ON IDENTITIES OF RUGGLES, HORADAM, HOWARD, AND YOUNG

CURTIS COOPER, STEVEN MILLER, PETER J. C. MOSES, MURAT SAHIN,
AND THOTSAPORN THANATIPANONDA

ABSTRACT. Ruggles (1963) discovered that for integers $n \geq 0$ and $k \geq 1$

$$F_{n+2k} = L_k F_{n+k} + (-1)^{k+1} F_n.$$

Horadam (1965), Howard (2001), and Young (2003) each expanded this identity to generalized linear recurrence relations of orders 2, 3, and integers $r \geq 2$, respectively. In this paper we let $r \geq 2$ be an integer and w_0, w_1, …, w_{r-1}, and p_1, p_2, …, $p_r \neq 0$ be integers. For $n \geq r$ set

$$w_n = p_1 w_{n-1} + p_2 w_{n-2} + \cdots + p_r w_{n-r}.$$

We find identities like those of Ruggles, Horadam, Howard, and Young, of the form

$$w_{n+rk} = R_k(r-1,r) w_{n+(r-1)k} + R_k(r-2,r) w_{n+(r-2)k} + \cdots + R_k(1,r) w_{n+k} + R_k(0,r) w_n,$$

where, by a result of Young, $R_k(i,r)$ is a linear recurrence relation of order $\binom{r}{i}$ for $i = 0, 1, \ldots, r-1$. Our proof uses the Cayley-Hamilton theorem. Next, we find the recurrences $R_k(0,r)$ and $R_k(r-1,r)$ for arbitrary r. Finally, we explicitly find identities for orders $r = 3$, $r = 4$ and $r = 5$.

1. INTRODUCTION

Let $\{F_n\}$ and $\{L_n\}$ be the Fibonacci and Lucas numbers, respectively. That is, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Ruggles [4] proved that for integers $n \geq 0$ and $k \geq 1$,

$$F_{n+2k} = L_k F_{n+k} + (-1)^{k+1} F_n.$$

Horadam [1] generalized this result to a general second order recurrence relation.

Theorem 1. *Let w_0, w_1, a, and $b \neq 0$ be integers. Let*

$$w_n = a w_{n-1} + b w_{n-2} \text{ for } n \geq 2.$$

In addition, let $x_0 = 2$, $x_1 = a$, and for $n \geq 2$,

$$x_n = a x_{n-1} + b x_{n-2}.$$

Then for integers $n \geq 0$ and $k \geq 1$,

$$w_{n+2k} = x_k w_{n+k} + (-1)^{k+1} b^k w_n.$$

Howard [2] generalized this result to third order recurrence relations. Young [6] generalized Howard's result for rth order recurrence relations, where $r \geq 2$ is an integer. In this paper we let $r \geq 2$ be an integer and let w_0, w_1, …, w_{r-1}, and p_1, p_2, …, $p_r \neq 0$ be integers. For $n \geq r$ set

$$w_n = p_1 w_{n-1} + p_2 w_{n-2} + \cdots + p_r w_{n-r}.$$

We find identities of the form

$$w_{n+rk} = R_k(r-1,r) w_{n+(r-1)k} + R_k(r-2,r) w_{n+(r-2)k} + \cdots + R_k(1,r) w_{n+k} + R_k(0,r) w_n,$$

where $R_k(i,r)$ is a linear recurrence sequence in k of order $\binom{r}{i}$ for $i = 0, 1, \ldots, r-1$. Our proof uses the Cayley-Hamilton theorem. In addition, we find the recurrences $R_k(0,r)$ and $R_k(r-1,r)$ for arbitrary r and we explicitly find identities for $r = 3$, $r = 4$ and $r = 5$.

2. General Equation and Lemma

To begin, we need a general equation and a useful lemma.

Let $r \geq 2$ be an integer. Let w_0, w_1, $\ldots$, w_{r-1} and p_1, p_2, $\ldots$, $p_r \neq 0$ be integers. Let

$$w_n = p_1 w_{n-1} + p_2 w_{n-2} + \cdots + p_r w_{n-r} \text{ for } n \geq r. \tag{1}$$

We now state our lemma.

Lemma 1. *Let $k \geq 1$ and $r \geq 2$ be an integers. Let $\{w_n\}$ be defined by (1). Let M be the $r \times r$ matrix given by*

$$\begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_{r-1} & p_r \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Let

$$p(x) = \det(xI - M^k) = \sum_{i=0}^{r} C_k(i,r) x^i$$

be the characteristic polynomial of M^k. Then

$$\sum_{i=0}^{r} C_k(i,r) w_{n+ik} = 0. \tag{2}$$

Proof. By the Cayley-Hamilton Theorem, every matrix satisfies its characteristic polynomial. Therefore,

$$p(M^k) = \det(M^k I - M^k) = \sum_{i=0}^{r} C_k(i,r)(M^k)^i = 0. \tag{3}$$

Multiplying both sides of (3) on the right by

$$\begin{pmatrix} w_n \\ w_{n-1} \\ \vdots \\ w_{n-r+1} \end{pmatrix}$$

gives

$$\sum_{i=0}^{r} C_k(i,r) M^{ik} \begin{pmatrix} w_n \\ w_{n-1} \\ \vdots \\ w_{n-r+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4}$$

It can be shown by a routine induction on m, that

$$\begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_{r-1} & p_r \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^m \begin{pmatrix} w_n \\ w_{n-1} \\ \vdots \\ w_{n-r+1} \end{pmatrix} = \begin{pmatrix} w_{n+m} \\ w_{n+m-1} \\ \vdots \\ w_{n+m-r+1} \end{pmatrix}. \tag{5}$$

Letting $m = ik$ in (5) and substituting the right-hand side of (5) into (4), we obtain

$$\sum_{i=0}^{r} C_k(i,r) \begin{pmatrix} w_{n+ik} \\ w_{n+ik-1} \\ \vdots \\ w_{n+ik-r+1} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{r} C_k(i,r) w_{n+ik} \\ \sum_{i=0}^{r} C_k(i,r) w_{n+ik-1} \\ \vdots \\ \sum_{i=0}^{r} C_k(i,r) w_{n+ik-r+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6}$$

Equating the first component of the two column vectors of (6) gives the result. □

Since the leading coefficient of the characteristic polynomial of M^k is 1, we have $C_k(r,r) = 1$. Therefore, we can rewrite (2) as

$$w_{n+rk} = -C_k(r-1,r) w_{n+(r-1)k} - C_k(r-2,r) w_{n+(r-2)k} - \cdots - C_k(0,r) w_n.$$

