v-PALINDROMES: AN ANALOGY TO THE PALINDROMES

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Abstract. Around the year 2007, one of the authors, Tsai, accidentally discovered a property of the number 198 he saw on the license plate of a car. Namely, if we take 198 and its reversal 891, which have prime factorizations $198 = 2 \cdot 3^2 \cdot 11$ and $891 = 3^4 \cdot 11$ respectively, and sum the numbers appearing in each factorization getting $2 + 3 + 2 + 11 = 18$ and $3 + 4 + 11 = 18$, both sums are 18. Such numbers were later named v-palindromes because they can be viewed as an analogy to the usual palindromes. In this article, we introduce the concept of a v-palindrome in base b and prove their existence for infinitely many bases. We also exhibit infinite families of v-palindromes in bases $p+1$ and p^2+1, for each odd prime p. Finally, we collect some conjectures and problems involving v-palindromes.

1. Introduction

If I (D. Tsai) recall correctly, it was in the year 2007 when I was 15 years old. My mother and younger brother were in a video rental shop near our home in Taipei and my father and I were waiting outside the shop, standing beside our parked car. I was a bit bored and glanced at the license plate of our car, which was 0198-QB. For no clear reason, I took the number 198 and did the following. I factorized $198 = 2 \cdot 3^2 \cdot 11$, reversed the digits of 198, and factorized $891 = 3^4 \cdot 11$. Then, I summed the numbers appearing in each factorization: $2 + 3 + 2 + 11 = 18$ and $3 + 4 + 11 = 18$, respectively. Surprisingly to me, they are equal! We also illustrate this pictorially in Figure (1). Afterwards I spent some time trying to prove there are infinitely many such numbers (we define them rigorously in Subsection 1.1), but could not show it.

In October 2018, I published a one-and-a-half page note [23] in the Sūgaku Seminar magazine, which is sort of like the American Mathematical Monthly of Japan, defining v-palindromes and showing their

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$$\begin{array}{ccccc} 198 & = & 2\cdot 3^2\cdot 11 & \longmapsto & 2+(3+2)+11 \\ \uparrow & & & & \parallel \\ \text{digit reversal} & & & & 18 \\ \downarrow & & & & \parallel \\ 891 & = & 3^4\cdot 11 & \longmapsto & (3+4)+11 \end{array}$$

FIGURE 1. 198 is a v-palindrome in base 10

infinitude. However, I recall knowing how to show their infinitude as early as the summer of 2015.

In March, 2021, I published the paper [24], first calling such numbers *v-palindromes*. I proved a general theorem [24, Theorem 1] describing a periodic phenomenon pertaining to v-palindromes. Then, in August of 2022, I published [20] (formerly [21] on arxiv), in which more in-depth investigations were done. I also wrote the manuscripts [25, 26, 22] ([26] is a former version of this manuscript) which are, at time of writing, preprints.

In Subsection 1.1 we define v-palindromes in a general base, briefly discuss prime v-palindromes, and explain why the v-palindromes can be considered an analogue to the usual palindromes. In Subsection 1.2, we give an outline of the rest of the paper. In Subsection 1.3, we review other related work.

1.1. **v-palindromes.** Recall the base b representation of a natural number where $b \geq 2$ is the base. For every natural number n, there exist unique integers $L \geq 1$ and $0 \leq a_0, a_1, \ldots, a_{L-1} < b$ with $a_{L-1} \neq 0$ such that

$$n = a_0 + a_1 b + \cdots + a_{L-1} b^{L-1}. \tag{1}$$

We also denote this as $n = (a_{L-1}, \cdots, a_1, a_0)_b$. Thus L is the number of base b digits of n. We define the *digit reversal* in base b of n to be

$$r_b(n) = a_{L-1} + a_{L-2} b + \cdots + a_0 b^{L-1}. \tag{2}$$

For instance, $r_{10}(18) = 81$, $r_{10}(2) = r_{10}(200) = 2$, and $r_2(2) = r_2((1,0)_2) = 1$. Next we define a function $v(n)$ to denote "summing the numbers appearing in the factorization".

Definition 1. Suppose that the prime factorization of the natural number n is

$$n = p_1^{\varepsilon_1} \cdots p_s^{\varepsilon_s} q_1 \cdots q_t, \tag{3}$$

where $s, t \geq 0$ and $\varepsilon_1, \ldots, \varepsilon_s \geq 2$ are integers and $p_1, \ldots, p_s, q_1, \ldots, q_t$ are distinct primes. Then we set

$$v(n) = \sum_{i=1}^{s}(p_i + \varepsilon_i) + \sum_{j=1}^{t} q_j. \tag{4}$$

Notice that $v(n)$ is an additive function, i.e., $v(mn) = v(m) + v(n)$ whenever m and n are relatively prime natural numbers. The values of $v(n)$ have been created as sequence A338038 in the On-Line Encyclopedia of Integer Sequences [13]. We can now make the following definition.

Definition 2 (v-palindrome)**.** Let $b \geq 2$ be an integer. A natural number n is a *v-palindrome* in base b if

(i) $b \nmid n$,
(ii) $n \neq r_b(n)$, and
(iii) $v(n) = v(r_b(n))$.

The set of v-palindromes in base b is denoted by $\mathbb{V}_b$.

Condition (i) is included merely for the aesthetic look of n and $r_b(n)$ having the same number of digits. Condition (ii) is included since if $n = r_b(n)$, then condition (iii) holds trivially, and so nothing is surprising. The sequence of v-palindromes in base ten has been created as sequence A338039 in [13]. A generalization of Figure (1), with the factorizing step omitted, would be as follows:

$$\begin{array}{ccc}
n & \longmapsto & v(n) \\
\uparrow & & \| \\
\text{digit reversal in base } b & & \text{same number} \\
\downarrow & & \| \\
r_b(n) & \longmapsto & v(r_b(n))
\end{array}$$

FIGURE 2. v-palindromes in base b

Only base ten is dealt with in [23, 24, 20, 25, 22], but most of the results and proofs therein generalize straightforwardly to a general base.

