## WALKING TO INFINITY ON PRIMES IN $\mathbb{Z}[\sqrt{2}]$

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ABSTRACT. An interesting question known as the Gaussian Moat problem asks whether it is possible to walk to infinity along the Gaussian primes with a bounded step size. We examine a similar version of this problem in the real quadratic integer ring  $\mathbb{Z}[\sqrt{2}]$  whose primes mostly cluster along the asymptotes  $y = \pm x/\sqrt{2}$  as compared to the Gaussian primes, which mainly cluster at the origin. A probabilistic model of primes  $a + b\sqrt{2}$  in  $\mathbb{Z}[\sqrt{2}]$  is then constructed according to their norms  $a^2 - 2b^2$  by applying the Prime Number Theorem and a combinatorial theorem for counting the number of lattice points in the region  $|a^2 - 2b^2| \leq n^2$ . Lastly, we perform a few moat calculations in  $\mathbb{Z}[\sqrt{2}]$  for various step sizes and make a conjecture about the existence of a prime walk to infinity.

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### 1. INTRODUCTION

1.1. **Background.** It is known that one cannot walk to infinity along real prime numbers with a bounded step size, i.e., there is no finite number N such that there exists an infinite sequence of increasing real primes  $p_1, p_2, ...$  where  $p_{i+1} - p_i \leq N$ for any  $i \in \mathbb{N}$ . This can be proven by considering the primorial p#, which is

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defined as the product of all primes  $p_i$  less than or equal to p, i.e.,

$$p\# := \prod_{p_i \text{ prime, } p_i \le p} p_i.$$

For any real prime p, consider the following sequence

$${p\#+j}_{j=2,3,\ldots,p}$$

All numbers in this sequence are composite, so we have a gap of length at least p-1 between subsequent primes. Thus, it is impossible to walk to infinity along primes using a bounded gap.

This classical problem is clearly one-dimensional as we only need to find an arbitrarily large gap on the number line. However, one can ask a more flavorful question of whether there exists a prime walk to infinity in *two dimensions*, for example, in a real or an imaginary quadratic ring. In this case, a gap is considered as a moat around the starting point. This idea gives rise to the Gaussian Moat problem, where we ask if there exists a moat of arbitrary large width surrounding the origin.

1.2. Gaussian prime walks and our motivation. To understand the Gaussian Moat problem, we first refer to the definition of quadratic integer rings.

**Definition 1.1.** For a square-free integer d, we define its quadratic integer ring as

$$\mathbb{Z}[\sqrt{d}] := \begin{cases} \{a + b\frac{1+\sqrt{d}}{2}, a, b \in \mathbb{Z}\}, & d \equiv 1 \pmod{4} \\ \{a + b\sqrt{d}, a, b \in \mathbb{Z}\}, & \text{otherwise.} \end{cases}$$

For both choices of d, the norm of any element in  $\mathbb{Z}[\sqrt{d}]$  is defined as  $a^2 - b^2 d$ . We call an element in  $\mathbb{Z}[\sqrt{d}]$  a *unit* if its norm is  $\pm 1$ , and call two distinct elements *associate* with each other if their norms are the same.

In particular, the set of Gaussian integers which is defined as  $\mathbb{Z}[i] := \{a + bi, a, b \in \mathbb{Z}\}$  is a special case when d = -1. Thus, the norm of a + bi is defined as  $a^2 + b^2$ , and any element a + bi is a unit if its norm is 1 as  $a^2 + b^2 \ge 0$ . Then, the Gaussian primes are defined as follows.

**Definition 1.2.** An element  $a + bi \in \mathbb{Z}[i]$  is a Gaussian prime if it satisfies one of the following requirements up to associates:

1)  $a, b \neq 0$  and  $a^2 + b^2$  is a real prime,

2) a = 0 and |b| is an ordinary prime such that  $b \equiv 3 \mod 4$ ,

3) b = 0 and |a| is an ordinary prime such that  $a \equiv 3 \mod 4$ .

Figure 1, which we refer from Gethner [EWW, p. 328], shows all the Gaussian primes with norm less than 1000. Note that Gaussian primes and primes in  $\mathbb{Z}[\sqrt{2}]$  both have 8-fold symmetry, so it suffices to consider only primes in the first quadrant with  $a, b \geq 0$ . The Gaussian Moat problem asks whether one can walk to infinity in  $\mathbb{Z}[\sqrt{i}]$  with a bounded step size, or equivalently, whether there exists k-moat for any finite real k, where the k-moat is a region of composite numbers with width k. Jordan and Rabung [JR] constructed a  $\sqrt{10}-moat$ , while Gethner [EWW] proved the existence of some larger moats up to the  $\sqrt{26}-moat$ .



