Allgebra -- Making it Fun and Accessible to All

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http://www.williams.edu/Mathematics/sjmiller/public_html

Teachers As Scholars: March 2026

TOPICS

- General Code
- Hedging Bets
- Double Plus Ungood
- I Love Rectangles
- Triangle Game
- Geometry: Seating, Pooled Testing, Tomography
- What do you mean?
- Three Hat Game
- Pirates and Math
- Egg Drop Mathematics
- Cross That Moat
- Legal 21

Coronavirus Lectures (see also my math riddles page: https://mathriddles.williams.edu/)

1. From Pythagoras to Pi: Part I: https://youtu.be/JHvmP-1KXYc (slides here). Given to 7th grade son who is in Algebra I. 41 minutes
2. From Pythagoras to Pi: Part II: https://youtu.be/ISo8kXDP-6U (slides here). Given to 7th grade son who is in Algebra I. 27 minutes
    1. Unified Teachers as Scholars Lecture: https://youtu.be/nuKYR-UCGjY (3/18/2020): 39 minutes
3. What do you MEAN? https://youtu.be/jBKZaCxpgSE (word file here, pdf here) (3/19/2020): Comfort with Algebra sufficient: 40 minutes
4. Shoe size and age: https://youtu.be/aDxdDifvlKM (word file here, pdf here) (3/20/2020): Just need to be able to multiply and add, say grade 1 or 2 and up: 20 minutes
5. Mono-variants: https://youtu.be/PbZhVLXyatY (powerpoint here, pdf here) (3/23/2020): Grade 5 and up: 28 minutes
6. I Love Rectangles Game: https://youtu.be/JHtrzARHwHU (powerpoint here, pdf here) (3/24/2020): Aimed for K, should be good for all ages. 13 minutes
7. From Pascal to Calculus: Part I: https://youtu.be/dv15VTyEWyQ (powerpoint here, pdf here) (3/25/2020): For those knowing Algebra I (equations of lines): 52 minutes
8. From Pascal to Calculus: Part II: https://youtu.be/D6OnleQJ1XM (powerpoint here, pdf here) (3/26/2020): For those knowing Algebra I (equations of lines): 40 minutes
9. Games: Tic Tac Toe, Chocolate Bar, Coins, Devil: https://Mathyoutu.be/4Kvx-JHBvs0 (slides here) (3/27/2020): Much of it can be done with K, 2nd and higher to be safe: 38 minutes
10. Induction and Sums: Part I: Induction https://youtu.be/3orVsQECaag (powerpoint here, pdf here) (3/30/2020): Assuming Algebra I: 32 minutes
11. Induction and Sums: Part II: Geometric Series Formula https://youtu.be/CJjgIF65x6g (powerpoint here, pdf here): (3/31/2020): Assuming Algebra I: 25 minutes
12. Induction and Sums: Part III: From the Geometric Series Formula to Primes https://youtu.be/UWNM8EtzoMI (powerpoint here, pdf here) (4/4/2020): Assuming Algebra: 22 minutes