It is conjectured in [23, (a)] that there are no prime v-palindromes in base ten. Recently, Boran et al. [2] characterized prime v-palindromes in base ten and showed that they are precisely the primes of the form

$5\cdot 10^m-1$ such that $5\cdot 10^m-3$ is also prime. Thus, prime v-palindromes in base ten, if any exist, must be twin primes.

When we consider an arbitrary base, however, we have been able to find several prime v-palindromes. For example, 109 is a prime v-palindrome in base 16. We have that $v(109) = 109$ since 109 is prime, and $r_{16}(109) = r_{16}((6,13)_{16}) = (13,6)_{16} = 214$. Then $v(r_{16}(109)) = v(214) = 107 + 2 = 109$, so 109 is a v-palindrome in base 16. Notice that 109 happens to be a twin prime, but we can also find examples of prime v-palindromes that are not twin primes. For example, 467 is a prime v-palindrome in base 276, yet 467 is not a twin prime. It remains an open problem to classify which bases are similar to base 10 where any prime v-palindrome must necessarily be a twin prime.

We now explain how v-palindromes can be viewed as an analogy to usual palindromes. Recall the following definition of the usual palindrome.

Definition 3 (palindrome)**.** Let $b \geq 2$ be an integer. A natural number n is a *palindrome* in base b if $n = r_b(n)$.

The definition of v-palindromes can be obtained from that of the usual palindromes by applying v to the equality $n = r_b(n)$ and then including the conditions (i) and (ii) for the reasons explained earlier. The mere application of v to the equality $n = r_b(n)$ causes palindromes and v-palindromes to behave very differently. Although the function v is a specific as function defined above, there is nothing special about it. It is equally conceivable to use any other function $f\colon \mathbb{N} \to \mathbb{C}$ instead in condition (iii), calling the defined kind of numbers f-palindromes.

1.2. **Outline and main results of this paper.** In Section 2, we recall infinitely many more examples of v-palindromes in base ten from the previous works [23, 24, 25]. In Section 3, we state analogous results in a general base b of results in base ten from [24, 20], as well as prove that if there exists a v-palindrome in a base b, then there exists infinitely many (Theorem 5). In Section 4, we exhibit v-palindromes in bases $p+1$ and p^2+1, for each odd prime p. In Section 5, we prove that v-palindromes exist in infinitely many bases b (Corollary 22). Finally in Section 6, we discuss further problems pertaining to v-palindromes in a general base.

1.3. **Other related work.** In this section, we give references to other works related to v-palindromes in various ways. For results on the usual palindromes, we refer the reader to Goins [5], Hernández

Hernández and Luca [8], Pongsriiam [15], Pongsriiam and Subwattanachai [16], and the references therein, though there is much more literature available on palindromes.

Digit reversal has been studied in the past. In Hardy [7], it is mentioned that $4 \cdot 2178 = 8712$ and $9 \cdot 1089 = 9801$. Therefore the numbers 2178 and 1089 have the property that their digit reversal is a multiple of (at least twice) themselves. Following Sutcliffe [19], in general we are solving $kn = r_b(n)$ for integers $k \geq 2$, $n \geq 1$, and $b \geq 2$. If n has one base b digit, then $n = r_b(n)$ and there is no solution. Hence n has at least two base b digits. Then by the generalization of [2, Lemma 2.3] to a general base, n^2 has more base b digits than n. Now, as $r_b(n)$ has no more base b digits than n, we have that n^2 has more base b digits than $r_b(n)$. Consequently, $kn = r_b(n)$ implies that $kn = r_b(n) < n^2$, and so $k < n$. In [19], the case of n having less than four base b digits is addressed completely, whereas the case of n having four base b digits is partially addressed. Klosinski and Smolarski [9] also considered this problem, and mentioned that $4 \cdot 219 \cdots 978 = 879 \cdots 912$ for any number of 9's in between, generalizing the aforementioned $4 \cdot 2178 = 8712$ nicely.

Other functions similar to $v(n)$ have been studied by many authors. Alladi and Erdös [1] and Lal [10] studied the function

$$A(n) = \sum_{i=1}^{s} p_i \varepsilon_i + \sum_{j=1}^{t} q_j, \tag{5}$$

following the same notation as Equation (3). In [1], analytical and other aspects of $A(n)$ are studied. In [10], iterates of $A(n)$ are investigated. Mullin [12] and Gordon and Robertson [6] studied the function

$$\psi(n) = \prod_{i=1}^{s} p_i \varepsilon_i \cdot \prod_{j=1}^{t} q_j. \tag{6}$$

In [12], research problems on $\psi(n)$ were posed, and [6] proved two theorems on $\psi(n)$.

Similar equalities to the condition $v(n) = v(r_b(n))$ from the definition of v-palindromes, with another function in place of v, have been studied by several authors. Following Spiegelhofer [17], we define *Stern's diatomic sequence* $s(n)$ by $s(0) = 0$, $s(1) = 1$, and $s(2n) = s(n)$ and $s(2n+1) = s(n) + s(n+1)$ for all $n \geq 1$. In Dijkstra [3], a problem is given asking to show $s(n) = s(r_2(n))$ for all $n \geq 1$. Dijkstra also proved this equality in [4, pp. 230–232]. Following [17] again, we define the function $b(n)$ introduced by Northshield as $b(0) = 0$, $b(1) = 1$, and

