The Accelerated Zeckendorf Game

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About Me

- Bengali-American, New Yorker
- Undergrad at Stony Brook University
- Currently 2nd year PhD student at CUNY Graduate Center
- Instructor at Hunter College
- DRP organizer
- Research interest: Number Theory!

About DRP

https://sites.google.com/view/cunydrp

About Polymath

Polymath Jr.

Collaborative mathematical research for undergraduate students

https://geometrynyc.wixsite.com/polymathreu

About this project

- Part of the 2022 Summer Polymath Jr. program
- Faculty mentor: Steven Miller (Williams College)
- Graduate student mentor: me
- Undergraduate researchers:
  - Diego Garcia-Fernandezsesma (Boston University)
  - Thomas Rascon (UCSD)
  - Risa Vandergrift (University of Minnesota)

Zeckendorf Decomposition

Fibonaccis: $F_0 = 1, F_1 = 1, F_{n+2} = F_{n+1} + F_n$.

1

Zeckendorf Decomposition

Fibonaccis: $F_0 = 1, F_1 = 1, F_{n+2} = F_{n+1} + F_n$.

1, 2

Zeckendorf Decomposition

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1, 2, 3

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1, 2, 3, 5, 8, 13...

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

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Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

Zeckendorf Decomposition: Greedy Algorithm

Greedy Algorithm

- Start with $n > 0$.
- Subtract off the largest Fibbonacci number less than n.
- Repeat.

Example:
$2023 = 1597 + 377 + 34 + 13 + 2 = F_{16} + F_{13} + F_8 + F_6 + F_2$.

Zeckendorf Decomposition: Greedy Algorithm

Greedy Algorithm

- Start with $n > 0$.
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- Repeat.

Example:
$2023 = 1597 + 377 + 34 + 13 + 2 = F_{16} + F_{13} + F_8 + F_6 + F_2$.

Justification:

- We will never subtract off consecutive Fibonacci numbers (say F_j and then F_{j-1}) because we could have subtracted off their sum $F_{j+1} = F_j + F_{j-1}$ to begin with.
- We will never subtract off the same Fibonacci number twice because $F_{j+1} < 2 \cdot F_j$.

Zeckendorf Decomposition: Uniqueness

Uniqueness of Zeckendorf Decomposition:
Proof by Contradiction

- Exercise.
- Use Lemma: The sum of distinct, non-consecutive Fibonacci numbers all less than F_j is less than F_{j+1}.
  - Prove Lemma by induction.

The Zeckendorf Game

Zeckendorf Game

The Zeckendorf Game: Rules

- Two player game, alternate turns, last to move wins.

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- Bins $F_1, F_2, F_3, \dots$, start with N pieces in F_1 and others empty.

The Zeckendorf Game: Rules

- Two player game, alternate turns, last to move wins.
- Bins $F_1, F_2, F_3, \dots$, start with N pieces in F_1 and others empty.
- A turn is one of the following moves:
  - ◇ If have two pieces on F_k can remove and put one piece at F_{k+1} and one at F_{k-2}
    - (if $k = 1$ then $2F_1$ becomes $1F_2$)
    - (if $k = 2$ then $2F_2$ becomes $1F_3 + 1F_1$)
  - ◇ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

Questions

Questions:

- Does the game end? How long?
- For each N who has the winning strategy?
- What is the winning strategy?

Sample Game

Start with 10 pieces at F_1, rest empty.

10	0	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

8	1	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

6	2	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $2F_2 = F_3 + F_1$

Sample Game

Start with 10 pieces at F_1, rest empty.

7	0	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

5	1	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_2 + F_3 = F_4$.

Sample Game

Start with 10 pieces at F_1, rest empty.

5	0	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

3	1	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

1	2	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_2 = F_3$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	1	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_3 + F_4 = F_5$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

No moves left, Player One wins.

Sample Game

Player One won in 9 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Sample Game

Player Two won in 10 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
2	0	1	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Games end

Theorem (Baird-Smith, Epstein, Flynt and Miller)

All games end in finitely many moves.