13. The Three Hat Problem and Error Correcting Codes:https://youtu.be/oMeKf7AhAa4 (powerpoint here, pdf here) (4/6/2020): Aimed for 3rd Grade and up: 26 minutes
14. How to Attack Problems I: We WILL Cross That Bridge: https://youtu.be/JH6uxmgWoFQ (powerpoint here, pdf here) (4/8/2020): Aimed for 4th Grade and up: 20 minutes
15. How to Attack Problems II: Legal 21: https://youtu.be/dlBVLlt4PPA (powerpoint here, pdf here) (4/9/2020): Aimed for 4th/5th Grade and up: 14 minutes
16. Polynomials and Applications: Part I: Lecture 1: Lines: https://youtu.be/NNxBhUTzexA (powerpoint here, pdf here) (4/10/2020): Aimed at 6th/7th grade and up: 9 minutes
17. Polynomials and Applications: Part 2: Lecture 2: Introduction to Quadratics: Plotting, Simple Examples and Roots: https://youtu.be/bkIWsNcw-bU (powerpoint here, pdf here) (4/13/2020): Aimed at 7th grade and up: 17 minutes
18. Polynomials and Applications: Part 2: Lecture 3: Introduction to Quadratics: Plotting, Simple Examples and Roots: https://youtu.be/u8Xx7r0VrJ4 (powerpoint here, pdf here) (4/14/2020): Aimed at 7th grade and up: 18 minutes.
19. Polynomials and Applications: Part 2: Lecture 4: Application 1: Trajectories: https://youtu.be/hAw9vSNMzhg (powerpoint here, pdf here) (4/15/2020): Aimed at 7th grade and up: 24 minutes.
20. Polynomials and Applications: Part 3: Lecture 5: Application 2: Fibonacci Numbers: https://youtu.be/WB9gLTASXCw (powerpoint here, pdf here) (4/16/2020): Aimed at 7th grade and up: 24 minutes.
21. Polynomials and Applications: Part 4: Lecture 6: Application 3: Recurrence Relations and Gambling! https://youtu.be/Esa2TYwDmwA (4/17/2020): Aimed at 7th grade and up: 7 minutes
22. Polynomials and Applications: Part 5: Lecture 6: Application 4: Finding Trajectories (revisited) https://youtu.be/7bfeNy4XW5I (powerpoint here, pdf here) (4/21/2020): Aimed at 7th grade and up: 10 minutes
23. Polynomials and Applications: Part 6: Lecture 7: Application 5: Codes: https://youtu.be/wSEXv5PXxu0 (powerpoint here, pdf here) (4/24/2020): Aimed at 7th grade and up: 11 minutes

24. Introduction to Probability: Part 1: The Factorial Function: https://youtu.be/Un3tWIX-kDY (powerpoint here, pdf here) (4/27/2020): Aimed at 7th grade and up: 13 minutes
25. Introduction to Probability: Part 2: Permutations: https://youtu.be/hhYtkpcQQAY (powerpoint here, pdf here) (4/29/2020): Aimed at 7th grade and up: 11 minutes
26. Introduction to Probability: Part 3: Combinations: https://youtu.be/8w8HWvhWM34 (powerpoint here, pdf here) (4/29/2020): Aimed at 7th grade and up: 19 minutes
27. Introduction to Probability: Part 4: Darth Vader Problem: https://youtu.be/qsUYmXGgngE (powerpoint here, pdf here) (5/4/2020): Aimed at 7th grade and up: 19 minutes
28. Introduction to Probability: Part 5: Double Sixes Problem: https://youtu.be/zJTaXORiH9o (powerpoint here, pdf here) (5/6/2020): Aimed at 7th grade and up: 15 minutes (re-recorded; no TAs)
29. Introduction to Probability: Part 6: Long Suits: https://youtu.be/-iWWLLlCRy4 (powerpoint here, pdf here) (5/8/2020): Aimed at 7th grade and up: 18 minutes
30. Introduction to Probability: Part 7: Long Suits: https://youtu.be/ke5RLxIW6v4 (powerpoint here, pdf here) (5/11/2020): Aimed at 7th grade and up: 18 minutes
31. Introduction to Probability: Part 8: All trump, Poker: https://youtu.be/InhvzliUb4I (powerpoint here, pdf here) (5/13/2020): Aimed at 7th grade and up: 15 minutes
32. Introduction to Probability: Part 9: Advanced Combinatorics: https://youtu.be/WFcEQeCoG5k (powerpoint here, pdf here) (5/19/2020): Aimed at 7th grade and up: 16 minutes

https://sites.williams.edu/mathriddles

General Code

Consider an army with 10 generals. One wants a security system such that any three of them can determine the code to launch nuclear missiles, but no two of them can. It is possible to devise such a system by using a quadratic polynomial, such as $ax^2 + bx + c$; to launch the missiles, one must input (a,b,c). One cannot just tell each general one of a, b, or c (as then it is possible that some subset of three generals won't know a, b and c); however, if a general knows two of (a,b,c), then a set of two generals can launch the missiles! What information should be given to the generals so that any three can find (a,b,c) but no two can? What about the general situation with N generals and any M can launch (but no set of M-1) can?