FIGURE 1. Gaussian primes with norms less than 1000.

Tsuchimura [T] has shown the existence of a 6-moat in 2004. Based on these results, we speculate that there is no Gaussian prime walks to infinity with a bounded step size.

In this paper, we therefore investigate another quadratic integer ring,  $\mathbb{Z}[\sqrt{2}]$ , which is more likely to have such prime walks. The prime elements in  $\mathbb{Z}[\sqrt{2}]$  are defined as followed.

**Definition 1.3.** An element  $a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$  is a prime if it satisfies one of the following requirements up to associates: 1)  $\sqrt{2}$ ,

2)  $a^2 - 2b^2$  is an ordinary prime and  $a^2 - 2b^2 \equiv 1,7 \mod 8$ ,

3) b = 0 and a is an ordinary prime such that  $a \equiv 3, 5 \mod 8$ .

Figure 2 shows all the prime elements in  $\mathbb{Z}[\sqrt{2}]$  with Euclidean norms less than 800.

1.3. **Results.** Note that most primes tend to cluster along the asymptotes  $y = \pm x/\sqrt{2}$  in  $\mathbb{Z}[\sqrt{2}]$ . Figure 3 shows the comparison between the number of primes in a disk of radius n in  $\mathbb{Z}[\sqrt{2}]$  and  $\mathbb{Z}[i]$ . Following this trend, we prove in Section 2 that there exist more primes in  $\mathbb{Z}[\sqrt{2}]$  than in  $\mathbb{Z}[i]$ . We can also deduce from Theorem 2.13 and Figure 2 that a prime walk in  $\mathbb{Z}[\sqrt{2}]$  in the first quadrant is more likely to be found near the asymptote  $y = x/\sqrt{2}$ . Lastly, the main theorem of this paper, Theorem 3.1, is presented in Section 3 showing that it is impossible to perform a walk of a bounded step size to infinity in  $\mathbb{Z}[\sqrt{2}]$  if the walk remains within some bounded distance from the asymptote  $y = x/\sqrt{2}$  after some point. In Section 4, we then present some evidence that the longest walk possible must stay close to the asymptote. We therefore remove our assumption and state that it's impossible to have a walk in  $\mathbb{Z}[\sqrt{2}]$ , which is Conjecture 4.1.



FIGURE 2. Visualization of primes in the first quadrant of  $\mathbb{Z}[\sqrt{2}]$ .



FIGURE 3. Comparison of primes in  $\mathbb{Z}[\sqrt{2}]$  (top) and  $\mathbb{Z}[i]$  (bottom).

# 2. Modeling primes in $\mathbb{Z}[\sqrt{2}]$

2.1. Estimating the number of primes. Given the random structure of primes in  $\mathbb{Z}[\sqrt{2}]$ , we would like to approximate the number of primes within certain bounds. For primes in  $\mathbb{Z}$ , this is achieved with the Prime Number Theorem (PNT).

**Theorem 2.1** (Prime Number Theorem). The interval [1, n] contains about  $n/\log n$  primes.

We can extend the Prime Number Theorem to other integer rings by connecting primes in  $\mathbb{Z}$  with those in the Gaussian primes, or  $\mathbb{Z}[i]$ . Before doing so, we first refer to a known theorem related to the Dirichlet density.

**Definition 2.2** (Dirichlet density). For  $S \subseteq T$  two sets of positive integers, with  $\sum_{n \in T} n^{-1}$  divergent, d(S,T), the Dirichlet density of S in T, exists if and only if

$$\lim_{s \to 1^+} \sup \frac{\sum_{n \in S} n^{-s}}{\sum_{n \in T} n^{-s}} = d(S, T) = \lim_{s \to 1^+} \inf \frac{\sum_{n \in S} n^{-s}}{\sum_{n \in T} n^{-s}}.$$

**Theorem 2.3.** Suppose  $a, m \in \mathbb{Z}$ , with (a, m) = 1. Let  $\mathbb{P}(a; m)$  be the set of positive primes p such that  $p \equiv a \pmod{m}$  and  $\mathbb{P}$  the set of all positive primes. Then,  $d(\mathbb{P}(a; m), \mathbb{P}) = 1/\varphi(m)$ , where  $\varphi$  is the Euler Totient function.