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Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: $-\sqrt{k-2} - \sqrt{k-1} + \sqrt{k} < 0.$
- Splitting: $-2\sqrt{k} + \sqrt{k-2} + \sqrt{k+1} < 0.$
- Adding 1's: $-2\sqrt{1} + \sqrt{2} < 0.$
- Splitting 2's: $-2\sqrt{2} + \sqrt{3} + \sqrt{1} < 0.$

Games end

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All games end at the Zeckendorf decomposition.

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All games end at the Zeckendorf decomposition.

Proof.

If it terminated elsewhere, there would either be duplicate terms or the recurrence would apply, by definition. So, there would still be a valid move and the game would not have terminated. □

Games Lengths: I

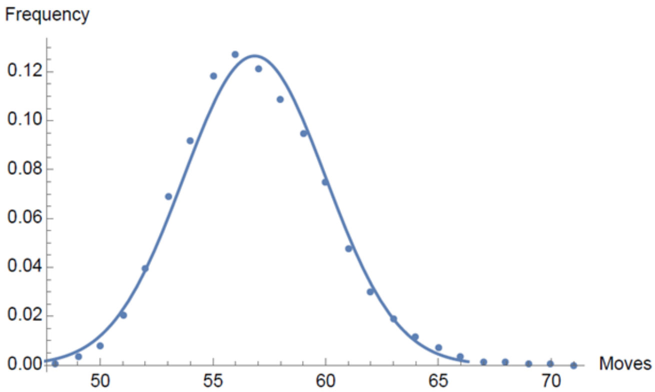
Theorem (Baird-Smith, Epstein, Flynt and Miller)

Upper bound: At most $n \log_\phi \left(n\sqrt{5} + 1/2\right)$ *moves.*

Fastest game: $n - Z(n)$ *moves (*$Z(n)$ *is the number of summands in n's Zeckendorf decomposition).*

From always moving on the largest summand possible (deterministic).

Games Lengths: II



Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian.
Natural conjecture....

Winning Strategy

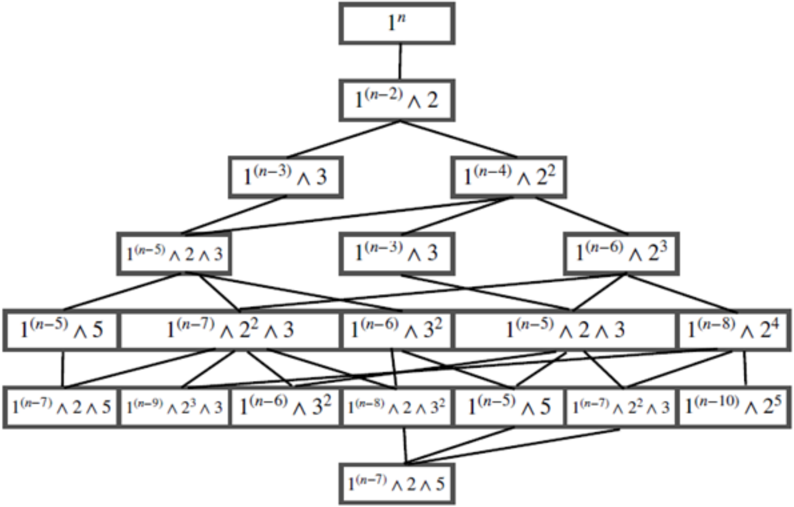
Theorem (Baird-Smith, Epstein, Flynt and Miller)

Player Two Has a Winning Strategy

Idea is to show if not, Player Two could steal Player One's strategy.

Non-constructive!

Sketch of Proof for Player Two's Winning Strategy



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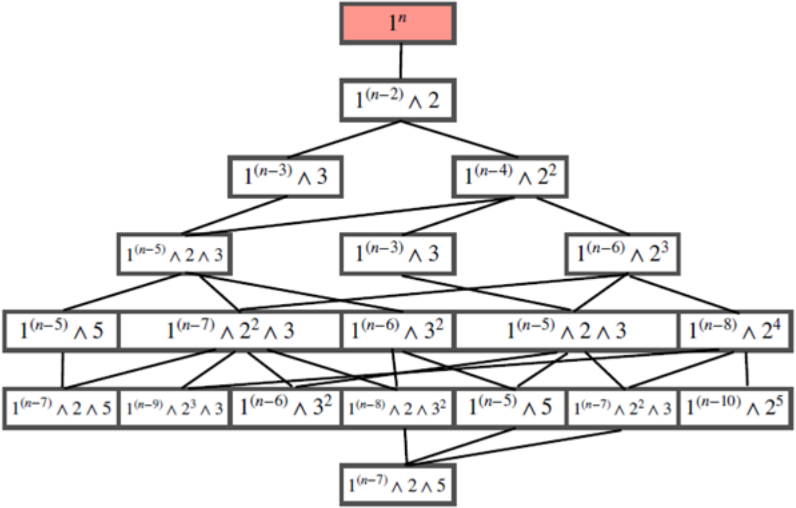