By letting $R_k(i,r) = -C_k(i,r)$ for $i = 0, 1, \ldots, r-1$, this identity takes the form

$$w_{n+rk} = R_k(r-1,r) w_{n+(r-1)k} + R_k(r-2,r) w_{n+(r-2)k} + \cdots + R_k(0,r) w_n.$$

First, we find this identity for the Tribonacci sequence. Then, we determine the sequences $R_k(r-1,r)$ and $R_k(0,r)$ for general r. Finally, using a computer algebra system and a result of Young [6], who proved that each sequence $R_k(i,r)$ is a recurrence relation of order $\binom{r}{i}$, we explicitly find the recurrence relations for the sequences $R_k(1,3)$, $R_k(1,4)$, $R_k(2,5)$, $R_k(3,5)$, $R_k(2,4)$ and $R_k(1,5)$.

3. Howard's Identity for the Tribonacci Sequence

In the following section we demonstrate use of Lemma 1 on the Tribonacci sequence [3, A000073], defined by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \text{ for } n \geq 3, \tag{7}$$

with initial conditions $T_0 = 0$, $T_1 = 0$, and $T_2 = 1$.

The polynomials producing $R_k(2,3)$, $R_k(1,3)$, and $R_k(0,3)$ for (7) are the following.

$$\det(xI - I) = \det \begin{pmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{pmatrix} = x^3 - 3x^2 + 3x - 1.$$

$$\det(xI - M) = \det \begin{pmatrix} x-1 & -1 & -1 \\ -1 & x & 0 \\ 0 & -1 & x \end{pmatrix} = x^3 - x^2 - x - 1.$$

$$\det(xI - M^2) = \det \begin{pmatrix} x-2 & -2 & -1 \\ -1 & x-1 & -1 \\ -1 & 0 & x \end{pmatrix} = x^3 - 3x^2 - x - 1.$$

$$\det(xI - M^3) = \det \begin{pmatrix} x-4 & -3 & -2 \\ -2 & x-2 & -1 \\ -1 & -1 & x-1 \end{pmatrix} = x^3 - 7x^2 + 5x - 1.$$

$$\det(xI - M^4) = \det \begin{pmatrix} x-7 & -6 & -4 \\ -4 & x-3 & -2 \\ -2 & -2 & x-1 \end{pmatrix} = x^3 - 11x^2 - 5x - 1.$$

$$\det(xI - M^5) = \det \begin{pmatrix} x-13 & -11 & -7 \\ -7 & x-6 & -4 \\ -4 & -3 & x-2 \end{pmatrix} = x^3 - 21x^2 - x - 1.$$

$$\det(xI - M^6) = \det \begin{pmatrix} x-24 & -20 & -13 \\ -13 & x-11 & -7 \\ -7 & -6 & x-4 \end{pmatrix} = x^3 - 39x^2 + 11x - 1.$$

Here are the initial values of these sequences.

Table 1. Values of Specific Third Order Sequences

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
T_k	0	0	1	1	2	4	7	13	24	44	81	149	274	504	927	1705
$R_k(2,3)$	3	1	3	7	11	21	39	71	131	241	443	815	1499	2757	5071	9327
$R_k(1,3)$	-3	1	1	-5	5	1	-11	15	-3	-23	41	-21	-43	105	-83	-65
$R_k(0,3)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Let

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \text{ for } n \geq 3 \tag{8}$$

with initial conditions $a_0 = 3$, $a_1 = 1$, and $a_2 = 3$. This is [3, A001644].

Let

$$b_n = -b_{n-1} - b_{n-2} + b_{n-3} \text{ for } n \geq 3 \tag{9}$$

with initial conditions $b_0 = -3$, $b_1 = 1$, and $b_2 = 1$. This is [3, A073145].

We now have the following theorem.

Theorem 2. *Let $n \geq 0$ and $k \geq 1$. Let $\{T_n\}$, $\{a_n\}$ and $\{b_n\}$ be defined by (7), (8), and (9), respectively. Then*

$$T_{n+3k} = a_k T_{n+2k} + b_k T_{n+k} + T_n.$$

4. The Recurrence $R_k(r-1, r)$

In this section, we determine the sequence $R_k(r-1, r)$ for arbitrary r.

Let $r \geq 2$ be a positive integer and let $p_1, p_2, \ldots, p_r \neq 0$ be integers. Let

$$a_n = p_1 a_{n-1} + p_2 a_{n-2} + \cdots + p_r a_{n-r} \text{ for } n \geq r \tag{10}$$

with initial conditions $a_0 = 0$, $a_1 = 0$, $\ldots$, $a_{r-2} = 0$ and $a_{r-1} = 1$.

We begin with a lemma.

Lemma 2. *Let k be a positive integer and $\{a_n\}$ be defined by (10). Then*

$$M^k = \begin{pmatrix} a_{k+r-1} & p_2a_{k+r-2}+p_3a_{k+r-3}+\cdots+p_ra_k & p_3a_{k+r-2}+\cdots+p_ra_{k+1} & \cdots & p_ra_{k+r-2} \\ a_{k+r-2} & p_2a_{k+r-3}+p_3a_{k+r-4}+\cdots+p_ra_{k-1} & p_3a_{k+r-3}+\cdots+p_ra_k & \cdots & p_ra_{k+r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_k & p_2a_{k-1}+p_3a_{k-2}+\cdots+p_ra_{k-r+1} & p_3a_{k-1}+\cdots+p_ra_{k-r-2} & \cdots & p_ra_{k-1} \end{pmatrix}.$$

Proof. The proof of the lemma is by induction on k. □

For a positive integer k, the characteristic polynomial of M^k is

$$\det(xI - M^k) = \det \begin{pmatrix} x-a_{k+r-1} & -p_2a_{k+r-2}-p_3a_{k+r-3}-\cdots-p_ra_k & \cdots & -p_ra_{k+r-2} \\ -a_{k+r-2} & x-p_2a_{k+r-3}-p_3a_{k+r-4}-\cdots-p_ra_{k-1} & \cdots & -p_ra_{k+r-3} \\ \cdots & \cdots & \cdots & \cdots \\ -a_k & -p_2a_{k-1}-p_3a_{k-2}-\cdots-p_ra_{k-r+1} & \cdots & x-p_ra_{k-1} \end{pmatrix}.$$

By examining the $-x^{r-1}$ term of the determinant we observe that the sequence $R_k(r-1,r)$ is

$$\begin{aligned} & a_{k+r-1} + (p_2a_{k+r-3}+\cdots+p_ra_{k-1}) + (p_3a_{k+r-4}+\cdots+p_ra_{k-1}) + \cdots + p_ra_{k-1} \\ &= a_{k+r-1} + p_2a_{k+r-3} + 2p_3a_{k+r-4} + \cdots + (r-1)p_ra_{k-1}, \end{aligned}$$

where k is a positive integer.