General Code

- Cannot give one person all 3
- If Mary has (a, b), no one can know c or 2 could teach.
- Helen = a, Lisa = b, Stephanie = c, Steve = ? nothing works

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$f(x) = ax^2 + bx + c = a(x - r_1)(x - r_2)$

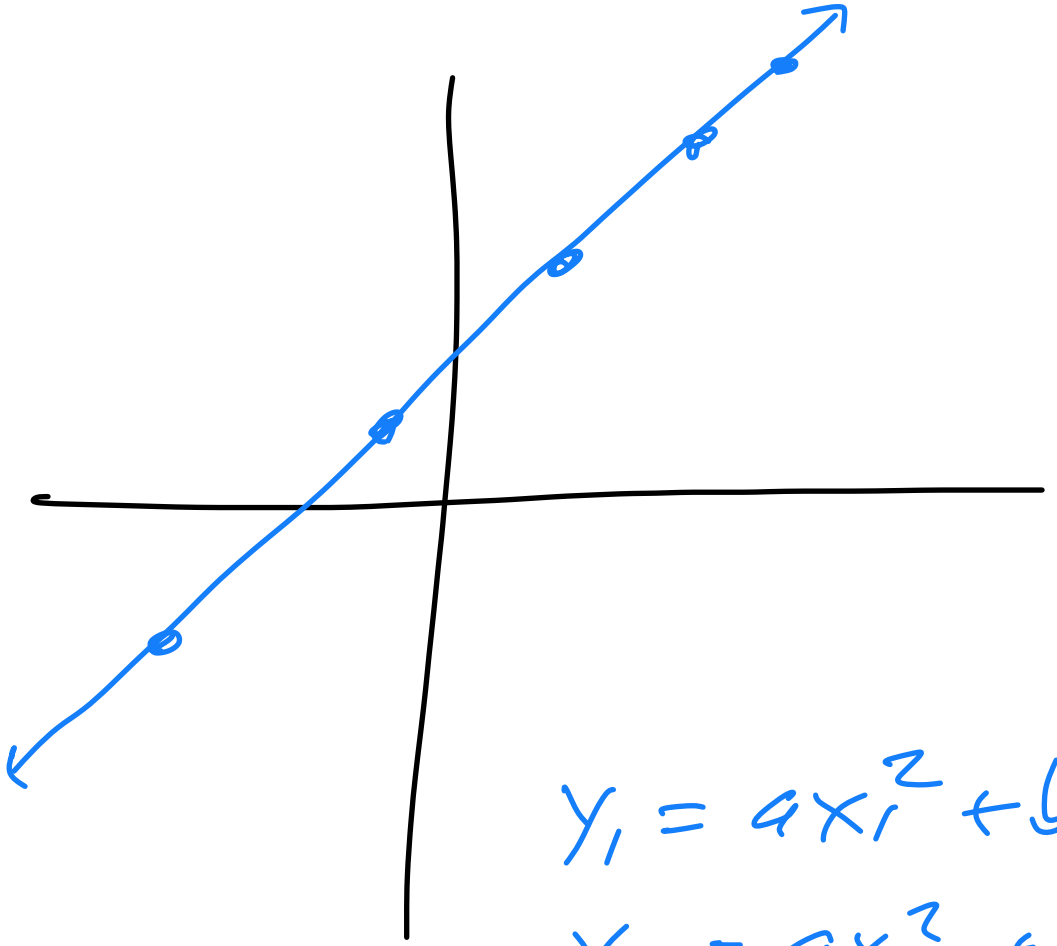
Helen = a, Lisa = b, Steve = r_1 : does this give c?