Then, the following theorem which we refer its proof from, for example, [HW, ch. XV] and [R, p. 188], can be obtained.

**Theorem 2.4.** The number of Gaussian primes contained within the disk of radius r about the origin is approximately

$$\frac{2r^2}{\log r} + \frac{2r}{\log r} + 4$$

*Proof.* Theorem 2.3 implies the densities of primes of the form 4k+1 and 4k+3 are roughly the same since  $d(\mathbb{P}(1;4),\mathbb{P}) = d(\mathbb{P}(3;4),\mathbb{P}) = 1/\varphi(4) = 1/2$ . We consider primes of the form 4k+1 for when a prime is a sum of squares. As in Definition 1.2, all the Gaussian primes can be derived from real primes as follows.

- (1) The real prime 2 gives us four Gaussian primes: 1 + i, 1 i, -1 + i, -1 i, -1
- (2) p = 4k + 3 in real primes gives us four Gaussian primes: p, -p, pi, -pi,
- (3) p = 4k+1 in real primes gives us eight Gaussian primes as p can be written as (a+bi)(a-bi), i.e., each p gives a+bi, -(a+bi), (a+bi)i, -(a+bi)i, a-bi, -(a-bi)i, (a-bi)i, -(a-bi)i.

Consider a disk centered at 0 and of radius r. For any real (4k + 3)-prime, there are four Gaussian primes on the real and imaginary axes of the complex plane. From the PNT and Theorem 2.3, there are approximately  $r/(\varphi(4) \log r)$  (4k + 3)-primes, so we have approximately

$$4 \cdot \frac{r}{\varphi(4)\log r} = \frac{2r}{\log r} \tag{2.1}$$

Gaussian primes lying on the axes.

Now, consider the disk in the first quarter, not including the axes. For positive integers  $1, 2, \ldots, r^2$ , there are approximately  $r^2/\log r^2$  primes. We consider the integers from 1 to  $r^2$  since each lattice point in the disk has norm at most  $r^2$ . By Lemma 2.3, approximately half of them are primes of the form 4k + 1. From (3), these primes give us about

$$8 \cdot \frac{r^2}{2\log r^2} = \frac{4r^2}{\log r^2} = \frac{2r^2}{\log r}$$
(2.2)

Gaussian primes.

Note that in generalizing the PNT to the Gaussian primes, we consider primes whose norm lies within an interval. In other words, we estimate the number of Gaussian primes of the form a + bi such that  $a^2 + b^2 \leq r^2$ . We follow this line of thinking in generalizing further to primes in  $\mathbb{Z}[\sqrt{2}]$ , where we take the absolute



FIGURE 4. Norm regions NR(1) (blue) and NR(2) (brown), and the asymptotes  $y = \pm x/\sqrt{2}$  (green).

value of the norm as it can be negative. Here, we estimate the number of primes of the form  $a + b\sqrt{2}$  such that

$$|a^2 - 2b^2| \le r^2. (2.3)$$

**Definition 2.5.** Let  $NR(r^2)$  be the norm region defined by  $|x^2 - 2y^2| \le r^2$  where  $x, y \in \mathbb{R}$ .

Clearly,  $NR(r^2)$  contains all the points  $(a, b) \in \mathbb{Z}[\sqrt{2}]$  that satisfy (2.3), so our goal is then to estimate the number of primes within  $NR(r^2)$ . However, unlike the disk norm region  $x^2 + y^2 \leq r^2$  we use for counting the Gaussian primes,  $NR(r^2)$  is unbounded with infinite area as shown in Figure 4. Thus, there might be infinitely many primes within  $NR(r^2)$ . To verify this statement, we require the following definition.

**Definition 2.6.** Let NC(c) denote the "c-norm curve" or the graph of the equation  $x^2 - 2y^2 = c$ .

**Lemma 2.7.** If there exists a prime in some NC(c) for some c, then there are infinitely many primes in the same curve.