To make the notation easier to write, we introduce the following sequence.

Let $\{a_n\}$ be defined by (10). Let $x_0 = r$ and for any positive integer k, let

$$x_k = a_{k+r-1} + p_2a_{k+r-3} + 2p_3a_{k+r-4} + \cdots + (r-1)p_ra_{k-1}. \tag{11}$$

The following theorem shows that x_k is a linear recurrence of order r and gives its recurrence.

Theorem 3. *Let $n \geq r+1$ be an integer and $\{x_n\}$ be defined by (11). Then*

$$x_n = p_1x_{n-1} + p_2x_{n-2} + \cdots + p_rx_{n-r}.$$

Proof. Let $n \geq r+1$ be an integer. From the definition of the sequence $\{x_k\}$, for $k = n-1, \ldots, n-r$ we have that

$$x_k = a_{k+r-1} + p_2a_{k+r-3} + 2p_3a_{k+r-4} + \cdots + (r-1)p_ra_{k-1}. \tag{12}$$

For $k = n-1, \ldots, n-r$, multiply the right-hand side of (12) by $p_1, p_2, \ldots, p_r$, respectively. Adding the first terms of each of the r expressions, we have

$$p_1a_{n+r-2} + p_2a_{n+r-3} + \cdots + p_ra_{n-1} = a_{n+r-1}.$$

Adding the second terms of each of the r expressions, we have

$$p_2(p_1a_{n+r-4} + p_2a_{n+r-5} + \cdots + p_ra_{n-3}) = p_2a_{n+r-3}.$$

Adding the third terms of each of the r expressions, we have

$$2p_3(p_1a_{n+r-5} + p_2a_{n+r-6} + \cdots + p_ra_{n-4}) = 2p_3a_{n+r-4}.$$

Continue this process until the rth terms of each of the r expressions is reached.

The final result is

$$a_{n+r-1} + p_2a_{n+r-3} + \cdots + (r-1)p_ra_n = x_n.$$

which is what we wanted to prove. □

5. The Recurrence $R_k(0,r)$

In this section we determine the sequence $R_k(0,r)$ for arbitrary r. We prove the following theorem.

Theorem 4. *Let k be a non-negative integer and $\{a_n\}$ be defined by (10). Then*

$$R_k(0,r) = \begin{cases} p_r^k, & \text{if } r \text{ is odd;} \\ (-1)^{k+1} p_r^k, & \text{if } r \text{ is even.} \end{cases}$$

To obtain the recurrence $R_k(0,r)$, we evaluate $\det(xI - M^k)$ at $x = 0$. In general, this sequence is

$$\det \begin{pmatrix} -a_{k+r-1} & -p_2 a_{k+r-2} - p_3 a_{k+r-3} - \cdots - p_r a_k & -p_3 a_{k+r-2} - \cdots - p_r a_{k+1} & \cdots & -p_r a_{k+r-2} \\ -a_{k+r-2} & -p_2 a_{k+r-3} - p_3 a_{k+r-4} - \cdots - p_r a_{k-1} & -p_3 a_{k+r-3} - \cdots - p_r a_k & \cdots & -p_r a_{k+r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_k & -p_2 a_{k-1} - p_3 a_{k-2} - \cdots - p_r a_{k-r+1} & -p_3 a_{k-1} - \cdots - p_r a_{k-r-2} & \cdots & -p_r a_{k-1} \end{pmatrix}. \tag{13}$$

To continue the computation of (13), we need the following standard lemma (see Turnbull [5, p. 31]).

Lemma 3. *Let $r \geq 2$ be an integer. An $r \times r$ determinant is unaltered in value by adding to one of its columns any linear combination of its other columns.*

Now we compute the determinant in (13) with the help of two lemmas.

Lemma 4. *Let k be a positive integer and $\{a_n\}$ be defined by (10). Then*

$$\det \begin{pmatrix} -a_{k+r-1} & -p_2 a_{k+r-2} - p_3 a_{k+r-3} - \cdots - p_r a_k & -p_3 a_{k+r-2} - \cdots - p_r a_{k+1} & \cdots & -p_r a_{k+r-2} \\ -a_{k+r-2} & -p_2 a_{k+r-3} - p_3 a_{k+r-4} - \cdots - p_r a_{k-1} & -p_3 a_{k+r-3} - \cdots - p_r a_k & \cdots & -p_r a_{k+r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_k & -p_2 a_{k-1} - p_3 a_{k-2} - \cdots - p_r a_{k-r+1} & -p_3 a_{k-1} - \cdots - p_r a_{k-r-2} & \cdots & -p_r a_{k-1} \end{pmatrix}$$
$$= -p_r^{r-1} \det \begin{pmatrix} a_k & a_{k+1} & \cdots & a_{k+r-1} \\ a_{k-1} & a_k & \cdots & a_{k+r-2} \\ \cdots & \cdots & \cdots & \\ a_{k-r+1} & a_{k-r+2} & \cdots & a_k \end{pmatrix}.$$

Proof. First of all, we factor (-1) from every column of the matrix. Therefore, our initial determinant is equal to

$$(-1)^r \det \begin{pmatrix} a_{k+r-1} & p_2 a_{k+r-2} + p_3 a_{k+r-3} + \cdots + p_r a_k & p_3 a_{k+r-2} + \cdots + p_r a_{k+1} & \cdots & p_r a_{k+r-2} \\ a_{k+r-2} & p_2 a_{k+r-3} + p_3 a_{k+r-4} + \cdots + p_r a_{k-1} & p_3 a_{k+r-3} + \cdots + p_r a_k & \cdots & p_r a_{k+r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_k & p_2 a_{k-1} + p_3 a_{k-2} + \cdots + p_r a_{k-r+1} & p_3 a_{k-1} + \cdots + p_r a_{k-r-2} & \cdots & p_r a_{k-1} \end{pmatrix}. \tag{14}$$

We now start with the determinant

$$\det \begin{pmatrix} a_k & a_{k+1} & \cdots & a_{k+r-1} \\ a_{k-1} & a_k & \cdots & a_{k+r-2} \\ \cdots & \cdots & \cdots & \\ a_{k-r+1} & a_{k-r+2} & \cdots & a_k \end{pmatrix}$$

and work our way backwards to (14).

