Chapter 1
A characterization of prime v-palindromes

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Abstract An integer $n \geq 1$ is a v-palindrome if it is not a multiple of 10, nor a decimal palindrome, and such that the sum of the prime factors and corresponding exponents larger than 1 in the prime factorization of n is equal to that of the integer formed by reversing the decimal digits of n. For example, if we take 198 and its reversal 891, their prime factorizations are $198 = 2 \cdot 3^2 \cdot 11$ and $891 = 3^4 \cdot 11$ respectively, and summing the numbers appearing in each factorization both give 18. This means that 198 and 891 are v-palindromes. We establish a characterization of prime v-palindromes: they are precisely the larger of twin prime pairs of the form $(5 \cdot 10^m - 3, 5 \cdot 10^m - 1)$, and thus standard conjectures on the distribution of twin primes imply that there are only finitely many prime v-palindromes.[1]

Key words: Prime v-palindromes, Iverson bracket, Cramér model

1.1 Introduction

1.1.1 Related Work

There have been many papers studying properties shared by numbers and their reversals. We first set some notation.

Definition 1.1.1 *Let $b \geq 2$, $L \geq 1$, and $0 \leq a_0, a_1, \ldots, a_{L-1} < b$ be any integers. We denote*

$$(a_{L-1} \cdots a_1 a_0)_b \;:=\; \sum_{i=0}^{L-1} a_i b^i. \tag{1.1}$$

We also write $(a_{L-1}, \ldots, a_1, a_0)_b$ to make it clear which are each digit.

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Definition 1.1.2 *Let the base $b \geq 2$ representation of an integer $n \geq 1$ be $(a_{L-1} \cdots a_1 a_0)_b$, where $a_{L-1} \neq 0$. The b-*reverse *of n is defined to be*

$$r_b(n) \;:=\; (a_0 a_1 \cdots a_{L-1})_b. \tag{1.2}$$

We write $r(n)$ for $r_{10}(n)$.

So for example $r(198) = 891$.

In *A Mathematician's Apology* [Har], G. H. Hardy states that "8712 and 9801 are the only 4-digit numbers which are integral multiples of their decimal reversal":

$$8712 \;=\; 4 \cdot r(8712), \;\; 9801 \;=\; 9 \cdot r(9801). \tag{1.3}$$

In 1966, A. Sutcliffe [Su] generalized this observation and studied all integer solutions of the equation

$$k \cdot r_b(n) \;=\; n, \tag{1.4}$$

where $b \geq 2$ is the base and $0 < k < n$. In [KS], numbers n such that n divides $r(n)$ are mentioned. In particular, numbers of the form

$$2178, \quad 21978, \quad 219978, \quad 2199978, \quad \ldots, \tag{1.5}$$

with any number of 9's in the middle, all satisfy $4n = r(n)$.

Suppose that the prime factorization of an integer $n \geq 1$ is

$$n \;=\; p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \tag{1.6}$$

where $p_1 < \cdots < p_k$ are primes and $\alpha_1, \ldots, \alpha_k \geq 1$ integers. In 1977, P. Erdős and K. Alladi [AE] studied the function

$$A(n) \;=\; \sum_{i=1}^{k} p_i \cdot \alpha_i. \tag{1.7}$$

The entries A008474 and A000026 from the OEIS [OEIS] are

$$F(n) \;=\; \sum_{i=1}^{k} (p_i + \alpha_i), \tag{1.8}$$

$$G(n) \;=\; \prod_{i=1}^{k} p_i \cdot \alpha_i, \tag{1.9}$$

respectively. These functions are somehow similar in expression. We introduce an arithmetic function denoted by $v(n)$ which is obtained from $F(n)$ by replacing α_i with 0 when $\alpha_i = 1$. In other words,

$$v(n) \;=\; \sum_{\substack{1 \leq i \leq k, \\ \alpha_i = 1}} p_i + \sum_{\substack{1 \leq i \leq k, \\ \alpha_i \geq 2}} (p_i + \alpha_i). \tag{1.10}$$

1.1.2 v-palindromes

The concept of v-palindromes was introduced by Tsai in [Tsai0, Tsai1] and explored further in four later manuscripts [Tsai2, Tsai3, Tsai4, Tsai5]. As in the abstract, consider the number 198 whose digit reversal is 891. Their prime factorizations are

$$198 = 2 \cdot 3^2 \cdot 11, \tag{1.11}$$

$$891 = 3^4 \cdot 11, \tag{1.12}$$

and we have

$$2 + (3+2) + 11 = (3+4) + 11. \tag{1.13}$$

In other words, the sum of the numbers "appearing" on the right-hand side of (1.11) equals that of (1.12). We first give the following definitions.

Definition 1.1.3 *The additive function $v\colon \mathbb{N} \to \mathbb{Z}$ is defined by setting $v(p) := p$ for primes p and $v(p^\alpha) := p + \alpha$ for prime powers p^α with $\alpha \geq 2$.*

Notice that this definition of $v(n)$ agrees with (1.10). We define v-palindromes as follows.

Definition 1.1.4 *Let $n \geq 1$ and $b \geq 2$ be integers. Then n is a v-*palindrome in base *b if $b \nmid n$, $n \neq r_b(n)$, and $v(n) = v(r_b(n))$. A v-palindrome in base 10 is simply called a v-*palindrome.

Thus 198 and 891 are v-palindromes. The following are infinite sequences of v-palindromes [Tsai0, Tsai1]:

$$18,\ 198,\ 1998,\ 19998,\ 199998,\ 1999998,\ \ldots, \tag{1.14}$$

$$18,\ 1818,\ 181818,\ 18181818,\ 1818181818,\ 181818181818,\ \ldots. \tag{1.15}$$

In (1.14), we simply keep increasing the number of 9's in the middle; in (1.15), we simply keep appending another 18. The sequences (1.14) and (1.15) are actually parts of a larger family of v-palindromes derived in [Tsai3, Theorem 3]. In particular, there are infinitely many v-palindromes. According to [Tsai0], the v-palindromes $n \leq 10^5$ with $n < r(n)$ are

18, 198, 576, 819, 1131, 1304, 1818, 1998, 2262, 3393, 4154, 4636, 8749, 12441, 14269, 14344, 15167, 15602, 16237, 18018, 18449, 18977, 19998, 23843, 24882, 26677, 26892, 27225, 29925, 31229, 36679, 38967, 39169, 42788, 45694, 46215, 46655, 47259, 48048, 52416, 56056, 60147, 62218, 66218, 79689, 97999.

