LINEAR RECURRENCES OF ORDER AT MOST TWO IN NONTRIVIAL SMALL DIVISORS AND LARGE DIVISORS

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ABSTRACT. For each positive integer N, define

$$S'_N \;=\; \{1 < d < \sqrt{N} : d|N\} \text{ and } L'_N \;=\; \{\sqrt{N} < d < N : d|N\}.$$

Recently, Chentouf characterized all positive integers N such that the set of small divisors $\{d \le \sqrt{N} : d|N\}$ satisfies a linear recurrence of order at most two. We nontrivially extend the result by excluding the trivial divisor 1 from consideration, which dramatically increases the analysis complexity. Our first result characterizes all positive integers N such that S'_N satisfies a linear recurrence of order at most two. Moreover, our second result characterizes all positive N such that L'_N satisfies a linear recurrence of order at most two, thus extending considerably a recent result that characterizes N with L'_N being in an arithmetic progression.

CONTENTS

1. INTRODUCTION

For a positive integer N, the set of small divisors of N is

$$S_N \;:=\; \{d \,:\, 1 \le d \le \sqrt{N}, d \text{ divides } N\}.$$

Since the case $N = 1$ is trivial, we assume throughout the paper that $N > 1$. In 2018, Iannucci characterized all positive integers N whose S_N forms an arithmetic progression (or AP, for short). Iannucci's key idea was to show that if S_N forms an AP, then the size $|S_N|$ cannot exceed 6. Observing that the trivial divisor 1 plays an important role in Iannucci's proofs (see [4, Lemma 3 and Theorem 4]), Chu [3] excluded both 1

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and $\sqrt{N}$ from the definition of S_N to obtain a more general theorem that characterizes all N whose

$$S'_N \ := \ \{d \ : \ 1 < d < \sqrt{N}, d \text{ divides } N\}$$

is in an AP. Interestingly, with the trivial divisor 1 excluded, [2, Theorem 1.1] still gives that $|S'_N| \leq 5$. Recently, Chentouf generalized Iannucci's result from a different perspective by characterizing all N whose S_N satisfies a linear recurrence of order at most two. In particular, for each tuple $(u, v, a, b) \in \mathbb{Z}^4$, there is an integral linear recurrence, denoted by $U(u, v, a, b)$, of order at most two, given by

$$n_i \ = \ \begin{cases} u & \text{if } i = 1, \\ v & \text{if } i = 2, \\ an_{i-1} + bn_{i-2} & \text{if } i \geq 3. \end{cases}$$

Noting that the appearance of the trivial divisor 1 contributes nontrivially to the proof of [1, Theorem 3, Lemma 8, Theorem 10], we generalize Chentouf's result in the same manner as [2, Theorem 1.1] generalizes [4, Theorem 4]: we characterize all positive integers N whose S'_N satisfies a linear recurrence of order at most two.

Definition 1.1. A positive integer N is said to be small recurrent if S'_N satisfies a linear recurrence of order at most two. When $|S'_N| \leq 2$, N is vacuously small recurrent.

Theorem 1.2. *Let p, q, r denote prime numbers such that $p < q < r$ and k be some positive integer. A positive integer $N > 1$ is small recurrent if and only if N belongs to one of the following forms.*

(1) *$N = p^k$ for some $k \geq 1$. In this case, $S'_N = \{p, p^2, \ldots, p^{\lfloor (k-1)/2 \rfloor}\}$ satisfies $U(p, p^2, p, 0)$.*
(2) *$N = p^k q$ or $N = pq^k$ for some $1 \leq k \leq 3$. A restriction for $N = p^3 q$ is that either $p < q < p^2$ or $p^3 < q$.*
(3) *$N = p^k q$ for some $k \geq 4$ and $q > p^k$. In this case, $S'_N = \{p, p^2, p^3, \ldots, p^k\}$ satisfies $U(p, p^2, p, 0)$.*
(4) *$N = p^k q$ for some $k \geq 4$ and $\sqrt{q} < p < q$. In this case, $S'_N = \{p, q, p^2, pq, \ldots\}$ satisfies $U(p, q, 0, p)$.*
(5) *$N = pq^k$ for some $k \geq 4$ and $p < q$. In this case, $S'_N = \{p, q, pq, q^2, \ldots\}$ satisfies $U(p, q, 0, q)$.*
(6) *$N = pq^k r$ for some $k \geq 2$, $p < q$, and $r > pq^k$. In this case, $S'_N = \{p, q, pq, q^2, \ldots, pq^{k-1}, q^k, pq^k\}$ satisfies $U(p, q, 0, q)$.*
(7) *$N = p^2 q^2$ for some $p < q < p^2$. In this case, $S'_N = \{p, q, p^2\}$.*
(8) *$N = pqr$ for some $p < q < r$. If $r < pq$, then $S'_N = \{p, q, r\}$. If $r > pq$, then $S'_N = \{p, q, pq\}$.*
(9) *$N = p^3 q^2$ for some $p^{3/2} < q < p^2$. In this case, $S'_N = \{p, q, p^2, pq, p^3\}$ satisfies $U(p, q, 0, p)$.*
(10) *$N = p^2 qr$, where $p < q < p^2 < r < pq$, $(q^2 - p^3) | (pq - r)$, $(q^2 - p^3) | (rq - p^4)$, and $r = pq - \sqrt{(q^2 - p^3)(p^2 - q)}$. In this case, $S'_N = \{p, q, p^2, r, pq\}$ satisfies $U\left(p, q, \frac{p(pq-r)}{q^2-p^3}, \frac{rq-p^4}{q^2-p^3}\right)$.*

Next, consider the set of large divisors of N

$$L_N \ := \ \{d \ : \ d \geq \sqrt{N}, d \text{ divides } N\} \text{ and } L'_N \ := \ \{d \ : \ \sqrt{N} < d < N, d \text{ divides } N\}.$$

The second result of this paper is the characterization of all positive integers N whose L'_N satisfies a linear recurrence of order at most two. This considerably extends [2, Theorem 1].

Definition 1.3. A positive integer N is said to be large recurrent if L'_N satisfies a linear recurrence of order at most two. When $|L'_N| \leq 2$, N is vacuously large recurrent.

Theorem 1.4. *Let p, q, r denote prime numbers such that $p < q < r$ and k be some positive integer. A positive integer $N > 1$ is large recurrent if and only if N belongs to one of the following forms.*

(1) $N = p^k$ *for some* $k \geq 1$*. In this case,* $L'_N = \{p^{\lceil (k-1)/2 \rceil +1}, p^{\lceil (k-1)/2 \rceil +2}, \ldots, p^{k-1}\}$ *satisfies* $U(p^{\lceil (k-1)/2 \rceil +1}, p^{\lceil (k-1)/2 \rceil +2}, p, 0)$.
(2) $N = p^k q$ *for some* $k \geq 1$ *and* $q > p^k$*. In this case,* $L'_N = \{q, pq, p^2q, \ldots, p^{k-1}q\}$ *satisfies* $U(q, pq, p, 0)$.
(3) $N = p^k q$ *for some* $k \geq 2$ *and* $p^{k-1} < q < p^k$*. Then*

$$L'_N \;=\; \{p^k, pq, p^2q, \ldots, p^{k-1}q\}$$

satisfies $U(p^k, pq, p, 0)$.
(4) $N = p^k q$ *some for* $k \geq 3$ *and* $p < q < p^2$*. In this case,*

$$L'_N \;=\; \begin{cases} \{p^{k/2+1}, p^{k/2}q, p^{k/2+2}, \ldots, p^{k-1}q\} & \text{if } 2|k, \\ \{p^{(k-1)/2}q, p^{(k+3)/2}, p^{(k+1)/2}q, \ldots, p^{k-1}q\} & \text{if } 2 \nmid k. \end{cases}$$

Observe that L'_N *satisfies* $U(p^{k/2+1}, p^{k/2}q, 0, p)$ *and* $U(p^{(k-1)/2}q, p^{(k+3)/2}, 0, p)$ *for even and odd* k*, respectively.*
(5) $N = p^4 q$ *with* $p^2 < q < p^3$*,* $(p^5 - q^2)|(p^2 - q)$*, and* $(p^5 - q^2)|(p^3 - q)$*. In this case,* $L'_N \;=\; \{pq, p^4, p^2q, p^3q\}$.
(6) $N = p^3 q^2$ *for* $p < q < p^2$*. In this case,* $L'_N = \{q^2, p^2q, pq^2, p^3q, p^2q^2\}$ *satisfies* $U(q^2, p^2q, 0, p)$.
(7) $N = p^2 q^2$ *for some* $p < q$*. If* $p < q < p^2$*, then* $L'_N = \{q^2, p^2q, pq^2\}$.
(8) $N = pq^k$ *for some* $k \geq 2$ *and* $p < q$*. In this case,*

$$L'_N \;=\; \begin{cases} \{pq^{\frac{k}{2}}, q^{\frac{k}{2}+1}, \ldots, q^k\} & \text{if } 2|k, \\ \{q^{\frac{k+1}{2}}, pq^{\frac{k+1}{2}}, \ldots, q^k\} & \text{if } 2 \nmid k. \end{cases}$$

Observe that L'_N *satisfies* $U(pq^{k/2}, q^{k/2+1}, 0, q)$ *and* $U(q^{(k+1)/2}, pq^{(k+1)/2}, 0, q)$ *for even and odd* k*, respectively.*
(9) $N \;=\; pq^k r$ *for some* $k \;\geq\; 1$ *and* $p \;<\; q \;<\; pq^k \;<\; r$*. In this case,* $L'_N \;=\; \{r, pr, qr, pqr, q^2r, \ldots, q^k r\}$ *satisfies* $U(r, pr, 0, q)$.

