$. Set $\ell_1 < \dots < \ell_t$ and $\ell'_1 < \dots < \ell'_t$ such that $I_i = \{j : \eta_S(\ell_i, j) = \zeta_{\ell_i}\}$ and $J_i = \{j : \eta_P(\ell'_i, j) = \zeta_{\ell'_i}\}$. Lastly, define $j_i = \sum_{k=1}^{\ell_i} \lambda_k$ and $j'_i = \sum_{k=1}^{\ell'_i} \lambda'_k$.

Without loss of generality, we may assume $\zeta_{\ell_1} = \zeta_{\ell'_1}$ or else we may replace each ϵ_i with $-\epsilon_i$. Since (I_1, \dots, I_t) and (J_1, \dots, J_t) have the same structure, for each i , we have

$$|I_i \cap [j_i + 1, j_{i+1}]| = |J_i \cap [j'_i + 1, j'_{i+1}]| \quad \text{and} \quad (5.57)$$

$$|I_{i+1} \cap [j_i + 1, j_{i+1}]| = |J_{i+1} \cap [j'_i + 1, j'_{i+1}]|. \quad (5.58)$$

Let $\tau \in S_n$ be the permutation which maps the k th smallest element of $|I_i \cap [j_i + 1, j_{i+1}]|$ to the k th smallest element of $|J_i \cap [j'_i + 1, j'_{i+1}]|$ and the k th smallest element of $|I_{i+1} \cap [j_i + 1, j_{i+1}]|$ to the k th smallest element of $|J_{i+1} \cap [j'_i + 1, j'_{i+1}]|$.

Since $\tau([j_i + 1, j_{i+1}]) = [j'_i + 1, j'_{i+1}]$, for all $i \in [1, t]$ and $j \in [1, n]$ we have $\eta(\ell_i, j) = \eta(\ell'_i, \tau(j))$. Since (I_1, \dots, I_t) and (J_1, \dots, J_t) have the same transition value T and we assumed $\zeta_{\ell_1} = \zeta_{\ell'_1}$, we have $\zeta_{\ell_i} = \zeta_{\ell'_i}$ for all i . Moreover, for $j \in [j_i + 1, j_{i+1}]$ we have $\eta(\ell_i, j)\epsilon_j\zeta_{\ell_i} = \eta(\ell'_i, \tau(j))\epsilon'_{\tau(j)}\zeta_{\ell'_i}$ so that $\epsilon_j = \epsilon'_{\tau(j)}$. But this is true for all i , so in fact $\epsilon_j = \epsilon'_{\tau(j)}$ for all $j \in [1, n]$. So, for all $i \in [1, t]$ and $j \in [1, n]$ we have $\eta(\ell_i, j)\epsilon_j\zeta_{\ell_i} = \eta(\ell'_i, \tau(j))\epsilon'_{\tau(j)}\zeta_{\ell'_i}$. It follows that $\tau(I_1, \dots, I_t) = (J_1, \dots, J_t)$ so $(J_1, \dots, J_t) \in C$. \square

Lemma 5.23 shows that if a structure is valid for C , then all tuples with that structure are in C . Thus in order to calculate $\sum T(S, C) A(S)$, we can first sum over all valid structures for C and then count tuples and s.o.p.s with that structure. All that remains is to determine when $(I_1, \dots, I_t) \in I(S)$.

Lemma 5.24. *Suppose $I_1, \dots, I_t \in J(S)$. Then, $I_1, \dots, I_t \in I(S)$ if and only if for each $1 \leq i \leq t$, $[j_i - \lambda_{\ell_i} + 1, j_i] \not\subseteq I_i$ and $[j_i + 1, j_i + \lambda_{\ell_{i+1}}] \not\subseteq I_i$.*

Proof. Note that $I_i \in J(S)$ implies $\#I_i \leq a - 1$. So, this is an immediate corollary of Lemma 5.14 which says $I_i \in I(S)$ if and only if $[j_i - \lambda_{\ell_i} + 1, j_i] \not\subseteq I_i$ and $[j_i + 1, j_i + \lambda_{\ell_{i+1}}] \not\subseteq I_i$. \square

Now we are ready to calculate $\sum T(S, C) A(S)$.

Lemma 5.25. *Let C be a valid t -class with $t \geq 2$. Then*

$$\sum_{\text{s.o.p.'s } S} T(S, C) A(S) \quad (5.59)$$

is a sum of terms of the form

$$\sum_{d=1}^f \sum_{\substack{\mu_1 + \dots + \mu_d = f \\ \mu_i \geq 1}} \frac{(-1)^d}{\mu_1! \cdots \mu_d!} H(f, g, \mu_1, \mu_d), \quad (5.60)$$

for some f and g , where

$$H(f, g, \mu_1, \mu_d) := \binom{f}{g} - \binom{f - \mu_1}{g - \mu_1} - \binom{f - \mu_d}{g} + \binom{f - \mu_1 - \mu_d}{g - \mu_1}. \quad (5.61)$$

Proof. Let C be a valid t -class. By Lemma 5.23, when summing over all s.o.p.s, we can first sum over all valid structures, and then over all s.o.p.s and tuples with that structure. To do this, we can sum over all m , then over all possible values of ℓ_1, \dots, ℓ_t , then over all $\lambda_1, \dots, \lambda_m$ such that $\lambda_1 + \dots + \lambda_m = n$ and $\lambda_{\ell_{i+1}} + \dots + \lambda_{\ell_{i+1}} = r_i + s_i$ for each $i \neq T - 1$. Now we use Lemma 5.24 to determine the summand. We can pick the elements of $\bigcap_{k=1}^t I_k$, which by Lemma 5.21 is a subset of $[j_{T-1} + 1, j_T]$, in $G(j_T - j_{T-1}, r_T, \lambda_{\ell_{T-1}}, \lambda_{\ell_T})$ ways, where G is defined as in (5.38). Next, we choose the r_i elements of I_i contained in the interval $[j_i + 1, j_{i+1}]$ in $H(r_i + s_i, r_i, \lambda_{\ell_{i+1}}, \lambda_{\ell_{i+1}})$ ways. Then, there are two possible choices for the sign ζ_{ℓ_1} of I_1 and then the signs for the rest of the I_i 's follow because the point of transition T is fixed. The choice of ζ_{ℓ_1} and each r_i -element set $[j_i + 1, j_{i+1}] \cap I_i$ determines all ϵ_j , so they determine exactly the same data as the I_i .

Lastly, we multiply by $A(S)$. We have that

$$\begin{aligned} \sum_{s.o.p.'s\ S} T(S, C) \cdot A(S) &= \sum_{\substack{(T, r_1, s_1, \dots, r_t, s_t) \\ \text{a valid structure for } C}} \sum_{m=1}^n \sum_{1 \leq \ell_1 < \dots < \ell_t \leq m} \sum_{\substack{\lambda_1 + \dots + \lambda_m = n \\ \lambda_{\ell_i+1} + \dots + \lambda_{\ell_{i+1}} = r_{i+1} + s_{i+1} \text{ for each } i \neq T \\ \lambda_i \geq 1}} \\ &\times 2G(u, v, \lambda_{\ell_1}, \lambda_{\ell_t+1}) \prod_{\substack{1 \leq i \leq t \\ i \neq T-1}} H(r_i + s_i, r_i, \lambda_{\ell_i+1}, \lambda_{\ell_{i+1}}) \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \dots \lambda_m!}. \end{aligned} \quad (5.62)$$

For each structure, we can fix some $i \neq T-1$, which exists since $t \geq 2$, to see that this is a sum of terms of the form

$$\sum_{d=1}^{r_i+s_i} \sum_{\substack{\mu_1 + \dots + \mu_d = r_i+s_i \\ \mu_i \geq 1}} \frac{(-1)^d}{\mu_1! \dots \mu_d!} H(r_i + s_i, r_i, \mu_1, \mu_d). \quad (5.63)$$

□

We finish the calculation with the following combinatorial lemma, proven in Appendix B.2.

Lemma 5.26. *Fix f, g and let $H(f, g, \mu_1, \mu_d)$ be as in (5.61). Then*

$$\sum_{d=1}^f \sum_{\substack{\mu_1 + \dots + \mu_d = f \\ \mu_i \geq 1}} \frac{(-1)^d}{\mu_1! \dots \mu_d!} H(f, g, \mu_1, \mu_d) = 0. \quad (5.64)$$

Combining Lemmas 5.25 and 5.26 completes the proof of Lemma 5.19. □

5.4. Computing the integral piece. In this section we complete the proof of Proposition 5.2 by calculating the integral $\int C dy$ appearing in (5.33). Applying Lemmas 5.19 and 5.18 to (5.11) gives

$$Q_n(\phi) = 2^{n-1}(-1)^n \sum_{\ell=0}^{a-1} (-1)^\ell \binom{n}{\ell} \int_0^\infty \dots \int_0^\infty \hat{\phi}(y_1) \dots \hat{\phi}(y_n) \chi_{\{n-\ell+1, \dots, n\}} dy_1 \dots dy_n. \quad (5.65)$$

Next we define

$$\xi_\ell(\phi) := \int_0^\infty \dots \int_0^\infty \hat{\phi}(y_1) \dots \hat{\phi}(y_n) \chi_{\{y_1 + \dots + y_{n-\ell} - y_{n-\ell+1} - \dots - y_n > 1\}} dy_1 \dots dy_n \quad (5.66)$$

and

$$\bar{\xi}_\ell(\phi) := \int_{-\infty}^\infty \dots \int_{-\infty}^\infty \hat{\phi}(y_1) \dots \hat{\phi}(y_n) \chi_{\{|y_1 + \dots + y_{n-\ell}| - |y_{n-\ell+1}| - \dots - |y_n| > 1\}} dy_1 \dots dy_n. \quad (5.67)$$

We have that (5.65) equals

$$Q_n(\phi) = 2^{n-1}(-1)^n \sum_{\ell=0}^{a-1} (-1)^\ell \binom{n}{\ell} \xi_\ell(\phi). \quad (5.68)$$

We express $Q_n(\phi)$ in terms of $\bar{\xi}_\ell(\phi)$ with the following lemma.