$$b(3n) = b(n), \tag{7}$$

$$b(3n+1) = \sqrt{2} \cdot b(n) + b(n+1), \tag{8}$$

$$b(3n+2) = b(n) + \sqrt{2} \cdot b(n+1), \tag{9}$$

for all $n \geq 0$. Then it is proved that $b(n) = b(r_3(n))$ for all $n \geq 1$ [17, Theorem 1]. [17, Theorem 2] gives a slightly intricate sufficient condition for a complex-valued function $f(n)$ and integer $b \geq 2$ to satisfy $f(n) = f(r_b(n))$ for all $n \geq 1$, to which [17, Theorem 1] is a corollary. Following Spiegelhofer [18], we define an analogue of Stern's diatomic sequence, the *Stern polynomials* $s_n(x, y)$, by $s_1(x, y) = 1$ and $s_{2n}(x, y) = s_n(x, y)$ and $s_{2n+1}(x, y) = xs_n(x, y) + ys_{n+1}(x, y)$ for $n \geq 1$. It was proved that $s_n(x, y) = s_{r_2(n)}(x, y)$ for all $n \geq 1$ [18, Theorem 1]. We introduce a final equality of this type from Morgenbesser and Spiegelhofer [11]. Let σ_b be the sum-of-digits function in base $b \geq 2$. For $\alpha \in \mathbb{R}$ and integers $n \geq 1$, define

$$\gamma(\alpha, n) = \lim_{x \to \infty} \frac{1}{x} \sum_{k<x} e^{2\pi i \alpha(\sigma_b(k+n) - \sigma_b(k))}. \tag{10}$$

Then it is proved that $\gamma(\alpha, n) = \gamma(\alpha, r_b(n))$ for all $n \geq 1$ [11, Theorem 1].

2. More v-palindromes in base ten

We illustrated in Figure (1) that 198 is a v-palindrome in base ten. That there are infinitely many v-palindromes in base ten is shown in [23] by specifically showing that all numbers

$$18, 198, 1998, \ldots, \tag{11}$$

with any number of nines in the middle are v-palindromes in base ten. Also, [23] mentions that all numbers of the form

$$18, 1818, 181818, \ldots, \tag{12}$$

with any number of 18s concatenated, are v-palindromes in base ten. In fact, the main theorem of [24] is inspired by Equation (12). Both Equations (11) and (12) seem to be derived from the number 18. They are subsets of the following more general family.

Theorem 1 ([25, Theorem 3]). *If ρ is a palindrome in base ten consisting entirely of the digits* 0 *and* 1, *then* 18ρ *is a v-palindrome in base ten.*

This theorem relates the usual palindromes with the v-palindromes. If we take ρ to be a repunit, then we get Equation (11). If we take ρ

to have alternating digits of 0 and 1, then we get Equation (12). If we take ρ to have only the first and last digits being 1 and at least one 0 in between, then we deduce the family of v-palindromes in base ten

$$1818, 18018, 180018, \ldots, \tag{13}$$

with any number of 0's in between two 18's.

Thus the infinitude of v-palindromes in base ten is well-established.

3. Past results for a general base

In this section we state some past results from [24, 20], which were just for base ten, for a general base. The base ten proofs generalize straightforwardly to a general base. Finally we show that if there exists a v-palindrome in a base b, then there exists infinitely many (Theorem 5).

3.1. **A periodic phenomenon.**

We state the main theorem of [24], which describes a periodic phenomenon involving v-palindromes and repeated concatenations in base ten, for a general base. The proof in [24] is only for base ten, but is easily adapted for a general base. Before that, we provide notation for repeated concatenations.

Definition 4. Suppose that $n = (a_{L-1}, \ldots, a_1, a_0)_b$ is a base b representation and $k \geq 1$ is an integer, then we denote the repeated concatenation of the base b digits of n consisting of k copies of n by $n(k)_b$. That is,

$$\begin{aligned} n(k)_b &= (\underbrace{a_{L-1}, \ldots, a_1, a_0, a_{L-1}, \ldots, a_1, a_0, \ldots\ldots, a_{L-1}, \ldots, a_1, a_0}_{k \text{ copies of } a_{L-1}, \ldots, a_1, a_0})_b \\ &= n(1 + b^L + \cdots + b^{(k-1)L}) = n \cdot \frac{1 - b^{Lk}}{1 - b^L}. \end{aligned} \tag{14}$$

For instance, $18(3)_{10} = 181818$ and $201(4)_{10} = 201201201201$. Now we can state the main theorem of [24] for a general base as follows.

Theorem 2 ([24, Theorem 1] for a general base)**.** *Let $b \geq 2$ be an integer. For every natural number n with $b \nmid n$ and $n \neq r_b(n)$, there exists an integer $\omega \geq 1$ such that for all integers $k \geq 1$,*

$$n(k)_b \in \mathbb{V}_b \quad \textit{if and only if} \quad n(k + \omega)_b \in \mathbb{V}_b. \tag{15}$$

Based on this theorem, we can make the following definitions.

Definition 5. The smallest possible ω in the above theorem is denoted by $\omega_0(n)_b$. If the base b digits of n can be repeatedly concatenated to form a v-palindrome in base b, i.e., if there exists an integer $k \geq 1$ such that $n(k)_b \in \mathbb{V}_b$, then the smallest k is denoted by $c(k)_b$; otherwise we set $c(n)_b = \infty$.

The sequence of numbers n such that $c(n)_{10} < \infty$ has been created as sequence A338371 in the On-Line Encyclopedia of Integer Sequences [13]. Hence there remains the problem of finding $\omega_0(n)_b$ and $c(n)_b$. [20] solves this problem for $b = 10$ by associating to each n a periodic function $\mathbb{Z} \to \{0, 1\}$ which we describe in the next subsection.