The paper is structured as follows: Section 2 studies the case when N has a small number of divisors and establish some preliminary results; Section 3 characterizes small recurrent numbers, while Section 4 characterizes large recurrent numbers.

2. Preliminaries

For each $N \in \mathbb{N}$ with the prime factorization $\prod_{i=1}^{\ell} p_i^{a_i}$, the divisor-counting function is

$$\tau(N) \ := \ \sum_{d|N} 1 \ = \ \prod_{i=1}^{\ell}(a_i+1). \tag{2.1}$$

It is easy to verify that for $N > 1$,

$$\tau(N) \ := \ \begin{cases} 2|S'_N|+3 \ = \ 2|L'_N|+3 & \text{if } N \text{ is a square,} \\ 2|S'_N|+2 \ = \ 2|L'_N|+2 & \text{otherwise.} \end{cases} \tag{2.2}$$

Using (2.1), we can characterize all N with $\tau(N) \le 9$; equivalently, $|S'_N|, |L'_N| \le 3$.

(i) If $\tau(N) = 2$ or 3, (2.1) gives that $N = p$ or p^2 for some prime p,
(ii) If $\tau(N) = 4$ or 5, $N = pq, p^3, p^4$ for some primes $p < q$,
(iii) If $\tau(N) = 6$ or 7, $N = p^5, pq^2, p^2q, p^6$ for some primes $p < q$,
(iv) If $\tau(N) = 8$ or 9, $N = pqr, pq^3, p^3q, p^7, p^2q^2, p^8$ for some primes $p < q < r$.

2.1. Regarding S'_N.

Proposition 2.1. *If $|S'_N| \ge 2$, then N cannot have two (not necessarily distinct) prime factors p_1 and p_2 at least $\sqrt{N}$*

Proof. Suppose that N has two prime factors p_1 and p_2 at least $\sqrt{N}$. Then $p_1p_2 \ge N$ and p_1p_2 divides N; hence, $N = p_1p_2$, which contradicts that $|S'_N| \ge 2$. □

Proposition 2.2. *If all elements of S'_N are divisible by some prime p and $|S'_N| \ge 4$, then either $N = p^k$ or $N = p^kq$ for some $k \ge 1$ and some prime $q > p^k$.*

Proof. If all divisors (except 1) of N are divisible by p, then $n = p^k$ for some $k \ge 1$. Assume that N has a prime factor $q \ne p$. Then $q \ge \sqrt{N}$. Proposition 2.1 implies that N cannot have another prime factor at least $\sqrt{N}$. Hence, $N = p^kq$ for some $k \ge 1$ and $q > p^k$. □

Let $p < q < r < s$ be distinct prime numbers. Write $S'_N = \{d_2, d_3, d_4, d_5, \ldots\}$. (We start with d_2 since the smallest divisor of N is usually denoted by $d_1 = 1$, which is excluded from S'_N.)

Lemma 2.3. *Suppose that the first 4 numbers in S'_N are $p < q < p^2 < r$. If N is small recurrent with $U(p, q, a, b)$, the following hold:*

i) $\gcd(a, b) = 1$.
ii) *if $d_{2i} \in S'_N$, then $p|d_{2i}$; however, if $d_{2i-1} \in S'_N$, then $p \nmid d_{2i-1}$.*
iii) *for $d_i, d_{2i-1} \in S'_N$, we have* $\gcd(b, d_i) = \gcd(a, d_{2i-1}) = 1$.
iv) *for $d_i, d_{i+1} \in S'_N$, we have* $\gcd(d_i, d_{i+1}) = 1$.
v) *for $d_{2i-1}, d_{2i+1} \in S'_N$, we have* $\gcd(d_{2i-1}, d_{2i+1}) = 1$.

Proof. i) Since $r = ap^2 + bq$, we have $\gcd(a, b)|r$. Observe that $\gcd(a, b) = r$ contradicts $p^2 = aq + bp$. Hence, $\gcd(a, b) = 1$.

ii) We prove by induction. The claim is true for $i \leq 2$. Assume that the claim holds for $i = j \geq 2$. We show that it holds for $i = j+1$. Since $p^2 = aq + bp$, we know that $p|a$. By item i), $p \nmid b$. Write

$$\begin{aligned} d_{2(j+1)} &= ad_{2(j+1)-1} + bd_{2(j+1)-2} = ad_{2j+1} + bd_{2j} \\ &= (a^2+b)d_{2j} + abd_{2j-1}. \end{aligned}$$

By the inductive hypothesis, $p|d_{2j}$. Since $p|a$, we obtain $p|d_{2(j+1)}$. Furthermore, write

$$d_{2(j+1)-1} = ad_{2(j+1)-2} + bd_{2(j+1)-3} = ad_{2j} + bd_{2j-1}.$$

Since $p \nmid bd_{2j-1}$ by the inductive hypothesis and $p|ad_{2j}$, we obtain $p \nmid d_{2(j+1)-1}$. This completes our proof.

iii) Suppose that $k = \gcd(b, d_i) > 1$ for some i. If $i = 2$, then $p|b$, which contradicts that $p|a$ and $\gcd(a,b) = 1$. If $i = 3$, then $q|b$, which contradicts $p^2 = aq + bp$. If $i \geq 4$, write $d_i = ad_{i-1} + bd_{i-2}$. Since $k|bd_{i-2}$ and $\gcd(a,b) = 1$, we get $k|d_{i-1}$ and so, $k|\gcd(b, d_{i-1})$. By induction, we obtain $k|\gcd(b, d_3)$, which has been shown to be impossible.

Next, suppose that $k = \gcd(a, d_{2i-1}) > 1$ for some i. If $i = 2$, then $q|a$, contradicting $r = ap^2 + bq$. If $i \geq 3$,

$$k \mid d_{2i-1},\ d_{2i-1} = ad_{2i-2} + bd_{2i-3}, \text{ and } \gcd(a,b) = 1 \implies k \mid d_{2i-3}.$$

By induction, $k|d_3$; that is, $\gcd(a, d_3) > 1$, which has been shown to be impossible.

iv) The claim holds for $i \leq 4$. For $i \geq 5$, using the recurrence $d_{i+1} = ad_i + bd_{i-1}$, we have

$$\gcd(d_i, d_{i+1}) = \gcd(d_i, bd_{i-1}) \overset{\text{iii)}}{=} \gcd(d_i, d_{i-1}).$$

By induction, we obtain $\gcd(d_i, d_{i+1}) = 1$.

v) The claim holds for $i \leq 2$. For $i \geq 3$, we have

$$\gcd(d_{2i-1}, d_{2i+1}) = \gcd(d_{2i-1}, ad_{2i}) \overset{\text{iii)}}{=} \gcd(d_{2i-1}, d_{2i}) \overset{\text{iv)}}{=} 1.$$

This completes our proof. □

Lemma 2.4. *Suppose that the first* 4 *numbers in* S'_N *are* $p < q < r < p^2$. *If* N *is small recurrent with U(p, q, a, b), the following hold:*

i) $\gcd(a,b) = 1$ *and* $p \nmid a$.
ii) *For all* $d_i \in S'_N$, $\gcd(b, d_i) = 1$.
iii) *For all* $d_i, d_{i+1} \in S'_N$, $\gcd(d_i, d_{i+1}) = 1$.
iv) *Let* $d_i \in S'_N$. *Then* $p|d_i$ *if and only if* $i \equiv 2 \bmod 3$.

Proof. i) We have $\gcd(a,b)$ divides r because $r = aq + bp$. Furthermore, $\gcd(a,b)$ divides p because $p^2 = ar + bq$. Hence, $\gcd(a,b) = 1$.