Lemma 5.27. *Let $\text{supp}(\hat{\phi}) \subseteq \left[-\frac{1}{n-a}, \frac{1}{n-a}\right]$. We have*

$$Q_n(\phi) = 2^{n-2}(-1)^n \sum_{t=0}^{a-1} (-1)^t \binom{n}{t} \bar{\xi}_t(\phi). \quad (5.69)$$

Proof. Given $\text{supp}(\widehat{\phi}) \subseteq \left[-\frac{1}{n-a}, \frac{1}{n-a}\right]$ and $t \leq a-1$, if $|y_1 + \cdots + y_{n-t}| - |y_{n-t+1}| - \cdots - |y_n| > 1$, then either at most $i \leq a-1-t$ of the y_j 's in the first absolute value are nonnegative and the rest are negative or at most $i \leq a-1-t$ of the y_j 's in the first absolute value are nonpositive or zero and the rest are positive. Moreover, the sign of $y_1 + \cdots + y_{n-t}$ matches the second group. There are $\binom{n-t}{i}$ ways to choose these indices and we introduce a factor of 2 from choosing the sign of $y_1 + \cdots + y_{n-t}$. Lastly, since $\widehat{\phi}$ is even, we multiply by a factor of 2^t to account for changing the limits of integration over y_{n-t+1}, \dots, y_n . Thus we have

$$\begin{aligned} \overline{\xi}_t(\phi) &= 2^{t+1} \int_0^\infty \cdots \int_0^\infty \widehat{\phi}(y_1) \cdots \widehat{\phi}(y_n) \left[\sum_{i=0}^{a-1-t} \binom{n-t}{i} \chi_{\{y_1 + \cdots + y_{n-i-t} - y_{n-i-t+1} - \cdots - y_n > 1\}} \right] dy_1 \cdots dy_n \\ &= 2^{t+1} \sum_{i=0}^{a-1-t} \binom{n-t}{i} \xi_{i+t}(\phi). \end{aligned} \quad (5.70)$$

Applying the identity

$$\sum_{t=0}^{\ell} (-2)^t \binom{n}{t} \binom{n-t}{\ell-t} = \binom{n}{\ell} \sum_{t=0}^{\ell} (-2)^t \binom{\ell}{t} = \binom{n}{\ell} (1-2)^\ell = \binom{n}{\ell} (-1)^\ell \quad (5.71)$$

to (5.68) gives

$$Q_n(\phi) = 2^{n-1} (-1)^n \sum_{\ell=0}^{a-1} \xi_\ell(\phi) \sum_{t=0}^{\ell} (-2)^t \binom{n}{t} \binom{n-t}{\ell-t}. \quad (5.72)$$

Switching the order of summation and setting $i = \ell - t$ gives

$$Q_n(\phi) = 2^{n-2} (-1)^n \sum_{t=0}^{a-1} (-1)^t \binom{n}{t} \left[2^{t+1} \sum_{i=0}^{a-1-t} \binom{n-t}{i} \xi_{i+t}(\phi) \right]. \quad (5.73)$$

Applying (5.70) gives the desired result. \square

We complete the evaluation of $Q_n(\phi)$ by computing $\overline{\xi}_t(\phi)$.

Lemma 5.28. *Let ϕ be an even Schwartz function with $\text{supp}(\widehat{\phi}) \subseteq \left[-\frac{1}{n-a}, \frac{1}{n-a}\right]$. Then, for $\ell \leq a-1$, we have*

$$\begin{aligned} \overline{\xi}_\ell(\phi) &= \phi^n(0) \\ &\quad - 2 \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \widehat{\phi}(x_{\ell+1}) \cdots \widehat{\phi}(x_2) \int_{-\infty}^\infty \phi^{n-\ell}(x_1) \frac{\sin(2\pi x_1(1 + |x_2| + \cdots + |x_{\ell+1}|))}{2\pi x_1} dx_1 \cdots dx_{\ell+1}. \end{aligned} \quad (5.74)$$

Proof. We apply a change of variables given by

$$\begin{array}{ll} x_1 = y_1 & y_1 = x_1 \\ x_2 = y_1 + y_2 & y_2 = x_2 - x_1 \\ \vdots & \vdots \\ x_{n-\ell} = \sum_{j=1}^{n-\ell} y_j & y_{n-\ell} = x_{n-\ell} - x_{n-\ell-1} \\ x_{n-\ell+1} = y_{n-\ell+1} & y_{n-\ell+1} = x_{n-\ell+1} \\ \vdots & \vdots \\ x_n = y_n & y_n = x_n \end{array} \quad (5.75)$$

to (5.67), giving

$$\begin{aligned} \overline{\xi}_\ell(\phi) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{\phi}(x_1) \hat{\phi}(x_2 - x_1) \cdots \hat{\phi}(x_{n-\ell} - x_{n-\ell-1}) \\ &\quad \times \hat{\phi}(x_{n-\ell+1}) \cdots \hat{\phi}(x_n) \chi_{\{|x_{n-\ell}| - (|x_{n-\ell+1}| + \cdots + |x_n|) > 1\}} dx_1 \cdots dx_n. \end{aligned} \quad (5.76)$$

Repeatedly applying the identity $\int_{-\infty}^{\infty} \hat{f}(v) \hat{g}(u - v) dv = \widehat{fg}(u)$ (which arises from the convolution theorem) to (5.76) gives

$$\overline{\xi}_\ell(\phi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi^{n-\ell}}(x_{n-\ell}) \hat{\phi}(x_{n-\ell+1}) \cdots \hat{\phi}(x_n) \chi_{\{|x_{n-\ell}| - (|x_{n-\ell+1}| + \cdots + |x_n|) > 1\}} dx_{n-\ell} \cdots dx_n. \quad (5.77)$$

We rename $x_{n-\ell}$ to x_1 , $x_{n-\ell+1}$ to x_2 , and so on until x_n to $x_{\ell+1}$. This and the identity

$$\chi_{\{|x_1| - (|x_2| + \cdots + |x_{\ell+1}|) > 1\}} = 1 - \chi_{\{|x_1| \leq 1 + |x_2| + \cdots + |x_{\ell+1}|\}} \quad (5.78)$$

gives

$$\overline{\xi}_\ell(\phi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi^{n-\ell}}(x_1) \hat{\phi}(x_2) \cdots \hat{\phi}(x_{\ell+1}) (1 - \chi_{\{|x_1| \leq 1 + |x_2| + \cdots + |x_{\ell+1}|\}}) dx_1 \cdots dx_{\ell+1}. \quad (5.79)$$

Distributing and using the identity $\phi(0) = \int_{-\infty}^{\infty} \hat{\phi}(x) dx$, we have that

$$\overline{\xi}_\ell(\phi) = \phi^n(0) - \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi^{n-\ell}}(x_1) \hat{\phi}(x_2) \cdots \hat{\phi}(x_{\ell+1}) \chi_{\{|x_1| \leq 1 + |x_2| + \cdots + |x_{\ell+1}|\}} dx_1 \cdots dx_{\ell+1}. \quad (5.80)$$

Fix $x_2, \dots, x_{\ell+1}$ and set $S_\ell(x_1) = \sin(2\pi x_1(1 + |x_2| + \cdots + |x_{\ell+1}|)) / (2\pi x_1)$. We have the identity

$$\chi_{\{|x_1| \leq 1 + |x_2| + \cdots + |x_{\ell+1}|\}}(x_1) = 2\widehat{S}_\ell(x_1), \quad (5.81)$$

which follows from the Fourier pair

$$\frac{\sin(2\pi Ax)}{2\pi x} = \int_{-\infty}^{\infty} \frac{1}{2} \chi_{\{|u| \leq A\}} e^{2\pi i x u} du. \quad (5.82)$$

Thus Plancherel's theorem gives us that

$$\begin{aligned} \overline{\xi}_\ell(\phi) &= \phi^n(0) \\ &\quad - 2 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{\phi}(x_{\ell+1}) \cdots \hat{\phi}(x_2) \int_{-\infty}^{\infty} \phi^{n-\ell}(x_1) \frac{\sin(2\pi x_1(1 + |x_2| + \cdots + |x_{\ell+1}|))}{2\pi x_1} dx_1 \cdots dx_{\ell+1} \end{aligned} \quad (5.83)$$

as desired. \square

Applying Lemma 5.28 to (5.69) and comparing with (1.9) completes the proof of Proposition 5.2.

APPENDIX A. PROOFS OF LEMMAS IN SECTIONS 3 AND 4

In this section, we prove the lemmas stated in Sections 3 and 4.

A.1. Proof of Lemma 3.1.

Proof. We sum over n primes p_1, \dots, p_n in (3.1). Utilizing (2.19) we see that $S_2^{(n)}$ is made up of terms of the form

$$i^k \sqrt{N} \sum_{\substack{q_1 \nmid N, \dots, q_\ell \nmid N \\ q_j \text{ distinct}}} \prod_{j=1}^{\ell} \psi(q_j)^{n_j} \langle \lambda_f(N q_1^{m_1} \cdots q_\ell^{m_\ell}) \rangle_*, \quad (\text{A.1})$$

where $m_j \leq n_j$, $m_j \equiv n_j \pmod{2}$ for each j and

$$\psi(q_j) := \hat{\phi} \left(\frac{\log q_j}{\log R} \right) \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right). \quad (\text{A.2})$$

Additionally, we have that $\sum_{j=1}^{\ell} n_j = n$. From (A.1) we may write $p_1 \cdots p_n = q_1^{n_1} \cdots q_\ell^{n_\ell}$. With an appropriate reindexing of the p_i 's and q_j 's, there exists some $n' \leq n$, $\omega < \ell$ such that $p_{n-n'+1} \cdots p_n = q_1^{n_1} \cdots q_\omega^{n_\omega}$ and $p_1 \cdots p_{n'} = q_{\omega+1}^{n_{\omega+1}} \cdots q_\ell^{n_\ell}$, where $m_j < n_j$ for $1 \leq j \leq \omega$ and $m_j = n_j$ for $\omega+1 \leq j \leq \ell$. Then we can write the summand in (A.1) as

$$\prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right) \right)^{n_j} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \left(\frac{2 \log p_i}{\sqrt{p_i} \log R} \right) \langle \lambda_f(N p_1 \cdots p_{n-n'} q_1^{m_1} \cdots q_\omega^{m_\omega}) \rangle_*, \quad (\text{A.3})$$

where each $m_j < n_j$. Combining this expression with (A.1), we see that $S_2^{(n)}$ is made up of terms of the form

$$i^k \sqrt{N} \sum_{\substack{q_1 \nmid N, \dots, q_\omega \nmid N \\ q_j \text{ distinct}}} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right) \right)^{n_j} \sum_{\substack{p_1, \dots, p_{n-n'} \nmid N \\ p_i \neq q_j}} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \left(\frac{2 \log p_i}{\sqrt{p_i} \log R} \right) \times \langle \lambda_f(N p_1 \cdots p_{n-n'} q_1^{m_1} \cdots q_\omega^{m_\omega}) \rangle_*. \quad (\text{A.4})$$

Notice $\sum_{j=1}^{\omega} n_j = n'$, and the inner sum is over $n - n'$ primes p_i , where $p_i \neq q_j$ for all q_j and p_i , which do not divide N .