3.2. **Associated periodic function.** Fix a base $b \geq 2$ and a natural number n with $b \nmid n$ and $n \neq r_b(n)$ throughout this subsection. To have a clearer picture of the periodic phenomenon illustrated in Theorem 2, we define the function $I_b^n \colon \mathbb{N} \to \{0, 1\}$ by setting

$$I_b^n(k) = \begin{cases} 0 & \text{if } n(k)_b \notin \mathbb{V}_b, \\ 1 & \text{if } n(k)_b \in \mathbb{V}_b. \end{cases} \tag{16}$$

Then by Theorem 2 I_b^n is a periodic function. It therefore has a unique periodic extension $I_b^n \colon \mathbb{Z} \to \{0, 1\}$ which we give the same notation. By [20, Theorem 11], I_b^n can be expressed as a linear combination when $b = 10$, and the same holds for a general base. We first give notation for certain functions used to form the linear combination.

Definition 6. For a natural number a, denote by $I_a \colon \mathbb{Z} \to \{0, 1\}$ the function defined by

$$I_a(k) = \begin{cases} 0 & \text{if } a \nmid k \\ 1 & \text{if } a \mid k. \end{cases} \tag{17}$$

That is, I_a is the indicator function of $a\mathbb{Z}$ in $\mathbb{Z}$.

We can now state the linear combination as follows.

Theorem 3 ([20, Theorem 11] for a general base)**.** *The function I_b^n can be expressed in the form*

$$I_b^n = \lambda_1 I_{a_1} + \lambda_2 I_{a_2} + \cdots + \lambda_u I_{a_u}, \tag{18}$$

where the $u \geq 0$, $1 \leq a_1 < a_2 < \cdots < a_u$, and $\lambda_1, \lambda_2, \ldots, \lambda_u \neq 0$ are integers.

Having expressed the function I_b^n in the form of Equation (18), we use the following result to find $\omega_0(n)_b$ and $c(n)_b$.

Theorem 4 ([20, Corollaries 4 and 5] for a general base)**.** *The smallest period $\omega_0(n)_b$ and $c(n)_b$ can be found from the expression* (18) *by*

$$\omega_0(n)_b = \mathrm{lcm}\{a_1, a_2, \ldots, a_u\}, \tag{19}$$

$$c(n)_b = \inf\{a_1, a_2, \ldots, a_u\} = \begin{cases} \infty & \text{if } u = 0 \\ a_1 & \text{if } u \geq 1. \end{cases} \tag{20}$$

This infimum is thought of as that in the extended real number system.

We did not say how to express I_b^n in the form of Equation (18). A definite procedure for doing this, for $b = 10$, is described in [20], and is easily adapted for a general base.

3.3. **One implies infinitely many.** We show that if there exists a v-palindrome in base b, then there exist infinitely many.

Theorem 5. *Let $b \geq 2$ be an integer. If there exists a v-palindrome in base b, then there exist infinitely many v-palindromes in base b.*

Proof. Suppose that n is a v-palindrome in base b. We have the associated function I_b^n from the previous subsection. If n is a v-palindrome in base b that means $I_b^n(1) = 1$. Since I_b^n is periodic, say with period ω, we see that

$$I_b^n(1) = I_b^n(1 + \omega) = I_b^n(1 + 2\omega) = \cdots . \tag{21}$$

Consequently,

$$n(1)_b, n(1 + \omega)_b, n(1 + 2\omega)_b, \ldots \tag{22}$$

are all v-palindromes in base b. □

4. v-palindromes in bases $p + 1$ and $p^2 + 1$

In this section we give more examples of v-palindromes in bases other than ten. In Subsection 4.1, we give examples of v-palindromes in bases $p + 1$, for each odd prime p. In Subsection 4.2, we give examples of v-palindromes in bases $p^2 + 1$, for each odd prime p. We first prove the following lemmas.

Lemma 6. *Let $n \in \mathbb{N}$. Then $r_{n+1}(2n) = n^2$.*

Proof. We have

$$\begin{aligned} r_{n+1}(2n) &= r_{n+1}(2(n+1-1)) \\ &= r_{n+1}(2n+2-2) \\ &= r_{n+1}(n+1+(n+1-2)) \\ &= (n+1-2)\cdot(n+1)+1 \\ &= (n+1)^2-2(n+1)+1 \\ &= (n+1-1)^2 \\ &= n^2. \end{aligned}$$

□

Lemma 7. *Let p be an odd prime. Then $v(2p) = v(p^2)$.*

Proof. We find that

$$v(2p) \;=\; v(2)+v(p) \;=\; 2+p \;=\; v(p^2). \tag{23}$$

□

4.1. **v-palindromes in base $p+1$.** We have the following theorem.

Theorem 8. *Let p be an odd prime. Then $2p \in \mathbb{V}_{p+1}$.*

Proof. We have $2p = (1, p-1)_{p+1}$ and so

$$r_{p+1}(2p) = (p-1, 1)_{p+1} = (p-1)(p+1)+1 = p^2. \tag{24}$$

It is clear from the base $p+1$ representation of $2p$, namely $(1, p-1)_{p+1}$, that we have $p+1 \nmid 2p$ and $2p \neq r_{p+1}(2p)$. Finally, because p is an odd prime, and using Equation (24), we have

$$v(2p) = 2+p = v(p^2) = v(r_{p+1}(2p)). \tag{25}$$

This shows that $2p \in \mathbb{V}_{p+1}$. □

Next we consider repeated concatenations of $2p$ in base $p+1$, where p is an odd prime. As in the proof of Theorem 8, both $2p$ and p^2 are two digits long in base $p+1$. Using similar notation to [24], we define

$$\rho_{k,2} \;:=\; \sum_{i=0}^{k-1}(p+1)^{2i} \;=\; 1+\sum_{i=1}^{k-1}(p+1)^{2i}, \tag{26}$$

for integers $k \geq 1$. We find that $2 \mid p+1$ as p is odd, so $2 \mid \sum_{i=1}^{k-1}(p+1)^{2i}$, and hence $2 \nmid \rho_{k,2}$. We also want to know when p is coprime with $\rho_{k,2}$.

