The Accelerated Zeckendorf Game

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Zeckendorf Decomposition

Defining the Fibonacci numbers as
$F_1 = 1, F_2 = 2, F_n = F_{n-1} + F_{n-2}$,

Theorem (Zeckendorf)

Every positive integer has a unique representation as a sum of non-adjacent Fibonacci numbers. This is referred to as that number's Zeckendorf decomposition.

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- Bins $F_1, F_2, F_3, \dots$, start with n pieces in F_1.
- Two players, alternating moves, last to move wins.
- Two types of moves: combining and splitting.

Move Types

Two types of combining moves:

- Take two pieces in F_1 and place one piece in F_2, denoted C_1.
- For $k \geq 2$, take a piece from F_{k-1} and a piece from F_k and place a piece in F_{k+1}, denoted C_k.

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Two types of splitting moves:

- Take two pieces in F_2 and place one piece in F_1 and one piece in F_3, denoted S_2.
- For $k \geq 3$, take two pieces from F_k and place one piece in F_{k-2} and one piece in F_{k+1}, denoted S_k.

Same setup as the Zeckendorf Game, with the addition of *accelerated moves*:
A player can repeat their move multiple times during their turn, denoted $m \cdot C_k$ or $m \cdot S_k$, where C_k/S_k is the move and m is the number of times it's performed.

Sample Game

Start with $n = 8$ pieces in F_1.

0	0	0	0	8
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move : Player 1, $3 \cdot C_1$.

Sample Game

Start with $n = 8$ pieces in F_1.

0	0	0	3	2
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move : Player 2, $1 \cdot C_1$.

Sample Game

Start with $n = 8$ pieces in F_1.

0	0	0	4	0
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move : Player 1, $2 \cdot S_2$.

Sample Game

Start with $n = 8$ pieces in F_1.

0	0	2	0	2
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

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Next move : Player 1, $1 \cdot S_3$.

Sample Game

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Next move : Player 1, $1 \cdot C_4$.

Sample Game

Start with $n = 8$ pieces in F_1.

1	0	0	0	0
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Player 1 wins!

Helpful Terminology

- $i_{\max}(n)$ is the index of the largest Fibonacci number in the Zeckendorf decomposition of n.
- $\delta_i(n)$ denotes the number of F_i's in the Zeckendorf decomposition of n.
- $Z(n)$ denotes the number of terms in the Zeckendorf decomposition of n.
- $IZ(n)$ denotes the sum of the indices of the terms in the Zeckendorf decomposition of n; i.e. $IZ(n) = \sum_{i=1}^{i_{\max}(n)} i \cdot \delta_i(n)$.
- A game state will be represented by $(a_{i_{\max}(n)}, a_{i_{\max}(n)-1}, \ldots, a_3, a_2, a_1)$, where a_j is the current number of pieces in F_j.

Results That Carried Over

Every Accelerated game has an associated base Zeckendorf game by decoupling all accelerated moves. As such, there are a few findings that carry over from the Zeckendorf Game into the Accelerated Zeckendorf Game:

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Every game terminates within a finite number of moves at the Zeckendorf decomposition.

Theorem (Cusenza et al.)

Let $a_i = F_{i+2} - i - 2$. The upper bound of the game is given by $\sum_{i=1}^{i_{max}(n)} a_i \cdot \delta_i(n)$, which is at most $\frac{3+\sqrt{5}}{2}n - IZ(n) - \frac{1+\sqrt{5}}{2}Z(n)$.

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Theorem (Garcia-Fernandezsesma et al.)

$i_{max}(n) - 1$ *is a sharp lower bound on the number of moves in the Accelerated Zeckendorf Game.*

For contrast, in the Zeckendorf Game,

Theorem (Baird-Smith, Epstein, Flint and Miller)

The shortest game, achieved by greedy algorithm, arrives at the Zeckendorf decomposition in $n - Z(n)$ *moves.*

Sharp Lower Bound Proof: Part 1

Proof: First, we showed $i_{\max}(n) - 1$ is a lower bound:

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All moves advance a piece at most 1 bin forward. As such, at least $i_{\max}(n) - 1$ moves need to be made to get a piece from F_1 to $F_{i_{\max}(n)}$.

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- $F_{k-2} \cdot C_1$ brings $(0, \ldots, 0, 0, F_k)$ to $(0, \ldots, 0, F_{k-2}, F_{k-3})$.
- $F_{j-1} \cdot C_i$ brings $(0, \ldots, 0, 0, F_j, F_{j-1}, 0, \ldots, 0)$ to $(0, \ldots, 0, F_{j-1}, F_{j-2}, 0, 0, \ldots, 0)$.