We now remove the condition $p_i \neq q_j$ through inclusion-exclusion. In order to do this, we subtract off terms of the following form, where we fix a constant $\beta_1 > 0$ and values n'_j and m'_j for each j so that for each j , $n'_j \geq n_j$, $n'_j - n_j = m'_j - m_j$ and $\sum_{j=1}^{\omega} n'_j = n' + \beta_1$:

$$i^k \sqrt{N} \sum_{\substack{q_1 \nmid N, \dots, q_\omega \nmid N \\ q_j \text{ distinct}}} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right) \right)^{n'_j} \sum_{\substack{p_1, \dots, p_{n-n'-\beta_1} \nmid N \\ p_i \neq q_j}} \prod_{i=n'+\beta_1+1}^n \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \left(\frac{2 \log p_i}{\sqrt{p_i} \log R} \right) \times \langle \lambda_f(N p_1 \cdots p_{n-n'-\beta_1} q_1^{m'_1} \cdots q_\omega^{m'_\omega}) \rangle_*. \quad (\text{A.5})$$

These terms emerge when β_1 of the p_i 's in (A.4) are equal to some of the q_j 's. These terms are of the same form as in (A.4), so we again apply inclusion-exclusion to them to remove the $p_i \neq q_j$ condition. We repeat this process until every term has the $p_i \neq q_j$ condition removed. In particular, since $\beta_1 > 0$, this process may take at most $n - n'$ steps. Thus, $S_2^{(n)}$ can be written as the sum of terms of the form

$$i^k \sqrt{N} \sum_{\substack{q_1 \nmid N, \dots, q_\omega \nmid N \\ q_j \text{ distinct}}} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right) \right)^{n_j} \sum_{p_1, \dots, p_{n-n'} \nmid N} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \left(\frac{2 \log p_i}{\sqrt{p_i} \log R} \right) \times \langle \lambda_f(N p_1 \cdots p_{n-n'} q_1^{m_1} \cdots q_\omega^{m_\omega}) \rangle_*. \quad (\text{A.6})$$

We can remove the distinctness condition on the q_j 's in (A.6) by using inclusion-exclusion. We subtract off terms when some of the q_j 's are equal, which have the following form:

$$i^k \sqrt{N} \sum_{\substack{q_1 \nmid N, \dots, q_{\omega'} \nmid N \\ q_j \text{ distinct}}} \prod_{j=1}^{\omega'} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right) \right)^{n'_j} \sum_{p_1, \dots, p_{n-n'} \nmid N} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \left(\frac{2 \log p_i}{\sqrt{p_i} \log R} \right) \\ \times \left\langle \lambda_f(N p_1 \cdots p_{n-n'} q_1^{m'_1} \cdots q_{\omega'}^{m'_{\omega'}}) \right\rangle_*. \quad (\text{A.7})$$

Here, we have some fixed $\omega' < \omega$ and $\sum_{j=1}^{\omega'} n'_j = n'$. These terms appear when some of the q_j 's in (A.6) are equal to each other. In particular, if $q_{j_1} = q_{j_2}$ for $j_1 \neq j_2$, the exponent of q_{j_1} in this term will be $n'_{j_1} = n_{j_1} + n_{j_2}$ and $m'_{j_1} = m_{j_1} + m_{j_2}$. Notice that $m_{j_1} + m_{j_2} < n_{j_1} + n_{j_2}$, so the condition $m'_j < n'_j$ is preserved. Thus, the terms (A.7) are of the same form as (A.6), so we may apply inclusion exclusion to them again. We repeat this process until all of the sums over q_j have the distinctness condition removed or sum over only one q_j , in which case we can remove the distinctness condition immediately.

Thus, $S_2^{(n)}$ can be written as the sum of terms of the form

$$i^k \sqrt{N} \sum_{q_1 \nmid N, \dots, q_{\omega} \nmid N} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right) \right)^{n_j} \sum_{p_1, \dots, p_{n-n'} \nmid N} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \left(\frac{2 \log p_i}{\sqrt{p_i} \log R} \right) \langle \lambda_f(N p_1 \cdots p_{n-n'} q_1^{m_1} \cdots q_{\omega}^{m_{\omega}}) \rangle_*, \quad (\text{A.8})$$

where each $n_j > m_j$. \square

A.2. Proof of Lemma 3.2.

Proof. Let $\text{supp}(\hat{\phi}) \subset (-\sigma, \sigma)$ with $\sigma < 1/(n-a)$. Using Lemma 2.12, we may use the prime number theorem and partial summation to bound the sum over $p_1, \dots, p_{n-n'}$ in (3.4) by

$$N^{-3/2+\epsilon} (q_1^{m_1} \cdots q_{\omega}^{m_{\omega}})^{1/2} R^{\sigma(n-n')}. \quad (\text{A.9})$$

Next, since $n_j > m_j$ for $1 \leq j \leq \omega$ and $n_j - m_j$ is even, we have that $n_j - m_j \geq 2$ so we can again use the prime number theorem and partial summation on the sums over q_1, \dots, q_{ω} in (3.4) to bound (3.4) by $N^{-1+\epsilon} R^{\sigma(n-n')}$. This is negligible when $n' \geq a$, since in this case $\sigma(n-n') < 1$. \square

A.3. Proof of Lemma 3.3.

Proof. Let $\text{supp}(\hat{\phi}) \subset (-\sigma, \sigma)$. We set $b = cN$ and sum over all $c \geq 1$. The sum over all such b in (3.5) is then

$$\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{q_1 \nmid N, \dots, q_{\omega} \nmid N} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \frac{\log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \sum_{p_1 \nmid N, \dots, p_{n-n'} \nmid N} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i}{\sqrt{p_i} \log R} \\ \times \sum_{m \leq N^{\epsilon}} \frac{1}{m} \sum_{c=1}^{\infty} \frac{S(m^2, NQ; N^2 c)}{Nc} J_{k-1} \left(\frac{4\pi m}{cN\sqrt{N}} \right). \quad (\text{A.10})$$

We use the bound (2.5) to find that $S(m^2, NQ; N^2 c) \ll m^2 (Nc)^{1/2+\epsilon}$. Combining this with the bounds $J_{k-1}(x) \ll x$ and $m \leq N^{\epsilon}$, we find that the sum over c in (A.10) is bounded by $N^{-2+\epsilon} \sqrt{Q}$. Using the fact that $n_j - m_j \geq 2$, applying the prime number theorem and partial summation to (A.10) with our bound on the sum over c , we find that (A.10) is bounded by $N^{-5/2+\epsilon} R^{(n-n')\sigma}$, which is negligible when $\sigma < \frac{5}{2(n-n')}$. \square

A.4. Proof of Lemma 3.4.

Proof. Let $\text{supp}(\hat{\phi}) \subset (-\sigma, \sigma)$. The sum over all such $b \geq N^{2022}$ in (3.5) is

$$\begin{aligned} & \frac{2^{n+1}\pi}{\sqrt{N}} \sum_{q_1 \nmid N, \dots, q_{\omega \nmid N}} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \frac{\log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \sum_{p_1 \nmid N, \dots, p_{n-n'} \nmid N} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i}{\sqrt{p_i} \log R} \\ & \times \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{b \geq N^{2022}} \frac{S(m^2, NQ; Nb)}{b} J_{k-1} \left(\frac{4\pi m}{b\sqrt{N}} \right). \end{aligned} \quad (\text{A.11})$$

We use the bound (2.5) to find that $S(m^2, NQ; Nb) \ll m^2 b^{1/2+\epsilon} N^\epsilon$. Combining this with the bounds $J_{k-1}(x) \ll x$ and $m \leq N^\epsilon$, we find that the sum over b in (A.11) is bounded by

$$N^{-1/2+\epsilon} \sum_{b \geq N^{2022}} b^{-3/2+\epsilon}. \quad (\text{A.12})$$

Using the integral comparison test, we find that (A.12) is $b^{-1011-1/2+\epsilon}$. We can use the fact that $n_j - m_j \geq 2$ and apply the prime number theorem and partial summation to (A.11) with our bound on the sum over b to find that (A.11) is bounded by $N^{-1012+\epsilon} R^{(n-n')\sigma}$, which is $O(N^{-12})$ when $\sigma < \frac{1000}{n-n'}$. \square

A.5. Proof of Lemma 3.5.