We first replace the first column by p_r times the first column plus p_{r-1} times the second column, plus $\cdots$ plus p_2 times the next to last column. Next, we replace the second column by p_r times the second column plus $\cdots$ plus p_3 times the next to last column. Continuing this process, we finally replace the next to last column by p_r times the next to last column. By Lemma 3 the value of the determinant is unchanged.

Once we have this new matrix, we swap columns r and $r-1$, then columns $r-1$ and $r-2$. We continue this process until we finally swap columns 2 and 1.

Counting the number of swaps and number of times we multiplied by p_r, we have the result. $\square$

To continue the proof we need the following lemma.

Lemma 5. *Let $r \geq 2$ be an integer and $\{a_n\}$ be defined by (10). Then for $k \geq r-1$,*

$$\det \begin{pmatrix} a_k & a_{k+1} & \cdots & a_{k+r-1} \\ a_{k-1} & a_k & \cdots & a_{k+r-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-r+1} & a_{k-r+2} & \cdots & a_k \end{pmatrix} = \begin{cases} p_r^{k-r+1}, & \text{if } r \text{ is odd;} \\ (-1)^{k+1} p_r^{k-r+1}, & \text{if } r \text{ is even.} \end{cases}$$

Proof. The proof of the lemma will be by induction on k. For $k = r-1$, we have

$$\det \begin{pmatrix} a_{r-1} & a_r & \cdots & a_{2r-2} \\ a_{r-2} & a_{r-1} & \cdots & a_{2r-3} \\ \cdots & \cdots & \cdots & \cdots \\ a_0 & a_1 & \cdots & a_{r-1} \end{pmatrix} = 1$$

so the base step is true.

Next, we assume the result is true for some $k-1 \geq r-1$ and attempt to prove the result is true for k. We start with the determinant

$$\det \begin{pmatrix} a_k & a_{k+1} & \cdots & a_{k+r-1} \\ a_{k-1} & a_k & \cdots & a_{k+r-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-r+1} & a_{k-r+2} & \cdots & a_k \end{pmatrix}.$$

In this matrix we replace the last column by the right side of (10) with $n = k+r-1$, $k+r-2$, $\ldots$, k, obtaining

$$\det \begin{pmatrix} a_k & a_{k+1} & \cdots & p_1 a_{k+r-2} + p_2 a_{k+r-3} + \cdots + p_r a_{k-1} \\ a_{k-1} & a_k & \cdots & p_1 a_{k+r-3} + p_2 a_{k+r-4} + \cdots + p_r a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-r+1} & a_{k-r+2} & \cdots & p_1 a_{k-1} + p_2 a_{k-2} + \cdots + p_r a_{k-r} \end{pmatrix}.$$

By Lemma 3 the value of the determinant remains the same if we subtract from the last column p_1 times the 2nd to last column, p_2 times the 3rd to last column, $\ldots$, and p_{r-1} times the first column.

$$\det \begin{pmatrix} a_k & a_{k+1} & \cdots & p_r a_{k-1} \\ a_{k-1} & a_k & \cdots & p_r a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-r+1} & a_{k-r+2} & \cdots & p_r a_{k-r} \end{pmatrix}.$$

If in the resulting matrix we now swap columns r and $r-1$, $r-1$ and $r-2$, ..., and columns 2 and 1 and factor out p_r from the last column the resulting determinant is

$$p_r(-1)^{r-1}\det\begin{pmatrix} a_{k-1} & a_k & \cdots & a_{k+r-3} \\ a_{k-2} & a_{k-1} & \cdots & a_{k+r-4} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-r} & a_{k-r+1} & \cdots & a_{k-1} \end{pmatrix}.$$

The result is true for k independent of the parity of r. Therefore, by the principle of mathematical induction, the result is true for all $k \geq r-1$. $\square$

Putting both of these lemmas together and using the fact the $R_k(0,r)$ is the coefficient of $-x^0 = -1$, we drop the minus sign to obtain the result.

Therefore, the sequence $R_k(0,r)$ is

$$R_k(0,r) = p_r^{r-1} \cdot \begin{cases} p_r^{k-r+1}, & \text{if } r \text{ is odd;} \\ (-1)^{k+1}p_r^{k-r+1}, & \text{if } r \text{ is even.} \end{cases} = \begin{cases} p_r^{k}, & \text{if } r \text{ is odd;} \\ (-1)^{k+1}p_r^{k}, & \text{if } r \text{ is even.} \end{cases}$$

This is the statement of the theorem. $\square$

6. An Explicit Formula for Howard's Third Order Recurrence

We next state the sequences we need to find an explicit formula for Howard's third order result.

Let

$$w_n = aw_{n-1} + bw_{n-2} + cw_{n-3} \text{ for } n \geq 3. \tag{15}$$

where w_0, w_1, w_2, a, b, and $c \neq 0$ are integers.

Using Lemma 1, Young's result, and a computer algebra system, we can calculate the sequences $R_k(2,3)$, $R_k(1,3)$, and $R_k(0,3)$. This leads to the following sequences and theorem.

Let a, b, and $c \neq 0$ be integers. Let

$$x_n = ax_{n-1} + bx_{n-2} + cx_{n-3} \text{ for } n \geq 3, \tag{16}$$

with initial conditions $x_0 = 3$, $x_1 = a$, and $x_2 = a^2 + 2b$.

Let

$$y_n = -by_{n-1} - acy_{n-2} + c^2y_{n-3} \text{ for } n \geq 3, \tag{17}$$

with initial conditions $y_0 = -3$, $y_1 = b$, and $y_2 = 2ac - b^2$.

Theorem 5. *Let $n \geq 0$ and $k \geq 1$ be integers. Let $\{w_n\}$, $\{x_n\}$, and $\{y_n\}$ be defined in (15), (16), and (17), respectively. Then*

$$w_{n+3k} = x_k w_{n+2k} + y_k w_{n+k} + c^k w_n.$$

7. An Explicit Formula for Young's Fourth Order Result

We next state the definitions we need to find an explicit formula for Young's fourth order result.