Since $r = aq + bp$ and r is a prime, $p \nmid a$.

ii) Suppose that $k = \gcd(b, d_i) > 1$ for some i. If $i = 2$, then $k = p$ and $p|b$. It follows from $p^2 = ar + bq$ that $p|a$, which contradicts that $\gcd(a,b) = 1$. If $i = 3$, then $k = q$ and $q|b$. It follows from $r = aq + bp$ that $q|r$, a contradiction. Hence, $i \geq 4$. Write

$$1 < \gcd(b, d_i) = \gcd(b, ad_{i-1} + bd_{i-2}) = \gcd(b, d_{i-1}).$$

By induction, we obtain $\gcd(b, d_3) > 1$, which has been shown to be impossible.

iii) The claim holds for $i \leq 4$. Let $d_i, d_{i+1} \in S'_N$ and $i \geq 5$. Then

$$\gcd(d_i, d_{i+1}) = \gcd(d_i, ad_i + bd_{i-1}) = \gcd(d_i, bd_{i-1}) \overset{\text{ii)}}{=} \gcd(d_i, d_{i-1}).$$

By induction, $\gcd(d_i, d_{i+1}) = 1$.

iv) The claim holds for $i \leq 4$. Suppose that the claim holds for all $i \leq j$ for some $j \geq 5$. We show that it also holds for $i = j+1$ under the assumption that $d_{j+1} \in S'_N$. We have

$$p^2 = ar + bq = a(aq + bp) + bq = (a^2 + b)q + abp.$$

Hence, $p|(a^2 + b)$. Write

$$d_{j+1} = ad_j + bd_{j-1} = a(ad_{j-1} + bd_{j-2}) + bd_{j-1} = (a^2 + b)d_{j-1} + abd_{j-2}.$$

Therefore, $p|d_{j+1}$ if and only if $p|d_{j-2}$. By the inductive hypothesis, we know that $p|d_{j+1}$ if and only if $j + 1 \equiv 2 \bmod 3$. □

The proofs of the next two lemmas are similar to those of Lemmas 2.3 and 2.4. Thus, we move their proofs to the Appendix.

Lemma 2.5. *Suppose that the first* 4 *numbers in* S'_N *are* $p < q < r < pq$*. If* N *is small recurrent with* $U(p, q, a, b)$*, the following hold:*

i) $\gcd(a, b) = 1$ *and* $p \nmid a$.
ii) *For all* $d_i \in S'_N$, $\gcd(b, d_i) = 1$.
iii) *For all* $d_i, d_{i+1} \in S'_N$, $\gcd(d_i, d_{i+1}) = 1$.
iv) *Let* $d_i \in S'_N$*. Then* $p|d_i$ *if and only if* $i \equiv 2 \bmod 3$.
v) *Let* $d_i \in S'_N$*. Then* $q|d_i$ *if and only if* i *is odd.*

Lemma 2.6. *Suppose that the first* 4 *numbers in* S'_N *are* $p < q < r < s$*. If* N *is small recurrent with* $U(p, q, a, b)$*, the following hold:*

i) $\gcd(a, b) = 1$.
ii) *For all* $d_i \in S'_N$ *with* $i \geq 3$, $\gcd(b, d_i) = 1$.
iii) *For* $d_i, d_{i+1} \in S'_N$, $\gcd(d_i, d_{i+1}) = 1$.
iv *For* $d_i \in S'_N$, $\gcd(a, d_i) = 1$.
v) *For* $d_i, d_{i+2} \in S'_N$, $\gcd(d_i, d_{i+2}) = 1$.

3. Small recurrent numbers

We first find all small recurrent numbers with $|S'_N| \geq 4$. Then we check which N is small recurrent out of all N with $|S'_N| \leq 3$ at the end of this section. As we rely heavily on case analysis, we underline possible forms of N throughout our analysis for the ease of later summary.

3.1. **The case** $|S'_N| \geq 4$**.** Let $d_2 = p$ for some prime p. Then d_3 is either p^2 or q for some prime $q > p$. If $d_3 = p^2$, according to Proposition 2.2, we know that $\underline{N = p^k \text{ or } N = p^k q \text{ for some } k \geq 1 \text{ and } q > p^k}$.

Assume, for the rest of this subsection, that $d_3 = q$ for some prime $q > p$. Then $d_4 = pq, p^2, r$ for some prime $r > q$.

3.1.1. *When* $d_4 = pq$. Since S'_N satisfies $U(p,q,a,b)$, we get $pq = aq + bp$. So, $p|a$ and $q|b$. Write $a = pm$ for some $m \in \mathbb{Z}$ and get $b = (1-m)q$. Since $p^2 \nmid N$ and q divides $d_5 = apq + bq$, we can write

$$d_5 \;=\; p^s q^t r_1^{\ell_1} \cdots {r_k}^{\ell_k},$$

where $s \le 1$, $t \ge 1$, and r_i's are primes strictly greater than pq. If some $\ell_i \ge 1$, then $pq < r_i < d_5$ and $r_i \in S'_N$, a contradiction. Hence, $\ell_i = 0$ for all $i \le k$ and $d_5 = p^s q^t$. Since $d_5 > pq$ and $s \le 1$, we know that $t \ge 2$.

a) If $s = 1$, $d_5 = pq^2 > q^2 > d_4$ and $q^2 | d_5$, so $q^2 \in S'_N$, a contradiction.

b) If $s = 0$, $d_5 = q^t$ for some $t \ge 2$. Since d_5 is the next number after d_4 in increasing order, $d_5 = q^2$. Using the linear recurrence, we obtain $q^2 = apq + bq$, so $q = ap + b = p^2 m + (1-m)q$. It follows that $p^2 m = mq$. We arrive at $m = 0$, $a = 0$, and $b = q$. Hence, all elements of S'_N are divisible by either p or q. If N has a prime factor $r \ge \sqrt{N}$, by Proposition 2.1, r is unique. We conclude that $N = pq^k$ or $pq^k r$ for some $k \ge 2$, $p < q$, and $pq^k < r$.

3.1.2. *When* $d_4 = p^2$. The first few divisors of N are $1 < p < q < p^2$. We have $p^2 = aq + bp$, so $p|a$. Write $a = pm$ and get $b = p - mq$. We argue for possible forms of d_5. Let r be the largest prime factor of d_5. If $r > q$, then $r > p^2$ and $d_5 = r$. Otherwise, if $r \le q$, then $d_5 = p^\ell q^k$ for some $\ell, k \ge 0$. Suppose that $k \ge 2$. We get

$$d_5 \;\ge\; q^2 \;>\; pq \;>\; d_4 \text{ and } pq \mid N,$$

a contradiction. Hence, $k \le 1$. If $k = 0$, then

$$d_5 \;=\; p^3 \;>\; pq \;>\; d_4 \text{ and } pq \mid N,$$

another contradiction. Therefore, $k = 1$ and $d_5 = pq$. We conclude that either $d_5 = r$ for some $r > q$ or $d_5 = pq$.

a) If $d_5 = pq$, then $bq + ap^2 = pq$. So, $(p - mq)q + mp^3 = pq$, which gives $mq^2 = mp^3$. Hence, $m = 0$, $a = 0$, and $b = p$. We know that elements of S'_N are divisible by either p or q. If N has a prime factor r' at least $\sqrt{N}$, by Proposition 2.1, r' is unique. Hence, either $N = p^\ell q^k$ or $p^\ell q^k r'$ for some prime $r' > p^\ell q^k$, $\ell \ge 2$, and $k \ge 1$.

Case a.i) $N = p^\ell q^k r'$. We claim that $k = 1$. Indeed, if $k \ge 2$, then

$$q^4 \;<\; q^2 r' \;<\; N \;\Longrightarrow\; q^2 \;<\; \sqrt{N} \;\Longrightarrow\; q^2 \in S'_N.$$

Since $b = p$, we know that $p|d$ for all $d \ge d_4$ and $d \in S'_N$, which contradicts $q^2 \in S'_N$. Hence, $N = p^\ell q r'$ for some $\ell \ge 2$, some prime $r' > p^\ell q$, and $\sqrt{q} < p < q$.

Case a.ii) $N = p^\ell q^k$. As above, $q^2 \notin S'_N$. If $k \ge 2$, then

$$q^2 \;\ge\; \sqrt{N} \;\Longrightarrow\; q^4 \;\ge\; N \;=\; p^\ell q^k \;=\; p^{\ell-2} p^2 q^k \;>\; q^{k+1}.$$

Hence, $k < 3$, which implies that $k = 2$. In this case, $N = p^\ell q^2$ and

$$q^2 \;>\; p^\ell \;>\; q^{\ell/2} \;\Longrightarrow\; \ell \;\le 3.$$

We conclude that one of the following holds:

- $N = p^\ell q$ for some $\ell \ge 2$ and $\sqrt{q} < p < q$,
- $N = p^2 q^2$ for some $p < q < p^2$,

- $\underline{N = p^3q^2 \text{ for some } p^{3/2} < q < p^2}$.

b) $d_5 = r$.