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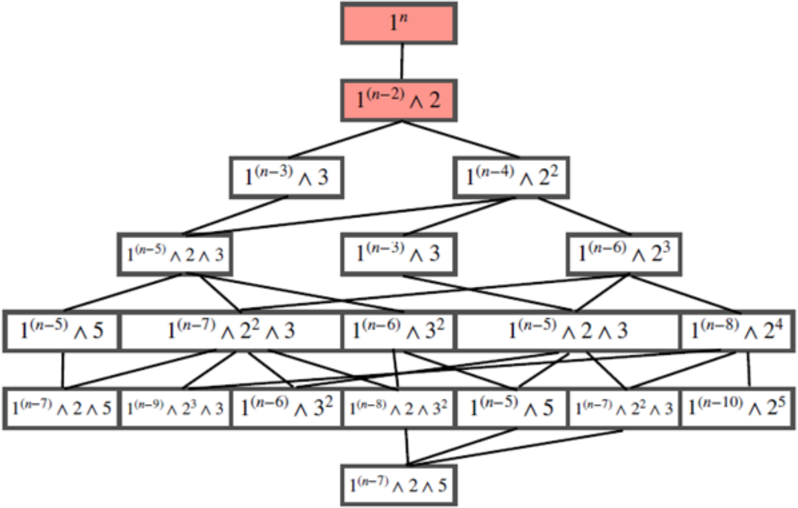

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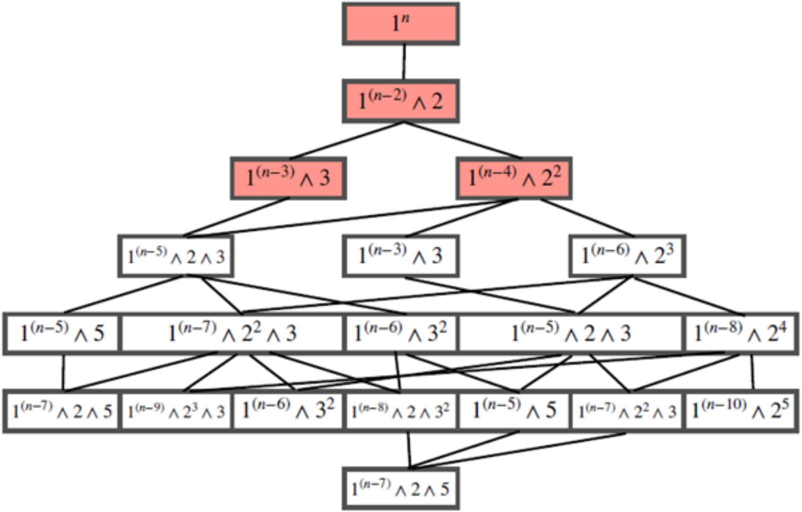

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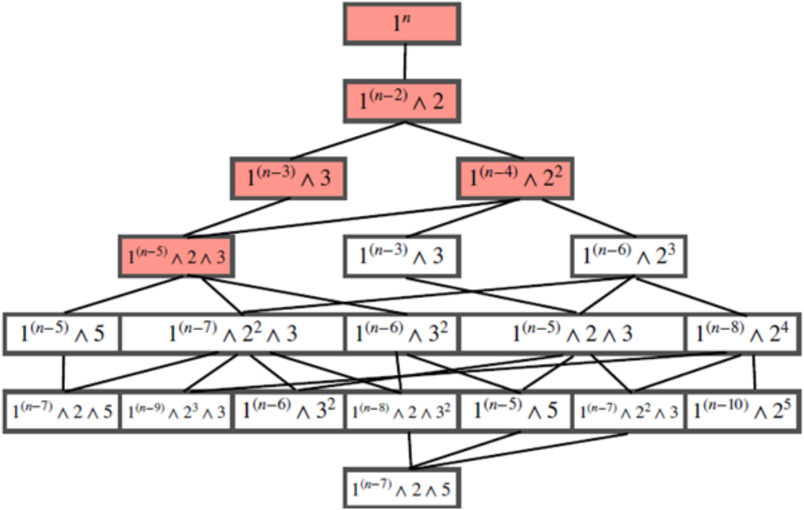