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This gives us the series of $i_{\max}(F_k) - 1$ moves $(F_{k-2} \cdot C_1, F_{k-3} \cdot C_2, \dots, F_1 \cdot C_{k-2}, F_0 \cdot C_{k-1})$ that takes us to $(F_0, F_{-1}, 0, \dots, 0) = (1, 0, 0, \dots, 0)$, the Zeckendorf decomposition of F_k.

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$F_{k_1-2} \cdot C_1, \ldots, F_0 \cdot C_{k_1-1},$
$F_{k_2-2} \cdot C_1, \ldots, F_0 \cdot C_{k_2-1},$
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$\vdots$
$F_{k_m-2} \cdot C_1, \ldots, F_0 \cdot C_{k_m-1}$

We can then group together all like combine moves:

$(\sum_{j=1}^{m} F_{k_j-2}) \cdot C_1, \ldots, (\sum_{j=1}^{m} F_{k_j-k_m}) \cdot C_{k_m-1},$
$(\sum_{j=1}^{m-1} F_{k_j-k_m+1}) \cdot C_{k_m}, \ldots, (\sum_{j=1}^{m-1} F_{k_j-k_{m-1}}) \cdot C_{k_{m-1}-1},$
$\vdots$
$F_{k_1-k_2+1} \cdot C_{k_2}, \ldots, F_0 \cdot C_{k_1-1}$

Sharp Lower Bound Example Game

Start with $n = 11 = 8 + 3$ pieces in F_1.

0	0	0	0	11
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move : Player 1, $(3 + 1) \cdot C_1$.

Sharp Lower Bound Example Game

Start with $n = 11 = 8 + 3$ pieces in F_1.

0	0	0	4	3
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move : Player 2, $(2 + 1) \cdot C_2$.

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0	0	3	1	0
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move : Player 1, $1 \cdot C_3$.

Sharp Lower Bound Example Game

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0	1	2	0	0
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Start with $n = 11 = 8 + 3$ pieces in F_1.

1	0	1	0	0
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Game ended in 4 moves.

Conjecture on Winning Strategy

Conjecture (Garcia-Fernandezsesma et al.)

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Tested up to $n = 140$ by computer (see https://github.com/ThomasRascon/Accelerated-Zeckendorf-Game), and in stark contrast to the Zeckendorf Game:

Theorem (Baird-Smith, Epstein, Flint and Miller)

For all $n > 2$, Player 2 has the winning strategy for the Zeckendorf Game.

Accelerated Zeckendorf Game Strategy Methodology

We considered winning and losing states; states where the current player does or doesn’t have a winning strategy, respectively.

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Allows for colorings of game state graphs:

- All states that lead to a losing state are winning states.
- A losing state leads to only winning states.
- A winning state leads to at least one losing state.
- The final game state is a losing state.

Steps Towards Proving Conjecture

Conjecture (Garcia-Fernandezsesma et al.)

If Player 2 has the winning strategy, then all game states of the form $(0, \ldots, 0, k, 0, n - 3k)$ *are losing states.*

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Theorem (Garcia-Fernandezsesma et al.)

The above conjecture implies Player 1's winning strategy for $n > 9$.

Proof of Conjecture Implication: Auxiliary Lemma

Proof: We first prove an auxiliary lemma:

Lemma (Garcia-Fernandezsesma et al.)

Let k be an odd positive integer and let $n \geq 2k$. Assume all game states of the form $(0, \ldots, 0, i, 0, n - 3i)$ with $i < k$ are losing states. Then $n \geq 3k$ and $(0, \ldots, 0, k, 0, n - 3k)$ is the only losing state reachable from $(0, \ldots, 0, k, n - 2k)$ by a single accelerated move.

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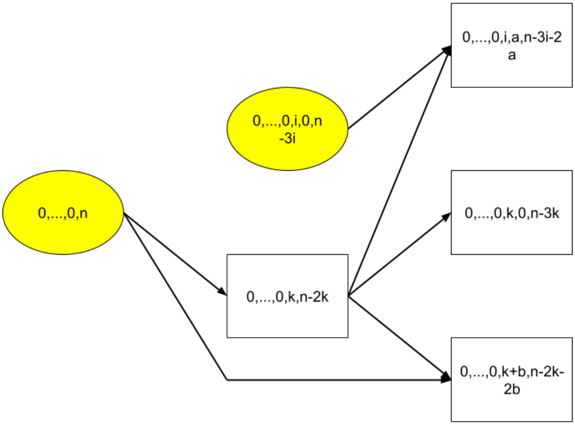
This can be shown graphically.