Let

$$w_n = aw_{n-1} + bw_{n-2} + cw_{n-3} + dw_{n-4} \text{ for } n \geq 4, \tag{18}$$

where w_0, w_1, w_2, w_3, a, b, c, and $d \neq 0$ are integers.

Again, using Lemma 1, Young's result, and a computation using a computer algebra system, we can calculate the sequences $R_k(3,4)$, $R_k(2,4)$, $R_k(1,4)$, and $R_k(0,4)$. This leads to the following sequences and theorem.

Let a, b, c, and $d \neq 0$ be integers. Let

$$x_n = ax_{n-1} + bx_{n-2} + cx_{n-3} + dx_{n-4} \text{ for } n \geq 4, \tag{19}$$

with initial conditions $x_0 = 4$, $x_1 = a$, $x_2 = a^2 + 2b$, and $x_3 = a^3 + 3ab + 3c$.

Let

$$y_n = -by_{n-1} - (d+ac)y_{n-2} + (c^2 - 2bd - a^2d)y_{n-3} + d(d+ac)y_{n-4} - bd^2y_{n-5} + d^3y_{n-6}, \tag{20}$$

for $n \geq 6$ with initial conditions $y_0 = -6$, $y_1 = b$, $y_2 = 2ac - b^2 + 2d$, $y_3 = 3a^2d + b^3 + 3bd - 3abc - 3c^2$, $y_4 = -4a^2bd - 2a^2c^2 + 4ab^2c - 8acd - b^4 - 4b^2d + 4bc^2 - 6d^2$, and $y_5 = -5a^3cd + 5a^2b^2d + 5a^2bc^2 - 5a^2d^2 - 5ab^3c + 5abcd + 5ac^3 + b^5 + 5b^3d - 5b^2c^2 + 5bd^2 + 5c^2d$.

Let

$$z_n = cz_{n-1} - bdz_{n-2} + ad^2z_{n-3} + d^3z_{n-4} \text{ for } n \geq 4, \tag{21}$$

with initial conditions $z_0 = 4$, $z_1 = c$, $z_2 = c^2 - 2bd$, and $z_3 = 3ad^2 + c^3 - 3bcd$.

Theorem 6. *Let $n \geq 0$ and $k \geq 1$. Let $\{w_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by (18), (19), (20), and (21), respectively. Then*

$$w_{n+4k} = x_kw_{n+3k} + y_kw_{n+2k} + z_kw_{n+k} + (-1)^{k+1}d^kw_n.$$

8. An Explicit Formula for Young's Fifth Order Result

We next state the definitions we need to find an explicit formula for Young's fourth order result.

Let

$$w_n = aw_{n-1} + bw_{n-2} + cw_{n-3} + dw_{n-4} + ew_{n-5} \text{ for } n \geq 5, \tag{22}$$

where w_0, w_1, w_2, w_3, w_4, a, b, c, d, and $e \neq 0$ are integers

Again, using Lemma 1, Young's result, and an extensive computation using a computer algebra system, we can calculate the sequences $R_k(4,5)$, $R_k(3,5)$, $R_k(2,5)$, $R_k(1,5)$, and $R_k(0,5)$. This leads to the following definitions and theorem. The calculations and sequences can be found in Appendix I. With the definitions in Appendix I, we have the following result.

Theorem 7. *Let $n \geq 0$ and $k \geq 1$. Let $\{w_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{v_n\}$ be defined by (22) and Appendix I. Then*