The sequence of v-palindromes n (whether $n < r(n)$ or not) is A338039 in the OEIS [OEIS]. In [Tsai0], it is said that extensive computer calculations suggest the following.

Conjecture 1.1.5 *There are no prime v-palindromes.*

1.1.3 New results

We are able to make significant progress towards a possible proof of Conjecture 1.1.5 by proving the following characterization of prime v-palindromes.

Theorem 1.1.6 *The prime v-palindromes are precisely the primes of the form*

$$5 \cdot 10^m - 1 \;=\; 4\underbrace{9\cdots9}_{m}, \tag{1.16}$$

for some integer $m \geq 4$, such that

$$5 \cdot 10^m - 3 \;=\; 4\underbrace{9\cdots9}_{m-1}7 \tag{1.17}$$

is also prime.

Here, the $m \geq 4$ is not very significant and only means that it has been checked that there are no prime v-palindromes of fewer than 5 decimal digits, and thus conceivably can be improved with more checking for small values of m. From this characterization of prime v-palindromes, it is a consequence of standard models for primes (such as the Cramér model, though weaker assumptions suffice) that there are only finitely many prime v-palindromes. In particular, we have the following.

Theorem 1.1.7 *Assume that the probability n and $n+2$ are both prime is bounded by $C/\log^2 n$ for some C. Then there are only finitely many prime v-palindromes.*

The main purpose of this paper is to prove Theorem 1.1.6. In Section 1.2, we give various definitions and lemmas to be used throughout the rest of the paper. The proof of the forward direction of Theorem 1.1.6 consists of Sections 1.3 to 1.6. The proof of the converse consists of just Section 1.7. In Section 1.8, we elaborate on the above-mentioned heuristics that there are only finitely many prime v-palindromes.

1.2 Preliminaries

We start with some useful definitions.

Definition 1.2.1 *Let P denote any mathematical statement. Then the* Iverson bracket *is defined by*

$$[P] \;:=\; \begin{cases} 1, & \text{if } P \text{ is true},\\ 0, & \text{if } P \text{ is false}. \end{cases} \tag{1.1}$$

Definition 1.2.2 *For integers $\alpha \geq 1$, denote $\iota(\alpha) := \alpha[\alpha > 1]$. That is,*

$$\iota(\alpha) \;:=\; \begin{cases} 0, & \text{if } \alpha \;=\; 1,\\ \alpha, & \text{if } \alpha \;>\; 1. \end{cases} \tag{1.2}$$

With this notation, the additive function $v\colon \mathbb{N} \to \mathbb{Z}$ can be defined by setting in one stroke

$$v(p^\alpha) \;:=\; p + \iota(\alpha) \tag{1.3}$$

for all prime powers p^α.

Definition 1.2.3 *Let the decimal representation of an integer $n \geq 1$ be $(a_{L-1}\cdots a_1 a_0)_{10}$, where $a_{L-1} \neq 0$. Then we denote*

$$a_i(n) \;:=\; a_i \quad \text{for } i \;=\; 0,1,\ldots,L-1, \tag{1.4}$$

$$L(n) \;:=\; L, \tag{1.5}$$

to indicate dependence on n. We also by convention denote $L(0) := 0$.

For example,

$$a_0(198) \;=\; 8, \quad a_1(198) \;=\; 9, \quad a_2(198) \;=\; 1, \tag{1.6}$$

$$L(198) \;=\; 3. \tag{1.7}$$

Hence we have defined a function $L\colon \mathbb{N}\cup\{0\} \to \mathbb{Z}$, where L stands for length. We then have the following lemmas the first two of which are obvious and follow immediately from definitions.

Lemma 1.2.4 *Let $0 \leq m \leq n$ be integers. Then $L(m) \leq L(n)$.*

Lemma 1.2.5 *Let $n, \ell \geq 1$ be integers. Then $L(n) = \ell$ if and only if $10^{\ell-1} \leq n < 10^\ell$.*

Lemma 1.2.6 *Let $m,n \geq 1$ be integers. Then*

$$L(mn) \;=\; L(m) + L(n) - [mn \;<\; 10^{L(m)+L(n)-1}]. \tag{1.8}$$

In particular,

$$L(m) + L(n) - 1 \;\leq\; L(mn) \;\leq\; L(m) + L(n). \tag{1.9}$$

Proof. By Lemma 1.2.5, $10^{L(m)-1} \leq m < 10^{L(m)}$ and $10^{L(n)-1} \leq n < 10^{L(n)}$. Therefore

$$10^{L(m)+L(n)-2} \;\leq\; mn \;<\; 10^{L(m)+L(n)}. \tag{1.10}$$

If $mn < 10^{L(m)+L(n)-1}$, then by Lemma 1.2.5,

$$L(mn) \;=\; L(m)+L(n)-1 \;=\; L(m)+L(n) - [mn \;<\; 10^{L(m)+L(n)-1}]. \tag{1.11}$$

If $10^{L(m)+L(n)-1} \leq mn$, then by Lemma 1.2.5,

$$L(mn) \;=\; L(m)+L(n) \;=\; L(m)+L(n) - [mn \;<\; 10^{L(m)+L(n)-1}]. \tag{1.12}$$

This proves (1.8). Because an Iverson bracket is always 0 or 1, clearly (1.9) holds.

Lemma 1.2.7 *Let $n_1, \ldots, n_k \geq 1$ be integers. Then*

$$L(n_1\cdots n_k) \;\geq\; L(n_1) + \cdots + L(n_k) - (k-1). \tag{1.13}$$

Proof. This follows by repeated application of the left inequality in (1.9) in Lemma 1.2.6.

The following is an elementary inequality which essentially says that the sum is no greater than the product and which we do not prove.