Proof of Auxiliary Lemma



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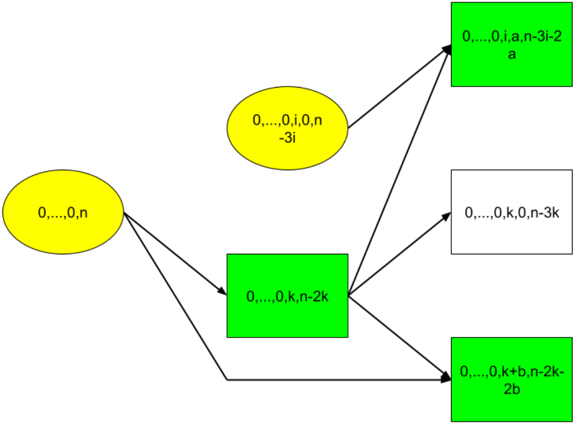

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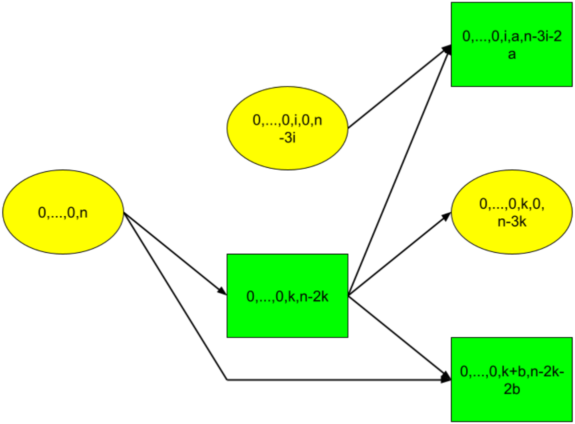

Proof of Conjecture Implication

Assume the losing state conjecture and assume there exists an $n > 9$ such that player 2 has the winning strategy. Since $n > 9$, there exists an odd k such that $2k \leq n < 3k$. By assumption, all states of the form $(0, \ldots, 0, i, 0, n - 3i)$ with $i < k$ are losing states. Therefore, $n \geq 3k$, contradiction.

Steps Towards Losing State Conjecture

If $(0, \ldots, 0, k, 0, n-3k)$ can be proven to be losing for even k, then with the auxiliary lemma the conjecture is proven.

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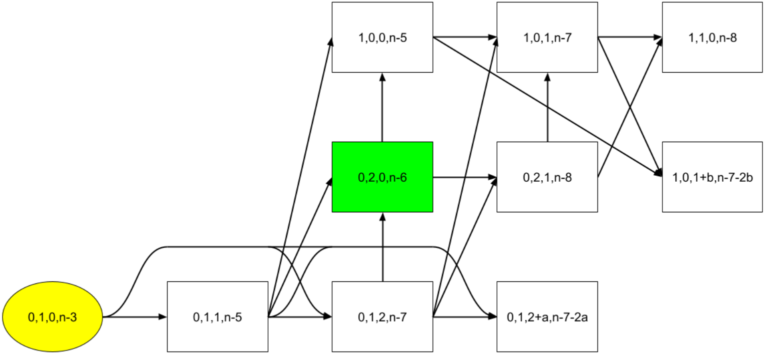
If $(0, \ldots, 0, k, 0, n-3k)$ can be proven to be losing for even k, then with the auxiliary lemma the conjecture is proven.
We were able to prove the losing state for $k = 2$ and $k = 4$ assuming Player 2's winning strategy by contradiction.

$(0, \ldots, 0, 2, n-6)$ Losing



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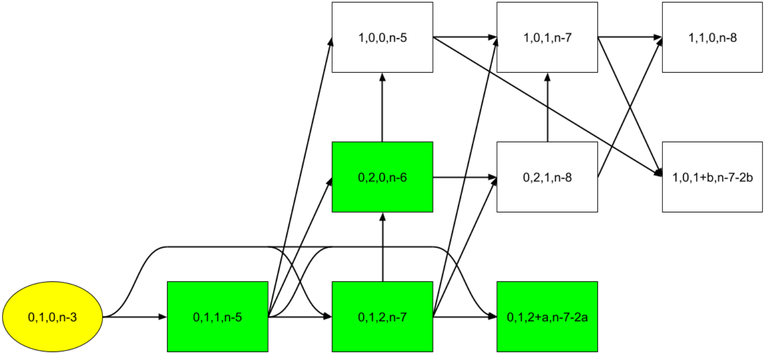