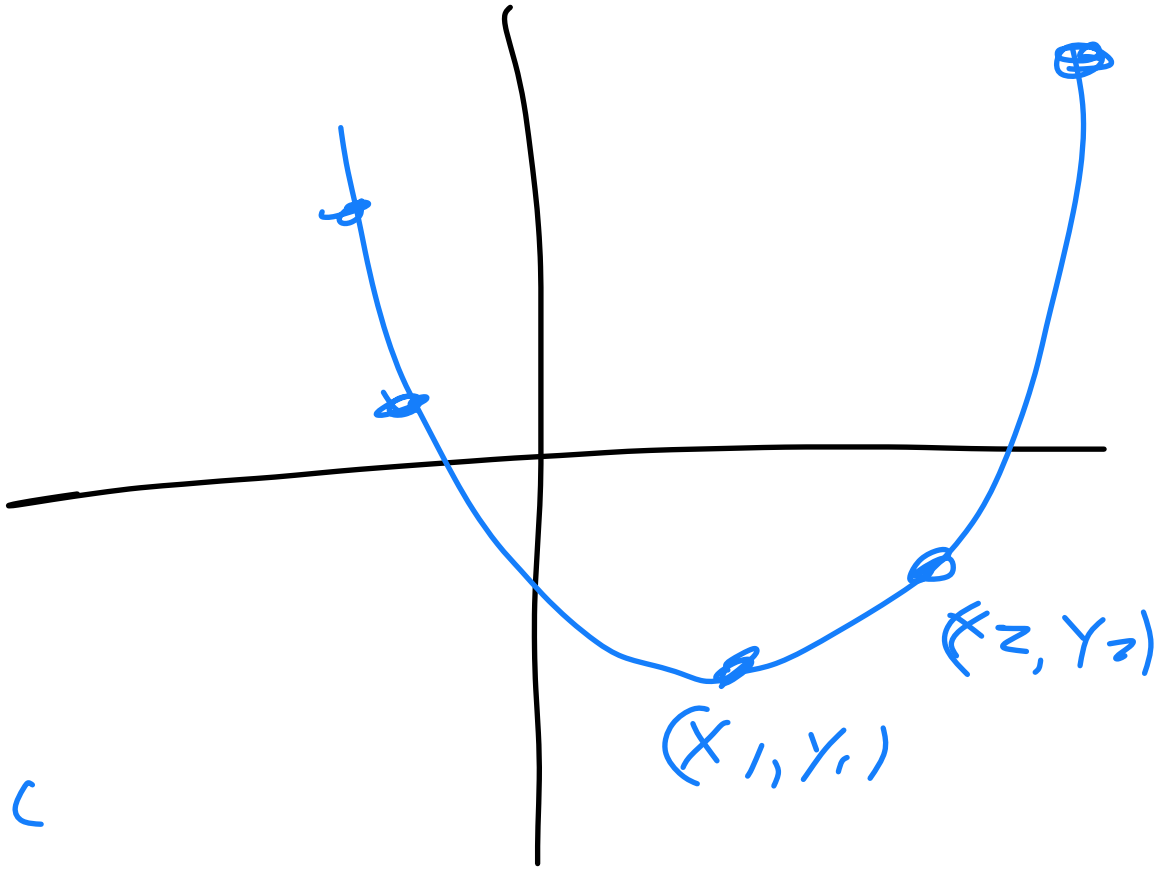
$ax^2 + bx + c = ax^2 - a(r_1 + r_2)x + a r_1 r_2$

$b = -a(r_1 + r_2)$

$c = a r_1 r_2$

General Code





$$y_1 = ax_1^2 + bx_1 + c$$
$$y_2 = ax_2^2 + bx_2 + c$$
$$y_3 = ax_3^2 + bx_3 + c$$

$$\begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Look at polynomials and applications: Lectures 16-23.
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/MathEd_QuadraticFormula_BaseballToGambling.pdf

Password is $N = pq$, p, q prime, $\approx 10^{200}$

Public N: password is p or q

Choose a random 5000 digit number, x

Password is x ARGH!