Note that

$$\rho_{k,2} \;=\; \sum_{i=0}^{k-1}(p+1)^{2i} \;\equiv\; \sum_{i=0}^{k-1}1^{2i} \;\equiv\; k \mod p. \tag{27}$$

Thus, $p \mid \rho_{k,2}$ if and only if $p \mid k$. We use these two facts to prove the following theorem, recalling that $(2p)(k)_{p+1}$ and $(p^2)(k)_{p+1}$ denote repeated concatenations according to Definition 4.

Theorem 9. *Let p be an odd prime and $k \in \mathbb{N}$. Then $(2p)(k)_{p+1} \in \mathbb{V}_{p+1}$ if and only if $p \nmid k$.*

Proof. Since $2p = (1, p-1)_{p+1}$, we have $r_{p+1}(2p) = (p-1, 1)_{p+1} = p^2$ and that

$$(2p)(k)_{p+1} = 2p \cdot \rho_{k,2}, \quad r_{p+1}(2p \cdot \rho_{k,2}) = p^2 \cdot \rho_{k,2}. \tag{28}$$

Since $\rho_{k,2}$ is an odd number and from Equation (27), given $p \nmid k$ we know p is coprime with $\rho_{k,2}$. Using the additive property of v, Lemma 7, and Equation (28), we find,

$$\begin{aligned} v(2p \cdot \rho_{k,2}) &= v\,(2p) + v(\rho_{k,2}) \\ &= v\left(p^2\right) + v(\rho_{k,2}) \\ &= v\left(p^2 \cdot \rho_{k,2}\right) \\ &= v\left(r_{p+1}\,(2p \cdot \rho_{k,2})\right). \end{aligned}$$

Since $2p \cdot \rho_{k,2} = (1, p-1, \ldots, 1, p-1)_{p+1}$, we know that $p+1 \nmid 2p \cdot \rho_{k,2}$. Lastly, we know that $2p \cdot \rho_{k,2} \neq r_{p+1}\,(2p \cdot \rho_{k,2}) = p^2 \cdot \rho_{k,2}$ as $2p \neq p^2$, therefore $(2p)(k)_{p+1} \in \mathbb{V}_{p+1}$.

Conversely, assume $(2p)(k)_{p+1} \in \mathbb{V}_{p+1}$. Suppose $p \mid k$. From Equation (28), $(2p)(k)_{p+1} = 2p \cdot \rho_{k,2}$ and $r_{p+1}((2p)(k)_{p+1}) = p^2 \cdot \rho_{k,2}$. From Equation (27) we know $\rho_{k,2} \equiv k \mod p$. Since $p \mid k$ we have

$$\rho_{k,2} \equiv 0 \mod p,$$

so $p \mid \rho_{k,2}$. Hence we can rewrite $\rho_{k,2}$ as $\rho_{k,2} = p^a \cdot n$, where $a, n \in \mathbb{N}$ and $p \nmid n$. Note that $\rho_{k,2}$ is odd so $2 \nmid n$. Applying v to $(2p)(k)_{p+1}$ we find

$$v((2p)(k)_{p+1}) = v(2p \cdot \rho_{k,2}) = v(2 \cdot p^{a+1} \cdot n) = 2 + p + a + 1 + v(n)$$

and

$$v(r_{p+1}((2p)(k)_{p+1})) = v(p^2 \cdot \rho_{k,2}) = v(p^{a+2} \cdot n) = p + a + 2 + v(n),$$

which contradicts the fact that $(2p)(k)_{p+1} \in \mathbb{V}_{p+1}$. Hence if $(2p)(k)_{p+1} \in \mathbb{V}_{p+1}$, then $p \nmid k$. □

For our last pattern of v-palindromes in these bases, we require the following lemma.

Lemma 10. *Let $n, k \in \mathbb{N}$ with $n \geq 2$ and $b = n + 1$. Then*

(1) $(1,\underbrace{n,\ldots,n}_{k},n-1)_{n+1} \;=\; 2n\cdot 1(k+1)_b$, *and*

(2) $(n-1,\underbrace{n,\ldots,n}_{k},1)_{n+1} \;=\; n^2\cdot 1(k+1)_b$.

Proof. Note that $2n=(1,n-1)_{n+1}$ and $n^2=(n-1,1)_{n+1}$. Using this, we find

(1)

$$\begin{aligned} 2n\cdot 1(k+1)_b &= (1,n-1)_{n+1}\cdot(\underbrace{1,\cdots,1}_{k+1})_{n+1} \\ &= (1,\underbrace{n-1+1,n-1+1,\ldots,n-1+1}_{k},n-1)_{n+1} \\ &= (1,\underbrace{n,\ldots,n}_{k},n-1)_{n+1}, \end{aligned}$$

(2)

$$\begin{aligned} n^2\cdot 1(k+1)_b &= (n-1,1)_{n+1}\cdot(\underbrace{1,\cdots,1}_{k+1})_{n+1} \\ &= (n-1,\underbrace{1+n-1,\ldots,1+n-1}_{k},1)_{n+1} \\ &= (n-1,\underbrace{n,\ldots,n}_{k},1)_{n+1}. \end{aligned}$$

□

We use this to prove the following theorem.

Theorem 11. *Let p be an odd prime, $k\in\mathbb{N}$, $p\nmid k+1$, and $b=p+1$. Then* $2p\cdot 1(k+1)_b\in\mathbb{V}_{p+1}$.