Lemma 1.2.8 *Let $x_1,\ldots,x_k \ge 2$ be real numbers. Then*

$$x_1+\cdots+x_k \ \le\ x_1\cdots x_k. \tag{1.14}$$

Lemma 1.2.9 *Let p be a prime and $\alpha \ge 0$ an integer. Then*

(i) *if $\alpha \in \{0,1\}$, then $v(p^\alpha) \le p^\alpha \le p+\alpha$;*
(ii) *if $\alpha > 1$, then $v(p^\alpha) \ =\ p+\alpha \le p^\alpha$.*

Proof. (i) can be easily checked. For (ii), we have, using Lemma 1.2.8 twice,

$$v(p^\alpha) \ =\ p+\alpha \ \le\ p\alpha \ =\ \underbrace{p+\cdots+p}_{\alpha} \ \le\ p^\alpha. \tag{1.15}$$

Lemma 1.2.10 *Let $n \ge 1$ be an integer. Then $v(n) \le n$ and $L(v(n)) \le L(n)$.*

Proof. We first prove that $v(n) \le n$. We have $v(1) = 0 \le 1$. Now assume that $n > 1$ has prime factorization $n = p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, where $p_1 < \cdots < p_k$ are primes and $\alpha_1,\ldots,\alpha_k \ge 1$ integers. Then using the fact that v is additive and Lemmas 1.2.9 and 1.2.8,

$$\begin{aligned} v(n) \ &=\ v(p_1^{\alpha_1}\cdots p_k^{\alpha_k}) \ =\ v(p_1^{\alpha_1})+\cdots+v(p_k^{\alpha_k}) \\ &\le\ p_1^{\alpha_1}+\cdots+p_k^{\alpha_k} \ \le\ p_1^{\alpha_1}\cdots p_k^{\alpha_k} \ =\ n. \end{aligned} \tag{1.16}$$

Now $L(v(n)) \le L(n)$ follows from $v(n) \le n$ with Lemma 1.2.4.

Lemma 1.2.11 *We have the following inequalities.*

(i) *If $x > -1$ is real, then $\log_2(10^{x+1}-1) < (x+1)\log_2 10$.*
(ii)*If $n \ge 2$ is an integer, then $(n+1)\log_2 10 < 10^{n-1}$.*

Proof. (i) Since $\log_2$ is strictly increasing, we have

$$\log_2(10^{x+1}-1) \ <\ \log_2(10^{x+1}) \ =\ (x+1)\log_2 10. \tag{1.17}$$

(ii)Define the function

$$q(n) \ =\ \frac{10^{n-1}}{n+1}, \quad \text{for integers } n \ge 0. \tag{1.18}$$

Then for any $n \ge 0$,

$$q(n+1) \ =\ \frac{10^n}{n+2} \ =\ \frac{10^{n-1}}{n+1}\cdot\frac{10(n+1)}{n+2} \ =\ q(n)\cdot\frac{10(n+1)}{n+2}. \tag{1.19}$$

As $10(n+1)/(n+2) > 1$ for $n \geq 0$, we see that $q(n)$ is strictly increasing. Now because

$$q(2) \;=\; \frac{10}{3} \;=\; 3.\overline{3} \;>\; 3.32\cdots \;=\; \log_2 10, \tag{1.20}$$

we see that $q(n) > \log_2 10$ for $n \geq 2$, which is exactly what is required.

1.3 Setup

It can be checked by a computer that there are no prime v-palindromes of fewer than 5 decimal digits. Therefore assume that p is a prime v-palindrome of $m+1$ decimal digits, where $m \geq 4$. In particular, according to Definition 1.1.4, $p \neq r(p)$ and $p = v(p) = v(r(p))$. Consequently,

$$10^m + 3 \;\leq\; p \;\leq\; 10^{m+1} - 3. \tag{1.1}$$

By the end of Section 1.6, we will have deduced that $p = 5 \cdot 10^m - 1$ and a bit more.

In the case $r(p) < p$, by Lemma 1.2.10, $v(r(p)) \leq r(p) < p$, and thus $p = v(r(p))$ cannot hold. Therefore we may assume that $r(p) > p$. Further, in the case $r(p)$ is prime, $v(r(p)) = r(p) > p$, and thus again $p = v(r(p))$ cannot hold. Therefore we may assume that $r(p)$ is composite. Suppose that

$$r(p) \;=\; fq^{\beta}, \tag{1.2}$$

where q is the largest prime factor of $r(p)$ and q^{β} the highest power of q dividing $r(p)$, namely, $q^{\beta} \parallel r(p)$. Let the number of decimal digits of q be denoted by ℓ, i.e., $L(q) = \ell$.

We shall assume the conditions and notation laid out in this section throughout the rest of this paper, without explicitly stating such assumptions in each lemma below.

Lemma 1.3.1 *We have the following:*

(i) $v(f) \;=\; p - v(q^{\beta}) \;=\; p - q - \iota(\beta)$,
(ii) $L(v(f)) \;=\; L(p - q - \iota(\beta))$,
(iii) $L(v(f)) \;\geq\; L(p - q - \beta)$.

Proof. (i) This follows by applying v to (1.2).
(ii) This follows by applying L to part (i).
(iii) If $\beta > 1$, then (ii) becomes $L(v(f)) = L(p - q - \beta)$. If $\beta = 1$, then because $r(p)$ is composite, $f > 1$, and so part (i) implies $2 \leq v(f) = p - q$. Therefore

$$v(f) \;>\; p - q - 1 \;\geq\; 1, \tag{1.3}$$

and by Lemma 1.2.4, $L(v(f)) \geq L(p - q - 1)$.

Lemma 1.3.2 *We have* $\beta \leq \log_2(10^{m+1}-1) < 10^{m-1}$.

Proof. Since $r(p) = fq^\beta$ in equation (1.2), we have $r(p) \geq q^\beta$. Also, because $r(p)$ has $m+1$ decimal digits, we have $r(p) \leq 10^{m+1}-1$. Consequently,

$$\beta \log q \;\leq\; \log r(p) \;\leq\; \log(10^{m+1}-1), \tag{1.4}$$

and so

$$\beta \;\leq\; \log_q(10^{m+1}-1) \;\leq\; \log_2(10^{m+1}-1). \tag{1.5}$$

That $\log_2(10^{m+1}-1) < 10^{m-1}$ follows from Lemma 1.2.11, using both parts.