Proposition 3.1. *Suppose that the first* 4 *numbers in* S'_N *are* $p < q < p^2 < r$. *Then* $|S_N| \leq 7$. *As a result,* $|S'_N| \leq 6$.

Proof. Assume that $|S_N| \geq 2i$ for some $i \geq 4$. We obtain a contradiction by showing that $|S_N| \geq 2i + 2$. By Lemma 2.3 item ii), $p \nmid d_{2i-1}$, $p|d_{2i-2}$, and $p \nmid d_{2i-3}$. By Lemma 2.3 item v), $\gcd(d_{2i-1}, d_{2i-3}) = 1$, so $p^2 d_{2i-1} d_{2i-3}$ divides N. Hence, $pd_{2i-3} \in S'_N$.

If $pd_{2i-3} = d_{2i-2}$, then

$$pd_{2i-3} \;=\; ad_{2i-3} + bd_{2i-4} \implies d_{2i-3} \mid bd_{2i-4},$$

which contradicts Lemma 2.3 items iii) and iv).

If $pd_{2i-3} = d_{2i}$, then

$$\begin{aligned} pd_{2i-3} \;=\; ad_{2i-1} + bd_{2i-2} \;&=\; a(ad_{2i-2} + bd_{2i-3}) + bd_{2i-2} \\ &=\; (a^2 + b)d_{2i-2} + abd_{2i-3}. \end{aligned}$$

Therefore, d_{2i-3} divides $a^2 + b$. It is easy to check that for $d_j \in S'_N$, the sequence $d_j \bmod a^2 + b$ is congruent to

$$1, p, q, p^2, abp, abq, abp^2, (ab)^2p, (ab)^2q, (ab)^2p^2, \dots.$$

Hence, we can write

$$d_{2i-3} \;=\; (a^2 + b)\ell + a^k b^k s,$$

for some $\ell \in \mathbb{Z}$, some $k \geq 1$, and some $s \in \{p, q, p^2\}$. Since $d_{2i-3}|(a^2 + b)$, $d_{2i-3}|a^k b^k s$. By Lemma 2.3 item iii), $d_{2i-3}|s$; that is, $d_{2i-3} \leq p^2$. However, $d_{2i-3} \geq d_5 > d_4 = p^2$, a contradiction.

We conclude that $pd_{2i-3} \geq d_{2i+2}$. Since $pd_{2i-3} \in S'_N$, we know that $d_{2i+2} \in S'_N$ and $|S_N| \geq 2i + 2$. □

Proposition 3.2. *Suppose that the first* 4 *numbers in* S'_N *are* $p < q < p^2 < r$. *If* N *is small recurrent, then* $|S'_N| \neq 4, 6$.

Proof. If $|S'_N| = 4$, then (2.2) gives $\tau(N) = 10$ or 11. Note that N has three distinct prime factors p, q, r and the power of p is at least 2. Since $2^3 \cdot 3 > 11$, N cannot have another prime factor besides p, q, r. Write $N = p^a q^b r^c$, for some $a \geq 2, b \geq 1, c \geq 1$. However, neither $(a + 1)(b + 1)(c + 1) = 10$ nor $(a + 1)(b + 1)(c + 1) = 11$ has a solution. Therefore, $|S'_N| \neq 4$. A similar argument gives $|S'_N| \neq 6$. □

By Propositions 3.1 and 3.2, we know that $|S'_N| = 5$; that is, $\tau(N) = 12$ or 13. Using the same reasoning as in the proof of Proposition 3.2, we know that $\tau(N) = 12$ and $\underline{N = p^2qr, \text{ where } p < q < p^2 < r}$.

3.1.3. *When $d_4 = r$ for some $r > q$.* The possible values for d_5 are p^2, pq, s for some prime $s > r$.

a) If $d_5 = p^2$, we can generalize the method by Chentouf.

Proposition 3.3. *If N is small recurrent and the first four numbers of S'_N are $p < q < r < p^2$, then $|S_N| \le 7$. As a result, $|S'_N| \le 6$.*

Proof. Suppose that $|S_N| \ge 8$. We show that $|S_N| \ge 3i+2$ for all $i \in \mathbb{N}$, which is a contradiction. The claim holds for $i = 2$. Assume that $|S_N| \ge 3j + 2$ for some $j \ge 2$. By Lemma 2.4, $p \nmid d_{3j}d_{3j+1}$ and $\gcd(d_{3j}, d_{3j+1}) = 1$. Hence, $p^2 d_{3j}d_{3j+1}$ divides N, which implies that $pd_{3j} \in S'_N$.

If $pd_{3j} = d_{3j+2} = ad_{3j+1} + bd_{3j}$, then d_{3j} divides ad_{3j+1}. By Lemma 2.4, $d_{3j}|a$. Observe that for $d_i \in S'_N$, the sequence $d_i \bmod a$ is

$$1, p, q, bp, bq, b^2p, b^2q, \ldots.$$

Write $d_{3j} = a\ell + b^k s$, for some $\ell \in \mathbb{Z}$, some $k \ge 0$, and some $s \in \{p, q\}$. We see that $d_{3j}|b^k s$ for some $k \ge 0$ and $s \in \{p, q\}$. By Lemma 2.4, $d_{3j} \le q$. However,

$$d_{3j} \ \ge \ d_6 \ > \ d_3 \ = \ q,$$

a contradiction.

If $pd_{3j} > d_{3j+2}$, then $pd_{3j} \ge d_{3(j+1)+2}$ by Lemma 2.4. Therefore, $|S_N| \ge 3(j+1) + 2$. □

Proposition 3.4. *There is no small recurrent N whose the first four numbers of S'_N are $p < q < r < p^2$.*

Proof. By Proposition 3.3, $|S'_N| \in \{4, 5, 6\}$. If $|S'_N| = 4$, then $\tau(N) = 10$ or 11, none of which can be written as a product of at least three integers, each of which is at least 2. This contradicts (2.2) and the fact that N has three distinct prime factors. We arrive at the same conclusion when $|S'_N| = 6$. For $|S'_N| = 5$, we obtain $N = p^2qr$ for some primes $p < q < r < p^2$. However, this poses another contradiction. Observe that $(pq)^2 < p^2qr$, so the divisors in S'_N are $p < q < r < p^2 < pq$. Since $pq = ap^2 + br$, we get $p|b$, which contradicts Lemma 2.4 item ii). □

b) Suppose that $d_5 = pq$.

Proposition 3.5. *There is no small recurrent number N such that the first four numbers of S'_N are $p < q < r < pq$.*

Proof. Assume that $|S'_N| \ge 8$. Since $p^2 \notin S'_N$, p divides N exactly. By Lemma 2.5, d_6 is divisible neither by p nor q. Hence, $d_6 = s$ for some prime $s > pq$. The divisor d_7 is divisble by q; hence, $d_7 = qr$ or q^2. The divisor d_8 is divisible by p but not by q. So, $d_8 = pr$, which gives that d_7 must be q^2 because $d_7 < d_8$. Now $q|d_9$ and $p \nmid d_9 \implies d_9 = qr$. However, that $\gcd(d_8, d_9) = r$ contradicts Lemma 2.5. Therefore, $|S'_N| \in \{4, 5, 6, 7\}$. Using the same argument as in the proof of Proposition 3.4, we know that $|S'_N| \ne 4, 6$ and so, $|S'_N| \in \{5, 7\}$. By the above argument, if $|S'_N| \ge 5$, then d_6 is a prime greater than pq. Hence, N has at least 4 distinct prime factors, so $\tau(N)$ can be written as a product of at least 4 integers greater than 1. Clearly, (2.1) rules out the case $|S'_N| = 5$. If $|S'_N| = 7$, the above argument shows that $q^2|N$; hence, $\tau(N)$ can be written as a product of at least 4 integers greater than 1, one of which is greater than 2. This cannot happen as $\tau(N) \in \{16, 17\}$. □

c) Suppose that $d_5 = s$.

Proposition 3.6. *There is no small recurrent number N such that the first four numbers of S'_N are $p < q < r < s$.*

Proof. Observe that pq and pr are in S'_N. Let $d_j = pv$ be the largest element of S'_N that is divisible by p. Clearly, $v > p$ and $j \geq 7$. By Lemma 2.6, d_j, d_{j-1}, and d_{j-2} are pairwise coprime. Hence, $pvd_{j-1}d_{j-2}$ divides N, so $pd_{j-2} \in S'_N$.

If $pd_{j-2} = d_{j-1}$, then $p|d_{j-1}$ and so, $p|\gcd(d_{j-1}, d_j)$, which contradicts Lemma 2.6 item iii).

If $pd_{j-2} = d_j$, then $d_{j-2} = v$ and $\gcd(d_{j-2}, d_j) = v > 1$, which contradicts Lemma 2.6 item v).