$$w_{n+5k} = x_kw_{n+4k} + y_kw_{n+3k} + z_kw_{n+2k} + v_kw_{n+k} + e^kw_n.$$

9. Appendix I

Let a, b, c, d, and $e \neq 0$ be integers. Let M be the 5×5 matrix

$$\begin{pmatrix} a & b & c & d & e \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let x_k, y_k, z_k and v_k be the coefficient of $-x^4$, $-x^3$, $-x^2$, and $-x^1$ in the $\det(xI - M^k)$, respectively. We compute the first 10 terms of each sequence using a computer algebra system.

```
#################################################
det(x*I-I)
= x^5 - 5*x^4 + 10*x^3 - 10*x^2 + 5*x - 1

#################################################
det(x*I-M)
= x^5 - a*x^4 - b*x^3 - c*x^2 - d*x - e

#################################################
det(x*I-M^2)
= x^5 + (-a^2 - 2*b)*x^4 + (-2*c*a + (b^2 - 2*d))*x^3 + (-2*e*a + (2*d*b - c^
2))*x^2 + (-2*e*c + d^2)*x - e^2

#################################################
det(x*I-M^3)
= x^5 + (-a^3 - 3*b*a - 3*c)*x^4 + (-3*d*a^2 + (3*c*b - 3*e)*a + (-b^3 - 3*d*
b + 3*c^2))*x^3 + ((3*e*c - 3*d^2)*a + (-3*e*b^2 + 3*d*c*b + (-c^3 - 3*e*d)))*x^
2 + (-3*e^2*b + (3*e*d*c - d^3))*x - e^3

#################################################
det(x*I-M^4)
= x^5 + (-a^4 - 4*b*a^2 - 4*c*a + (-2*b^2 - 4*d))*x^4 + (-4*e*a^3 + (4*d*b +
2*c^2)*a^2 + (-4*c*b^2 - 8*e*b + 8*d*c)*a + (b^4 + 4*d*b^2 - 4*c^2*b + (-4*e*c +
 6*d^2)))*x^3 + (-6*e^2*a^2 + (8*e*d*b + (4*e*c^2 - 4*d^2*c))*a + ((-4*e*c - 2*d
^2)*b^2 + (4*d*c^2 - 4*e^2)*b + (-c^4 + 8*e*d*c - 4*d^3)))*x^2 + (-4*e^3*a + (4*
e^2*d*b + (2*e^2*c^2 - 4*e*d^2*c + d^4)))*x - e^4

#################################################
det(x*I-M^5)
= x^5 + (-a^5 - 5*b*a^3 - 5*c*a^2 + (-5*b^2 - 5*d)*a + (-5*c*b - 5*e))*x^4 +
 ((5*e*b + 5*d*c)*a^3 + (-5*d*b^2 - 5*c^2*b + (10*e*c + 5*d^2))*a^2 + (5*c*b^3 +
 10*e*b^2 - 5*d*c*b + (-5*c^3 + 15*e*d))*a + (-b^5 - 5*d*b^3 + 5*c^2*b^2 + (15*e
*c - 5*d^2)*b + (-5*d*c^2 + 10*e^2)))*x^3 + ((-5*e^2*c - 5*e*d^2)*a^2 + (-5*e^2*
b^2 + (5*e*d*c + 5*d^3)*b + (5*e*c^3 - 5*d^2*c^2 - 15*e^2*d))*a + (5*e*d*b^3 + (
-5*e*c^2 - 5*d^2*c)*b^2 + (5*d*c^3 - 15*e^2*c + 10*e*d^2)*b + (-c^5 + 10*e*d*c^2
 - 5*d^3*c - 10*e^3)))*x^2 + (5*e^3*d*a + ((5*e^3*c - 5*e^2*d^2)*b + (-5*e^2*d*c
^2 + 5*e*d^3*c + (-d^5 + 5*e^4))))*x - e^5

#################################################
det(x*I-M^6)
= x^5 + (-a^6 - 6*b*a^4 - 6*c*a^3 + (-9*b^2 - 6*d)*a^2 + (-12*c*b - 6*e)*a +
 (-2*b^3 - 6*d*b - 3*c^2))*x^4 + ((6*e*c + 3*d^2)*a^4 + (-6*e*b^2 - 12*d*c*b + (
-2*c^3 + 12*e*d))*a^3 + (6*d*b^3 + 9*c^2*b^2 + (-18*d*c^2 + 9*e^2))*a^2 + (-6*c*
b^4 - 12*e*b^3 + 12*c^3*b - 18*d^2*c)*a + (b^6 + 6*d*b^4 - 6*c^2*b^3 + (-18*e*c
+ 9*d^2)*b^2 - 6*e^2*b + (3*c^4 - 12*e*d*c - 2*d^3)))*x^3 + (-2*e^3*a^3 + (18*e^
2*d*b + (-9*e^2*c^2 - 3*d^4))*a^2 + (-18*e*d^2*b^2 + (12*d^3*c + 12*e^3)*b + (6*
e*c^4 - 6*d^2*c^3 - 12*e*d^3))*a + (-3*e^2*b^4 + (12*e*d*c + 2*d^3)*b^3 + (-6*e*
```

```
c^3 - 9*d^2*c^2)*b^2 + (6*d*c^4 - 18*e^2*c^2 + 6*d^4)*b + (-c^6 + 12*e*d*c^3 - 6
*d^3*c^2 - 6*e^3*c - 9*e^2*d^2)))*x^2 + ((6*e^4*c - 6*e^3*d^2)*a + (3*e^4*b^2 +
(-12*e^3*d*c + 6*e^2*d^3)*b + (-2*e^3*c^3 + 9*e^2*d^2*c^2 - 6*e*d^4*c + (d^6 - 6
*e^4*d))))*x - e^6

##################################################
det(x*I-M^7)
= x^5 + (-a^7 - 7*b*a^5 - 7*c*a^4 + (-14*b^2 - 7*d)*a^3 + (-21*c*b - 7*e)*a^
2 + (-7*b^3 - 14*d*b - 7*c^2)*a + (-7*c*b^2 - 7*e*b - 7*d*c))*x^4 + (7*e*d*a^5 +
 ((-14*e*c - 7*d^2)*b + (-7*d*c^2 + 7*e^2))*a^4 + (7*e*b^3 + 21*d*c*b^2 + (7*c^3
 + 7*e*d)*b + (-21*e*c^2 - 21*d^2*c))*a^3 + (-7*d*b^4 - 14*c^2*b^3 + (-14*e*c -
7*d^2)*b^2 + (35*d*c^2 + 7*e^2)*b + (7*c^4 - 35*e*d*c - 14*d^3))*a^2 + (7*c*b^5
+ 14*e*b^4 + 7*d*c*b^3 + (-21*c^3 + 14*e*d)*b^2 + (-35*e*c^2 + 14*d^2*c)*b + (21
*d*c^3 - 21*e^2*c - 21*e*d^2))*a + (-b^7 - 7*d*b^5 + 7*c^2*b^4 + -7 (2*d^2
- 3 c*e)*b^3 + (7*d*c^2 + 14*e^2)*b^2 + (-7*c^4 + 7*e*d*c - 7*d^3)*b + (-7*e*c^3
 + 14*d^2*c^2 - 7*e^2*d)))*x^3 + ((7*e^3*c - 14*e^2*d^2)*a^3 + (-14*e^3*b^2 +
 (14*e^2*d*c + 21*e*d^3)*b + (-14*e^2*c^3 + 7*e*d^2*c^2 - 7*d^4*c
 - 21*e^3*d))*a^2 + (21*e^2*d*b^3 + (7*e^2*c^2 - 35*e*d^2*c - 7*d^4)*b^2
 + (-7*e*d*c^3 + 21*d^3*c^2 + 7*e^3*c + 35*e^2*d^2)*b + (7*e*c^5 - 7*d^2*c^4
 - 14*e^2*d*c^2 + 7*e*d^3*c + (-7*d^5 - 7*e^4)))*a + ((-7*e^2*c - 7*e*d^2)*b^4
 + (21*e*d*c^2 + 7*d^3*c - 7*e^3)*b^3+ (-7*e*c^4 - 14*d^2*c^3 + 35*e^2*d*c
 - 21*e*d^3)*b^2 + (7*d*c^5 - 21*e^2*c^3 - 14*e*d^2*c^2 + 14*d^4*c
 + 21*e^3*d)*b + (-c^7 + 14*e*d*c^4 - 7*d^3*c^3 - 14*e^3*c^2 + 7*e^2*d^2*c
 - 7*e*d^4)))*x^2 + ((7*e^5*b + (-14*e^4*d*c + 7*e^3*d^3))*a +
(-7*e^4*d*b^2 + (-7*e^4*c^2 + 21*e^3*d^2*c - 7*e^2*d^4)*b + (7*e^3*d*c^3 - 14*e^