Lemma 1.3.3 *If* $\ell \leq m-1$, *then*

(i) $L(p-q-\beta) \;\geq\; m$,
(ii) $L(p-q-\iota(\beta)) \;\geq\; m$,
(iii) $L(v(f)) \;\geq\; m$.

Proof. (i) Since q has ℓ digits, $q \leq 10^\ell - 1$. Together with (1.1) and Lemma 1.3.2, we have

$$\begin{aligned} p-q-\beta \;&\geq\; 10^m+3-10^\ell+1-10^{m-1}+1 \\ &\geq\; 10^m+3-10^{m-1}+1-10^{m-1}+1 \\ &\geq\; 8\cdot 10^{m-1}+5. \end{aligned} \tag{1.6}$$

Therefore $p-q-\beta$ has at least m decimal digits.
(ii) This is because $p-q-\iota(\beta) \geq p-q-\beta$.
(iii)This is by combining Lemma 1.3.1(iii) and part (i).

Lemma 1.3.4 *We have*

(i) $L(f) \;=\; m+1-L(q^\beta)+[r(p) \;<\; 10^{L(f)+L(q^\beta)-1}]$,
(ii) $L(q^\beta) \;=\; m+1-L(f)+[r(p) \;<\; 10^{L(f)+L(q^\beta)-1}]$,
(iii) $L(q^\beta) \;\geq\; \ell$,
(iv) $L(f) \;\leq\; m+1$,
(v) $L(f) \;\leq\; m+2-\ell$,
(vi) $L(f) \;\leq\; m+1-\beta(\ell-1)$, *and*
(vii) $L(p-q-\beta) \;\leq\; m+2-\ell$.

Proof. (i) By (1.2) and Lemma 1.2.6, we have

$$\begin{aligned} m+1 \;&=\; L(r(p)) \;=\; L(fq^\beta) \;=\; L(f)+L(q^\beta)-[fq^\beta \;<\; 10^{L(f)+L(q^\beta)-1}] \\ &=\; L(f)+L(q^\beta)-[r(p) \;<\; 10^{L(f)+L(q^\beta)-1}]. \end{aligned} \tag{1.7}$$

The required equality then follows by rearranging.
(ii) This follows by rearranging part (i).
(iii)Since $q^\beta \geq q$, by Lemma 1.2.4 we have $L(q^\beta) \geq L(q) = \ell$.

(iv)By (1.2), we have $f \le r(p)$. Thus by Lemma 1.2.4, we have $L(f) \le L(r(p)) = m+1$.

(v)By parts (i) and (iii) and the fact that an Iverson bracket must be no greater than 1, we have

$$\begin{aligned} L(f) &= m+1-L(q^{\beta})+[r(p) < 10^{L(f)+L(q^{\beta})-1}] \\ &\le m+1-\ell+1 = m+2-\ell. \end{aligned} \tag{1.8}$$

(vi)By Lemma 1.2.7, we have

$$L(q^{\beta}) \ge \beta L(q)-(\beta-1) = \beta\ell-(\beta-1) = \beta(\ell-1)+1. \tag{1.9}$$

Consequently from part (i),

$$\begin{aligned} L(f) &= m+1-L(q^{\beta})+[r(p) < 10^{L(f)+L(q^{\beta})-1}] \\ &\le m+1-\beta(\ell-1)-1+1 = m+1-\beta(\ell-1). \end{aligned} \tag{1.10}$$

(vii)By Lemma 1.3.1(iii), Lemma 1.2.10, and part (v),

$$L(p-q-\beta) \le L(v(f)) \le L(f) \le m+2-\ell. \tag{1.11}$$

1.4 The case $\ell \le m$

Since $r(p) = fq^{\beta}$ and the number of decimal digits of $r(p)$ and q are $m+1$ and ℓ, respectively, clearly $m+1 \ge \ell$. In this section we consider the case $\ell \le m$, dividing it into four cases corresponding to the four subsections below, and in each case show that a contradiction results. This means that necessarily $\ell = m+1$, which we consider in the next section.

1.4.1 Case $\ell = 1$

Since $\ell = 1 \le m-1$, by Lemma 1.3.3(iii), $L(v(f)) \ge m$. By Lemma 1.2.10 and Lemma 1.3.4(iv),

$$m \le L(v(f)) \le L(f) \le m+1. \tag{1.1}$$

By Lemma 1.3.4(ii),

$$L(q^{\beta}) = m+1-L(f) + [r(p) < 10^{L(f)+L(q^{\beta})-1}] \le 1+[r(p) < 10^{L(f)+L(q^{\beta})-1}] \le 2. \tag{1.2}$$

So we have $L(q^\beta) \le 2$. If $q = 2$, then because q is the largest prime factor of $r(p)$, necessarily $f = 1$. This implies that $r(p) = 2^\beta$ has at most 2 decimal digits, which is a contradiction. Hence $q \in \{3,5,7\}$. There remains only 8 possibilities for q^β and by checking one by one, it can be seen that $3 \le v(q^\beta) \le 9$. By Lemma 1.3.1(i) and (1.1),

$$10^m - 6 \;=\; 10^m + 3 - 9 \;\le\; v(f) \;=\; p - v(q^\beta) \;\le\; 10^{m+1} - 3 - 3 \;=\; 10^{m+1} - 6. \quad (1.3)$$

Thus we have

$$10^m - 6 \;\le\; v(f) \;\le\; 10^{m+1} - 6. \quad (1.4)$$

In the remainder of this subsection we discuss the cases $q = 3$, $q = 5$, and $q = 7$, one by one, showing that each case leads to a contradiction. This means that the whole case $\ell = 1$ leads to a contradiction.

Sub case $q = 3$: We must have $r(p) = 2^\gamma 3^\beta$, where $\gamma \ge 0$ is an integer. Since $L(3^\beta) \le 2$, we have $3^\beta \le 81$. Therefore $10^4 \le r(p) \le 2^\gamma \cdot 81$, and so $\gamma \ge 7$. Thus because $f = 2^\gamma$, (1.4) and Lemma 1.2.9 implies

$$10^m - 6 \;\le\; v(f) \;=\; v(2^\gamma) \;\le\; 2 + \gamma. \quad (1.5)$$

Consequently,

$$10^m - 8 \;\le\; \gamma. \quad (1.6)$$

Hence

$$r(p) \;=\; 2^\gamma 3^\beta \;\ge\; 2^{10^m - 8} \cdot 3 \;\ge\; 10^{m+1} \quad (1.7)$$

(the last inequality can be shown to hold for $m \ge 2$). This contradicts the fact that $L(r(p)) = m + 1$.