*Proof.* Suppose that there is a prime  $a + b\sqrt{2} \in NC(c)$  for some c. This implies  $a^2 - 2b^2 = c$ . Since the norm in  $\mathbb{Z}[\sqrt{2}]$  is multiplicative, we know that for any non-negative integer k,

$$N((a+b\sqrt{2})(1+\sqrt{2})^{2k}) = N(a+b\sqrt{2})N(1+\sqrt{2})^{2k}$$
  
=  $c(-1)^{2k}$   
=  $c.$ 

Hence,  $a + b\sqrt{2}$  and  $(a + b\sqrt{2})(1 + \sqrt{2})^{2k}$  are associates. By Definition 1.3, we know that  $(a + b\sqrt{2})(1 + \sqrt{2})^{2k}$  is also a prime for any non-negative integer k.  $\Box$ 

**Corollary 2.8.** If  $r^2 \ge 2$ , there exists infinitely many primes within the norm region  $NR(r^2)$ .

*Proof.* Notice that the first norm curve that contains at least a prime is NC(2) because  $\sqrt{2}$  lies on the curve. Then, Lemma 2.7 implies that there are infinitely many primes on the same curve. Hence, as any norm region  $NR(r^2)$  where  $r^2 \ge 2$  contains NC(2), there are infinitely many primes within  $NR(r^2)$ .

According to Corollary 2.8, it is impossible to approximate the number of primes in  $\mathbb{Z}[\sqrt{2}]$ . However, as we notice that the number of norm curves within  $NR(r^2)$ is finite, we now shift our goal to estimate the number of norm curves within norm region  $NR(r^2)$  that contains primes in  $\mathbb{Z}[\sqrt{2}]$ . For convenience, if any elements lie on the same norm curve, we say that they are in the same *family*.

**Theorem 2.9.** The number of families of primes in  $\mathbb{Z}[\sqrt{2}]$  within  $NR(r^2)$  is about

$$\frac{r^2}{\log r} + \frac{r}{2\log r} + 1.$$

*Proof.* According to Definition 1.3 (3), the number of primes in  $\mathbb{Z}[\sqrt{2}]$  within  $NR(r^2)$  that lie on the positive real axis is the same as the number of ordinary primes  $\equiv 3, 5 \pmod{8}$  that are less than r. Hence, by the PNT and Theorem 2.3, there are approximately

$$2 \cdot \frac{r}{\varphi(8)\log r} = \frac{r}{2\log r} \tag{2.4}$$

such primes. Since, for each prime p on positive real axis, -p and p supply the same family  $NC(p^2)$ , there are exactly  $r/2\log r$  families within  $NR(r^2)$  when considering the entire real axis.

Similarly, as for Definition 1.3 for primes of the form (2), there are approximately

$$2 \cdot 2 \cdot \frac{r^2}{\varphi(8)\log r^2} = \frac{r^2}{\log r}$$
(2.5)

ordinary primes  $\equiv 1,7 \pmod{8}$  from  $-r^2$  to  $r^2$ . In this case, each norm p of the  $r^2/\log r$  norms supplies a unique family NC(p).

Combining (2.4), (2.5), and another family of primes containing  $\pm \sqrt{2}$ , NC(-2), there are approximately

$$\frac{r^2}{\log r} + \frac{r}{2\log r} + 1$$
 (2.6)

distinct families of primes within the region  $NR(r^2)$ .

This analysis confirms our observation that primes in  $\mathbb{Z}[\sqrt{2}]$  are most dense along the asymptotes  $y = \pm x/\sqrt{2}$ , since the region  $NR(r^2)$  for any r all straddle the asymptotes. The distribution of primes in  $\mathbb{Z}[\sqrt{2}]$  differs from the Gaussian primes which are most dense at the origin. This fact may have implications towards how to construct the longest walk possible in  $\mathbb{Z}[\sqrt{2}]$ .

2.2. Estimating the total number of integers. Now that we have an estimate on the number of families of primes within  $NR(r^2)$ , we want to determine the total number of families of integers within this region to make a statement about the probability of encountering a family of primes.

Similar to the number of primes in  $NR(r^2)$ , the number of integers within  $NR(r^2)$  is also infinite, so we aim to count the number of families instead. In order to do so, we refer to the following theorem from Bernays [B].

