Avoiding 3-term Geometric Progressions in Non-Commutative Settings

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History

In 1961: Rankin introduced subsets of $\mathbb{N}$ avoiding 3-term geometric progressions: $\{n, nr, nr^2\}$ and $r \in \mathbb{N} \setminus \{1\}$.

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Modification by McNew improves to about $.722130$, using some exclusions and inclusions that yield a higher density overall

Improved bounds on the greatest possible upper density of such a set, between $.81841$ and $.81922$.

Goals of Talk

Study geometric-progression-free subsets in non-commutative algebra.

Our choice is the Hurwitz quaternion.

Goals of Talk

Consider bounds on supremum of upper densities of subset avoiding 3-term progression.

Calculate asymptotic density of Rankin's quaternion greedy set.

Attempt to construct and find density of greedy set avoiding 3-term progression.

Quaternions

Types of Quaternions

Definition

Quaternions constitute the algebra over the reals generated by units i, j, and k such that

$$i^2 = j^2 = k^2 = ijk = -1.$$

Written as $a + bi + cj + dk$ for $a, b, c, d \in \mathbb{R}$.

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Definition

$a + bi + cj + dk$ is in the Hurwitz Order, $\mathcal{H}$, if a, b, c, d are all integers or halves of odd integers.

Types of Quaternions

Examples

$1 + 2i - k$ and $\frac{3}{2} - \frac{1}{2}i + \frac{1}{2}j + \frac{5}{2}k$ are in $\mathcal{H}$, whereas $\frac{1}{2} + i$ is not.

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Definition

The Norm of $Q = a + bi + cj + dk$ is given by
$\text{Nm}(Q) = a^2 + b^2 + c^2 + d^2$.

Units and Factorization

There are 24 units in the Hurwitz Order, namely $\pm 1, \pm i, \pm j, \pm k$ and $\pm \frac{1}{2} \pm \frac{1}{2}i \pm \frac{1}{2}j \pm \frac{1}{2}k$.

Theorem

Let $Q \in \mathcal{H}$ of norm q. For any factorization of q into a product $p_0 p_1 \cdots p_k$ of integer primes, there is a factorization

$$Q = P_0 P_1 \cdots P_k$$

where $P_i \in \mathcal{H}$ and $\mathrm{Nm}(P_i) = p_i$. We call such factorization modelled on $p_0 p_1 \cdots p_k$.

Furthermore, if Q is primitive (not divisible by $n > 1$), then the factorization is unique up to unit-migration, i.e. other factorizations modelled on $p_0 p_1 \cdots p_k$ are of the form $Q = P_0 U_1 U_1^{-1} P_1 U_2 \cdots U_k^{-1} P_k$, where $U_i's$ are units.

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Modelled on $3 \cdot 2$: $Q = (\frac{3}{2} - \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k)(1+i)$, others: $(\frac{3}{2} - \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k)u'u'^{-1}(1+i)$, where u' is a unit.

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In total, 48 different factorizations.

Counting Quaternions

Lemma

The number of Hurwitz quaternions of norm N is

$$|S(\{N\})| = 24 \sum_{2 \nmid d | N} d,$$

the sum of the odd divisors of N multiplied by 24.

Counting Quaternions

$\Longrightarrow$ Calculate the number of Hurwitz quaternions up to certain norm.

Lemma

The number of Hurwitz quaternions with norm less than or equal to $N \in \mathbb{R}^+$ is

$$|S(N)| \; = \; \pi^2 N^2 + O(N \log N)$$

Goal

Goal

Construct and bound large subsets of Hurwitz quaternions free of three-term geometric progressions, i.e. exclude any

$$Q,\ QR,\ QR^2$$

where $Q, R \in \mathcal{H}$ and $\mathrm{Nm}(R) \neq 1$.

Rankin's Quaternion Greedy Set

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Consider the set of HQs, norm in $G_3^*(\mathbb{Z})$ – the greedy set of non-negative integers avoiding 3-term geometric progressions.