Proof. We find that

$$\begin{aligned} (1,\underbrace{p,\ldots,p}_{k},p-1)_{p+1} &\neq r_{p+1}((1,\underbrace{p,\ldots,p}_{k},p-1)_{p+1}) \\ &= (p-1,\underbrace{p,\ldots,p}_{k},1)_{p+1} \end{aligned}$$

(this also shows $r_{p+1}(2p\cdot 1(k+1)_{p+1})=p^2\cdot 1(k+1)_{p+1}$). We have

$$1(k+1)_b \;=\; \sum_{i=0}^{k}(p+1)^i. \tag{29}$$

From this, we find

$$1(k+1)_b \equiv \sum_{i=0}^{k}(p+1)^i \equiv \sum_{i=0}^{k}(1)^i \equiv k+1 \mod p. \tag{30}$$

Since $p \nmid k+1$ we know $p \nmid 1(k+1)_b$. Additionally, we know $2 \nmid 1(k+1)_b$. Further, we find $p+1 \nmid 2\cdot 1(k+1)_b$ so $p+1 \nmid 2p\cdot 1(k+1)_b$. Finally we show the numbers are v-palindromes by using the additivity of v, Lemma 7, and Equation (28). We find that

$$\begin{aligned} v((1,\underbrace{p,\dots,p}_{k},p-1)_{p+1}) &= v\left(2p\cdot 1(k+1)_b\right) \\ &= v(2p)+v(1(k+1)_b) \\ &= v(p^2)+v(1(k+1)_b) \\ &= v(p^2\cdot 1(k+1)_b) \\ &= v((p-1,\underbrace{p,\dots,p}_{k},1)_{p+1}). \end{aligned}$$

This shows that $2p\cdot 1(k+1)_b \in \mathbb{V}_{p+1}$. □

4.2. **v-palindromes in base p^2+1.** Recall [25, Theorem 3], which applies to a base $b = 3^2+1$. We begin by generalizing this Theorem to all bases one greater then an odd prime squared. We set base $b = p^2+1$ as our base, keeping p as an odd prime for the remainder of this section.

Theorem 12. *Let p be an odd prime. If ρ is a palindrome in base $b = p^2+1$ consisting entirely of the digits 0 and 1, then $2p^2\rho \in \mathbb{V}_{p^2+1}$.*

Proof. We begin by noting that $b \nmid \rho$, since if ρ has a last digit of 0 then it has a leading digit of 0, however this means ρ is not a palindrome as any leading digits of 0 are ignored making ρ of the form $1,\dots,0$. Thus we know the last digit of ρ is 1. Further, since we know p is odd, $p+1$ is even thus any number with last digit 1 is odd, so we know $2 \nmid \rho$.

When read from left to right, ρ must be formed by a_1 ones, followed by a_2 zeros, followed by a_3 ones, and so on until lastly, a_{2r-1} ones, where $r, a_1, a_2, \dots, a_{2r-1} \in \mathbb{N}$ such that $a_i = a_{2r-i}$ for integers $i \in [1, 2r-1]$. Writing out ρ we get

$$\rho = \underbrace{1\dots1}_{a_1}\overbrace{0\dots0}^{a_2}\underbrace{1\dots1}_{a_3}\dots\underbrace{1\dots1}_{a_3}\overbrace{0\dots0}^{a_2}\underbrace{1\dots1}_{a_1}.$$

Using the equalities $2p^2 = (1, p^2-1)_{p^2+1}$ and $p^4 = (p^2-1, 1)_{p^2+1}$ we find

$$2p^2\rho =$$
$$(1, \underbrace{p^2, \ldots, p^2}_{a_1-1}, q, \overbrace{0, \ldots, 0}^{a_2-1}, 1, \underbrace{p^2, \ldots, p^2}_{a_3-1}, q, \ldots, 1, \underbrace{p^2, \ldots, p^2}_{a_3-1}, q, \overbrace{0, \ldots, 0}^{a_2-1}, 1, \underbrace{p^2, \ldots, p^2}_{a_1-1}, q)_{p^2+1}$$

and

$$p^4\rho =$$
$$(q, \underbrace{p^2, \ldots, p^2}_{a_1-1}, 1, \overbrace{0, \ldots, 0}^{a_2-1}, q, \underbrace{p^2, \ldots, p^2}_{a_3-1}, 1, \ldots, q, \underbrace{p^2, \ldots, p^2}_{a_3-1}, 1, \overbrace{0, \ldots, 0}^{a_2-1}, q, \underbrace{p^2, \ldots, p^2}_{a_1-1}, 1)_{p^2+1}$$

where $q = p^2 - 1$ has been substituted to save space. From this we clearly see that $p^2 + 1 \nmid 2p^2\rho$ and that $p^4\rho = r_{p^2+1}(2p^2\rho) \neq 2p^2\rho$. Let $\alpha \geq 0$ and $n \geq 1$ be integers such that $\rho = p^\alpha n$ and $(p, n) = 1$. Then

$$\begin{aligned}
v(2p^2\rho) &= v(2p^2 \cdot p^\alpha n) \\
&= v(2p^{2+\alpha} n) \\
&= 2 + p + 2 + \alpha + v(n) \\
&= v(p^{4+\alpha} n) \\
&= v(p^4 \cdot p^\alpha n) \\
&= v(r_{p^2+1}(2p^2\rho)).
\end{aligned}$$

This shows that $2p^2\rho \in \mathbb{V}_{p^2+1}$.

□

Next we prove three Corollaries to Theorem 12 that mirror the three theorems proved in Subsection 4.1.

Corollary 13. *Let p be an odd prime. Then $2p^2 \in \mathbb{V}_{p^2+1}$.*

Proof. Note that $2p^2 = 2p^2 \cdot 1$. Since 1 is a palindrome consisting only of the digit 1, by Theorem 12, we have $2p^2 \in \mathbb{V}_{p^2+1}$. □

Corollary 14. *Let p be an odd prime and $k \in \mathbb{N}$. Then $(2p^2)(k)_{p^2+1} \in \mathbb{V}_{p^2+1}$.*

Proof. We note that $(2p^2)(k)_{p^2+1} = 2p^2 \cdot \rho_k$, where

$$\rho_k := (\underbrace{1, 0, 1, 0, \ldots, 0, 1, 0, 1}_{2k-1})_{p^2+1}; \tag{31}$$

ρ_k is a palindrome consisting entirely of the digits 0 and 1. Thus, by Theorem 12 we know $(2p^2)(k)_{p^2+1} \in \mathbb{V}_{p^2+1}$ □

Corollary 15. *Let p be an odd prime, $k \in \mathbb{N}$ and $b = p^2 + 1$. Then* $2p^2 \cdot 1(k+1)_b \in \mathbb{V}_{p^2+1}$.