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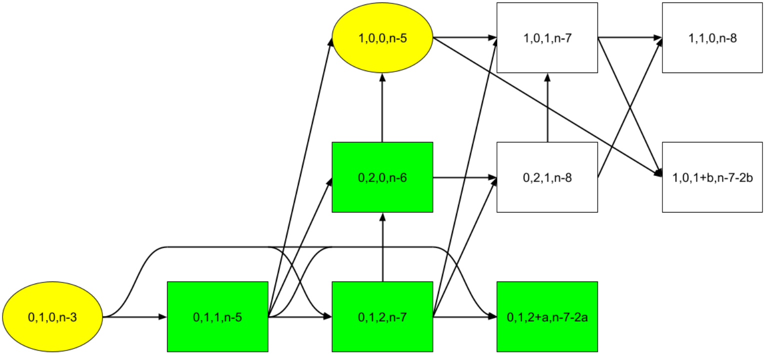

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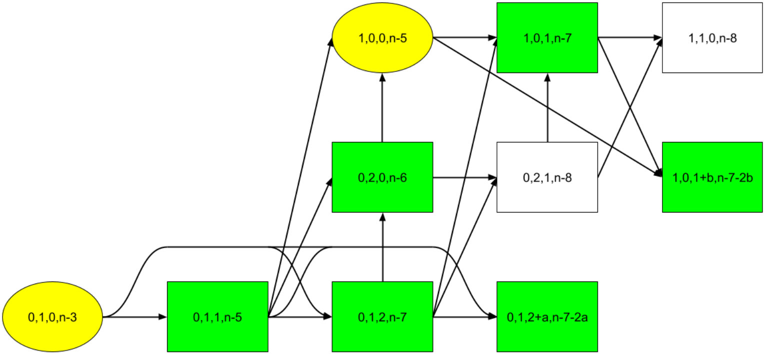

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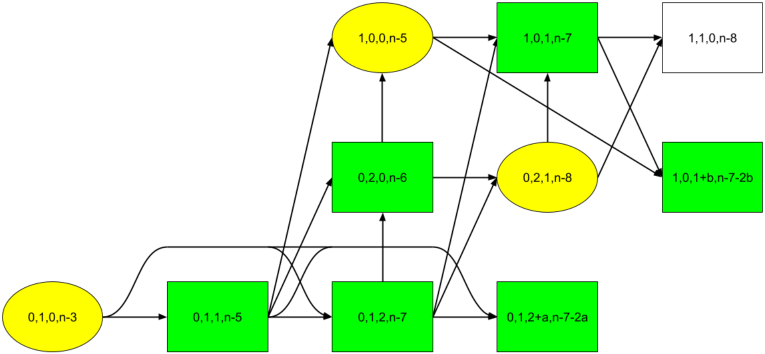

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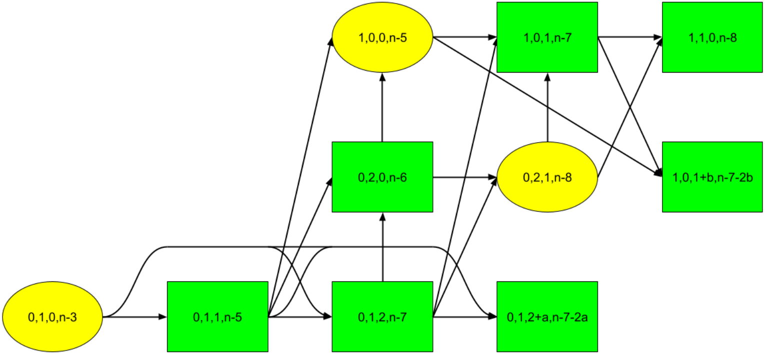