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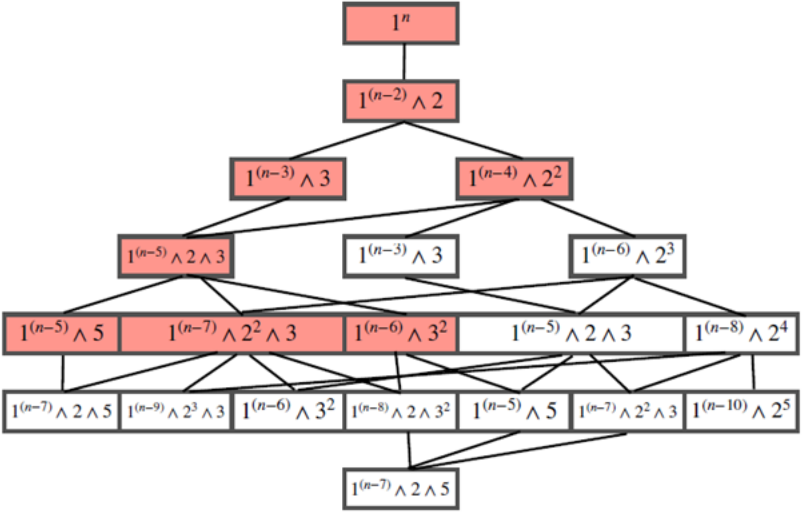

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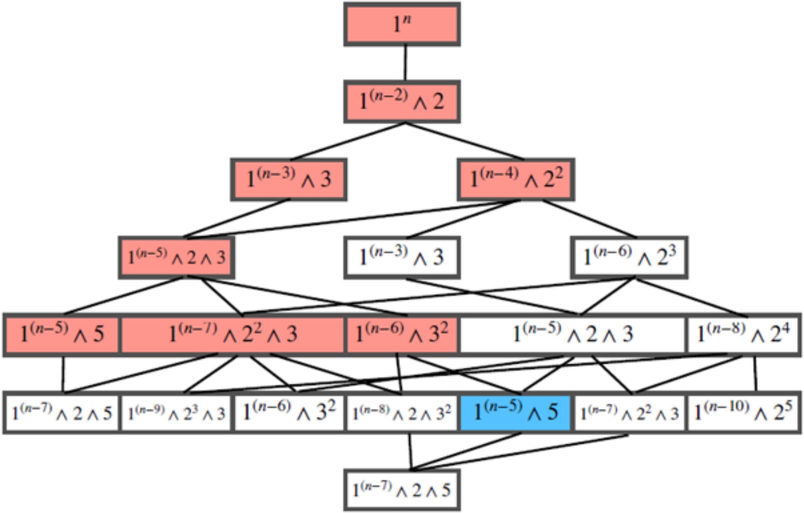

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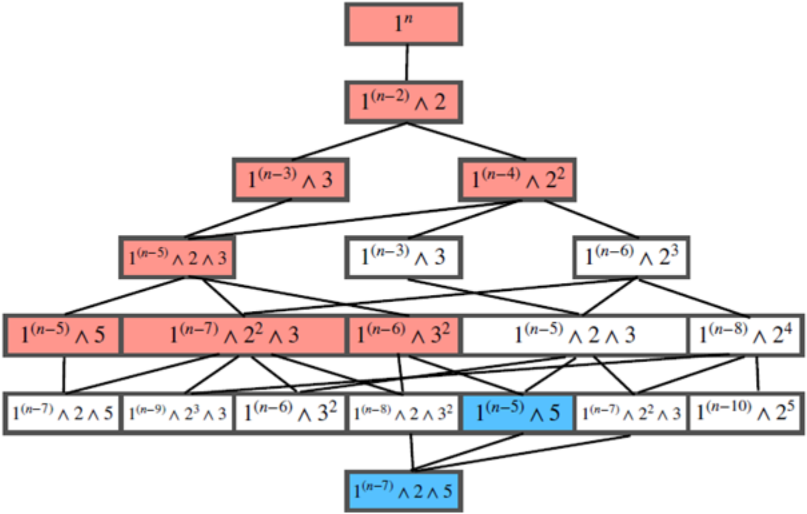