Sub case $q = 5$: We must have $r(p) = 2^\delta 3^\gamma 5^\beta$, where $\gamma, \delta \ge 0$ are integers. Thus because $f = 2^\delta 3^\gamma$, by (1.4) and Lemma 1.2.9,

$$10^m - 6 \;\le\; v(f) \;=\; v(2^\delta) + v(3^\gamma) \;\le\; 2 + \delta + 3 + \gamma. \quad (1.8)$$

Consequently,

$$\delta + \gamma \;\ge\; 10^m - 11. \quad (1.9)$$

Hence

$$r(p) \;=\; 2^\delta 3^\gamma 5^\beta \;\ge\; 2^{\delta+\gamma} \cdot 5 \;\ge\; 2^{10^m - 11} \cdot 5 \;\ge\; 10^{m+1} \quad (1.10)$$

(the last inequality can be shown to hold for $m \ge 2$). This contradicts the fact that $L(r(p)) = m + 1$.

Sub case $q = 7$: We must have $r(p) = 2^\eta 3^\delta 5^\gamma 7^\beta$, where $\gamma, \delta, \eta \ge 0$ are integers. Thus because $f = 2^\eta 3^\delta 5^\gamma$, by (1.4) and Lemma 1.2.9,

$$10^m - 6 \;\le\; v(f) \;=\; v(2^\eta) + v(3^\delta) + v(5^\gamma) \;\le\; 10 + \eta + \delta + \gamma. \quad (1.11)$$

Consequently,

$$\eta+\delta+\gamma \;\geq\; 10^m-16. \tag{1.12}$$

We then have

$$r(p) \;\geq\; 2^{\eta+\delta+\gamma}\cdot 7 \;\geq\; 2^{10^m-16}\cdot 7 \;\geq\; 10^{m+1} \tag{1.13}$$

(the last inequality can be shown to hold for $m \geq 2$). This contradicts the fact that $L(r(p)) = m+1$.

1.4.2 Case $\ell = 2$

By Lemma 1.3.4(iii), $L(q^\beta) \geq 2$, and by Lemma 1.3.4(v), $L(f) \leq m$. Since $\ell = 2 \leq m-1$, by Lemma 1.3.3(iii), $L(v(f)) \geq m$. By Lemma 1.2.10,

$$m \;\leq\; L(v(f)) \;\leq\; L(f) \;\leq\; m. \tag{1.14}$$

Therefore $L(f) = L(v(f)) = m$. By Lemma 1.3.4(vi),

$$m \;=\; L(f) \;\leq\; m+1-\beta, \tag{1.15}$$

and thus $\beta = 1$. Hence (1.2) simplifies to $r(p) = fq$. Since $\ell = 2$, we have $11 \leq q \leq 97$. Therefore because $L(r(p)) = m+1$,

$$f \;=\; \frac{r(p)}{q} \;<\; \frac{10^{m+1}}{11} \;\leq\; 10^m-100 \tag{1.16}$$

(it can be shown that the rightmost inequality holds for $m \geq 4$). By taking the v of $r(p) = fq$, we have

$$p \;=\; v(f)+q \;\leq\; f+97 \;<\; 10^m-100+97 \;=\; 10^m-3. \tag{1.17}$$

This implies that $L(p) < m+1$, which is a contradiction.

1.4.3 Case $3 \leq \ell \leq m-1$

Since $\ell \leq m-1$, by Lemma 1.3.3(i) and Lemma 1.3.4(vii),

$$m \;\leq\; L(p-q-\beta) \;\leq\; m+2-\ell. \tag{1.18}$$

This implies that $\ell \leq 2$, a contradiction. Hence this case is impossible.

1.4.4 Case $\ell = m$

By Lemma 1.3.4(iii), $L(q^{\beta}) \geq m$, and by Lemma 1.3.4(v), $L(f) \leq 2$. Therefore $L(f) \in \{1,2\}$. By Lemma 1.3.4(vi),

$$1 \ \leq\ L(f)\ \leq\ m+1-\beta(m-1). \tag{1.19}$$

This implies that

$$\beta\ \leq\ \frac{m}{m-1}\ =\ 1+\frac{1}{m-1}, \tag{1.20}$$

and so $\beta = 1$. Therefore $r(p) = fq$. If $q = 2$, then $f = 1$ and so $r(p) = 2$, which contradicts $L(r(p)) = m+1 \geq 5$. Hence q is an odd prime. In addition, if $f = 1$, then $r(p) = q$ is prime, contrary to our assumption that $r(p)$ is composite. Hence $f > 1$. By Lemma 1.3.1(i), $v(f) = p-q$, and so $v(f)$ is even. In the following we consider the cases $L(f) = 1$ and $L(f) = 2$ separately, showing that each leads to a contradiction and so ultimately this case $\ell = m$ is also impossible.

Sub case $L(f) = 1$: Since $v(f)$ is even and $2 \leq f \leq 9$, we have $f \in \{2,4\}$. In the case $f = 2$, we have $2 = p-q$. Then by (1.1),

$$q\ =\ p-2\ \geq\ 10^m+3-2\ =\ 10^m+1, \tag{1.21}$$

contradicting that $L(q) = m$. In the case $f = 4$, we have $4 = p-q$. Similarly by (1.1),

$$q\ =\ p-4\ \geq\ 10^m+3-4\ =\ 10^m-1. \tag{1.22}$$

As $L(q) = m$, we have $q = 10^m-1$, contradicting the primeness of q.