Definition

Define Q_{Ran} as the set of Hurwitz quaternions whose norms are in $G_3^*(\mathbb{Z})$.

Rankin's Quaternion Greedy Set

Since norms of HQs in Q_{Ran} (which is $G_3^*(\mathbb{Z})$) form no 3-term geometric progressions, then HQs themselves in Q_{Ran} form no 3-term geometric progressions.

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Since norms of HQs in Q_{Ran} (which is $G_3^*(\mathbb{Z})$) form no 3-term geometric progressions, then HQs themselves in Q_{Ran} form no 3-term geometric progressions.

An integer in $G_3^*(\mathbb{Z}) \Leftrightarrow$ its prime powers in $A_3^*(\mathbb{Z})$, the greedy set of non-negative integers avoiding 3-term arithmetic progressions.

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Lemma

If p odd, the proportion of Hurwitz quaternions whose norm is divisible by p^n but not p^{n+1} is

$$\frac{p^{n+3} - p^{n+2} - p^2 + 1}{(p-1)p^{2n+2}}.$$

If $p = 2$, this proportion is instead

$$\frac{2^2 - 1}{2^2 2^{2n}} = \frac{3}{2^{2n+2}}.$$

Rankin's Quaternion Greedy Set

Proof:

$$\frac{\text{\# of HQ with norm div by } p^n \text{ - \# of HQ with norm div by } p^{n+1}}{\text{\# of HQ with norm} \leq N}$$

$$= \frac{(\text{\# of HQ with norm } p^n)(\text{\# of HQ with norm} \leq N/p^n)}{24 \cdot (\text{\# of HQ with norm} \leq N)}$$

$$- \frac{(\text{\# of HQ with norm } p^{n+1})(\text{\# of HQ with norm} \leq N/p^{n+1})}{24 \cdot (\text{\# of HQ with norm} \leq N)}$$

$$= \frac{\left(\sum_{2\nmid d|p^n} d\right) \pi^2 (N/p^n)^2 - \left(\sum_{2\nmid d|p^{n+1}} d\right) \pi^2 (N/p^{n+1})^2}{\pi^2 N^2} + \text{error},$$

For p odd

$$\sum_{2\nmid d|p^n} d = 1 + \cdots + p^n = (p^{n+1} - 1)/(p-1).$$

For $p = 2$, the quantity is 1.

Rankin's Quaternion Greedy Set

Taking limit $N \to \infty$ gives the porportion of HQs whose norm is exactly divisible by p^n. For p odd, the proportion is

$$\frac{p^{n+3} - p^{n+2} - p^2 + 1}{(p-1)p^{2n+2}},$$

for $p = 2$, the proportion is

$$\frac{2^2 - 1}{2^2 2^{2n}} = \frac{3}{2^{2n+2}}.$$

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Theorem

The asymptotic density of Q_{Ran} is

$$d(Q_{\text{Ran}}) = \left(\prod_{\underline{p odd}} \left[\sum_{n \in A_3^*(\mathbb{Z})} \frac{p^{n+3} - p^{n+2} - p^2 + 1}{(p-1)p^{2n+2}} \right] \right) \left(\sum_{n \in A_3^*(\mathbb{Z})} \frac{3}{2^{2n+2}} \right)$$

which is converging and estimated at .771245.

The proof involves a Chinese Remainder Theorem-type argument.

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The proof involves a Chinese Remainder Theorem-type argument.

Higher than .71974 from Rankin's greedy set for int case, due to var in num of HQs per norm.

Supremum of Upper Density

Apart from studying Q_{Ran}, also study the supremum of upper densities of sets avoiding 3-term progression.

Definition (Upper Density)

The upper density of a set $A \subset \mathcal{H}$ is

$$\limsup_{N\to\infty} \frac{|A \cap \{q \in \mathcal{H} : \text{Norm}[q] \leq N\}|}{|\{q \in \mathcal{H} : \text{Norm}[q] \leq N\}|}$$

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Estimate the lower and upper bounds on the supremum.