Proof. Set $r = (Q, b^\infty)$ and $Q' = Q/r$. Then $(Q', b) = 1$. Then we have that

$$\begin{aligned} S(m^2, NQ; Nb) &= \frac{1}{\varphi(b)} \sum_{\chi(b)} \sum_{a(b)}^* \chi(a) \bar{\chi}(Q') S(m^2, Nra; Nb) \\ &= \frac{1}{\varphi(b)} \sum_{\chi(b)} \sum_{a(b)}^* \chi(a) \bar{\chi}(Q') \sum_{d(Nb)}^* e \left(\frac{m^2 d}{Nb} \right) e \left(\frac{Nra\bar{d}}{Nb} \right) \\ &= \frac{1}{\varphi(b)} \sum_{\chi(b)} \bar{\chi}(Q') \sum_{d(Nb)}^* e \left(\frac{m^2 d}{Nb} \right) \sum_{a(b)}^* \chi(a) \left(\frac{rad}{b} \right). \end{aligned} \quad (\text{A.13})$$

As $(d, b) = 1$, we can do a change of variables from a to ad in the inner sum:

$$S(m^2, NQ; Nb) = \frac{1}{\varphi(b)} \sum_{\chi(b)} \bar{\chi}(Q') G_\chi(r) \sum_{d(Nb)}^* \chi(d) e \left(\frac{m^2 d}{Nb} \right). \quad (\text{A.14})$$

In the inner sum, we rewrite $d = u_1 N + u_2 b$ with $(u_1, b) = 1$ and $(u_2, N) = 1$ since $(N, b) = 1$. We have that $\chi(u_1 N + u_2 b) = \chi(u_1) \chi(N)$ which gives

$$\begin{aligned} \sum_{d(Nb)}^* \chi(d) e \left(\frac{m^2 d}{Nb} \right) &= \sum_{u_1(b)}^* \sum_{u_2(N)}^* \chi(u_1) \chi(N) e \left(\frac{m^2 u_1 N}{Nb} \right) e \left(\frac{m^2 u_2 b}{Nb} \right) \\ &= \sum_{u_1(b)}^* \chi(u_1) \chi(N) e \left(\frac{m^2 u_1}{b} \right) \sum_{u_2(N)}^* e \left(\frac{m^2 u_2}{N} \right). \end{aligned} \quad (\text{A.15})$$

Since $(m^2, N) = 1$, the inner sum equals -1 so we have

$$\begin{aligned} \sum_{d(Nb)}^* \chi(d) e \left(\frac{m^2 d}{Nb} \right) &= -\chi(N) \sum_{u_1(b)}^* \chi(u_1) e \left(\frac{m^2 u_1}{b} \right) \\ &= -\chi(N) G_\chi(m^2). \end{aligned} \quad (\text{A.16})$$

Applying this to (A.14) gives

$$S(m^2, NQ; Nb) = -\frac{1}{\varphi(b)} \sum_{\chi(b)} G_\chi(m^2) G_\chi(r) \overline{\chi}(Q') \chi(N) \quad (\text{A.17})$$

as desired. \square

A.6. Proof of Lemma 3.6. We first prove an auxiliary lemma.

Lemma A.1. *Assuming GRH for Dirichlet L -functions, if χ is a primitive character of modulus b , then for real x and t we have that*

$$\sum_{n \leq x} \Lambda(n) \chi(n) n^{-it} = O(x^{1/2} (bxt)^\epsilon). \quad (\text{A.18})$$

Proof of Lemma A.1. Let χ be a primitive character modulo b and let $L(s, \chi)$ be its L -function. For ease of notation, define

$$\psi(\chi, x, t) := \sum_{n \leq x} \Lambda(n) \chi(n) n^{-it}. \quad (\text{A.19})$$

By Proposition 5.54 of [IK04] and Mellin inversion,

$$\sum_{n \leq x} \Lambda(n) \chi(n) n^{-it} = \int_{3/2-ix}^{3/2+ix} \frac{L'}{L}(s+it, \chi) \frac{x^s}{s} ds + O(x^{1/2}). \quad (\text{A.20})$$

We evaluate the integral on the right hand side of (A.20). We complete the contour by integrating counter-clockwise around a box with vertices at $3/2-ix, 3/2+ix, -1/2+ix, -1/2-ix$. In doing so, we pick up residues at the nontrivial zeros of $L(s+it, \chi)$. To bound the contribution from the three sides, we use the following bound for $\frac{L'}{L}(s, \chi)$ from page 116 of [Dav80], valid for $-1 \leq \Re(s) \leq 2$:

$$\frac{L'}{L}(s, \chi) \ll (\log b |s|)^2. \quad (\text{A.21})$$

Applying this bound gives

$$\sum_{n \leq x} \Lambda(n) \chi(n) n^{-it} = \sum_{|\gamma-t| \leq x} \frac{x^{\rho-it}}{\rho-it} + O(x^{1/2} (bxt)^\epsilon). \quad (\text{A.22})$$

where the sum is over the nontrivial zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$. Assuming GRH for Dirichlet L -functions we have that $\rho = 1/2 + i\gamma$. Taking absolute values in (A.22) gives

$$\psi(\chi, x, t) \leq x^{1/2} \sum_{|\gamma-t| \leq x} \frac{1}{|1/2 + i(\gamma-t)|} + O(x^{1/2} (bxt)^\epsilon) \quad (\text{A.23})$$

since $|x^{i\gamma-it}| = 1$. By equation (1) on page 101 of [Dav80], the number of zeros satisfying $u \leq \gamma - t \leq u + 1$ is $\ll \log(b(|u| + |t|))$. Thus we can bound (A.23) by

$$\psi(\chi, x, t) \ll x^{1/2} \log(b(|x| + |t|)) \log(|x| + |t|) + O(x^{1/2} (bxt)^\epsilon) \quad (\text{A.24})$$

$$\ll x^{1/2} (bxt)^\epsilon \quad (\text{A.25})$$

as desired. \square

Proof of Lemma 3.6. We show that subterms involving non-principal characters do not contribute to $S_2^{(n)}$ in the limit. Suppose $\hat{\phi}$ has support in $(-\sigma, \sigma)$. The sum of subterms over non-principal characters in (3.8) is

$$\begin{aligned} & -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{q_1, \dots, q_\omega} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \frac{\log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \sum_{p_1, \dots, p_{n-n'}} \prod_{i=1}^{n-n'} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i}{\sqrt{p_i} \log R} \\ & \times \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{\substack{(b, N)=1 \\ b < N^{2022}}} \frac{1}{b\varphi(b)} \sum_{\substack{\chi(b) \\ \chi \neq \chi_0}} G_\chi(m^2) G_\chi((Q, b^\infty)) \bar{\chi} \left(\frac{Q}{(Q, b^\infty)} \right) \chi(N) J_{k-1} \left(\frac{4\pi m \sqrt{Q}}{b\sqrt{N}} \right). \end{aligned} \quad (\text{A.26})$$

where $Q = p_1 \cdots p_{n-n'} q_1^{m_1} \cdots q_\omega^{m_\omega}$. We case on the value of (Q, b^∞) . Without loss of generality, set $(Q, b^\infty) = p_1 \cdots p_\alpha q_1^{m_1} \cdots q_\theta^{m_\theta}$ for some $\alpha \leq n - n'$, $\theta \leq \omega$. Thus (A.26) can be written as a sum of terms of the form

$$\begin{aligned} S &:= -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{q_1, \dots, q_\omega} \prod_{j=1}^{\omega} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right) \frac{\log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \sum_{p_1, \dots, p_\alpha} \prod_{i=1}^{\alpha} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i}{\sqrt{p_i} \log R} \\ & \times \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{\substack{(b, N)=1 \\ b < N^{2022}}} \frac{1}{b\varphi(b)} \sum_{\substack{\chi(b) \\ \chi \neq \chi_0}} G_\chi(m^2) G_\chi((Q, b^\infty)) \bar{\chi} (q_{\theta+1}^{m_{\theta+1}} \cdots q_\omega^{m_\omega}) \chi(N) \\ & \times \sum_{p_{\alpha+1}, \dots, p_{n-n'}} J_{k-1} \left(\frac{4\pi m \sqrt{Q}}{b\sqrt{N}} \right) \prod_{i=1}^{\alpha} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i \bar{\chi}(p_i)}{\sqrt{p_i} \log R}. \end{aligned} \quad (\text{A.27})$$

We first evaluate the sum over $p_{\alpha+1}, \dots, p_{n-n'}$ in (A.27), which we denote by T . First, we write J_{k-1} as an integral along the line $\Re(s) = -1 + \epsilon$ using the inverse Mellin transform (2.10):

$$T = \sum_{p_{\alpha+1}, \dots, p_{n-n'}} \frac{1}{2\pi i} \int_{\Re(s)=-1+\epsilon} \left(\frac{4\pi m \sqrt{Q}}{b\sqrt{N}} \right)^{-s} G_{k-1}(s) \prod_{i=1}^{\alpha} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i \bar{\chi}(p_i)}{\sqrt{p_i} \log R} ds, \quad (\text{A.28})$$

where $G_{k-1}(s)$ is the Mellin transform of the Bessel function. Now, we may swap the sum and integral by Fubini's theorem and set $s = -1 + \epsilon + it$ to simplify (A.28) as

$$T = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{4\pi m \sqrt{Q'}}{b\sqrt{N}} \right)^{1-\epsilon-it} G_{k-1}(1-\epsilon+it) \left[\sum_p \hat{\phi} \left(\frac{\log p}{\log R} \right) \frac{\log p \bar{\chi}(p)}{p^{(\epsilon+it)/2} \log R} \right]^{n-n'-\alpha} dt, \quad (\text{A.29})$$

where $Q' = p_1 \cdots p_\alpha q_1^{m_1} \cdots q_\omega^{m_\omega}$. Using partial summation and Lemma A.1, we find that

$$\sum_p \hat{\phi} \left(\frac{\log p}{\log R} \right) \frac{\log p \bar{\chi}(p)}{p^{(\epsilon+it)/2} \log R} \ll R^{\sigma/2} (Rbt)^{\epsilon'}. \quad (\text{A.30})$$

Applying this bound to (A.29) gives

$$T \ll \left(\frac{m\sqrt{Q'}}{b\sqrt{N}} \right)^{1-\epsilon} R^{\sigma(n-n'-\alpha)/2} (Rb)^{\epsilon'} \int_{-\infty}^{\infty} |G_{k-1}(1-\epsilon+it) t^{\epsilon'}| dt. \quad (\text{A.31})$$

From (4.42) of [HM07] we have that $|G_{k-1}(1-\epsilon+it)| \ll (1+|t|)^{2-\epsilon}$. Applying this bound to (A.29) we see that the integral converges and

$$T \ll \left(\frac{m\sqrt{Q'}}{b\sqrt{N}} \right)^{1-\epsilon} R^{\sigma(n-n'-\alpha)/2} (Rb)^{\epsilon'}. \quad (\text{A.32})$$

Now, applying (A.32) to (A.27) and using the bound

$$\frac{1}{\varphi(b)} \sum_{\chi(b)} |G_\chi(x) G_\chi(y)| \ll b \quad (\text{A.33})$$

gives

$$\begin{aligned} S &\ll N^{1-\epsilon} R^{\sigma(n-n'-\alpha)/2} R^{\epsilon'} \sum_{q_1, \dots, q_\omega} \prod_{j=1}^{\omega} \hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \frac{\log^{n_j} q_j}{q_j^{(n_j-m_j)/2} \log^{n_j} R} \sum_{p_1, \dots, p_\alpha} \prod_{i=1}^{\alpha} \hat{\phi} \left(\frac{\log p_i}{\log R} \right) \frac{\log p_i}{\log R} \\ &\times \sum_{\substack{(b,N)=1 \\ b < N^{2022} \\ p_1, \dots, p_\alpha, q_1, \dots, q_\theta | b}} b^{-1+\epsilon+\epsilon'}. \end{aligned} \quad (\text{A.34})$$

The sum over b in (A.34) is $N^{\epsilon''} (p_1 \cdots p_\alpha q_1 \cdots q_\theta)^{-1}$. Applying this bound, partial summation and the prime number theorem to (A.34) we finally have that

$$S \ll N^{1+\epsilon''} R^{\sigma(n-n'-\alpha)/2}, \quad (\text{A.35})$$

which is negligible if $\sigma < 2/(n-n')$ since $\alpha \geq 0$. \square

A.7. Proof of Lemma 4.4.