Sub case $L(f) = 2$: Since $v(f)$ is even and $10 \leq f \leq 99$, we see that f must be one of

$$\begin{gathered}15,\ 16,\ 21,\ 24,\ 27,\ 30,\ 33,\ 35,\ 39,\ 40,\ 42,\ 45,\ 51,\ 54,\ 55,\ 56,\ 57,\ 60,\ 63,\\ 64, 65,\ 66,\ 69, 70, 72,\ 75,\ 77,\ 78,\ 84,\ 85,\ 87,\ 88,\ 90,\ 91,\ 93,\ 95,\ 96,\ 99,\end{gathered} \tag{1.23}$$

with $v(f)$ being one of

$$6,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18,\ 20,\ 22,\ 24,\ 26,\ 32,\ 34. \tag{1.24}$$

Consequently, by (1.1),

$$q \;=\; p - v(f) \;\geq\; 10^m + 3 - 34 \;=\; 10^m - 31, \tag{1.25}$$

and so

$$r(p) \;=\; fq \;\geq\; 15(10^m - 31) \;\geq\; 10^{m+1} \tag{1.26}$$

(it can be shown that the rightmost inequality holds for $m \geq 2$). This contradicts the fact that $L(r(p)) = m+1$.

1.5 The case $\ell \;=\; m+1$

In this section we consider the case $\ell = m+1$ and narrow down the potentially possible values of p more, i.e., deduce more necessary conditions.

By Lemma 1.3.4(v), $L(f) = 1$. By Lemma 1.3.4(vi),

$$1 \;=\; L(f) \;\leq\; m+1-\beta m. \tag{1.1}$$

This implies that $\beta \leq 1$, and so $\beta = 1$. Therefore $r(p) = fq$. If $q = 2$, then $f = 1$ and so $r(p) = 2$, which contradicts $L(r(p)) = m+1 \geq 5$. Hence q is an odd prime. In addition, if $f = 1$, then $r(p) = q$ is prime, contrary to our assumption that $r(p)$ is composite. Hence $f > 1$. By Lemma 1.3.1(i), $v(f) = p - q$, and so $v(f)$ is even. As $L(f) = 1$, we see that $v(f) = f \in \{2,4\}$. Consequently, $r(p)$ must be even.

Let the decimal representations of p, $r(p)$, and q be

$$p \;=\; (a_m \cdots a_0)_{10}, \tag{1.2}$$
$$r(p) \;=\; (a_0 \cdots a_m)_{10}, \tag{1.3}$$
$$q \;=\; (b_m \cdots b_0)_{10}, \tag{1.4}$$

where $a_m, b_m \neq 0$. As p is odd and prime, $a_0 \in \{1,3,7,9\}$, and as $r(p)$ is even, $a_m \in \{2,4,6,8\}$. Since $v(f) = p - q$, we have $q = p - v(f)$, and so

$$(b_m \cdots b_0)_{10} \;=\; (a_m \cdots a_0)_{10} - v(f). \tag{1.5}$$

Consequently, because $v(f) \in \{2,4\}$,

$$a_m \cdot 10^m - 4 \;\leq\; (b_m \cdots b_0)_{10} \;<\; (a_m \cdots a_0)_{10}, \tag{1.6}$$

and so $b_m \in \{a_m - 1, a_m\}$. Since $r(p) = fq$,

$$(a_0 \cdots a_m)_{10} \;=\; f(b_m \cdots b_0)_{10}. \tag{1.7}$$

This implies that

$$fb_m \;\leq\; a_0. \tag{1.8}$$

Hence

$$a_m - 1 \ \le\ b_m \ \le\ \frac{a_0}{f}, \quad \text{which implies that} \quad a_m \ \le\ 1 + \frac{a_0}{f} \ \le\ 1 + \frac{9}{f}. \tag{1.9}$$

Notice that from (1.5) we have

$$b_0 \ \equiv\ a_0 - v(f) \pmod{10}, \tag{1.10}$$

and that from (1.7) we have

$$a_m \ \equiv\ f b_0 \pmod{10}. \tag{1.11}$$

In the following we consider the cases $f = 2$ and $f = 4$ separately, corresponding to two subsections. For $f = 2$, we show that necessarily $a_m = 4$, $a_0 = 9$, $b_m = 4$, and $b_0 = 7$; while for $f = 4$, we show that a contradiction results.

1.5.1 Case $f = 2$

(1.10) and (1.11) become respectively

$$b_0 \ \equiv\ a_0 - 2 \pmod{10}, \tag{1.12}$$

$$a_m \ \equiv\ 2b_0 \pmod{10}. \tag{1.13}$$

By (1.9), $a_m \le 1 + 9/2 = 5.5$ and so as $a_m \in \{2,4,6,8\}$, we have $a_m \in \{2,4\}$. In the following we consider the cases $a_m = 2$ and $a_m = 4$ separately. For $a_m = 2$, we show that a contradiction results; while for $a_m = 4$, we show that necessarily $a_0 = 9$, $b_m = 4$, and $b_0 = 7$.

Sub case $a_m = 2$: (1.13) becomes $2 \equiv 2b_0 \pmod{10}$, or equivalently, $b_0 \equiv 1 \pmod{5}$. Thus as $0 \le b_0 < 10$, we have $b_0 \in \{1,6\}$. By (1.12), we have modulo 10,

$$a_0 \ \equiv\ b_0 + 2 \ \equiv\ \begin{cases} 3, & \text{if } b_0 \ =\ 1, \\ 8, & \text{if } b_0 \ =\ 6. \end{cases} \tag{1.14}$$

Thus as $0 \le a_0 < 10$,

$$a_0 \ =\ \begin{cases} 3, & \text{if } b_0 \ =\ 1, \\ 8, & \text{if } b_0 \ =\ 6. \end{cases} \tag{1.15}$$

As $a_0 \in \{1,3,7,9\}$, we have $b_0 = 1$ and $a_0 = 3$. Consequently, (1.5) becomes

$$(b_m \cdots 1)_{10} \ =\ (2 \cdots 3)_{10} - 2, \tag{1.16}$$

which means that $b_m = 2$. Then however, (1.8) becomes $2 \cdot 2 \leq 3$, which is false and we have a contradiction.