**Theorem 2.10.** Let  $f(x, y) = rx^2 + sxy + ty^2$  be defined on  $\mathbb{Z}^2$  such that  $r, s, t \in \mathbb{Z}$ and  $s^2 - rt$  is not square. Then, the number of positive integers less than n that can be expressed as f(x, y) is

$$O\left(\frac{n}{\sqrt{\log n}}\right).$$

By Theorem 2.10, taking  $f(x, y) = x^2 - 2y^2$ , we verify that  $0^2 - 1 \cdot (-2) = 2$  is not a square. Then, we can apply the theorem to find the number of positive integers c < n such that

$$c = x^2 - 2y^2 = N(x + y\sqrt{2})$$

for some  $x + y\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ . In other words, we have an estimate for the number of *c*-norm curves with 0 < c < n. We determine that the number of *c*-norm curves with -n < c < 0 is the same using Lemma 2.11.

**Lemma 2.11.** There exist  $a_1, b_1 \in \mathbb{Z}$  such that  $a_1^2 - 2b_1^2 = c$  if and only if there exist  $a_2, b_2 \in \mathbb{Z}$  such that  $a_2^2 - 2b_2^2 = -c$ .

*Proof.* Suppose  $a_1, b_1 \in \mathbb{Z}$  satisfy  $a_1^2 - 2b_1^2 = c$ . Then consider

$$(a_1 + b_1\sqrt{2})(1 + \sqrt{2}) = (a_1 + 2b_1) + (a_1 + b_1)\sqrt{2}.$$

Taking  $a_2 = a_1 + 2b_1$  and  $b_2 = a_1 + b_1$ , we verify that

$$\begin{aligned} u_2^2 - 2b_2^2 &= (a_1 + 2b_1)^2 - 2(a_1 + b_1)^2 \\ &= -a_1^2 + 2b_1^2 \\ &= -(a_1^2 - 2b_1^2) \\ &= -c. \end{aligned}$$

Given any  $a_2, b_2 \in \mathbb{Z}$  such that  $a_2^2 - 2b_2^2 = -c$ , we can perform the same change of variables to find integers  $a_1, b_1$ .

**Remark 2.12.** Lemma 2.11 can also be proven using the fact that the norm in  $\mathbb{Z}[\sqrt{2}]$  is multiplicative. Since  $N(1 + \sqrt{2}) = -1$ , we simply have

$$N(a_1 + b_1\sqrt{2})N(1 + \sqrt{2}) = -c = N((a_1 + b_1\sqrt{2})(1 + \sqrt{2})).$$

In geometrical terms, Lemma 2.11 tells us that if we have an integer that lies on the norm curve NC(c), then there also exists an integer that lies on the related norm curve NC(-c). Therefore, for any  $r \in \mathbb{N}$ , the number of norm curves within  $NR(r^2)$  is

$$O\left(\frac{2r^2}{\sqrt{\log r^2}}\right) = O\left(\frac{r^2}{\sqrt{\log r}}\right).$$

2.3. Probability of a prime in  $\mathbb{Z}[\sqrt{2}]$ . Using the bound from Theorem 2.10, we can divide our estimate of the number of families of primes by the total number of families of integers in  $\mathbb{Z}[\sqrt{2}]$  to determine the probability of encountering a family of primes. Thus, our work culminates in the following estimate.

**Theorem 2.13.** The probability an integer  $z \in NR(r^2)$  lies on a prime c-norm curve is about

$$O\left(\frac{1}{\sqrt{\log r}}\right).$$

It is worth noting that analogous probabilities for  $\mathbb{Z}$  and  $\mathbb{Z}[i]$  are both

$$O\left(\frac{1}{\log r}\right),$$

which indicates a greater density of primes for  $\mathbb{Z}[\sqrt{2}]$  within its respective norm-region.

### 3. Proof of main results

Now we address the main question of this paper, which regards the possibility of a walk to infinity of a bounded step size along the primes in  $\mathbb{Z}[\sqrt{2}]$ . In the previous section, we have shown that the number of primes in  $\mathbb{Z}[\sqrt{2}]$  is infinite, yet the number of prime norm curves is not. In this section, we again view primes roughly as norm curves that converge to a single line. While these curves have infinitely many primes, their exponential growth renders them increasingly and negligibly sparse as our walk of linear growth rate progresses. This observation leads to the following theorem.

**Theorem 3.1.** It is impossible to perform a walk of a bounded step size k to infinity in  $\mathbb{Z}[\sqrt{2}]$  if the walk remains within some bounded distance r from the asymptote  $y = x/\sqrt{2}$ .