Therefore, we have $pd_{j-2} > d_j$, which, however, contradicts that d_j is the largest element of S'_N that is divisible by p. We conclude that there is no small recurrent number N such that the first four numbers of S'_N are $p < q < r < s$. □

From the above analysis, we arrive at the following proposition.

Proposition 3.7. *If N is small recurrent and $|S'_N| \geq 4$, then N belongs to one of the following forms.*

(S1) *$N = p^k$ or $N = p^k q$ for some $k \geq 1$ and $q > p^k$.*
(S2) *$N = pq^k$ or $pq^k r$ for some $k \geq 2$, $p < q$, and $pq^k < r$.*
(S3) *$N = p^k qr$ for some $k \geq 2$, some prime $r > p^k q$, and $\sqrt{q} < p < q$.*
(S4) *$N = p^k q$ for some $k \geq 2$ and $\sqrt{q} < p < q$.*
(S5) *$N = p^2 q^2$ for some $p < q < p^2$.*
(S6) *$N = p^3 q^2$ for some $p^{3/2} < q < p^2$.*
(S7) *$N = p^2 qr$, where the first four numbers in S'_N are $p < q < p^2 < r$.*

These forms together establish the necessary condition for a small recurrent N to have $|S'_N| \geq 4$. We now refine each form (if necessary) to obtain a necessary and sufficient condition.

(S1)
- If $N = p^k$, then N is small recurrent with $|S'_N| \geq 4$ if $k \geq 9$. In this case, $S'_N = \{p, p^2, p^3, \ldots, p^{\lfloor (k-1)/2 \rfloor}\}$ satisfies $U(p, p^2, p, 0)$.
- If $N = p^k q$ for some $k \geq 1$ and $q > p^k$, then N is small recurrent with $|S'_N| \geq 4$ if $k \geq 4$. In this case, $S'_N = \{p, p^2, p^3, \ldots, p^k\}$ satisfies $U(p, p^2, p, 0)$.

(S2)
- If $N = pq^k$ for some $k \geq 2$ and $p < q$, then N is small recurrent with $|S'_N| \geq 4$ if $\sqrt{N} = \sqrt{pq^k} > q^2$. Hence, $N = pq^k$ for some $k \geq 4$ and $p < q$. In this case, $S'_N = \{p, q, pq, q^2, \ldots\}$ satisfies $U(p, q, 0, q)$.
- If $N = pq^k r$ for some $k \geq 2$, $p < q$, and $r > pq^k$, then N is small recurrent with $|S'_N| \geq 4$. In this case, $S'_N = \{p, q, pq, q^2, \ldots, pq^{k-1}, q^k, pq^k\}$ satisfies $U(p, q, 0, q)$.

(S3) If N belongs to (S3), then $S'_N = \{p, q, p^2, pq, \dots, p^k, p^{k-1}q, p^kq\}$. Since $p^2 = aq + bp$, we know that $p|a$. Write $a = pm$ for some $m \in \mathbb{Z}$ and get $b = p - mq$. Hence,

$$pq \ = \ ap^2 + bq \ = \ p^3m + (p - mq)q \implies mq^2 = p^3m.$$

Therefore, $(m, a, b) = (0, 0, p)$. However, the largest element in S'_N, p^kq, is not equal to $p \cdot p^k$. We conclude that form (S3) does not give a small recurrent number.

(S4) If $N = p^kq$ for some $k \geq 2$ and $\sqrt{q} < p < q$, then the nontrivial divisors of N in increasing order is $p < q < p^2 < pq < \cdots$. In order that $|S'_N| \geq 4$, we need $(pq)^2 < p^kq$, so $q < p^{k-2}$. Hence, $k \geq 4$. In this case, $S'_N = \{p, q, p^2, pq, \dots\}$ satisfies $U(p, q, 0, p)$.

(S5) If $N = p^2q^2$ for some $p < q < p^2$, then $\tau(N) = 9$. However, if $|S'_N| \geq 4$, then $\tau(N) \geq 10$ by (2.2). We conclude that form (S5) does not give a small recurrent number.

(S6) If $N = p^3q^2$ for some $p^{3/2} < q < p^2$, then $S'_N = \{p, q, p^2, pq, p^3\}$ satisfies $U(p, q, 0, p)$.

(S7) Let N have form (S7). Since the first four numbers of S'_N are $p < q < p^2 < r$ and $\tau(N) = 12$, we know that the fifth number in S'_N must be pq. That $p < q < p^2 < r < pq$ satisfies some $U(p, q, a, b)$ gives $a = \frac{p(pq-r)}{q^2-p^3}$, $b = \frac{rq-p^4}{q^2-p^3}$, and $r = pq - \sqrt{(q^2 - p^3)(p^2 - q)}$. We conclude that a number of form (S7) is small recurrent if and only if $p < q < p^2 < r < pq$, $(q^2 - p^3)|(pq - r)$, $(q^2 - p^3)|(rq - p^4)$, and $r = pq - \sqrt{(q^2 - p^3)(p^2 - q)}$. An example is $(p, q, r) = (2, 3, 5)$. We do not know if $(2, 3, 5)$ is the only set of primes that satisfy all these conditions or not.

From the above analysis, we obtain the proposition, which is a refinement of Proposition 3.7.

Proposition 3.8. *Let p, q, r denote prime numbers and k be some positive integer. A positive integer N is small recurrent with $|S'_N| \geq 4$ if and only if N belongs to one of the following forms.*

(1) *$N = p^k$ for some $k \geq 9$. In this case, $S'_N = \{p, p^2, p^3, \dots, p^{\lfloor (k-1)/2 \rfloor}\}$ satisfies $U(p, p^2, p, 0)$.*
(2) *$N = p^kq$ for some $k \geq 4$ and $q > p^k$. In this case, $S'_N = \{p, p^2, p^3, \dots, p^k\}$ satisfies $U(p, p^2, p, 0)$.*
(3) *$N = pq^k$ for some $k \geq 4$ and $p < q$. In this case, $S'_N = \{p, q, pq, q^2, \dots\}$ satisfies $U(p, q, 0, q)$.*
(4) *$N = pq^kr$ for some $k \geq 2$, $p < q$, and $r > pq^k$. In this case, $S'_N = \{p, q, pq, q^2, \dots, pq^{k-1}, q^k, pq^k\}$ satisfies $U(p, q, 0, q)$.*
(5) *$N = p^kq$ for some $k \geq 4$ and $\sqrt{q} < p < q$. In this case, $S'_N = \{p, q, p^2, pq, \dots\}$ satisfies $U(p, q, 0, p)$.*
(6) *$N = p^3q^2$ for some $p^{3/2} < q < p^2$. In this case, $S'_N = \{p, q, p^2, pq, p^3\}$ satisfies $U(p, q, 0, p)$.*

(7) $N = p^2qr$, *where* $p < q < p^2 < r < pq$, $(q^2 - p^3)|(pq - r)$, $(q^2 - p^3)|(rq - p^4)$, *and* $r = pq - \sqrt{(q^2 - p^3)(p^2 - q)}$. *In this case,* $S'_N = \{p, q, p^2, r, pq\}$ *satisfies* $U\left(p, q, \frac{p(pq-r)}{q^2-p^3}, \frac{rq-p^4}{q^2-p^3}\right)$.

3.2. **The case** $|S'_N| \leq 3$**.** If $|S'_N| \leq 3$, then $\tau(N) \leq 9$. We use the classifications of those N from the introduction to obtain the following proposition.

Proposition 3.9. *Let* p, q, r *denote prime numbers and* k *be some positive integer. A positive integer* $N > 1$ *is small recurrent with* $|S'_N| \leq 3$ *if and only if* N *belongs to one of the following forms.*

(1) $N = p^k$ *for some* $k \leq 8$. *In this case,* $S'_N = \{p, p^2, \ldots, p^{\lfloor (k-1)/2 \rfloor}\}$ *satisfies* $U(p, p^2, p, 0)$.
(2) $N = pq$ *for some* $p < q$. *In this case,* $S'_N = \{p\}$.
(3) $N = pq^2$ *for some* $p < q$. *In this case,* $S'_N = \{p, q\}$.
(4) $N = p^2q$ *for some* $p < q$. *If* $q < p^2$, *then* $S'_N = \{p, q\}$. *If* $q > p^2$, *then* $S'_N = \{p, p^2\}$.
(5) $N = pq^3$ *for some* $p < q$. *In this case,* $S'_N = \{p, q, pq\}$.
(6) $N = p^3q$ *for some* $p < q$. *If* $p < q < p^2$, *then* $S'_N = \{p, q, p^2\}$. *If* $p^3 < q$, *then* $S'_N = \{p, p^2, p^3\}$. *(The case* $p^2 < q < p^3$ *is eliminated because the three elements in* S'_N *would be* $p < p^2 < q$. *However, there is no integral solution* (a, b) *to* $q = ap^2 + bp$.*)*
(7) $N = p^2q^2$ *for some* $p < q$. *If* $p < q < p^2$, *then* $S'_N = \{p, q, p^2\}$. *(The case* $p^2 < q$ *is eliminated due to the same reason as in item* (6).*)*
(8) $N = pqr$ *for some* $p < q < r$. *If* $r < pq$ *and there is an integral solution* (a, b) *to* $r = aq + bp$, *then* $S'_N = \{p, q, r\}$. *If* $r > pq$, *then* $S'_N = \{p, q, pq\}$.