Lower Bound on the Supremum

For lower bound, construct a set with large upper density. Consider
$T_N = \left(\frac{N}{48}, \frac{N}{45}\right] \cup \left(\frac{N}{40}, \frac{N}{36}\right] \cup \left(\frac{N}{32}, \frac{N}{27}\right] \cup \left(\frac{N}{24}, \frac{N}{12}\right] \cup \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$

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Use T_N to estimate the lower bound.

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$$\lim_{N\to\infty} \frac{\{q \in \mathcal{H} : \text{Norm}[q] \in S_N\}}{\{q \in \mathcal{H} : \text{Norm}[q] \le N\}} \approx .946589.$$

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Fix r, consider quaternions up to norm N. Suppose Nm$(b) \leq N/2^2$ and Nm(b) has no factor of 2, b, br, br^2 forms a progression and one excluded.

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Repeating this process, we get an upper bound of

$$\lim_{N\to\infty} 1 - (\text{Proportion not div by 2})\left(\frac{\text{\# of HQs up to } N/2^2}{\text{Number of HQs up to } N} + \frac{\text{\# of HQs up to } N/2^5}{\text{Number of HQs up to } N} + \frac{\text{\# of HQs up to } N/2^8}{\text{Number of HQs up to } N} + \cdots\right)$$

$$= 1 - \frac{3}{2^6} \cdot \sum_{i=0}^{\infty} \frac{1}{2^{6i}} \approx .952381.$$

Supremum of Upper Density

Theorem

Let m_{Hur} be supremum of upper densities of subsets of Q_{Hur} containing no 3-term geometric progressions. Then

$$.946589 \leq m_{\text{Hur}} \leq .952381.$$

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Quaternion Greedy Set

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Proposition

If $r \in \mathbb{Z}^+$ cannot be represented as the sum of three integer squares, then $\exists Q$ of norm $\text{Nm}(Q) = r^2$ s.t. $Q \neq UR^2$ for any unit $U \in Q_{\text{Hur}}$ and any Hurwitz quaternion R of norm $\text{Nm}(R) = r$.

Quaternion Greedy Set

Unlike Q_{Ran}, norms of some HQs form 3-term progressions with norms of those in $G_3^*(\text{Hur})$, HQs themselves don't form 3-term progressions with those in $G_3^*(\text{Hur})$.

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E.g. $7i$ has norm 49. 49 forms 3-term geometric progressions with 1 and 7, but $7i$ doesn't form 3-term geometric progressions.

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Inclusion of $7i$ and others $\Rightarrow$ exclusion of some HQs of norm $49 * 7 = 343$.

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Inclusion of $7i$ and others $\Rightarrow$ exclusion of some HQs of norm $49 * 7 = 343$.

343 is included in Q_{Ran}. Inclusions and exclusions continue for all powers of 7.

Quaternion Greedy Set

Any integer with an odd divisor greater than 23 poses the same problem.

Proposition

If $d|r$, $d > 23$ odd, then $\exists Q$ of $\mathrm{Nm}(Q) = r^2$ s.t. $Q \neq UR^2$ for any unit U and Hurwitz quaternion R of $\mathrm{Nm}(R) = r$.

Quaternion Greedy Set

These properties create large number of inclusions and exclusions between $G_3^*(\text{Hur})$ and Q_{Ran}, hard to keep track and unable to construct or bound $G_3^*(\text{Hur})$.

Future Work

Consider geometric-progression-free subsets of further noncommutative settings, e.g. matrix rings.

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Many questions left to be answered here. E.g. our investigation of the HQs naturally raises first two questions below.

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Can greedily constructed set of HQs avoiding progressions $G_3^*(\text{Hur})$ be described explicitly?

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Question

How about subsets of the Hurwitz quaternions which avoid geometric progressions with quaternion ratios not necessarily in Hurwitz order?

Question

Can we generalize to octonions, and thus lose associativity as well as commutativity?

Q&A

Any Questions?