*Proof.* Because  $\mathbb{Z}[\sqrt{2}]$  is a unique factorization domain, so there is only one prime with norm c up to associates for any  $c \in \mathbb{Z}$ . For each norm curve  $a^2 - 2b^2 = c$ , there is only 1 solution up to associates, but these associates are spread out exponentially, so we do not expect them to affect our walk which takes steps of a bounded size. Therefore, after we walk on a prime from a norm curve, we are likely not using that norm curve again.

First, we can examine the number of norm curves  $a^2 - 2b^2 = c$  that intersect our disk of possible next steps, where the disk is centered at the current prime with radius k. If we can show that the number of these norm curves that are accessible from the previous prime in the walk grows slower than the number of steps in the walk, then we know that we eventually run out of possible primes to walk to.

Since each norm curve  $a^2 - 2b^2 = c_1$  is closer to the asymptote than  $a^2 - 2b^2 = c_2$ if and only if  $|c_1| < |c_2|$ , we only need to find the largest positive c for which  $a^2 - 2b^2 = c$  is within distance r from the asymptote as in Figure 5. This is achieved by examining the growth of the vertical gap between  $x^2 - 2y^2 = c$  and  $y = x/\sqrt{2}$ , where c > 0. We can represent half the vertical gap length as a function



FIGURE 5. Assorted norm curves converging within bounded asymptote region  $|y - \frac{1}{\sqrt{2}}x| = k$  (red).

of x as  $g(x) = (x - \sqrt{x^2 - c})/\sqrt{2}$ . Then, an upper bound and a lower bound of g(x) can be found as follows.

$$g(x) = \frac{x - \sqrt{x^2 - c}}{\sqrt{2}} = \frac{c}{\sqrt{2}(x + \sqrt{x^2 - c})} \le \frac{c}{\sqrt{2}x},$$
 (3.1)

$$g(x) = \frac{x - \sqrt{x^2 - c}}{\sqrt{2}} = \frac{c}{\sqrt{2}(x + \sqrt{x^2 - c})} \ge \frac{c}{2\sqrt{2}x}.$$
 (3.2)

Thus, the gap closes at a rate of  $\Theta(1/x)$ . However, as the actual gap we want is not vertical, but rather perpendicular to the asymptote, we have to take into account the slope of the asymptote. This gives the actual gap to be  $\sqrt{2}/\sqrt{3}$  of its vertical distance. Letting g'(x) be the function of half of the perpendicular gap length, we have

$$\frac{c}{2\sqrt{3}x} \le g'(x) \le \frac{c}{\sqrt{3}x}.\tag{3.3}$$

To find the largest positive c for which  $a^2 - 2b^2 = c$  is within r, we solve

$$r \ge \frac{c}{2\sqrt{3}x},\tag{3.4}$$

which is true when  $c \leq 2\sqrt{3}rx$ , so we estimate roughly at most  $2 \cdot 2\sqrt{3}rx = 4\sqrt{3}rx$  norm curves and possible choices of points for the next step.

Moreover, when we are x units from the origin, we take as few as x/k steps. This is a rough lower bound that occurs when we step only along one of the axes, taking the maximum k-length step each time. Then, the upper bound is also in the order of x, which is found by taking steps of a unit length and finishing on the line y = x. Thus, the number of steps taken is  $\Theta(x)$ . Even though both the number of norm curves and the number of steps estimates have the same growth,  $\Theta(x)$ , not all of the norm curves contains primes, so we can narrow down the number of possible norm curves by using Theorem 2.9.

From (3.4), we know that for any x, the largest c such that the c-norm curve is within r is  $4\sqrt{3}rx$ . Then, at any x, all the norm curves we can step on lie within the norm region  $NR(4\sqrt{3}rx)$ . Thus, there are approximately

$$\frac{4\sqrt{3}rx}{\log\sqrt{4\sqrt{3}rx}} + \frac{\sqrt{4\sqrt{3}rx}}{2\log\sqrt{4\sqrt{3}rx}} + 1 = \Theta\left(\frac{x}{\log x}\right)$$
(3.5)

curves that we can actually step on.

We note that the growth of the number of possible curves we can walk,  $\Theta(x/\log x)$ , is less than the growth of the number of steps,  $\Theta(x)$ . This means that a walk to infinity is impossible because the number of options is not growing as fast as what is required for the walk.