Sub case $a_m = 4$: (1.13) becomes $4 \equiv 2b_0 \pmod{10}$, or equivalently, $b_0 \equiv 2 \pmod 5$. Thus as $0 \leq b_0 < 10$, we have $b_0 \in \{2, 7\}$. By (1.12), we have modulo 10,

$$a_0 \equiv b_0 + 2 \equiv \begin{cases} 4, & \text{if } b_0 = 2, \\ 9, & \text{if } b_0 = 7. \end{cases} \tag{1.17}$$

Thus as $0 \leq a_0 < 10$,

$$a_0 = \begin{cases} 4, & \text{if } b_0 = 2, \\ 9, & \text{if } b_0 = 7. \end{cases} \tag{1.18}$$

As $a_0 \in \{1, 3, 7, 9\}$, we have $b_0 = 7$ and $a_0 = 9$. Consequently, (1.5) becomes

$$(b_m \cdots 7)_{10} = (4 \cdots 9)_{10} - 2, \tag{1.19}$$

which means that $b_m = 4$. Then (1.8) becomes $2 \cdot 4 \leq 9$, which is true and so we do not have a contradiction like we just did in the case $a_m = 2$.

1.5.2 Case $f = 4$

(1.10) and (1.11) become respectively

$$b_0 \equiv a_0 - 4 \pmod{10}, \tag{1.20}$$

$$a_m \equiv 4b_0 \pmod{10}. \tag{1.21}$$

By (1.9), $a_m \leq 1 + 9/4 = 3.25$ and so as $a_m \in \{2, 4, 6, 8\}$, we have $a_m = 2$.

Thus (1.21) becomes $2 \equiv 4b_0 \pmod{10}$, or equivalently, $2b_0 \equiv 1 \pmod 5$, or equivalently, $b_0 \equiv 3 \pmod 5$. Thus as $0 \leq b_0 < 10$, we have $b_0 \in \{3, 8\}$. By (1.20), we have modulo 10,

$$a_0 \equiv b_0 + 4 \equiv \begin{cases} 7, & \text{if } b_0 = 3, \\ 12, & \text{if } b_0 = 8. \end{cases} \tag{1.22}$$

Thus as $0 \leq a_0 < 10$,

$$a_0 = \begin{cases} 7, & \text{if } b_0 = 3, \\ 2, & \text{if } b_0 = 8. \end{cases} \tag{1.23}$$

As $a_0 \in \{1, 3, 7, 9\}$, we have $b_0 = 3$ and $a_0 = 7$. Consequently, (1.5) becomes

$$(b_m \cdots 3)_{10} = (2 \cdots 7)_{10} - 4, \tag{1.24}$$

which means that $b_m = 2$. Then however, (1.8) becomes $4 \cdot 2 \leq 7$, which is false and we have a contradiction.

1.6 The other decimal digits of p

As a result of Sections 1.3 through 1.5, we see that if p is a prime v-palindrome, then the following are true:

(i) p has $m+1$ decimal digits, for some $m \geq 4$,
(ii)p is of the form $(4\cdots 9)_{10}$, i.e., its leftmost decimal digit is 4 and its rightmost decimal digit is 9,
(iii)$p-2$ is prime, and
(iv)$r(p) = 2(p-2)$.

In this section we show further that all other decimal digits of p must be 9's as well, and thus $p = 5 \cdot 10^m - 1$. Filling in what we know into (1.2), (1.3), and (1.4), we have

$$\begin{aligned} p &= (4, a_{m-1}, \ldots, a_1, 9)_{10}, & (1.1)\\ r(p) &= (9, a_1, \ldots, a_{m-1}, 4)_{10}, & (1.2)\\ p-2 &= (4, a_{m-1}, \ldots, a_1, 7)_{10}. & (1.3) \end{aligned}$$

Since $r(p) = 2(p-2)$,

$$(9, a_1, \ldots, a_{m-1}, 4)_{10} = 2(4, a_{m-1}, \ldots, a_1, 7)_{10}. \quad (1.4)$$

We need to prove that

$$a_i = a_{m-i} = 9, \quad \text{for } 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor. \quad (1.5)$$

For integers $0 \leq I \leq \left\lfloor \frac{m}{2} \right\rfloor$, let $S(I)$ be the statement that

$$a_i = a_{m-i} = 9, \quad \text{for } 1 \leq i \leq I. \quad (1.6)$$

We prove that $S(I)$ holds for all $0 \leq I \leq \left\lfloor \frac{m}{2} \right\rfloor$ inductively, which will imply in particular that $S(\left\lfloor \frac{m}{2} \right\rfloor)$, i.e., (1.5), holds. Firstly, notice that $S(0)$ holds vacuously. Next, suppose that $S(I)$ holds for some $0 \leq I < \left\lfloor \frac{m}{2} \right\rfloor$. We shall proceed to prove $S(I+1)$, which amounts to proving

$$a_{I+1} = a_{m-I-1} = 9. \quad (1.7)$$

We have

$$\begin{aligned} p &= (4, \{9\}^I, a_{m-I-1}, \ldots, a_{I+1}, \{9\}^{I+1})_{10}, & (1.8)\\ r(p) &= (\{9\}^{I+1}, a_{I+1}, \ldots, a_{m-I-1}, \{9\}^I, 4)_{10}, & (1.9)\\ p-2 &= (4, \{9\}^I, a_{m-I-1}, \ldots, a_{I+1}, \{9\}^I, 7)_{10}, & (1.10)\end{aligned}$$

where $\{9\}^j$ for some integer $j \geq 0$ means that there are j digits of 9 consecutively; $\{9\}^0$ means that there is nothing. (1.4) becomes

$$(\{9\}^{I+1}, a_{I+1}, \ldots, a_{m-I-1}, \{9\}^I, 4)_{10} \ = \ 2(4, \{9\}^I, a_{m-I-1}, \ldots, a_{I+1}, \{9\}^I, 7)_{10}. \quad (1.11)$$

If $a_{m-I-1} \leq 4$, then the right-hand side of (1.11) must be of the form

$$(\underbrace{9, \ldots, 9}_{I}, 8, \ldots)_{10}, \quad (1.12)$$

which cannot equal the left-hand side. Therefore necessarily $5 \leq a_{m-I-1} \leq 9$. Notice that the congruence

$$2a_{I+1} + 1 \ \equiv \ a_{m-I-1} \pmod{10} \quad (1.13)$$

follows from (1.11). Reducing this congruence to modulo 2, we see that a_{m-I-1} must be odd. Therefore necessarily $a_{m-I-1} \in \{5, 7, 9\}$. In the following we consider each such possible value of a_{m-I-1}, corresponding to three subsections.