Proof. Given nonnegative integers γ, n with $\gamma < n$, define

$$T(\gamma, n) := \frac{1}{2\pi i} \int_{\Re(s)=1} \left(\frac{4\pi m \sqrt{c}}{b\sqrt{N}} \right)^{-s} G_{k-1}(s) \left[\sum_{v=1}^{\infty} \hat{\phi} \left(\frac{\log v}{\log R} \right) \frac{\chi_0(v) \Lambda(v)}{v^{(1+s)/2} \log R} \right]^\gamma \phi \left(\frac{1-s}{4\pi i} \log R \right)^{n-\gamma} ds \quad (\text{A.36})$$

where $G_{k-1}(s)$ is the Mellin transform of the Bessel function (see (2.9)). Via a change of variable $s = 1 + it$ and the definition of the Mellin transform,

$$\begin{aligned} T(\gamma, n) &= \frac{b\sqrt{N}}{8\pi^2 m \sqrt{c}} \int_{t=-\infty}^{\infty} \phi \left(\frac{-t \log R}{4\pi} \right)^{n-\gamma} \left[\sum_{v_1, \dots, v_\gamma=1}^{\infty} \prod_{i=1}^{\gamma} \hat{\phi} \left(\frac{\log v_i}{\log R} \right) \frac{\chi_0(v_i) \Lambda(v_i)}{v_i^{it/2+1} \log R} \right] \\ &\times \left(\frac{4\pi m \sqrt{c}}{b\sqrt{N}} \right)^{-it} \int_{x=0}^{\infty} J_{k-1}(x) x^{it} dx dt. \end{aligned} \quad (\text{A.37})$$

Setting $\Phi_k(x) = \phi(x)^k$, applying Fubini, and then applying another change of variable $u = -t \log R / (4\pi)$ gives

$$\begin{aligned} T(\gamma, n) &= \frac{b\sqrt{N}}{2\pi m \sqrt{c}} \sum_{v_1, \dots, v_\gamma=1}^{\infty} \left[\prod_{i=1}^{\gamma} \hat{\phi} \left(\frac{\log v_i}{\log R} \right) \frac{\chi_0(v_i) \Lambda(v_i)}{v_i \log R} \right] \\ &\times \int_{x=0}^{\infty} J_{k-1}(x) \widehat{\Phi_{n-\gamma}} \left(\frac{2 \log(bx \sqrt{N/cv_1 \cdots v_\gamma} / 4\pi m)}{\log R} \right) \frac{dx}{\log R}. \end{aligned} \quad (\text{A.38})$$

We now consider the left hand side of the original statement in equation (4.13), which we denote by U . Using the Mellin transform of the Bessel function (2.9) gives

$$U = \frac{1}{2\pi i} \int_{\Re(s)=1} \left[\sum_{v=1}^{\infty} \hat{\phi} \left(\frac{\log v}{\log R} \right) \frac{\chi_0(v) \Lambda(v)}{v^{(1+s)/2} \log R} \right]^{n-\eta} \left(\frac{4\pi m \sqrt{c}}{b\sqrt{N}} \right)^{-s} G_{k-1}(s) ds. \quad (\text{A.39})$$

By Equation (4.34) of [HM07] we have that

$$\sum_{v=1}^{\infty} \hat{\phi} \left(\frac{\log v}{\log R} \right) \frac{\chi_0(v) \Lambda(v)}{v^{(1+s)/2} \log R} = \phi \left(\frac{1-s}{4\pi i} \log R \right) + \mathcal{E}(s), \quad (\text{A.40})$$

where

$$\mathcal{E}(s) := -\frac{1}{2\pi i} \int_{\Re(z)=3/4} \phi \left(\frac{(2z-1-s) \log R}{4\pi i} \right) \frac{L'}{L}(z, \chi_0) dz. \quad (\text{A.41})$$

We substitute this formula into (A.39) to find

$$\begin{aligned} U &= \frac{1}{2\pi i} \int_{\Re(s)=1} \left[\phi \left(\frac{1-s}{4\pi i} \log R \right) + \mathcal{E}(s) \right]^{n-\eta} \left(\frac{4\pi m \sqrt{c}}{b\sqrt{N}} \right)^{-s} G_{k-1}(s) ds \\ &= \sum_{\gamma=0}^{n-\eta} \binom{n-\eta}{\gamma} \frac{1}{2\pi i} \int_{\Re(s)=1} \phi \left(\frac{1-s}{4\pi i} \log R \right)^{n-\eta-\gamma} \mathcal{E}(s)^\gamma \left(\frac{4\pi m \sqrt{c}}{b\sqrt{N}} \right)^{-s} G_{k-1}(s) ds. \end{aligned} \quad (\text{A.42})$$

Equation (4.43) of [HM07] gives the bound

$$\frac{1}{2\pi i} \int_{\Re(s)=1} \phi \left(\frac{1-s}{4\pi i} \log R \right)^{n-\eta-\gamma} \mathcal{E}(s)^\gamma \left(\frac{4\pi m \sqrt{c}}{b\sqrt{N}} \right)^{-s} G_{k-1}(s) ds \ll N^{(n-\eta-\gamma)\sigma/2+\epsilon}. \quad (\text{A.43})$$

Since we have $\sigma < \frac{1}{n-a}$, when $\gamma > a - \eta - 1$ the term is be $O(N^{1/2-\epsilon})$. Thus we have that

$$U = \sum_{\gamma=0}^{a-\eta-1} \binom{n-\eta}{\gamma} \frac{1}{2\pi i} \int_{\Re(s)=1} \phi \left(\frac{1-s}{4\pi i} \log R \right)^{n-\eta-\gamma} \mathcal{E}(s)^\gamma \left(\frac{4\pi m \sqrt{c}}{b\sqrt{N}} \right)^{-s} G_{k-1}(s) ds + O(N^{1/2-\epsilon}). \quad (\text{A.44})$$

Using the formula for $\mathcal{E}(s)$ in (A.40) and the definition of T in (A.36) gives

$$U = \sum_{\gamma=0}^{a-\eta-1} \binom{n-\eta}{\gamma} \sum_{j=0}^{\gamma} (-1)^{j-\gamma} \binom{\gamma}{j} T(j, n-\eta) + O(N^{1/2-\epsilon}). \quad (\text{A.45})$$

Rearranging gives

$$U = \sum_{\gamma=0}^{a-\eta-1} \sum_{j=\gamma}^{a-\eta-1} (-1)^{j-\gamma} \binom{n-\eta}{j} \binom{j}{\gamma} T(\gamma, n-\eta) + O(N^{1/2-\epsilon}). \quad (\text{A.46})$$

Now, applying (A.38) to (A.46) completes the proof of the lemma. \square

A.8. Proof of Lemma 4.11.

Proof of Lemma 4.11. Set

$$S := \sum_{\substack{b=1 \\ (b,M)=1}}^{\infty} \frac{R(1,b)R(m^2,b)}{\varphi(b)} \int_0^{\infty} J_{k-1}(y) \hat{\phi} \left(\frac{2 \log(by\sqrt{Q}/4\pi m)}{\log R} \right) \frac{dy}{\log R}. \quad (\text{A.47})$$

First, note that for $\epsilon > 0$ (and via the change of variable $y = \frac{m\sqrt{R}}{b\sqrt{Q}}x$),

$$\begin{aligned} & \left| \frac{R(1,b)R(m^2,b)}{\varphi(b)b^\epsilon} \int_0^\infty J_{k-1}(y) \hat{\phi}\left(\frac{2\log(by\sqrt{Q}/4\pi m)}{\log R}\right) \frac{dy}{\log R} \right| \\ & \ll \frac{m^4}{\varphi(b)b^\epsilon} \left| \int_0^\infty J_{k-1}\left(\frac{m\sqrt{R}}{b\sqrt{Q}}x\right) \hat{\phi}\left(\frac{2\log(x\sqrt{R}/4\pi)}{\log R}\right) \frac{m\sqrt{R}}{b\sqrt{Q}} \frac{dx}{\log R} \right| \\ & \ll b^{-3/2}, \end{aligned} \quad (\text{A.48})$$

where we use the estimates $|R(1,b)| = 1$, $|R(m^2,b)| \leq m^4$, $J(x) \ll 1$ and $\varphi(b) \leq b$. Since $\sum_{(b,M)=1} b^{-3/2}$ converges, we have by the dominated convergence theorem that

$$S = \lim_{\epsilon \rightarrow 0} \sum_{(b,M)=1} \frac{R(1,b)R(m^2,b)}{\varphi(b)b^\epsilon} \int_0^\infty J_{k-1}(y) \hat{\phi}\left(\frac{2\log(by\sqrt{Q}/4\pi m)}{\log R}\right) \frac{dy}{\log R}. \quad (\text{A.49})$$

Using the Mellin transform of the Bessel function (2.9) gives

$$\int_{y=0}^\infty J_{k-1}(y) \hat{\phi}\left(\frac{2\log(by\sqrt{Q}/4\pi m)}{\log R}\right) \frac{dy}{\log R} = \int_{-\infty}^\infty \phi(x \log R) \left(\frac{2\pi m}{b\sqrt{Q}}\right)^{4\pi i x} \frac{\Gamma\left(\frac{k}{2} - 2\pi i x\right)}{\Gamma\left(\frac{k}{2} + 2\pi i x\right)} dx. \quad (\text{A.50})$$

Thus after interchanging the sum and integral in (A.49) using Fubini's theorem, we have

$$S = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^\infty \phi(x \log R) \left(\frac{2\pi m}{\sqrt{Q}}\right)^{4\pi i x} \frac{\Gamma\left(\frac{k}{2} - 2\pi i x\right)}{\Gamma\left(\frac{k}{2} + 2\pi i x\right)} \sum_{(b,M)=1} \frac{R(1,b)R(m^2,b)}{\varphi(b)b^{\epsilon+4\pi i x}} dx. \quad (\text{A.51})$$

We now define

$$\chi(s) := \sum_{(b,M)=1} \frac{R(1,b)R(m^2,b)}{\varphi(b)b^s} \quad (\text{A.52})$$

so that

$$S = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^\infty \phi(x \log R) \left(\frac{2\pi m}{\sqrt{Q}}\right)^{4\pi i x} \frac{\Gamma\left(\frac{k}{2} - 2\pi i x\right)}{\Gamma\left(\frac{k}{2} + 2\pi i x\right)} \chi(\epsilon + 4\pi i x) dx. \quad (\text{A.53})$$

To evaluate this expression, we break the integral into two pieces: one for x close to 0 and one for $|x|$ large. We will show that the part with $|x|$ large has insignificant contribution as a result of the rapid decay of ϕ and then use Laurent expansions to handle the portion where $|x|$ is small.