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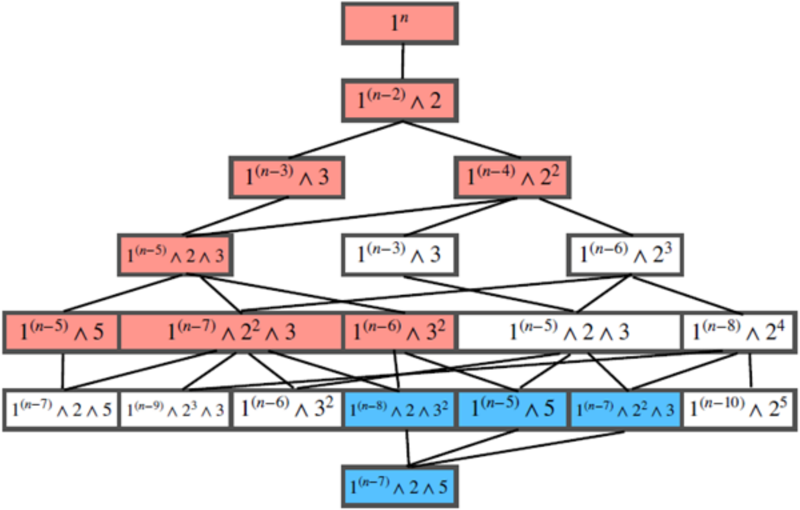

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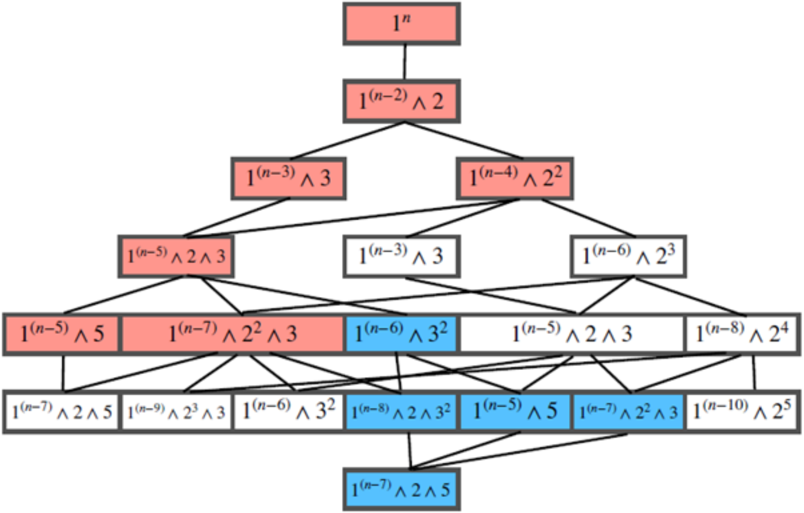

Further Questions

- What if $p \geq 3$ people play the Fibonacci game?
- Does the number of moves in random games converge to a Gaussian?
- Define k-nacci numbers by $S_{i+1} = S_i + S_{i-1} + \cdots + S_{i-k}$; game terminates but who has the winning strategy?

The Accelerated Zeckendorf Game

Accelerated Zeckendorf Game

The Zeckendorf Game

Definition (The two player Zeckendorf Game)

- Game states are represented $(a_k, a_{k-1}, a_{k-2}, \ldots, a_1)$ where a_j is the current number of copies of F_j.

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1. If the list contains two consecutive Fibonacci numbers, F_{i-1}, F_i, then a player can change these to F_{i+1}. We denote this move C_i.
2. If the list has two identical Fibonacci numbers, F_i, F_i, then
   1. if $i = 1$, a player can change F_1, F_1 to F_2, denoted C_1,
   2. if $i = 2$, a player can change F_2, F_2 to F_1, F_3, denoted S_2,
   3. if $i \geq 3$, a player can change F_i, F_i to F_{i-2}, F_{i+1}, denoted S_i.