Combining Propositions 3.8 and 3.9, we obtain Theorem 1.2.

4. Large recurrent numbers

Now we characterize all positive integers N whose L'_N satisfies a linear recurrence of order at most two. By a simple observation, instead of working directly with divisors in L'_N, we work with divisors in S'_N. Again, the set of divisors of a positive integer N is $1 = d_1 < d_2 < \cdots < d_{\tau(N)}$ and the set $S'_N = \{d_2, d_3, \ldots\}$.

4.1. **The case** $|L'_N| \geq 4$**.** Note that $|L'_N| \geq 4$ is equivalent to $|S'_N| \geq 4$.

Lemma 4.1. *For any* $d \in L'_N$, *we have* $N/d \in S'_N$. *If* N *is large recurrent with* $|L'_N| \geq 4$, *then*

$$ad_{i+2} + bd_{i+1} \ = \ \frac{d_{i+1}d_{i+2}}{d_i}, \forall d_i, d_{i+1}, d_{i+2} \in S'_N. \quad (4.1)$$

In particular, we have

$$ad_4 + bd_3 \ = \ \frac{d_3d_4}{d_2}. \quad (4.2)$$

Proof. If $d \in L'_N$, then $\sqrt{N} < d < N$. Then $1 < N/d < \sqrt{N}$, so $N/d \in S'_N$. Let

$$d'_i \ := \ d_{\tau(n)+1-i} \ = \ \frac{N}{d_i} \in L'_N, \forall d_i \in S'_N.$$

If N is large recurrent, then we have

$$d'_i = ad'_{i+1} + bd'_{i+2}, \forall d'_i, d'_{i+1}, d'_{i+2} \in L'_N.$$

Therefore,

$$\frac{N}{d_i} = a\frac{N}{d_{i+1}} + b\frac{N}{d_{i+2}}, \forall d_i, d_{i+1}, d_{i+2} \in S'_N,$$

which gives

$$ad_{i+2} + bd_{i+1} = \frac{d_{i+1}d_{i+2}}{d_i}, \forall d_i, d_{i+1}, d_{i+2} \in S'_N.$$

This completes our proof. □

Since d_2 is a prime number p and d_3 is either p^2 or a prime number $q > p$, we consider two cases.

4.1.1. *When* $d_3 = p^2$. Then d_4 is either p^3 or a prime number $q > p^2$.

a) If $d_4 = p^3$, then (4.2) implies that $p^2 = ap + b$.

Claim 4.2. *If* $p \neq a$, *then* $S'_N = \{p, p^2, \dots, p^k\}$ *for some* $k \geq 4$.

Proof. We need to show that if $d_i \in S'_N$, then $d_i = p^{i-1}$. Base case: the claim holds for $i \leq 4$. Suppose that there exists a $j \geq 4$ such that $d_i = p^{i-1}$ for all $i \leq j$. Using (4.1), we have

$$ad_{j+1} + bp^{j-1} = ad_{j+1} + bd_j = \frac{d_{j+1}d_j}{d_{j-1}} = \frac{d_{j+1}p^{j-1}}{p^{j-2}} = pd_{j+1},$$

which, combined with $p^2 = ap + b$, gives

$$(p-a)(d_{j+1} - p^j) = 0.$$

Since $p \neq a$, we obtain $d_{j+1} = p^j$, as desired. □

By Proposition 2.2, we know that when $p \neq a$, either $N = p^k$ or $N = p^k q$ for some $k \geq 1$ and some prime $q > p^k$.

Now suppose that $p = a$. Then $b = 0$. We can write elements in L'_N as $\{g_1, g_2, pg_2, p^2g_2, \dots, p^kg_2\}$ for some $k \geq 2$. Correspondingly, the set S'_N is $\{p, p^2, \dots, p^k, p^{k+1}, p^{k+1}g_2/g_1\}$. If $p^{k+1}g_2/g_1$ is a power of p, then we have the same conclusion about N as when $p \neq a$. If $p^{k+1}g_2/g_1$ is not a power of p, then

$$\frac{p^{k+1}g_2}{g_1} = q, \text{ for some prime } q > p \implies g_1 = p^{k+1}\frac{g_2}{q}.$$

Note that $g_2/q \in \mathbb{N}$. Furthermore, we claim that $g_2/q = p$. Indeed, since $1 < g_2/q < g_1$, we know that $g_2/q \in S'_N$. If $g_2/q = q$, then

$$pq < p^{k+1}\frac{g_2}{q} = g_1,$$

which implies that $pq \in S'_N$, a contradiction. If $g_2/q = p^j$ for some $j > 1$, then

$$p^{k+2} < p^{k+1+j} = p^{k+1}\frac{g_2}{q} = g_1,$$

which implies that $p^{k+2} \in S'_N$, another contradiction. Therefore, $g_2/q = p$, and we obtain $g_1 = p^{k+2}$ and $g_2 = pq$. We conclude that $\underline{N = p^{k+2}q \text{ for some } k \geq 2}$ $\underline{\text{and } p^{k+1} < q < p^{k+2}}$.

b) If $d_4 = q$, we claim that $a \neq p$. Suppose otherwise. Applying (4.2) to d_2, d_3, and d_4 gives $aq + bp^2 = pq$. Hence, $a = p$ implies that $b = 0$. However, applying (4.1) to d_3, d_4, and d_5 gives $(p^3 - q)d_5 = 0$, a contradiction. Therefore, $a \neq p$. By (4.2), we have

$$d_4 \ = \ \frac{bp^2}{p-a} \ = \ q \implies q|b \implies b = kq \text{ for some } k \in \mathbb{Z}\backslash\{0\}.$$

Hence, $a = p - kp^2$. By (4.1) applied to d_3, d_4, and d_5,

$$d_5 \ = \ \frac{kp^2q^2}{q - p^3 + kp^4}, \tag{4.3}$$

which implies that $p^2|d_5$ since $\gcd(p^2, q - p^3 + kp^4) = 1$. Hence, $d_5 = p^3$, and (4.3) gives $k = \frac{p(p^3-q)}{p^5-q^2}$.

Case b.i) If S'_N has exactly four elements, which are p, p^2, q, p^3, then $\tau(N) = 10$, which implies that $N = p^4q$. Hence, $L'_N = \{pq, p^4, p^2q, p^3q\}$ with $a = pq\frac{p^2-q}{p^5-q^2}$ and $b = pq\frac{p^3-q}{p^5-q^2}$. We conclude that $\underline{N = p^4q \text{ with } p^2 < q < p^3,}$ $\underline{(p^5 - q^2)|(p^2 - q), \text{ and } (p^5 - q^2)|(p^3 - q)}$.

Case b.ii) If $|S'_N| > 4$, then (4.1) gives

$$d_6 \ = \ \frac{bp^3q}{p^3 - aq} \implies q|d_6 \implies d_6 \ = \ pq.$$

However, since $(a, b) = (p(1 - kp), kq)$, we have

$$pq \ = \ d_6 \ = \ \frac{bp^3q}{p^3 - aq} \ = \ \frac{kp^2q^2}{p^2 - q(1 - kp)},$$

which gives $p^2 = q$, a contradiction.

We summarize our result when $d_3 = p^2$.

Proposition 4.3. *A number N is large recurrent with $|L'_N| \geq 4$ and $(d_2, d_3) = (p, p^2)$ for some prime p if and only if N belongs to one of the following forms.*

(1) *$N = p^k$ for some $k \geq 9$. In this case, $L'_N = \{p^{\lceil (k-1)/2 \rceil + 1}, p^{\lceil (k-1)/2 \rceil + 2}, \ldots, p^{k-1}\}$ satisfies $U(p^{\lceil (k-1)/2 \rceil + 1}, p^{\lceil (k-1)/2 \rceil + 2}, p, 0)$.*

(2) *$N = p^kq$ for some $k \geq 4$ and $q > p^k$. In this case, $L'_N = \{q, pq, p^2q, \ldots, p^{k-1}q\}$ satisfies $U(q, pq, p, 0)$.*

(3) *$N = p^kq$ for some $k \geq 4$ and $p^{k-1} < q < p^k$. Then*

$$L'_N \ = \ \{p^k, pq, p^2q, \ldots, p^{k-1}q\}$$

satisfies $U(p^k, pq, p, 0)$.