First, we note that our sum χ can be expressed as a product over primes as

$$\begin{aligned} \chi(s) &= \prod_{(p,M)=1} \sum_{t=0}^\infty \frac{R(1,p^t)R(m^2,p^t)}{\varphi(p^t)p^{ts}} \\ &= \prod_{(p,M)=1} \begin{cases} \left(1 + \frac{1}{(p-1)p^s}\right), & \text{if } (p,m) = 1, \\ \left(1 - \frac{1}{p^s}\right) & \text{if } (p,m) > 1, \end{cases} \end{aligned} \quad (\text{A.54})$$

since $R(1,b) = \mu(b)$, $R(m^2,1) = 1$, $R(m^2,p) = -1$ if $(m,p) = 1$, and $R(m^2,p) = \varphi(p) = p-1$ if $(p,m) > 1$. It is convenient to rewrite this formula as

$$\chi(s) = \prod_p \left(1 + \frac{1}{(p-1)p^s}\right) \prod_{p|M} \left(1 + \frac{1}{(p-1)p^s}\right)^{-1} \prod_{p \mid \frac{m}{(m,M^\infty)}} \left(1 - \frac{1}{p^s}\right) \left(1 + \frac{1}{(p-1)p^s}\right)^{-1}. \quad (\text{A.55})$$

We start by bounding the integral when $|x|$ is large. In particular, we consider the portion of the integral with $|x| \geq X$, where X is some function of R that we determine later. [Mil09] observed that

$$\begin{aligned} \prod_p \left(1 + \frac{1}{(p-1)p^s}\right) &= \prod_p \left(1 + \frac{1}{p^{1+s}}\right) \left(1 + \frac{1}{(p-1)(p^{1+s}+1)}\right) \\ &= \prod_p \frac{\left(1 + \frac{1}{p^{1+s}}\right) \left(1 - \frac{1}{p^{1+s}}\right)}{\left(1 - \frac{1}{p^{1+s}}\right)} \left(1 + \frac{1}{(p-1)(p^{1+s}+1)}\right) \\ &= \frac{\zeta(1+s)}{\zeta(2+2s)} \prod_p \left(1 + \frac{1}{(p-1)(p^{1+s}+1)}\right). \end{aligned} \quad (\text{A.56})$$

For $\Re(s) > -1$, Equation (7) from page 32 of [Dav80] gives that

$$\zeta(1+s) = \frac{1}{s} + 1 - (1+s) \int_1^\infty \{y\} y^{-2-s} dy, \quad (\text{A.57})$$

where $\{y\}$ denotes the fractional part of y . For $s = \epsilon + 4\pi ix$, $|x| > X$, this gives us that

$$|\zeta(1+s)| \leq \frac{1}{4\pi X} + 2 + |s|. \quad (\text{A.58})$$

We also have that

$$\left| \frac{1}{\zeta(2+2s)} \right| = \prod_p \left| 1 - \frac{1}{p^{2+s}} \right| \leq \prod_p \left(\frac{p^2+1}{p^2} \right) \leq \prod_p \left(\frac{p^2}{p^2-1} \right) = \zeta(2) \quad (\text{A.59})$$

and

$$\left| 1 + \frac{1}{(p-1)(p^{1+s}+1)} \right| \leq \left| 1 + \frac{1}{(p-1)^2} \right|, \quad (\text{A.60})$$

whence

$$\left| \prod_p \left(1 + \frac{1}{(p-1)(p^{1+s}+1)} \right) \right| \leq \prod_p \left(1 + \frac{1}{(p-1)^2} \right) \leq 2 \prod_p \left(1 + \frac{1}{p^2} \right) \leq 2\zeta(2). \quad (\text{A.61})$$

For the other products, it is useful to consider the $p = 2$ case separately, which gives us the alternate bound of

$$\left| \prod_p \left(1 + \frac{1}{(p-1)(p^{1+s}+1)} \right) \right| \leq \left| 1 + \frac{1}{2^{1+s}+1} \right| \cdot \zeta(2). \quad (\text{A.62})$$

Next we bound the product over $p|M$. Suppose M is the product of α primes. The magnitude of each term is maximized when p^s is a purely negative real. If $p_1, \dots, p_\alpha > 2$, we get the bound

$$\left| \prod_{p|M} \left(1 + \frac{1}{(p-1)p^s} \right)^{-1} \right| \leq \prod_{p|M} \left(1 - \frac{1}{(p-1)|p^s|} \right)^{-1} \leq \prod_{i=1}^\alpha \frac{p_i-1}{p_i-2} < 2^\alpha. \quad (\text{A.63})$$

If $p_n = 2$, we can combine this term with our previous estimate and instead bound

$$\left| \left(1 + \frac{1}{2^{1+s}+1} \right) \cdot \left(1 + \frac{1}{2^s} \right)^{-1} \right|. \quad (\text{A.64})$$

Writing $c = 2^s$ and noting that $1 < |c| < 2$, we have

$$\left| \left(1 + \frac{1}{2^{1+s} + 1} \right) / \left(1 + \frac{1}{2^s} \right) \right| = \left| \frac{1 + \frac{1}{2c+1}}{1 + \frac{1}{c}} \right| = \left| \frac{2c^2 + 2c}{2c^2 + 3c + 1} \right| = \left| \frac{2c}{2c + 1} \right| = \left| 1 - \frac{1}{2c + 1} \right| < 2. \quad (\text{A.65})$$

Finally, we bound the last product. We begin by noting that by our previous work and a trivial estimate, for all primes $p > 2$ we have

$$\left| \left(1 - \frac{1}{p^s} \right) \left(1 + \frac{1}{(p-1)p^s} \right)^{-1} \right| \leq 2 \cdot \frac{p-1}{p-2} \leq 4. \quad (\text{A.66})$$

For the case $p = 2$, we again proceed as in the product over $p|M$, this time getting the estimate

$$\left| \left(1 - \frac{1}{2^s} \right) \left(1 + \frac{1}{2^s} \right)^{-1} \left(1 + \frac{1}{2^{1+s} + 1} \right) \right| < 4. \quad (\text{A.67})$$

In any case, grouping all of the terms together, we find that for $s = \epsilon + 4\pi ix$ with $|x| > X$,

$$|\chi(s)| \leq \left(\frac{1}{4\pi X} + 3 + 4\pi|x| \right) 8\zeta(2)^2 \prod_{p|m} 4 \ll (X^{-1} + 1 + |x|) 4^{\log_{p_i}(m)+i} \quad (\text{A.68})$$

where p_i is the i th prime number. We can get an arbitrarily small exponent for m at the expense of a greater constant, so we have that

$$|\chi(s)| \leq \left(\frac{1}{4\pi X} + 3 + 4\pi|x| \right) O_{\epsilon'}(m^{\epsilon'}). \quad (\text{A.69})$$

Letting $X = (\log R)^{-1/2}$, we apply (A.69) and use the decay of ϕ to get

$$\begin{aligned} & \left| \int_X^\infty \phi(x \log R) \chi(\epsilon + 4\pi ix) \left(\frac{2\pi m}{\sqrt{Q}} \right)^{4\pi ix} \frac{\Gamma(\frac{k}{2} - 2\pi ix)}{\Gamma(\frac{k}{2} + 2\pi ix)} dx \right| \\ & \ll_{\epsilon'} m^{\epsilon'} \int_X^\infty (x \log R)^{-4} (X^{-1} + 1 + x) dx \\ & \ll_{\epsilon'} m^{\epsilon'} (\log R)^{-1}. \end{aligned} \quad (\text{A.70})$$

While [ILS99] gives an incorrect factorization for the Laurent expansion of $\chi(s)$ near $s = 0$, their result is still correct within the listed error terms. Specifically,

$$\begin{aligned} \prod_p \left(1 + \frac{1}{(p-1)p^s} \right) &= s^{-1} + O(1) \\ \prod_{p|M} \left(1 + \frac{1}{(p-1)p^s} \right)^{-1} &= \frac{\varphi(M)}{M} (1 + O(|s| \log \log 3M)) \\ \prod_{p \mid \frac{m}{(m, M^\infty)}} \left(1 - \frac{1}{p^s} \right) \left(1 + \frac{1}{(p-1)p^s} \right)^{-1} &= \delta \left(\frac{m}{(m, M^\infty)}, 1 \right) + O(|s| \log m). \end{aligned} \quad (\text{A.71})$$

Thus, we have that for $s \ll X = (\log R)^{-1/2}$,

$$\chi(s) = \frac{\varphi(M)}{sM} \delta \left(\frac{m}{(m, M^\infty)}, 1 \right) + O(\log(m \log M)), \quad (\text{A.72})$$

where we simplify the error term using $\varphi(M) \leq M$. (8.322) of [GR65] gives

$$\Gamma\left(\frac{k+s}{2}\right) = \Gamma\left(\frac{k-s}{2}\right) \left(\frac{k}{2}\right)^s \left[1 + O\left(\frac{|s|}{k}\right)\right]. \quad (\text{A.73})$$