2*d^3*c^2 + (7*e*d^5 - 7*e^5)*c + (-d^7 + 7*e^4*d^2))))*x - e^7

##################################################
det(x*I-M^8)
= x^5+(-a^8-8*b*a^6-8*c*a^5+(-20*b^2-8*d)*a^4+(-32*c*b-8*e)*a^3+(-16*b^3-24*d*b-12
*c^2)*a^2+(-24*c*b^2-16*e*b-16*d*c)*a+(-2*b^4-8*d*b^2-8*c^2*b+(-8*e*c-4*d^2)))*x
^4+(4*e^2*a^6+(-16*e*d*b+(-8*e*c^2-8*d^2*c))*a^5+((24*e*c+12*d^2)*b^2+(24*d*c^2+
8*e^2)*b+(2*c^4-48*e*d*c-8*d^3))*a^4+(-8*e*b^4-32*d*c*b^3+(-16*c^3-32*e*d)*b^2+(
32*e*c^2+32*d^2*c)*b+(32*d*c^3-16*e^2*c-48*e*d^2))*a^3+(8*d*b^5+20*c^2*b^4+(32*e
*c+16*d^2)*b^3+(-48*d*c^2+8*e^2)*b^2+(-24*c^4-32*e*d*c+16*d^3)*b+(16*e*c^3+56*d^
2*c^2-48*e^2*d))*a^2+(-8*c*b^6-16*e*b^5-16*d*c*b^4+(32*c^3-32*e*d)*b^3+(80*e*c^2
-16*d^2*c)*b^2+(-32*d*c^3+32*e^2*c-32*e*d^2)*b+(-8*c^5+32*e*d*c^2+32*d^3*c-8*e^3
))*a+(b^8+8*d*b^6-8*c^2*b^5+(-24*e*c+20*d^2)*b^4+(-16*d*c^2-16*e^2)*b^3+(12*c^4-
32*e*d*c+16*d^3)*b^2+(32*e*c^3-16*d^2*c^2-24*e^2*d)*b+(-8*d*c^4+20*e^2*c^2+8*e*d
^2*c+6*d^4)))*x^3+(-6*e^4*a^4+(32*e^3*d*b+(16*e^3*c^2-16*e^2*d^2*c-8*e*d^4))*a^3
+((-16*e^3*c-56*e^2*d^2)*b^2+(16*e^2*d*c^2+32*e*d^3*c+(8*d^5+8*e^4))*b+(-20*e^2*
c^4+16*e*d^2*c^3-12*d^4*c^2+32*e^3*d*c-48*e^2*d^3))*a^2+(-8*e^3*b^4+(32*e^2*d*c+
32*e*d^3)*b^3+(16*e^2*c^3-48*e*d^2*c^2-24*d^4*c-32*e^3*d)*b^2+(-16*e*d*c^4+32*d^
3*c^3+32*e^3*c^2-32*e^2*d^2*c+48*e*d^4)*b+(8*e*c^6-8*d^2*c^5-32*e^2*d*c^3+32*e*d
^3*c^2+(-16*d^5+24*e^4)*c-48*e^3*d^2))*a+(8*e^2*d*b^5+(-12*e^2*c^2-24*e*d^2*c-2*
d^4)*b^4+(32*e*d*c^3+16*d^3*c^2-32*e^3*c+16*e^2*d^2)*b^3+(-8*e*c^5-20*d^2*c^4+80
*e^2*d*c^2-32*e*d^3*c+(-8*d^5-20*e^4))*b^2+(8*d*c^6-24*e^2*c^4-32*e*d^2*c^3+24*d
```

```
^4*c^2+32*e^3*d*c+16*e^2*d^3)*b+(-c^8+16*e*d*c^5-8*d^3*c^4-16*e^3*c^3-8*e^2*d^2*
c^2+8*e*d^4*c+(-4*d^6-8*e^4*d))))*x^2+(4*e^6*a^2+(-16*e^5*d*b+(-8*e^5*c^2+24*e^4
*d^2*c-8*e^3*d^4))*a+((-8*e^5*c+12*e^4*d^2)*b^2+(24*e^4*d*c^2-32*e^3*d^3*c+(8*e^
2*d^5-8*e^6))*b+(2*e^4*c^4-16*e^3*d^2*c^3+20*e^2*d^4*c^2+(-8*e*d^6+16*e^5*d)*c+(
d^8-8*e^4*d^3))))*x-e^8

#####################################################
det(x*I-M^9)
= x^5+(-a^9-9*b*a^7-9*c*a^6+(-27*b^2-9*d)*a^5+(-45*c*b-9*e)*a^4+(-30*b^3-36*d*b-18
*c^2)*a^3+(-54*c*b^2-27*e*b-27*d*c)*a^2+(-9*b^4-27*d*b^2-27*c^2*b+(-18*e*c-9*d^2
))*a+(-9*c*b^3-9*e*b^2-18*d*c*b+(-3*c^3-9*e*d)))*x^4+((-9*e^2*b+(-18*e*d*c-3*d^3
))*a^6+(27*e*d*b^2+(27*e*c^2+27*d^2*c)*b+(9*d*c^3-27*e^2*c-27*e*d^2))*a^5+((-36*
e*c-18*d^2)*b^3+(-54*d*c^2-27*e^2)*b^2+(-9*c^4+54*e*d*c+9*d^3)*b+(36*e*c^3+54*d^
2*c^2-54*e^2*d))*a^4+(9*e*b^5+45*d*c*b^4+(30*c^3+63*e*d)*b^3+(-27*e*c^2-27*d^2*c
)*b^2+(-99*d*c^3-54*e^2*c+27*e*d^2)*b+(-9*c^5+108*e*d*c^2+63*d^3*c-30*e^3))*a^3+
(-9*d*b^6-27*c^2*b^5+(-54*e*c-27*d^2)*b^4+(54*d*c^2-27*e^2)*b^3+(54*c^4+81*e*d*c
-27*d^3)*b^2+(27*e*c^3-81*d^2*c^2-27*e^2*d)*b+(-54*d*c^4+27*e^2*c^2+135*e*d^2*c+
18*d^4))*a^2+(9*c*b^7+18*e*b^6+27*d*c*b^5+(-45*c^3+54*e*d)*b^4+(-135*e*c^2+27*d^
2*c)*b^3+(27*d*c^3-108*e^2*c+54*e*d^2)*b^2+(27*c^5+27*e*d*c^2-18*d^3*c-36*e^3)*b
+(9*e*c^4-63*d^2*c^3+54*e^2*d*c+45*e*d^3))*a+(-b^9-9*d*b^7+9*c^2*b^6+(27*e*c-27*
d^2)*b^5+(27*d*c^2+18*e^2)*b^4+(-18*c^4+63*e*d*c-30*d^3)*b^3+(-63*e*c^3+27*d^2*c
^2+27*e^2*d)*b^2+(9*d*c^4-54*e^2*c^2+54*e*d^2*c-9*d^4)*b+(3*c^6-9*e*d*c^3-18*d^3
*c^2-9*e^3*c+27*e^2*d^2)))*x^3+((-9*e^4*c-18*e^3*d^2)*a^4+(-18*e^4*b^2+(18*e^3*d
*c+63*e^2*d^3)*b+(30*e^3*c^3-27*e^2*d^2*c^2-9*e*d^4*c+(-3*d^6-45*e^4*d)))*a^3+(6
3*e^3*d*b^3+(-27*e^3*c^2-81*e^2*d^2*c-54*e*d^4)*b^2+(27*e^2*d*c^3+27*e*d^3*c^2+(
27*d^5-54*e^4)*c+135*e^3*d^2)*b+(-27*e^2*c^5+27*e*d^2*c^4-18*d^4*c^3+54*e^3*d*c^
2-27*e^2*d^3*c+(-27*e*d^5-27*e^5)))*a^2+((-9*e^3*c-54*e^2*d^2)*b^4+(27*e^2*d*c^2
+99*e*d^3*c+(9*d^5+9*e^4))*b^3+(27*e^2*c^4-54*e*d^2*c^3-54*d^4*c^2+27*e^3*d*c-10
8*e^2*d^3)*b^2+(-27*e*d*c^5+45*d^3*c^4+63*e^3*c^3-81*e^2*d^2*c^2+54*e*d^4*c+(18*
d^6+54*e^4*d))*b+(9*e*c^7-9*d^2*c^6-54*e^2*d*c^4+63*e*d^3*c^3+(-27*d^5+27*e^4)*c
^2+27*e^3*d^2*c-54*e^2*d^4))*a+(-3*e^3*b^6+(27*e^2*d*c+9*e*d^3)*b^5+(-18*e^2*c^3
-54*e*d^2*c^2-9*d^4*c+9*e^3*d)*b^4+(45*e*d*c^4+30*d^3*c^3-63*e^3*c^2-27*e^2*d^2*
c+36*e*d^4+45*a^2*d*e^3)*b^3+(-9*e*c^6-27*d^2*c^5+135*e^2*d*c^3-27*e*d^3*c^2+
(-27*d^5-54*e^4)*c-27*e^3*d^2-28*a*d^3*e^2)*b^2+(9*d*c^7-27*e^2*c^5-54*e*d^2*c^4
+36*d^4*c^3+108*e^3*d*c^2-54*e^2*d^3*c+(27*e*d^5-9*e^5))*b+(-c^9+18*e*d*c^6-9
*d^3*c^5-18*e^3*c^4-27*e^2*d^2*c^3+27*e*d^4*c^2+(-9*d^6+36*e^4*d)*c-30*e^3*d^3)))
*x^2+(-9*e^6*d*a^2+((-18*e^6*c+27*e^5*d^2)*b+(27*e^5*d*c^2-36*e^4*d^3*c+
(9*e^3*d^5-9*e^7)))*a+(-3*e^6*b^3+(27*e^5*d*c-18*e^4*d^3)*b^2+(9*e^5*c^3
-54*e^4*d^2*c^2+45*e^3*d^4*c+(-9*e^2*d^6+18*e^6*d))*b+(-9*e^4*d*c^4+30*e^3*d^3*c^3
+(-27*e^2*d^5+9*e^6)*c^2+(9*e*d^7-27*e^5*d^2)*c+(-d^9+9*e^4*d^4))))*x-e^9.
```