(4) *$N = p^4q$ with $p^2 < q < p^3$, $(p^5 - q^2)|(p^2 - q)$, and $(p^5 - q^2)|(p^3 - q)$. In this case, $L'_N = \{pq, p^4, p^2q, p^3q\}$.*

4.1.2. *When $d_3 = q$.* By (4.1),

$$p(ad_4 + bq) \;=\; d_4 q \implies p|d_4.$$

Write $d_4 = kp$ for some integer k. Since $d_2 = p$ and $d_3 = q$, d_4 must be either p^2 or pq.

a) If $d_4 = p^2$, (4.1) gives $ap^2 + bq = pq$. Hence, $q|a$ and $p|b$. Write $a = mq$ and $b = np$ for some integers m, n to get $mp + n = 1$. By (4.1), we see that

$$d_5 \;=\; \frac{bp^2q}{p^2 - aq} \implies q|d_5 \implies d_5 = pq.$$

Therefore,

$$\frac{bp^2q}{p^2 - aq} \;=\; \frac{np^3q}{p^2 - mq^2} \;=\; \frac{(1-mp)p^3q}{p^2 - mq^2} \;=\; pq \implies m(p^3 - q^2) = 0,$$

which gives $m = 0$ and so, $(a, b) = (0, p)$. By (4.1), $d_{i+2} = pd_i$ for all $d_i, d_{i+2} \in S'_N$ and

$$S'_N \;=\; \{p, q, p^2, pq, \ldots\}.$$

If $|S'_N| = 4$, then $\tau(N) = 10$ and $N = p^4q$ for $p < q < p^2$. Suppose that $|S'_N| \geq 5$, then $p^3 \in S'_N$.

Case a.i) If $q^2|N$, let $k \geq 2$ and $\ell \geq 3$ be the largest power such that $q^k|N$ and $p^k|N$, respectively. Since $q^2 \notin S'_N$, we know that

$$q^4 \;\geq\; N \;\geq\; p^3q^k \;>\; q^{k+3/2} \implies k \;<\; 5/2.$$

It follows that $k = 2$. That $q^2 < p^2q$ implies that

$$(p^2q)^2 \;>\; N \;\geq\; p^\ell q^2 \implies 3 \;\geq\; \ell \;\geq\; 3.$$

Hence, $\ell = 3$. If N does not have any other prime divisors besides p and q, then $N = p^3q^2$ for $p < q < p^2$. If N has a prime divisor $r \neq p, q$, then $r > \sqrt{N}$. So, r must be the unique prime divisor different from p and q. We have $N = p^3q^2r$ for $p < q < p^2$ and $r > p^3q^2$. Then $q^2 \in S'_N$, a contradiction.

Case a.ii) If $q^2 \nmid N$ and N has no prime divisors other than p and q, then $N = p^kq$ some for $k \geq 2$ and $p < q < p^2$.

Case a.iii) If $q^2 \nmid N$ and there exists a prime divisor r other than p or q, then $r > \sqrt{N}$ and r is the unique prime different from p and q. Therefore, $N = p^kqr$ for some $k \geq 2$ and $p < q < p^2 < p^kq < r$. Note that the two largest elements in S'_N are $p^{k-1}q$ and p^kq. Let d be the third largest divisor in S'_N. The relation $d_{i+2} = pd_i$ for all $d_i, d_{i+2} \in S'_N$ gives that $dp = p^kq$ and so, $d = p^{k-1}q$, which contradicts that $p^{k-1}q$ is the second largest in S'_N.

b) If $d_4 = pq$, then $p^2 \nmid N$ since $p^2 < pq$. By (4.2),

$$ap \;=\; q - b. \tag{4.4}$$

We see that d_5 is equal to q^2 or r, for some prime $r > pq$.

Case b.i) If $d_5 = q^2$, then (4.1) gives

$$bp \;=\; (p - a)q. \tag{4.5}$$

From (4.4) and (4.5), we obtain $a(p^2-q)=0$, so $(a,b)=(0,q)$. By (4.1), $d_{i+2}=qd_i$ for all $d_i,d_{i+2}\in S'_N$. Using [1, Proposition 5], we conclude that $N=pq^k$ or $N=pq^kr$ for some $k\geq 2$ and $p<q<pq^k<r$.

Case b.ii) If $d_5=r$, then we claim that $|S'_N|>4$. If not, $|S'_N|=4$ implies that $\tau(N)=10$, which contradicts that N has three distinct prime divisors. By (4.1), we see that

$$pq(ad_6+br)\ =\ d_6r,$$

so $pq|d_6$. So, $d_6\in\{pq^2,pqr\}$. If $d_6=pq^2$, then $q^2<d_6$, but q^2 does not appear before d_6 in S'_N, a contradiction. If $d_6=pqr$, then $pr<d_6$, but pr does not appear before d_6 in S'_N, again a contradiction.

Proposition 4.4. *A number N is large recurrent with $|L'_N|\geq 4$ and $(d_2,d_3)=(p,q)$ for some primes $p<q$ if and only if N belongs to one of the following forms.*

(1) *$N=p^3q^2$ for $p<q<p^2$. In this case, $L'_N=\{q^2,p^2q,pq^2,p^3q,p^2q^2\}$ satisfies $U(q^2,p^2q,0,p)$.*

(2) *$N=p^kq$ some for $k\geq 4$ and $p<q<p^2$. In this case,*

$$L'_N\ =\ \begin{cases}\{p^{k/2+1},p^{k/2}q,p^{k/2+2},\dots,p^{k-1}q\} & \text{if } 2|k,\\ \{p^{(k-1)/2}q,p^{(k+3)/2},p^{(k+1)/2}q,\dots,p^{k-1}q\} & \text{if } 2\nmid k.\end{cases}$$

Observe that L'_N satisfies $U(p^{k/2+1},p^{k/2}q,0,p)$ and $U(p^{(k-1)/2}q,p^{(k+3)/2},0,p)$ for even and odd k, respectively.

(3) *$N=pq^k$ for some $k\geq 4$ and $p<q$. In this case,*

$$L'_N\ =\ \begin{cases}\{pq^{\frac{k}{2}},q^{\frac{k}{2}+1},\dots,q^k\} & \text{if } 2|k,\\ \{q^{\frac{k+1}{2}},pq^{\frac{k+1}{2}},\dots,q^k\} & \text{if } 2\nmid k.\end{cases}$$

Observe that L'_N satisfies $U(pq^{k/2},q^{k/2+1},0,q)$ and $U(q^{(k+1)/2},pq^{(k+1)/2},0,q)$ for even and odd k, respectively.

(4) *$N=pq^kr$ for some $k\geq 2$ and $p<q<pq^k<r$. In this case, $L'_N=\{r,pr,qr,pqr,q^2r,\dots,q^kr\}$ satisfies $U(r,pr,0,q)$.*

Combining Propositions 4.3 and 4.4, we obtain the following.

Proposition 4.5. *A number N is large recurrent with $|L'_N|\geq 4$ if and only if N belongs to one of the following forms.*

(1) *$N=p^k$ for some $k\geq 9$. In this case, $L'_N=\{p^{\lceil(k-1)/2\rceil+1},p^{\lceil(k-1)/2\rceil+2},\dots,p^{k-1}\}$ satisfies $U(p^{\lceil(k-1)/2\rceil+1},p^{\lceil(k-1)/2\rceil+2},p,0)$.*

(2) *$N=p^kq$ for some $k\geq 4$ and $q>p^k$. In this case, $L'_N=\{q,pq,p^2q,\dots,p^{k-1}q\}$ satisfies $U(q,pq,p,0)$.*

(3) *$N=p^kq$ for some $k\geq 4$ and $p^{k-1}<q<p^k$. Then*

$$L'_N\ =\ \{p^k,pq,p^2q,\dots,p^{k-1}q\}$$

satisfies $U(p^k,pq,p,0)$.

(4) *$N=p^kq$ some for $k\geq 4$ and $p<q<p^2$. In this case,*

$$L'_N\ =\ \begin{cases}\{p^{k/2+1},p^{k/2}q,p^{k/2+2},\dots,p^{k-1}q\} & \text{if } 2|k,\\ \{p^{(k-1)/2}q,p^{(k+3)/2},p^{(k+1)/2}q,\dots,p^{k-1}q\} & \text{if } 2\nmid k.\end{cases}$$

Observe that L'_N *satisfies* $U(p^{k/2+1}, p^{k/2}q, 0, p)$ *and* $U(p^{(k-1)/2}q, p^{(k+3)/2}, 0, p)$ *for even and odd* k*, respectively.*

(5) $N = p^4q$ *with* $p^2 < q < p^3$, $(p^5 - q^2)|(p^2 - q)$, *and* $(p^5 - q^2)|(p^3 - q)$. *In this case,* $L'_N = \{pq, p^4, p^2q, p^3q\}$.