We have for small x and ϵ that

$$\chi(\epsilon + 4\pi ix) \frac{\Gamma(\frac{k}{2} - 2\pi ix)}{\Gamma(\frac{k}{2} + 2\pi ix)} = \frac{\varphi(M)}{sM} \delta\left(\frac{m}{(m, M^\infty)}, 1\right) \left(\frac{k}{2}\right)^{-4\pi ix} + O(\log(m \log M)). \quad (\text{A.74})$$

In particular, after changing x to $-x$ and noting the evenness of ϕ , the “small x ” integral is

$$\int_{-X}^X \phi(x \log R) \left(\frac{k\sqrt{Q}}{4\pi m}\right)^{4\pi ix} \frac{dx}{\epsilon - 4\pi ix} + O\left(\frac{\log(m \log M)}{\log R}\right). \quad (\text{A.75})$$

To extend the integral again, we use the decay of ϕ to find that

$$\left| \int_X^\infty \phi(x \log R) \left(\frac{k\sqrt{Q}}{4\pi m}\right)^{4\pi ix} \frac{dx}{\epsilon - 4\pi ix} \right| \ll \int_X^\infty \frac{1}{4\pi x} \phi(x \log R) dx \ll (X \log R)^{-2}. \quad (\text{A.76})$$

Since $X = (\log R)^{-1/2}$, the contribution from this term is absorbed into our error term. Adding both terms back in, keeping track of all of our error terms, and noting that ϕ is even, we have that

$$S = \delta\left(\frac{m}{(m, M^\infty)}, 1\right) \frac{\varphi(M)}{M} \lim_{\epsilon \downarrow 0} \int_{-\infty}^\infty \phi(x \log R) \left(\frac{k\sqrt{Q}}{4\pi m}\right)^{4\pi ix} \frac{dx}{\epsilon - 4\pi ix} + O_{\epsilon'}\left(\frac{m^{\epsilon'} \log \log M}{\log R}\right). \quad (\text{A.77})$$

Arguing as on page 100 of [ILS99], we find that this equals

$$\begin{aligned} S = & \delta\left(\frac{m}{(m, M^\infty)}, 1\right) \frac{\varphi(M)}{M} \left(-\frac{1}{2} \int_{-\infty}^\infty \phi(x) \sin\left(2\pi x \frac{\log(k^2 Q / 16\pi^2 m^2)}{\log R}\right) \frac{dx}{2\pi x} + \frac{1}{4} \phi(0) \right) \\ & + O_{\epsilon'}\left(\frac{m^{\epsilon'} \log \log M}{\log R}\right) \end{aligned} \quad (\text{A.78})$$

as desired. \square

APPENDIX B. PROOFS OF LEMMAS IN SECTION 5

B.1. Proof of Lemma 5.16.

Proof. We will consider each term appearing in (5.38) separately. First, define

$$\begin{aligned} g_1(n, f, c, d) &= \binom{n}{f}, & g_2(n, f, c, d) &= \binom{n-c}{f-c} \\ g_3(n, f, c, d) &= \binom{n-d}{f-d}, & g_4(n, f, c, d) &= \binom{n-c-d}{f-c-d} \end{aligned} \quad (\text{B.1})$$

and

$$G_i(n, f) = 2(-1)^n n! \sum_{\substack{c+d \leq n \\ c, d \geq 0}} (-1)^{c+d+1} \frac{g_i(n, f, c, d)}{(n-c-d)! c! d!}. \quad (\text{B.2})$$

We want to evaluate $G_1(n, f) - G_2(n, f) - G_3(n, f) + G_4(n, f)$. We set $\ell = c + d$ to rewrite (B.2) as

$$G_i(n, f) = 2(-1)^{n+1} n! \sum_{\ell=0}^n (-1)^\ell \sum_{c=0}^\ell \frac{g_i(n, f, c, \ell-c)}{(n-\ell)! (\ell-c)! c!}. \quad (\text{B.3})$$

To evaluate $G_1(n, f)$, we group the binomial coefficients to find that

$$\begin{aligned} G_1(n, f) &= 2(-1)^{n+1} \binom{n}{f} \sum_{\ell=0}^n (-1)^\ell \binom{n}{\ell} \sum_{c=0}^{\ell} \binom{\ell}{c} \\ &= 2(-1)^{n+1} \binom{n}{f} \sum_{\ell=0}^n (-2)^\ell \binom{n}{\ell} = -2 \binom{n}{f}. \end{aligned} \quad (\text{B.4})$$

Next, we note that $G_2(n, f) = G_3(n, f)$. We have that

$$\begin{aligned} G_2(n, f) &= 2(-1)^{n+1} \binom{n}{f} \sum_{\ell=0}^n (-1)^\ell \sum_{c=0}^{\ell} \binom{f}{c} \binom{n-c}{n-\ell} \\ &= 2(-1)^{n+1} \binom{n}{f} \sum_{c=0}^n \binom{f}{c} \sum_{\ell=c}^n (-1)^\ell \binom{n-c}{n-\ell}. \end{aligned} \quad (\text{B.5})$$

We reindex the sum by setting $\ell' = \ell - c$. Doing so, we see that sum over ℓ' is zero unless $n - c = 0$. However, in this case we have that $\binom{f}{c} = 0$ since $f \leq n/2 < n$. Thus each term vanishes and

$$G_2(n, f) = G_3(n, f) = 0. \quad (\text{B.6})$$

Lastly, again grouping terms into binomial coefficients gives

$$\begin{aligned} G_4(n, f) &= 2(-1)^{n+1} \binom{n}{f} \sum_{\ell=0}^n (-1)^\ell \binom{f}{\ell} \sum_{c=0}^{\ell} \binom{\ell}{c} \\ &= 2(-1)^{n+1} \binom{n}{f} \sum_{\ell=0}^n (-2)^\ell \binom{f}{\ell}. \end{aligned} \quad (\text{B.7})$$

We may restrict the sum in the last line to $0 \leq \ell \leq f$ since $f \leq n$ and $\binom{f}{\ell} = 0$ when $\ell > f$. Doing so, we find the sum over ℓ is $(-1)^f$ so

$$G_4(n, f) = 2(-1)^{n+f+1} \binom{n}{f}. \quad (\text{B.8})$$

Combining (B.4), (B.6) and (B.8) completes the proof of the lemma. \square

B.2. Proof of Lemma 5.26.

Proof. We consider each term appearing in (5.61) separately. First, define

$$\begin{aligned} h_1(f, g, \mu_1, \mu_d) &:= \binom{f}{g}, & h_2(f, g, \mu_1, \mu_d) &:= \binom{f - \mu_1}{g - \mu_1} \\ h_3(f, g, \mu_1, \mu_d) &:= \binom{f - \mu_d}{g}, & h_4(f, g, \mu_1, \mu_d) &:= \binom{f - \mu_1 - \mu_d}{g - \mu_1} \end{aligned} \quad (\text{B.9})$$

and

$$H_i(f, g) := \sum_{d=1}^f \sum_{\substack{\mu_1 + \dots + \mu_d = f \\ \mu_i \geq 1}} \frac{(-1)^d}{\mu_1! \dots \mu_d!} h_i(f, g, \mu_1, \mu_d) \quad (\text{B.10})$$

for $i \in \{1, 2, 3, 4\}$. We will show that $H_i(f, g) = (-1)^f / g! (f - g)!$ independent of i , so that $H_1 - H_2 - H_3 + H_4 = 0$ as desired. For H_1 the result follows immediately from comparing

coefficients of z^f in the identity (5.43). For H_2 , we pull out the μ_1 term to get

$$H_2(f, g) = - \sum_{\mu_1=1}^f \frac{1}{\mu_1!} \binom{f-\mu_1}{g-\mu_1} \sum_{d=1}^{f-\mu_1} \sum_{\substack{\mu_2+\dots+\mu_d=f-\mu_1 \\ \mu_i \geq 1}} \frac{(-1)^{d-1}}{\mu_2! \cdots \mu_d!}. \quad (\text{B.11})$$

Applying (5.43) and simplifying gives

$$\begin{aligned} H_2(f, g) &= - \sum_{\mu_1=1}^f \frac{1}{\mu_1!} \binom{f-\mu_1}{g-\mu_1} \frac{(-1)^{f-\mu_1}}{(f-\mu_1)!} \\ &= - \frac{(-1)^f}{(f-g)!} \sum_{\mu_1=1}^f \frac{(-1)^{\mu_1}}{\mu_1! (g-\mu_1)!} = \frac{(-1)^f}{g! (f-g)!}, \end{aligned} \quad (\text{B.12})$$

where the last step comes from restricting the summation to $1 \leq \mu_1 \leq g$ and using the binomial expansion of $(1-1)^g$. We can show the result for H_3 similarly. For H_4 , we pull out the μ_1 and μ_d terms to get

$$H_4(f, g) = \sum_{\mu_d=1}^f \sum_{\mu_1=1}^{f-\mu_d} \frac{1}{\mu_1! \mu_d!} \binom{f-\mu_1-\mu_d}{g-\mu_1} \sum_{d=1}^{f-\mu_1-\mu_d} \sum_{\substack{\mu_2+\dots+\mu_{d-1}=f-\mu_1-\mu_d \\ \mu_i \geq 1}} \frac{(-1)^{d-2}}{\mu_2! \cdots \mu_{d-1}!}. \quad (\text{B.13})$$

Applying (5.43) and simplifying gives

$$\begin{aligned} H_4(f, g) &= \sum_{\mu_1=1}^f \sum_{\mu_d=1}^{f-\mu_1} \frac{1}{\mu_1! \mu_d!} \binom{f-\mu_1-\mu_d}{g-\mu_1} \frac{(-1)^{f-\mu_1-\mu_d}}{(f-\mu_1-\mu_d)!} \\ &= (-1)^f \sum_{\mu_d=1}^f \frac{(-1)^{\mu_d}}{(f-g-\mu_d)! \mu_d!} \sum_{\mu_1=1}^{f-\mu_d} \frac{(-1)^{\mu_1}}{\mu_1! (g-\mu_1)!} \\ &= \frac{(-1)^{f+1}}{g!} \sum_{\mu_d=1}^f \frac{(-1)^{\mu_d}}{(f-g-\mu_d)! \mu_d!} = \frac{(-1)^f}{g! (f-g)!} \end{aligned} \quad (\text{B.14})$$

where the last two steps come from restricting the summation to $1 \leq \mu_1 \leq g$ and $1 \leq \mu_d \leq f-g$ and using the binomial expansion of $(1-1)^g$ and $(1-1)^{f-g}$. \square