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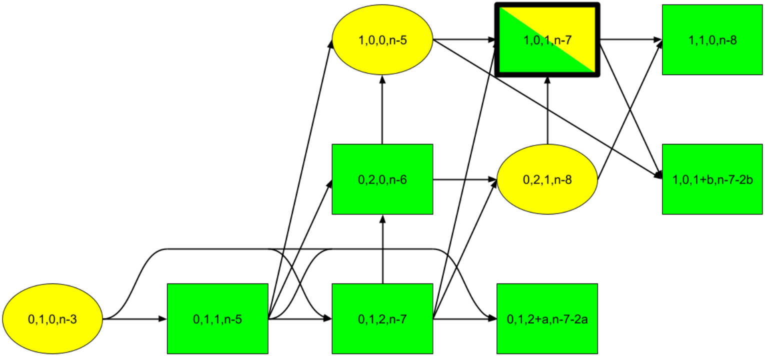
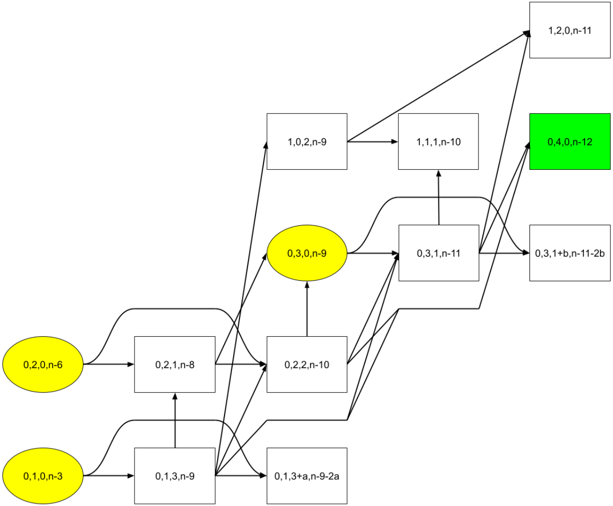


$(0, \ldots, 0, 4, n-12)$ **Losing**



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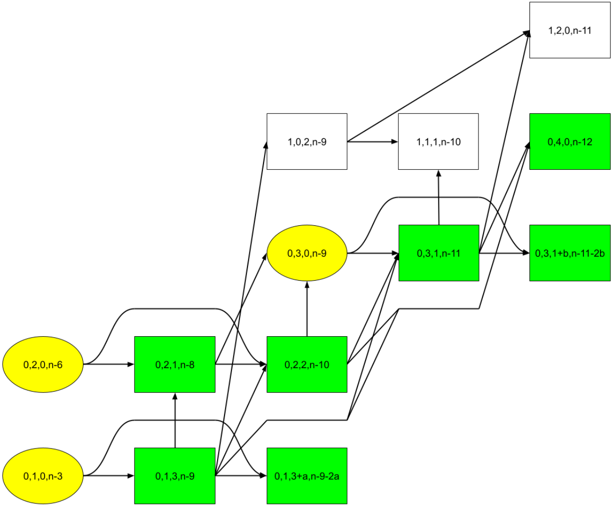

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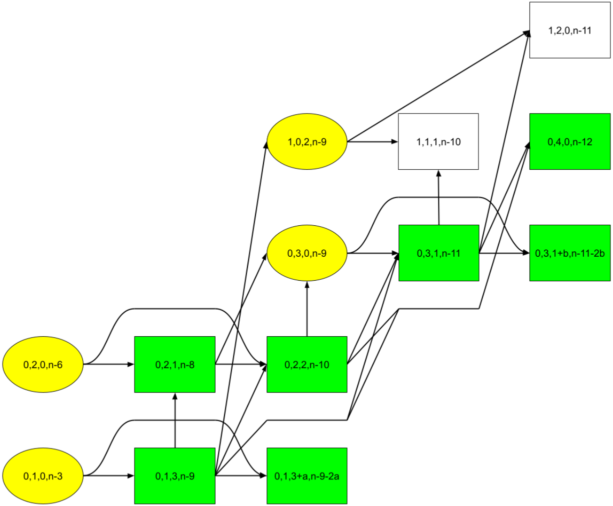

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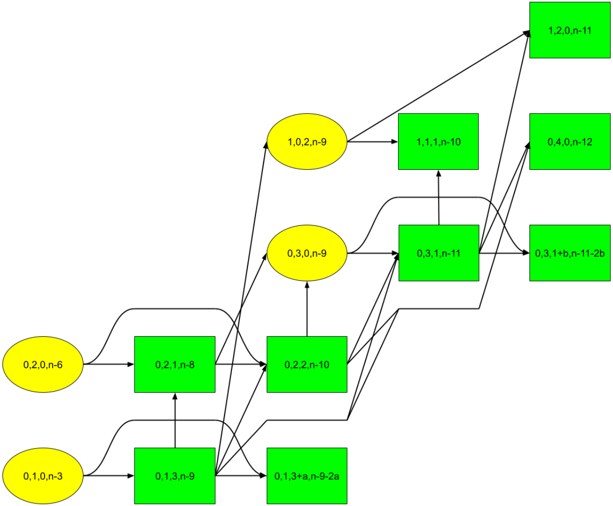