### 4. VISUALIZING PRIME WALKS

In the previous section, we have proven that it is impossible to have a prime walk to infinity in  $\mathbb{Z}[\sqrt{2}]$  if all the possible prime walks in  $\mathbb{Z}[\sqrt{2}]$  eventually remain within some bounded distance from the asymptote. In this section, we would like to present some evidence of why any possible walk is likely to stay close to the asymptote.

4.1. Random walks. We construct a random walk of primes in  $\mathbb{Z}[\sqrt{2}]$  with bounded step size k by starting at  $\sqrt{2}$  and searching randomly for the next prime within distance k with the previous one. We also require the next prime has a larger Euclidean norm than the previous one to let the walk always spread further away from the origin. Figure 6 shows a collection of such random walks with step size  $\sqrt{8}$  starting from the origin. It's interesting to discover that nearly all of the longest random walks are close to the asymptote  $x^2 - 2y^2 = 0$ . When the starting point is chosen randomly and not necessarily  $\sqrt{2}$ , the resulting random walks will also have the same property. In figure 7, we have a collection of random walks with step size  $\sqrt{8}$  starting from  $13 + 15\sqrt{2}$ . Again, the longest walk is very close to the asymptote in the first quadrant.

4.2. Prime walks for small step size. We now describe the algorithm used to visualize the actual prime walks for small step sizes  $(k = \sqrt{2}, \sqrt{8})$  starting from the origin. The algorithm consists of four main steps. The first step is to identify all the primes in a disk of radius n, which is achieved by checking whether each element in the disk satisfies Definition 1.2. Then, for each prime p, we find its k-neighbors which are the primes within the distance k from p. Next, we form the road network connecting all primes and their k-neighbors, and find the connected component of  $\sqrt{2}$  which is the prime closest to the origin. Finally, we can calculate a prime walk with a bounded step size k starting from the origin.

Figure 8 shows a prime walk with a step size  $\sqrt{2}$  from the origin, which is closed and ends at 31 + 24i. When we change k to  $\sqrt{8}$ , as in Figure 9, the prime walk does not end at x = 1500. This is not the best we can do but with this algorithm,



FIGURE 6. Random walks with  $k = \sqrt{8}$  starting at  $\sqrt{2}$ .



FIGURE 7. Prime walks with  $k = \sqrt{8}$  starting at  $13 + 15\sqrt{2}$  and the asymptote  $\sqrt{2}y - x = 0$  (straight line).

the time spent will increase dramatically if we go further. There is a clear trend that the longest walk is along the asymptote  $y = x/\sqrt{2}$ , which agrees with our assumption in Theorem 3.1. We have the following conjecture which removes such assumption.

**Conjecture 4.1.** It is impossible to perform a prime walk of a bounded step size to infinity in  $\mathbb{Z}[\sqrt{2}]$ .

### 5. Conclusion

By estimating the number of primes along the asymptote  $y = x/\sqrt{2}$ , we show that there is no prime walk to infinity with a bounded step size in  $\mathbb{Z}[\sqrt{2}]$  if any possible prime walk in  $\mathbb{Z}[\sqrt{2}]$  is guaranteed to stay within some bounded distance from  $y = x/\sqrt{2}$ . With some moat calculations, we demonstrate that the longest prime walk tend to cluster along the asymptote, which leads to Conjecture 4.1.



FIGURE 8. Prime walk with  $k = \sqrt{2}$  from the origin.



FIGURE 9. Prime walk with  $k = \sqrt{8}$  from the origin.

One direction for the future work is to prove Conjecture 4.1 without the assumption as in Theorem 3.1 and generalize this result to  $\mathbb{Q}[\sqrt{2}]$ . Figure 10, which we refer from Dekker [D, p. 13], shows that primes in  $\mathbb{Q}[\sqrt{2}]$  also cluster along the asymptote  $x^2 - 2y^2 = 0$  similar to the case in  $\mathbb{Z}[\sqrt{2}]$ , which suggests that there also exists no possible prime walk to infinity with a bounded step size in  $\mathbb{Q}[\sqrt{2}]$ . Another direction is to study prime walks in some quadratic integer rings which is not a unique factorization domain such as  $\mathbb{Z}[\sqrt{5}]$  since our proof of the main theorem requires the ring to be a UFD.



FIGURE 10. Primes (blue) and units (red) in  $\mathbb{Q}[\sqrt{2}]$ .

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