1.6.1 Case $a_{m-I-1} = 5$

(1.11) becomes

$$(\{9\}^{I+1}, a_{I+1}, \ldots, 5, \{9\}^I, 4)_{10} \ = \ 2(4, \{9\}^I, 5, \ldots, a_{I+1}, \{9\}^I, 7)_{10} \quad (1.14)$$

and (1.13) becomes

$$2a_{I+1} + 1 \ \equiv \ 5 \pmod{10}, \quad (1.15)$$

which forces $a_{I+1} \in \{2, 7\}$. However, in view of integer multiplication, we see that the digit of 10^{m-I-1} of the right-hand side of (1.14) must be 0 or 1. This means that we need to have $a_{I+1} \in \{0, 1\}$, which is impossible.

1.6.2 Case $a_{m-I-1} = 7$

(1.11) becomes

$$(\{9\}^{I+1}, a_{I+1}, \dots, 7, \{9\}^I, 4)_{10} \;=\; 2(4, \{9\}^I, 7, \dots, a_{I+1}, \{9\}^I, 7)_{10} \tag{1.16}$$

and (1.13) becomes

$$2a_{I+1} + 1 \;\equiv\; 7 \pmod{10}, \tag{1.17}$$

which forces $a_{I+1} \in \{3, 8\}$. However, in view of integer multiplication, we see that the digit of 10^{m-I-1} of the right-hand side of (1.16) must be 4 or 5. This means that we need to have $a_{I+1} \in \{4, 5\}$, which is impossible.

1.6.3 Case $a_{m-I-1} = 9$

(1.11) becomes

$$(\{9\}^{I+1}, a_{I+1}, \dots, \{9\}^{I+1}, 4)_{10} \;=\; 2(4, \{9\}^{I+1}, \dots, a_{I+1}, \{9\}^I, 7)_{10} \tag{1.18}$$

and (1.13) becomes

$$2a_{I+1} + 1 \;\equiv\; 9 \pmod{10}, \tag{1.19}$$

which forces $a_{I+1} \in \{4, 9\}$.

Assume that $a_{I+1} = 4$, then (1.18) becomes

$$(\{9\}^{I+1}, 4, \dots, \{9\}^{I+1}, 4)_{10} \;=\; 2(4, \{9\}^{I+1}, \dots, 4, \{9\}^I, 7)_{10}. \tag{1.20}$$

However, in view of integer multiplication, we see that the digit of 10^{m-I-1} of the right-hand side of (1.20) must be 8 or 9, in contrary to the left-hand side. Hence we must have $a_{I+1} = 9$.

Notice that this completes the induction because we are in the final case of $a_{m-I-1} = 9$.

1.7 Proof of the converse

Sections 1.3 through 1.6 proved the forward direction of Theorem 1.1.6. In this section we prove the converse.

Let $p = 5 \cdot 10^m - 1 = 4\underbrace{9 \cdots 9}_{m}$, for some integer $m \geq 4$, be a prime such that $p - 2 = 5 \cdot 10^m - 3$ is also prime. We show that p is a v-palindrome. Firstly, clearly $10 \nmid p$ and $p \neq r(p)$. We have

$$r(p) \;=\; r(4\underbrace{9 \cdots 9}_{m}) \;=\; \underbrace{9 \cdots 9}_{m}4 \;=\; 2 \cdot 4\underbrace{9 \cdots 9}_{m-1}7 \;=\; 2(p-2). \tag{1.1}$$

Consequently, as $p-2$ is an odd prime,

$$v(r(p)) \;=\; v(2(p-2)) \;=\; 2+(p-2) \;=\; p. \tag{1.2}$$

This completes the proof.

1.8 Number Of Prime v-palindromes

xTheorem 1.1.7 follows from standard models for prime numbers; we sketch below how a slightly weakened Cramér model, combined with our characterization of the form of prime v-palindromes, implies that there can only be finitely many.

For the standard Cramér model, one assumes that each integer n is prime with probability on the order of $1/\log n$, and the probability any two numbers are both prime is simply the product of the probabilities. This of course is clearly false, as we know if $n \geq 2$ is even then it cannot be prime, and if $n \equiv -2 \bmod p$ for any prime $p < n$ then $n+2$ cannot be prime. However, our goal is simply to provide support, and thus we ignore the more refined arguments one can do (see for example [Rub]). We assume instead that the probability n and $n+2$ are both prime is bounded by $C/\log^2 n$ for some fixed C; as we are only trying to prove there are at most finitely many prime v-palindromes, we are fine with a slightly larger but still finite upper bound.

Let T_n be the event that $5\cdot 10^n-3$ and $5\cdot 10^n-1$ are both prime, then the expected number of prime v-palindromes at most 10^{N+1} is

$$\sum_{n=1}^{N} 1\cdot \mathrm{Prob}(T_n). \tag{1.1}$$

As we are just concerned with supporting the conjecture that there are only finitely many, let us over-estimate and say

$$\mathrm{Prob}(T_n) \;\leq\; \frac{C}{\log^2(5\cdot 10^n-3)} \;\leq\; \frac{400C}{n^2\log^2 10} \;\leq\; \frac{100C}{n^2}. \tag{1.2}$$

As the sum of $1/n^2$ converges, the expected number of prime v-palindromes is finite.

Remark 1.8.1 *The Cramér model suggests we can take C to be around 1. With such an assumption, given that there are no prime v-palindromes for the first several candidates of the form* $5\cdot 10^m-1$*, the expected number of numbers of this form that are the larger in a twin prime pair is less than 1/2, and thus we do not expect there to be any prime v-palindromes.*

Remark 1.8.2 *While standard models predict the probability two integers of size x differing by 2 are both prime is on the order of* $1/\log^2 x$*, a significantly larger bound would still imply there are only finitely many v-primes. For example, if we instead* *had the probability bounded by a quantity of size* $1/\log^{1+\varepsilon} x$ *for any* $\varepsilon > 0$ *we would still get a finite sum in* (1.2).

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