Next, knowing the fact that the recurrences for each term are of order $\binom{5}{4}$, $\binom{5}{3}$, $\binom{5}{2}$, and $\binom{5}{1}$, respectively, we compute these using a computer algebra system.

```
##################################################
The recurrence for the constant term is $e^n$.

##################################################
The recurrence of v_n where
```

```
v_n are the coefficients of -x^1
of the det(X*I-M^n):

v(n) = -d*v(n-1)
 -(c*e)*v(n-2)
 -(b*e^2)*v(n-3)
 +(-e^3*a)*v(n-4)
 +e^4*v(n-5).

###############################################
The recurrence of z_n where
z_n are the coefficients of -x^2
of the det(X*I-M^n):

z(n) = c*z(n-1)
 +(a*e-b*d)*z(n-2)
 +(e*(b^2+d)-a*(2*c*e-d^2))*z(n-3)
 +(e^2*(a^2+b)+d^3-d*e*(a*b+3*c))*z(n-4)
 +(e*(2*e^2+2*a*d*e+e*c*(a^2+2*b)-b*d^2))*z(n-5)
 +(e^2*(d^2-c*e-a^3*e-3*a*b*e+a*c*d))*z(n-6)
 +(-e^3*(d*a^2+a*e-c^2+2*b*d))*z(n-7)
 +(-e^4*(a*c+d))*z(n-8)
 +(-e^5*b)*z(n-9)
 +(-e^6)*z(n-10).


###############################################
The recurrence of y_n where
y_n are the coefficients of -x^3
of the det(X*I-M^n):

y(n) = -b*y(n-1)
 +(-a*c-d)*y(n-2)
 +(-a*e+c^2-a^2*d-2*b*d)*y(n-3)
 +(-a^3*e-3*a*b*e+a*c*d-c*e+d^2 )*y(n-4)
 +(2*e^2+e*(2*a*d+a^2*c+2*b*c)-b*d^2 )*y(n-5)
 +(e^2*(a^2+b)+e*(-3*c*d-a*b*d)+d^3)*y(n-6)
 +(e*(b^2*e+a*d^2+d*e-2*a*c*e))*y(n-7)
 +(e^2*(a*e-b*d) )*y(n-8)
 +(c*e^3)*y(n-9)
 +(-e^4)*y(n-10).


 ###############################################
The recurrence of x_n where
x_n are the coefficients of x^4
of the det(X*I-M^n):
```

```
x(n) = a*x(n-1)
 +(b)*x(n-2)
 +(c)*x(n-3)
 +(d)*x(n-4)
 +(e)*x(n-5).
```

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Department of Mathematics and Computer Science, University of Central Missouri, Warrensburg, Mo 64093
E-mail address: cooper@ucmo.edu

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267
E-mail address: sjm1@williams.edu

Moparmatic Co., 1154 Evesham Road, Astwood Bank, Redditch, Worcestershire, England, B96 6DT
E-mail address: mows@mopar.freeserve.co.uk

Department of Mathematics, Ankara University, Ankara, Turkey
E-mail address: msahin@ankara.edu.tr

Science Division, Mahidol University International College, Nakornpathom, Thailand
E-mail address: thotsaporn@gmail.com