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    3. if $i \geq 3$, a player can change F_i, F_i to F_{i-2}, F_{i+1}, denoted S_i.

The players alternative moving. The game ends when no more moves can be made. The last player to move wins.

Accelerated Zeckendorf Game

Definition (The two player Accelerated Zeckendorf Game)

- Same rules as Zeckendorf game except players may play as many moves of the same type as possible on their turn.

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On each turn, a player can make one of the following moves.

1. If the list contains at least $m > 0$ copies of both F_{i-1} and F_i, then a player can change m copies of F_{i-1} and m copies of F_i to m copies of F_{i+1}. We denote this move $m \cdot C_i$.

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   2. if $i = 2$, a player can change $2m$ copies of F_2 to m copies of F_1 and m copies of F_3, denoted by $m \cdot S_2$
   3. if $i \geq 3$, a player can change $2m$ copies of F_i to m copies of F_{i-2} and m copies of F_{i+1}, denoted by $m \cdot S_i$.

Research Questions

Questions:

- Does the game end? How long?
- For each N who has the winning strategy?
- What is the winning strategy?

Sample Game

Start with 10 pieces at F_1, rest empty.
Game state: $(0, 0, 0, 0, 10)$

0	0	0	0	10
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move: Player 1: $5 \cdot C_1$.

Sample Game

Start with 10 pieces at F_1, rest empty.
Game state: $(0, 0, 0, 5, 0)$

0	0	0	5	0
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move: Player 2: $2 \cdot S_2$

Sample Game

Start with 10 pieces at F_1, rest empty.
Game state: $(0, 0, 2, 1, 2)$

0	0	2	1	2
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move: Player 1: $1 \cdot S_3$

Sample Game

Start with 10 pieces at F_1, rest empty.
Game state: (0,1,0,1,3)

0	1	0	1	3
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move: Player 2: $1 \cdot C_2$

Sample Game

Start with 10 pieces at F_1, rest empty.
Game state: $(0, 1, 1, 0, 2)$

0	1	1	0	2
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move: Player 1: $1 \cdot C_4$.

Sample Game

Start with 10 pieces at F_1, rest empty.
Game state: $(1, 0, 0, 0, 2)$

1	0	0	0	2
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Next move: Player 2: $1 \cdot C_1$.

Sample Game

Start with 10 pieces at F_1, rest empty.
Game state: $(1, 0, 0, 1, 0)$

1	0	0	1	0
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

No moves left, Player Two wins.

Sample Game

Player Two won in 6 moves.

0	0	0	0	10
0	0	0	5	0
0	0	2	1	2
0	1	0	1	3
0	1	1	0	2
1	0	0	0	2
1	0	0	1	0
$[F_5 = 8]$	$[F_4 = 5]$	$[F_3 = 3]$	$[F_2 = 2]$	$[F_1 = 1]$

Upper bound on Game Length

- Each game of the Accelerated Zeckendorf Game can be associated to a game of the Zeckendorf Game in which each $m \cdot C_i$ and $n \cdot S_j$ is replaced by m instances of a C_i move and n instances of a S_j move, respectively.

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Theorem (Garcia, Miller, Rascon, Vandegrift, Y.)

Every game terminates after a finite number of moves at the Zeckendorf decomposition of n.

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Theorem (Garcia, Miller, Rascon, Vandegrift, Y.)

Every game terminates after a finite number of moves at the Zeckendorf decomposition of n.

- Moreover, the same reasoning shows that the maximum number of moves in the Accelerated Zeckendorf Game on n is exactly equal to the maximum number of moves in the Zeckendorf Game on n.

Shortest Game

- Recall that the shortest Zeckendorf Game on n takes $n - Z(n)$ moves.

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Theorem (Garcia, Miller, Rascon, Vandegrift, Y.)

If k is the index of the greatest Fibonacci number in the Zeckendorf decomposition of n, then $k - 1$ is a sharp lower bound on the number of moves in the Accelerated Zeckendorf Game on n.

Thus the shortest Accelerated Zeckendorf Game on n is much shorter than the shortest Zeckendorf Game on n.

Winning Strategies

Theorem (Garcia, Miller, Rascon, Vandegrift, Y.)