First Eve: if Mark x is the password, if $N \div x$ is an INTEGER: you GO IN.

Football Wager

2007: Friend of a favorite student bet $500 at 1000:1 odds on Patriots going undefeated and winning the Superbowl.

Hedging Bets

Say bet $1000 on Pats to win, pays $5000 if win,

Could bet on pats to lose as a second bet

↳ Say Pats win with probability p

Say if Pats lose, for every $1 you bet,

you get $.25

How much do I make if I bet $X on Pats to lose:

Pats win: make $5000 lost $X = $5000 - X$

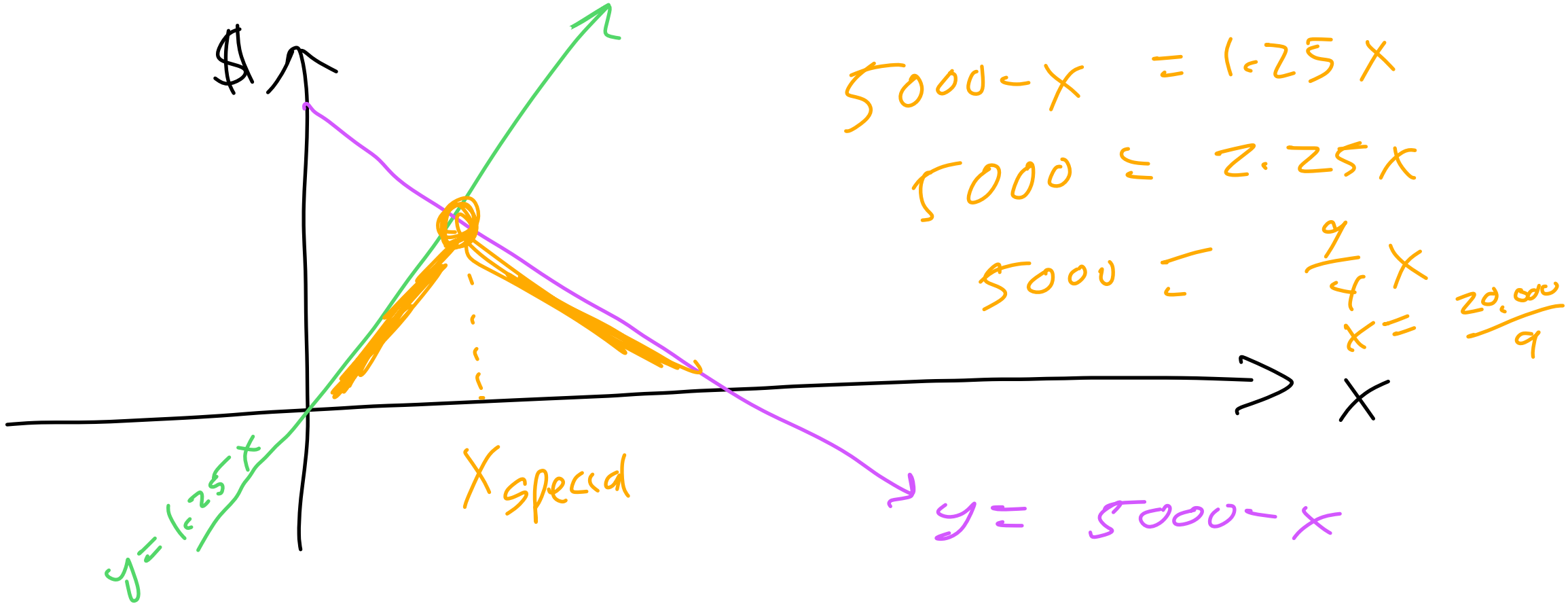
Pats lose: make $.25X + X = 1.25X$

$$f(X) = \begin{cases} 5000 - X & \text{if Pats win} \\ 1.25X & \text{if Pats lose} \end{cases}$$