Proof. As $1(k+1)_b$ consists only of the digit 1, we know it is a palindrome. Therefore, by Theorem 12 we know $2p^2 \cdot 1(k+1)_{p^2+1} \in \mathbb{V}_{p^2+1}$. □

5. Existence of v-palindromes for infinitely many bases

In this section we show the existence of v-palindromes (and therefore infinitely many v-palindromes by Theorem 5) for infinitely many bases. Everything is based on the simple fact that $v(5) = v(6)$. Since $v(n)$ is an additive function, for every integer $t \geq 1$ with $(t, 30) = 1$, we have $v(5t) = v(6t)$.

Imagine that we have a base $b \geq 2$ for which we would like to show that a v-palindrome exists. The first attempt would be to look at two-digit numbers. That is, numbers $(a, c)_b = ab + c$, where $1 \leq a < c < b$ are integers. By definition, $(a, c)_b$ is a v-palindrome in base b if and only if $v((a, c)_b) = v((c, a)_b)$, or equivalently,

$$v(ab + c) = v(cb + a). \tag{32}$$

This would hold if for some integer $t \geq 1$ with $(t, 30) = 1$,

$$\begin{cases} ab + c = 5t, \\ cb + a = 6t, \end{cases} \tag{33}$$

simply by the observation in the previous paragraph. To summarize, we have shown the following.

Lemma 16. *Let $b \geq 2$ be an integer. If there exists an ordered triple (a, c, t) of positive integers such that $a < c < b$, $(t, 30) = 1$, and Equation (33) holds, then the two-digit number $(a, c)_b$ is a v-palindrome in base b. Hence, there exists a v-palindrome in base b.*

Definition 7. We call a triple (a, c, t) in the premise of the above lemma a *permissible triple* for b.

Our strategy is to try to find permissible triples. The system (33) can be written in matrix from as

$$\begin{pmatrix} b & 1 \\ 1 & b \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = t \begin{pmatrix} 5 \\ 6 \end{pmatrix}. \tag{34}$$

Solving this we have

$$\begin{pmatrix} a \\ c \end{pmatrix} = t \begin{pmatrix} b & 1 \\ 1 & b \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{t}{b^2-1} \begin{pmatrix} b & -1 \\ -1 & b \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} \tag{35}$$

$$= \frac{t}{b^2-1} \begin{pmatrix} 5b-6 \\ -5+6b \end{pmatrix} = \begin{pmatrix} \frac{t(5b-6)}{b^2-1} \\ \frac{t(-5+6b)}{b^2-1} \end{pmatrix}. \tag{36}$$

We write this separately as

$$a = \frac{t(5b-6)}{b^2-1}, \quad c = \frac{t(6b-5)}{b^2-1}, \tag{37}$$

from which we also see that $0 < a < c$. Hence we have the following lemma.

Lemma 17. *Let $b \geq 2$ be an integer. For every integer $t \geq 1$, there exist unique rational numbers $a, c \in \mathbb{Q}$ such that Equation* (33) *holds, and they are given by Equation* (37)*. Moreover, $0 < a < c$.*

Hence the only possible permissible triples for b are

$$\left(\frac{t(5b-6)}{b^2-1}, \frac{t(-5+6b)}{b^2-1}, t \right), \tag{38}$$

for an integer $t \geq 1$ with $(t, 30) = 1$. The only missing conditions to fulfill are

$$\frac{t(5b-6)}{b^2-1}, \frac{t(-5+6b)}{b^2-1} \in \mathbb{Z}, \tag{39}$$

$$\text{and } \frac{t(-5+6b)}{b^2-1} < b. \tag{40}$$

We write

$$\frac{t(5b-6)}{b^2-1} = \frac{t(5b-6)/(5b-6, b^2-1)}{(b^2-1)/(5b-6, b^2-1)}, \tag{41}$$

$$\frac{t(-5+6b)}{b^2-1} = \frac{t(-5+6b)/(-5+6b, b^2-1)}{(b^2-1)/(-5+6b, b^2-1)}. \tag{42}$$

Hence we see that Equation (39) holds if and only if t is a multiple of

$$f(b) = \left[\frac{b^2-1}{(5b-6, b^2-1)}, \frac{b^2-1}{(-5+6b, b^2-1)} \right]; \tag{43}$$

here we also defined the function $f(b)$ for integers $b \geq 2$. Hence we have shown the following lemma.

Lemma 18. *Let $b \geq 2$ be an integer. Then the permissible triples of b are precisely the triples*

$$\left(\frac{t(5b-6)}{b^2-1}, \frac{t(-5+6b)}{b^2-1}, t \right), \tag{44}$$

where

$$t \in S(b) = \left\{ t \in \mathbb{N} \colon (t, 30) = 1,\ f(b) \mid t,\ t < \frac{b(b^2-1)}{-5+6b} \right\}; \tag{45}$$

where we also defined the set-valued function $S(b)$ for integers $b \geq 2$.

However, the above lemma does not promise that permissible triples exist, i.e., $S(b) \neq \varnothing$. However, we can get the following sufficient condition.

Lemma 19. *Let $b \geq 2$ be an integer. If*

$$(f(b), 30) = 1, \quad f(b) < \frac{b(b^2-1)}{-5+6b}, \tag{46}$$

then $f(b) \in S(b)$, and consequently there is a permissible triple for b.

Since $f(b) \mid b^2 - 1$, if $(b^2 - 1, 30) = 1$ then $(f(b), 30) = 1$. Hence the above lemma can be weakened to the following.