If Player 1 has the winning strategy, Player 1 has only one correct first move. In other words, there exists only one move that will maintain Player 1's winning strategy.

Conjecture on Winning Strategies

Conjecture (Garcia, Miller, Rascon, Vandegrift, Y.)

If $n > 9$, Player 1 has the winning strategy.

- To determine which player has the winning strategy for specific values of n, we created a program (see https://github.com/ThomasRascon/Accelerated-Zeckendorf-Game).
- We tested all games up to $n = 140$, and found that for all $n > 9$ Player 1 had the winning strategy.

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- We tested all games up to $n = 140$, and found that for all $n > 9$ Player 1 had the winning strategy.
- Note that Conjecture 1 is in stark contrast with the classical situation, as Player 2 always has the winning strategy in the Zeckendorf Game on n when $n > 2$.

Progress Towards Proving the Winning Strategies Conjecture

Conjecture (Garcia, Miller, Rascon, Vandegrift, Y.)

If Player 2 has the winning strategy, then all game states of the form $(0, \ldots, 0, k, 0, n-3k)$ *are losing states.*

Progress Towards Proving the Winning Strategies Conjecture

Conjecture (Garcia, Miller, Rascon, Vandegrift, Y.)

If Player 2 has the winning strategy, then all game states of the form $(0, \ldots, 0, k, 0, n - 3k)$ are losing states.

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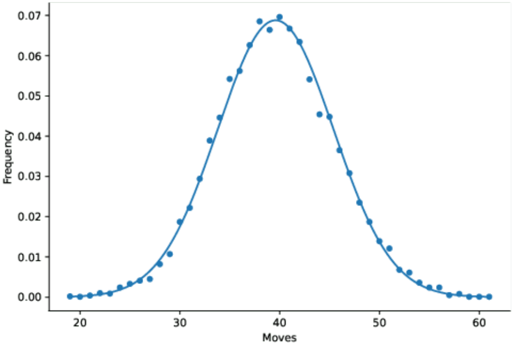
Lemma (Garcia, Miller, Rascon, Vandegrift, Y.)

Conjecture 2 is true for $k \in \{1, 2, 3, 4, 5\}$.

Game Length Data



Graph of the frequency of the number of moves in 9,999 simulations of the Accelerated Zeckendorf Game with random moves where each legal move has a uniform probability for $n = 100$ with the best fitting Gaussian (mean ≈ 39.6, STD ≈ 5.8).

Conjectures on Average Game Length

Conjecture (Garcia, Miller, Rascon, Vandegrift, Y.)

As n goes to infinity, the number of moves in a random Accelerated Zeckendorf Game on n, when all legal moves are equally likely, converges to a Gaussian.

- Same conjecture was made for the Zeckendorf Game.

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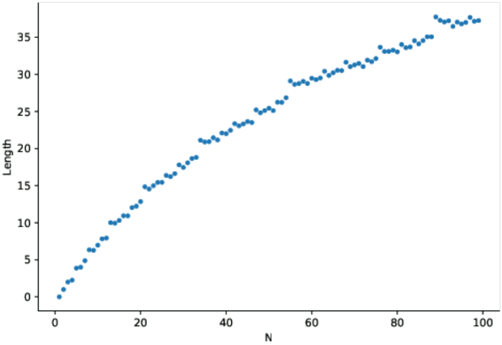
Conjecture

The average game length grows at a sub-linear rate with n.

- Contrast with the Zeckendorf Game, where the average game length appears to grow at a linear rate with n.

Average Game Length Data



Graph of the average number of moves in the Accelerated Zeckendorf Game with random uniform moves with 9,999 simulations with n varying from 1 to 99.

Future Work

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- Investigate the type of sub-linear growth, such as logarithmic versus n^δ for $\delta < 1$.
- Can the conjectured “Gaussianity of Random Zeckendorf Games” be related to the conjectured “Gaussianity of Random Accelerated Zeckendorf Games”?
    - Does one imply the other?

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Other Variants

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- Come up with your own variant and study it!

References

References

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- D. Garcia-Fernandezsesma, S. J. Miller, T. Rascon, R. Vandegrift, and A. Yamin, *The Accelerated Zeckendorf Game*. https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/zeckgame_accelerated_poly23v20.pdf