APPENDIX C. INCREASING SUPPORT FOR THE NON-SPLIT FAMILY

In this section, we show how to prove Theorem 1.4 without the condition $2k \geq n$. Arguing as in Appendix E of [HM07], we need to bound terms of the form

$$\overline{E} := 2\pi i^k \sum_{\substack{q_1, \dots, q_\ell \\ q_j \text{ distinct}}} \prod_{j=1}^{\ell} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \right) \sum_{m \leq N^\epsilon} \frac{1}{m} \sum_{b=1}^{\infty} \frac{S(m^2, Q; Nb)}{Nb} J_{k-1} \left(\frac{4\pi m \sqrt{Q}}{Nb} \right) \quad (\text{C.1})$$

where $Q = q_1^{m_1} \cdots q_\ell^{m_\ell}$ and $n_j \equiv m_j \pmod{2}$ for all j . Showing that these terms vanish as $N \rightarrow \infty$ for ϕ with $\text{supp } \hat{\phi} \subset (-\frac{2}{n}, \frac{2}{n})$ completes the proof of Theorem 1.4. These terms are very similar to the $E(\vec{n}, \vec{m})$ terms introduced in Section 3 (see (3.5) for example), and we are able to evaluate them in a similar fashion. We omit proofs as they are analogous to the proofs of the corresponding lemmas in Section 3, which we refer to. We will eventually prove the following lemma.

Lemma C.1. *Let \bar{E} be defined as in (C.1). Under GRH for Dirichlet L -functions, if $\text{supp}(\hat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$, then $\bar{E} \ll N^{-\epsilon}$ and thus does not contribute in the limit.*

First we restrict the sum over b as in Lemmas 3.3 and 3.4, which are proven in Appendices A.3 and A.4, respectively.

Lemma C.2. *Suppose $\text{supp}(\hat{\phi}) \subseteq (-\frac{7}{2n}, \frac{7}{2n})$. Then the subterms of \bar{E} in (C.1) for which $(b, N) > 1$ are $O(N^{-\epsilon})$.*

Lemma C.3. *Suppose $\text{supp}(\hat{\phi}) \subset (-\frac{1000}{n}, \frac{1000}{n})$. Then, the subterms of \bar{E} in (C.1) for which $b \geq N^{2022}$ are $O(N^{-12})$.*

Applying Lemmas C.2 and C.3 to (C.1) gives

$$\begin{aligned} \bar{E} &= 2\pi i^k \sum_{\substack{q_1, \dots, q_\ell \\ q_j \text{ distinct}}} \prod_{j=1}^{\ell} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \right) \sum_{m \leq N^\epsilon} \frac{1}{m} \\ &\quad \times \sum_{\substack{(b, N)=1 \\ b < N^{2022}}} \frac{S(m^2, Q; Nb)}{Nb} J_{k-1} \left(\frac{4\pi m \sqrt{Q}}{Nb} \right) + O(N^{-\epsilon}). \end{aligned} \quad (\text{C.2})$$

We convert the Kloosterman sums to sums over Gauss sums as in Lemma 3.5, which is proven in Appendix A.5.

Lemma C.4. *Let N be a prime not dividing b, Q, m . Then*

$$S(m^2, NQ; Nb) = -\frac{1}{\varphi(Nb)} \sum_{\chi(Nb)} G_\chi(m^2) G_\chi((Q, b^\infty)) \bar{\chi} \left(\frac{Q}{(Q, b^\infty)} \right). \quad (\text{C.3})$$

Applying Lemma C.4 to (C.2) gives

$$\begin{aligned} \bar{E} &= -2\pi i^k \sum_{\substack{q_1, \dots, q_\ell \\ q_j \text{ distinct}}} \prod_{j=1}^{\ell} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \right) \sum_{m \leq N^\epsilon} \frac{1}{m} \\ &\quad \times \sum_{\substack{(b, N)=1 \\ b < N^{2022}}} \frac{1}{Nb \varphi(Nb)} \sum_{\chi(Nb)} G_\chi(m^2) G_\chi((Q, b^\infty)) \bar{\chi} \left(\frac{Q}{(Q, b^\infty)} \right) J_{k-1} \left(\frac{4\pi m \sqrt{Q}}{Nb} \right) + O(N^{-\epsilon}). \end{aligned} \quad (\text{C.4})$$

Next, it holds that subterms involving non-principal characters in (C.4) are negligible in the limit. This leaves only subterms involving $\chi_0 = \bar{\chi}_0 \pmod{Nb}$ for each b . It holds that $G_{\chi_0}(x) = R(x, Nb)$, a Ramanujan sum.

Lemma C.5. *Assume GRH for Dirichlet L -functions and suppose that $\text{supp}(\hat{\phi}) \subset (-\frac{2}{n}, \frac{2}{n})$. Then the sum over all non-principal characters in (C.4) is $O(N^{-\epsilon})$.*

This lemma corresponds to Lemma 3.6, proven in Appendix A.6. Applying Lemma C.5 to (C.4) gives

$$\begin{aligned} \bar{E} = & -2\pi i^k \sum_{\substack{q_1, \dots, q_\ell \\ q_j \text{ distinct}}} \prod_{j=1}^{\ell} \left(\hat{\phi} \left(\frac{\log q_j}{\log R} \right)^{n_j} \left(\frac{2 \log q_j}{\sqrt{q_j} \log R} \right)^{n_j} \right) \sum_{m \leq N^\epsilon} \frac{1}{m} \\ & \times \sum_{\substack{(b, N)=1 \\ b < N^{2022}}} \frac{R(m^2, Nb) R((Q, b^\infty), Nb)}{Nb \varphi(Nb)} \chi_0 \left(\frac{Q}{(Q, b^\infty)} \right) J_{k-1} \left(\frac{4\pi m \sqrt{Q}}{Nb} \right) + O(N^{-\epsilon}). \end{aligned} \quad (\text{C.5})$$

Now, applying the bounds $R(m^2, Nb) \leq m^4$, $R(x, Nb) \leq \varphi(Nb)$, and $J_{k-1}(x) \ll x$ to (C.5) and using the fact that $\text{supp } \hat{\phi} \subset (-\frac{2}{n}, \frac{2}{n})$, we find that the main term is absorbed by the error term, completing the proof of Lemma C.1.

APPENDIX D. BOUNDING THE ORDER OF VANISHING AT THE CENTRAL POINT

In this section, we follow the arguments of Section 6 of [HM07] in order to bound the proportion of newforms with negative sign whose order of vanishing exceeds a certain threshold r . While they are conditional on GRH, our results surpass the best known conditional and unconditional bounds established in [ILS99], [HM07], and [BCD⁺20] when $r \geq 5$. We focus on the case $r = 5$, however our results may be easily generalized the case when $r > 5$. Additionally, we study the 4th-centered moment as it provides the best bounds for the case $r = 5$, but utilizing higher moments provides better bounds as r increases. Lastly, similar results may be obtained for the positive sign family. See [DM22] for a more in-depth analysis, where the results of this paper are used to find excellent bounds for vanishing to order r or more; specifically, for a fixed test function Dutta and Miller determine what level density gives the best bound.

We utilize Theorem 1.2 with $n = 4$ and

$$\phi(x) = \left(\frac{\sin \pi \sigma x}{\pi \sigma x} \right)^2, \quad \hat{\phi}(y) = \begin{cases} \frac{1}{\sigma} - \frac{|y|}{\sigma^2} & |y| < \sigma \\ 0 & |y| \geq \sigma. \end{cases} \quad (\text{D.1})$$

This test function is likely not optimal in general for minimizing the n^{th} -centered moment, and optimal test functions for the case $n = 1$ and $n = 2$ are found in [ILS99] and [BCD⁺20]. However, they are sufficient to surpass the bounds established in those papers. While Theorem 1.2 requires $\sigma < 0.5$ when $n = 4$, we may utilize the bounds given by $\sigma = 0.5$ by setting $\sigma = 0.5 - \epsilon$ and letting $\epsilon \rightarrow 0$. Now, Theorem 1.2 gives

$$\lim_{\substack{N \rightarrow \infty \\ N \text{ prime}}} \langle (D(f; \phi) - \langle D(f; \phi) \rangle_-)^4 \rangle_- = 3\sigma^4 - R(4, 2; \phi) = \frac{31}{105}. \quad (\text{D.2})$$

Now, if a newform f with negative sign has order of vanishing $r \geq 5$ at the central point, then by Theorem 1.4,

$$D(f; \phi) - \langle D(f; \phi) \rangle_- \geq r\phi(0) - \left(\hat{\phi}(0) + \frac{1}{2}\phi(0) \right) = r - \frac{5}{2} \geq \frac{5}{2}. \quad (\text{D.3})$$

Let $\text{Pr}(r \geq 5)$ be the proportion of newforms with negative sign whose order of vanishing at the central point is at least 5. Then (D.2) and (D.3) give

$$\text{Pr}(r \geq 5) \left(\frac{5}{2} \right)^4 \leq \frac{31}{105} \quad (\text{D.4})$$

so $\text{Pr}(r \geq 5) \leq \left(\frac{2}{5} \right)^4 \frac{31}{105} = \frac{496}{65625} \approx 0.00756$. [ILS99] and [HM07] obtain upper bounds of $\frac{1}{32} = 0.03125$ and $\frac{1}{49} \approx 0.02040$, respectively, our results surpass both of these. As the order of vanishing increases, our results are even better. For instance, taking $r = 19$ and $n = 20$, we find the proportion

of newforms with negative sign whose order of vanishing exceeds 19 is at most $2.86 \cdot 10^{-15}$, improving the upper bound $5.77 \cdot 10^{-6}$ given in [BCD⁺20] and the upper bound $3.29 \cdot 10^{-3}$ implicit in [ILS99].

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