Lemma 20. *Let $b \geq 2$ be an integer. If*

$$(b^2-1, 30) = 1, \quad f(b) < \frac{b(b^2-1)}{-5+6b}, \tag{47}$$

then $f(b) \in S(b)$, and consequently there is a permissible triple for b.

We now consider the condition $(b^2 - 1, 30) = 1$. It is easily shown that this is equivalent to having both $b \equiv 0 \pmod 6$ and $b \equiv 0, 2, 3 \pmod 5$. In particular, $b \equiv 0 \pmod{30}$ is a sufficient condition. Suppose that $k \geq 1$ is an integer, then

$$f(30k) = \left[\frac{(30k)^2-1}{(5(30k)-6, (30k)^2-1)}, \frac{(30k)^2-1}{(-5+6(30k), (30k)^2-1)} \right] \tag{48}$$

$$= \left[\frac{(30k)^2-1}{(6k-2, 11)}, \frac{(30k)^2-1}{(5k+2, 11)} \right], \tag{49}$$

where for the second equality we used a property of the greatest common divisor function to simplify. Because of the right inequality in Equation (47), we want $f(30k)$ to be small. Thus it might be good if we have $(6k - 2, 11) = (5k + 2, 11) = 11$, which is easily shown to be equivalent to $k \equiv 4 \pmod{11}$. If we assume that $k \equiv 4 \pmod{11}$, then

$$f(30k) = \frac{(30k)^2-1}{11}. \tag{50}$$

On the other hand, the right-hand-side of the right inequality (47) becomes

$$\frac{(30k)((30k)^2-1)}{-5+6(30k)}. \tag{51}$$

That $f(30k)$ is strictly less than the above quantity is equivalent to

$$-5+6(30k) < 11(30k), \tag{52}$$

which always holds. Hence the above lemma can be further weakened to the following.

Theorem 21. *Let $k \equiv 4 \pmod{11}$ be a positive integer, then*

$$\left(\frac{-6+150k}{11}, \frac{-5+180k}{11}, \frac{-1+900k^2}{11}\right) \tag{53}$$

is a permissible triple for the base $30k$. In particular, the two-digit number

$$\left(\frac{-6+150k}{11}, \frac{-5+180k}{11}\right)_{30k} \tag{54}$$

is a v-palindrome in base $30k$.

Hence we have proved the existence of v-palindromes for infinitely many bases, summarized as follows.

Corollary 22. *If $b \equiv 120 \pmod{330}$ is a positive integer, then there exists a v-palindrome in base b.*

In particular there is a positive density of bases $b \geq 2$ for which a v-palindrome exists.

6. Further problems

In this section we describe some directions for further investigation.

6.1. **Three conjectures.**

In the short note [23], three conjectures on v-palindromes in base ten have been proposed by commentators and we restate them as follows.

Conjecture 23 ([23, (a)])**.** *There does not exist a prime v-palindrome in base ten.*

Conjecture 24 ([23, (b)])**.** *There are infinitely many v-palindromes n in base ten such that both n and $r_{10}(n)$ are square-free.*

Conjecture 25 ([23, (c)])**.** *The only positive integer n such that $n \neq r_{10}(n)$ and $n = v(r(n))$ is 49.*

As mentioned in Subsection 1.1, the prime v-palindromes in base ten are characterized in [2]. This result, however, does not prove nor disprove Conjecture 23. Also noted in Subsection 1.1 is that there are prime v-palindromes in bases 16 and 276. Hence we may consider the following problem.

Problem 1. Let $b \geq 2$ be an integer. When does there exist a prime v-palindrome in base b?

We may also consider Conjectures 24 and 25 for a general base.

6.2. **Two problems.** While [16] provides an exact formula for the number of palindromes up to a given positive integer, the same can be considered for v-palindromes, namely the following.

Problem 2. Let $b \geq 2$ be an integer. Is there a formula for the number of v-palindromes in base b up to a given positive integer? If not, how can it be approximated?

From 199 until 575 are 377 consecutive positive integers which are not v-palindromes in base ten. Just as sequences of consecutive composite numbers can be arbitrarily long, we may consider the following problem.

Problem 3. Let $b \geq 2$ be an integer. Can a sequence of consecutive positive integers each not a v-palindrome in base b be arbitrarily long?

6.3. **Existence of v-palindromes in an arbitrary base.** Section 4 showed that v-palindromes exist in bases $p + 1$ and $p^2 + 1$ for any odd prime p. Section 5 showed that v-palindromes exist in all bases $b \equiv 120 \pmod{330}$. However we are still left with the problem of determining, for an arbitrary integer $b \geq 2$, whether a v-palindrome in base b exists.

The proof in the previous section is based on the equality $v(5) = v(6)$. It is conceivable that the same method basing on other common values of v will find other bases b for which a v-palindrome exists. For instance, we have

$$v(5) = v(6) = v(8) = v(9), \tag{55}$$

$$v(7) = v(10) = v(12) = v(18). \tag{56}$$

We give the following table of the smallest v-palindrome, i.e., $\min(\mathbb{V}_b)$, for the first few bases, calculated using PARI/GP [14].

TABLE 1. The smallest v-palindrome for bases $b \leq 19$.

b	$\min(\mathbb{V}_b)$ written in base 10	$\min(\mathbb{V}_b)$ written in base b
2	175	$1,0,1,0,1,1,1,1$
3	1280	$1,2,0,2,1,0,2$
4	6	$1,2$
5	288	$2,1,2,3$
6	10	$1,4$
7	731	$2,0,6,3$
8	14	$1,6$
9	93	$1,1,3$
10	18	$1,8$
11	135	$1,1,3$
12	22	$1,10$
13	63	$4,11$
14	26	$1,12$
15	291	$1,4,6$
16	109	$6,13$
17	581	$2,0,3$
18	34	$1,16$
19	144	$7,11$

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