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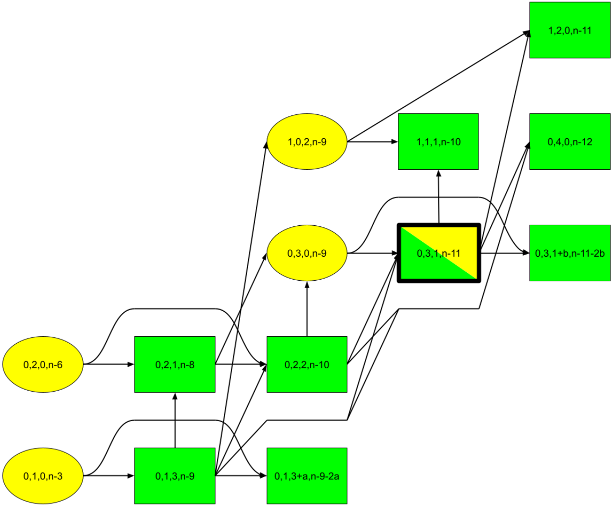

Statistical Conjectures

Conjecture (Garcia-Fernandezsesma et al.)

As n goes to infinity, the number of moves in a random Accelerated Zeckendorf Game on n, when all legal moves are equally likely, converges to a Gaussian.

Closely matches similar finding for the Zeckendorf Game.

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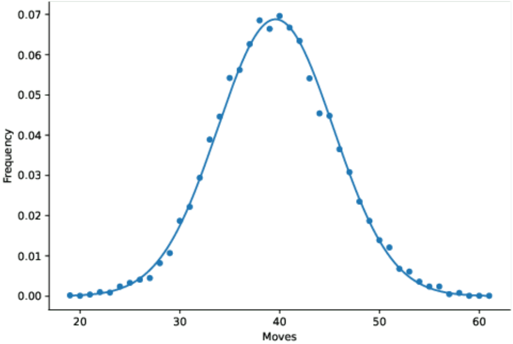
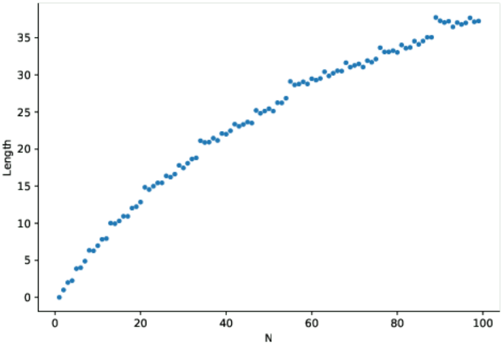
Conjectures based on simulations run by Thomas Rascon (see `https://github.com/ThomasRascon/Accelerated-Zeckendorf-Game`).

Gaussian Graph



Graph of the frequency of the number of moves in $9,999$ simulations of the Accelerated Zeckendorf Game with random move where each legal move has a uniform probability for $n = 100$ with the best fitting Gaussian (mean ≈ 39.6, STD ≈ 5.8).

Average Game Length Graph



Graph of the average number of moves in the Accelerated Zeckendorf Game with random uniform moves with 9,999 simulations with n varying from 1 to 99.

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- Extend Accelerated framework to other variants of the Zeckendorf Game.

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  - The Generalized Zeckendorf Game
  - The Fibonacci Quilt Game
  - The Reversed Zeckendorf Game

References

- P. Baird-Smith, A. Epstein, K. Flynt and S. J. Miller, *The Zeckendorf Game*, Combinatorial and Additive Number Theory III, CANT, New York, USA, 2017 and 2018, Springer Proceedings in Mathematics & Statistics, **297** (2020), 25–38. https://arxiv.org/pdf/1809.04881.

- A. Cusenza, A. Dunkelberg, K. Huffman, D. Ke, M. McClatchey , S. J. Miller, C. Mizgerd, V. Tiwari, J. Ye, and X. Zheng, *Bounds on Zeckendorf Games*, The Fibonacci Quarterly, **60.1** (2022), 57–71. https://fq.math.ca/Papers/60-1/miller01152021.pdf.

- D. Garcia-Fernandezsesma, S. J. Miller, T. Rascon, R. Vandegrift, and A. Yamin, *The Accelerated Zeckendorf Game*, The Fibonacci Quarterly, **62.1** (2024), 3–14. https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/zeckgame_accelerated_poly23v20.pdf