(6) $N = p^3q^2$ *for* $p < q < p^2$. *In this case,* $L'_N = \{q^2, p^2q, pq^2, p^3q, p^2q^2\}$ *satisfies* $U(q^2, p^2q, 0, p)$.

(7) $N = pq^k$ *for some* $k \geq 4$ *and* $p < q$. *In this case,*

$$L'_N = \begin{cases} \{pq^{\frac{k}{2}}, q^{\frac{k}{2}+1}, \ldots, q^k\} & \text{if } 2|k, \\ \{q^{\frac{k+1}{2}}, pq^{\frac{k+1}{2}}, \ldots, q^k\} & \text{if } 2 \nmid k. \end{cases}$$

Observe that L'_N *satisfies* $U(pq^{k/2}, q^{k/2+1}, 0, q)$ *and* $U(q^{(k+1)/2}, pq^{(k+1)/2}, 0, q)$ *for even and odd* k*, respectively.*

(8) $N = pq^kr$ *for some* $k \geq 2$ *and* $p < q < pq^k < r$. *In this case,* $L'_N = \{r, pr, qr, pqr, q^2r, \ldots, q^kr\}$ *satisfies* $U(r, pr, 0, q)$.

4.2. **The case** $|L'_N| \leq 3$**.** If $|L'_N| \leq 3$, then $\tau(N) \leq 9$. We use the classifications of those N from the introduction to obtain the following proposition

Proposition 4.6. *Let* p, q, r *denote prime numbers and* k *be some positive integer. A positive integer* $N > 1$ *is small recurrent with* $|L'_N| \leq 3$ *if and only if* N *belongs to one of the following forms.*

(1) $N = p^k$ *for some* $k \leq 8$. *In this case,* $L'_N = \{p^{\lceil (k-1)/2 \rceil + 1}, \ldots, p^{k-1}\}$ *satisfies* $U(p^{\lceil (k-1)/2 \rceil + 1}, p^{\lceil (k-1)/2 \rceil + 2}, p, 0)$.

(2) $N = pq$ *for some* $p < q$. *In this case,* $L'_N = \{q\}$.

(3) $N = pq^2$ *for some* $p < q$. *In this case,* $L'_N = \{pq, q^2\}$.

(4) $N = p^2q$ *for some* $p < q$. *If* $q < p^2$, *then* $L'_N = \{p^2, pq\}$. *If* $q > p^2$, *then* $L'_N = \{q, pq\}$.

(5) $N = pq^3$ *for some* $p < q$. *In this case,* $L'_N = \{q^2, pq^2, q^3\}$.

(6) $N = p^3q$ *for some* $p < q$. *If* $p < q < p^2$, *then* $L'_N = \{pq, p^3, p^2q\}$. *If* $p^3 < q$, *then* $L'_N = \{q, pq, p^2q\}$. *If* $p^2 < q < p^3$, *then* $L'_N = \{p^3, pq, p^2q\}$.

(7) $N = p^2q^2$ *for some* $p < q$. *If* $p < q < p^2$, *then* $L'_N = \{q^2, p^2q, pq^2\}$. *The case* $p^2 < q$ *is impossible as it gives* $L'_N = \{p^2q, pq^2, q^2\}$ *and there is no integral solution* (a, b) *to* $apq^2 + bp^2q = q^2$.

(8) $N = pqr$ *for some* $p < q < r$. *If* $r > pq$, *then* $L'_N = \{r, pr, qr\}$. *The case* $r < pq$ *is impossible as it gives* $L'_N = \{pq, pr, qr\}$ *and there is no integral solution* (a, b) *to* $apr + bpq = qr$.

Combining Propositions 4.5 and 4.6, we obtain Theorem 1.4.

5. Appendix

Proof of Lemma 2.5. i) Since $r = aq + bp$ and $pq = ar + bq$, we know that $\gcd(a, b)|r$ amd $\gcd(a, b)|pq$, respectively. Hence, $\gcd(a, b) = 1$.

Since $r = aq + bp$ and r is a prime, $p \nmid a$.

ii) Suppose that $k = \gcd(b, d_i) > 1$ for some $d_i \in S'_N$. If $d_i = d_2$, then $p|b$. Since $pq = ar + bq$, we get $p|a$, which contradicts $\gcd(a, b) = 1$. If $d_i = d_3$, then $q|b$. Since

$r = aq + bp$, we get $q|r$, a contradiction. If $d_i > d_3$, then write

$$\gcd(b, d_i) \ = \ \gcd(b, ad_{i-1} + bd_{i-2}) \ \stackrel{\text{i)}}{=} \ \gcd(b, d_{i-1}),$$

which, by induction, gives $1 < \gcd(b, d_i) = \gcd(b, d_3)$, which has been shown to be impossible.

iii) The claim holds for $i \leq 4$. Let $d_i, d_{i+1} \in S'_N$ for some $i \geq 5$. We have

$$\gcd(d_i, d_{i+1}) \ = \ \gcd(d_i, ad_i + bd_{i-1}) \ \stackrel{\text{ii)}}{=} \ \gcd(d_i, d_{i-1}).$$

By induction, we obtain $\gcd(d_i, d_{i+1}) = 1$.

iv) The claim holds for $i \leq 5$. Assume that it holds for all $i \leq j$ for some $j \geq 5$. We show that it holds for $i = j + 1$. We have

$$pq \ = \ ar + bq \ = \ a(aq + bp) + bq \ = \ (a^2 + b)q + abp.$$

Hence, $p|(a^2 + b)$. Write

$$d_{j+1} \ = \ ad_j + bd_{j-1} \ = \ a(ad_{j-1} + bd_{j-2}) + bd_{j-1} \ = \ (a^2 + b)d_{j-1} + abd_{j-2}.$$

Since $p|(a^2 + b)$ and $\gcd(p, ab) = 1$, we know that $p|d_{j+1}$ if and only if $p|d_{j-2}$. By the inductive hypothesis, $p|d_{j-2}$ if and only if $j - 2 \equiv 2 \bmod 3$, or equivalently, $j + 1 \equiv 2 \bmod 3$. By induction, we have the desired conclusion.

v) The claim holds for $i \leq 5$. Assume that it holds for all $i \leq j$ for some $j \geq 5$. We show that it holds for $i = j + 1$. That $pq = ar + bq$ implies that $q|a$. Write

$$d_{j+1} \ = \ ad_j + bd_{j-1}.$$

By ii), $q|d_{j+1}$ if and only if $q|d_{j-1}$. By the inductive hypothesis, $q|d_{j-1}$ if and only if $j + 1 \equiv 1 \bmod 2$. This completes our proof. □

Proof of Lemma 2.6. i) Same as the proof of Lemma 2.5 item i).

ii) Suppose, for a contradiction, that $\gcd(b, d_i) > 1$ for some $i \geq 3$. If $i = 3$, then $q|b$. We have $r = aq + bp$. Since $q|b$, we get $q|r$, a contradiction. If $i \geq 4$, write

$$\gcd(b, d_i) \ = \ \gcd(b, ad_{i-1} + bd_{i-2}) \ = \ \gcd(b, d_{i-1}).$$

By induction, $1 < \gcd(b, d_i) = \gcd(b, d_3)$, which has been shown to be impossible.

iii) The claim holds for $i \leq 4$. Pick $i \geq 5$. We have

$$\gcd(d_i, d_{i+1}) \ = \ \gcd(d_i, ad_i + bd_{i-1}) \ = \ \gcd(d_i, bd_{i-1}) \ \stackrel{\text{ii)}}{=} \ \gcd(d_i, d_{i-1}).$$

By induction, we obtain $\gcd(d_i, d_{i+1}) = \gcd(d_4, d_5) = 1$.

iv) Assume that $\gcd(a, d_i) > 1$ for some $i \geq 2$. If $i = 2$, then $p|a$, which contradicts the primality of r and the linear recurrence $r = aq + bp$. If $i = 3$, then $q|b$, which contradicts the primality of s and the linear recurrence $s = ar + bq$. Assume that $i \geq 4$. Write

$$\gcd(a, d_i) \ = \ \gcd(a, ad_{i-1} + bd_{i-2}) \ \stackrel{\text{i)}}{=} \ \gcd(a, d_{i-2}).$$

By induction, either $1 < \gcd(a, d_i) = \gcd(a, d_2)$ or $1 < \gcd(a, d_i) = \gcd(a, d_3)$, neither of which is possible.

v) The claims holds for $i \leq 3$. Pick $i \geq 4$ and suppose that $k = \gcd(d_i, d_{i+2}) > 1$. Since $d_{i+2} = ad_{i+1} + bd_i$, k divides ad_{i+1}. By iii), $\gcd(k, d_{i+1}) = 1$, so $k|a$. However, $\gcd(a, d_i) > 1$ contradicts iv). □

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