Hedging Bets

$$f(x) = \begin{cases} 5000 - x & \text{if Pats win} \\ 1.25x & \text{if Pats lose} \end{cases}$$

Worse case is get minimum of the two



$5000 - x = 1.25x$

$5000 = 2.25x$

$5000 = \frac{9}{4}x$

$x = \frac{20,000}{9}$

Hedging Bets

Lectures from Math/Stat 341: Spring 2025: (slides)

Lecture 01: 2/05/25: Introduction, Birthday Problem, Hoops Game: https://youtu.be/bSdLy3-BVvQ (slides)

DOUBLE PLUS UNGOOD:
From Fibonacci to Roulette and Modeling

Steven J. Miller (sjm1@williams.edu)
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https://web.williams.edu/Mathematics/sjmiller/public_html/

DOUBLE PLUS UNGOOD: From Fibonacci to Roulette and Modeling, SIMIODE EXPO 2025 conference, February 2025. pdf See also shorter version: https://www.youtube.com/watch?v=Esa2TYwDmwA&t=147s

GOALS

- ***Excite students about mathematics.***
- ***Introduce data collection and modeling.***
- ***Discuss simulation versus theory.***

ROULETTE

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36
1 to 18
EVEN
1st 12
2nd 12
3rd 12
ODD
19 to 36
2 to 1
2 to 1
2 to 1
0:47 / 6:42 • Introduction >

Double Plus One:

Bet $1 on red, if red up $1, else down $1.

If lost bet $2 red, if red up $1, else down $3.

If lost bet $4 red, if red up $1, else down $7

~~$4~~

Double Plus One: Analysis

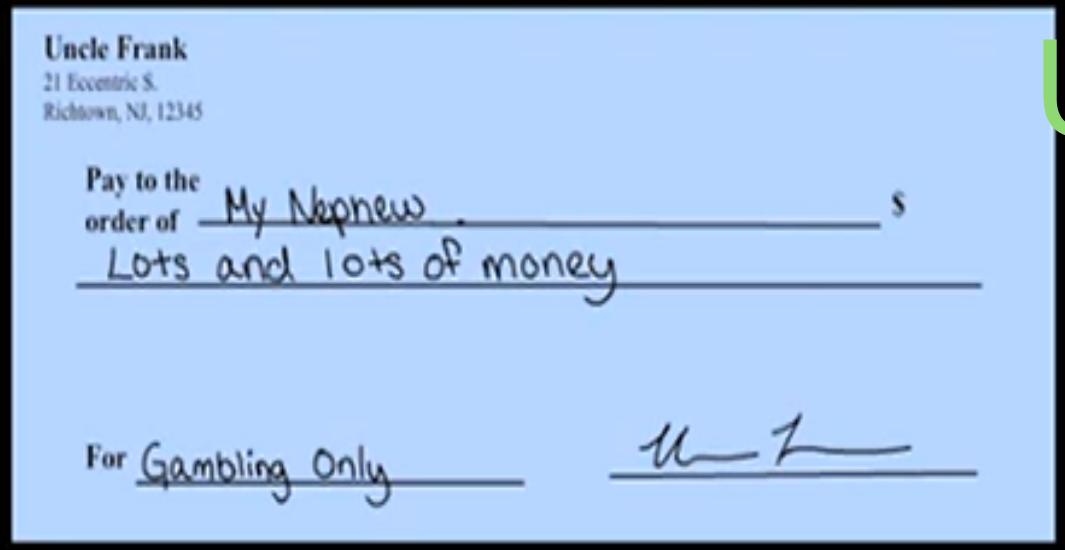
Eventually win, what can go wrong?

Let students think….

Write code to explore.

# of times black is Rolled	Next Bet
1	$2
2	$4
3	$8
4	$16
5	$32
6	$64
7	$128
8	$256
9	$512
10	$1024
11	$2048
12	$4096
13	$8192
14	$16384
15	$32768
16	$65536
17	$131072

ERU/AH:
The Eccentric Rich Uncle/Aunt Hypothesis



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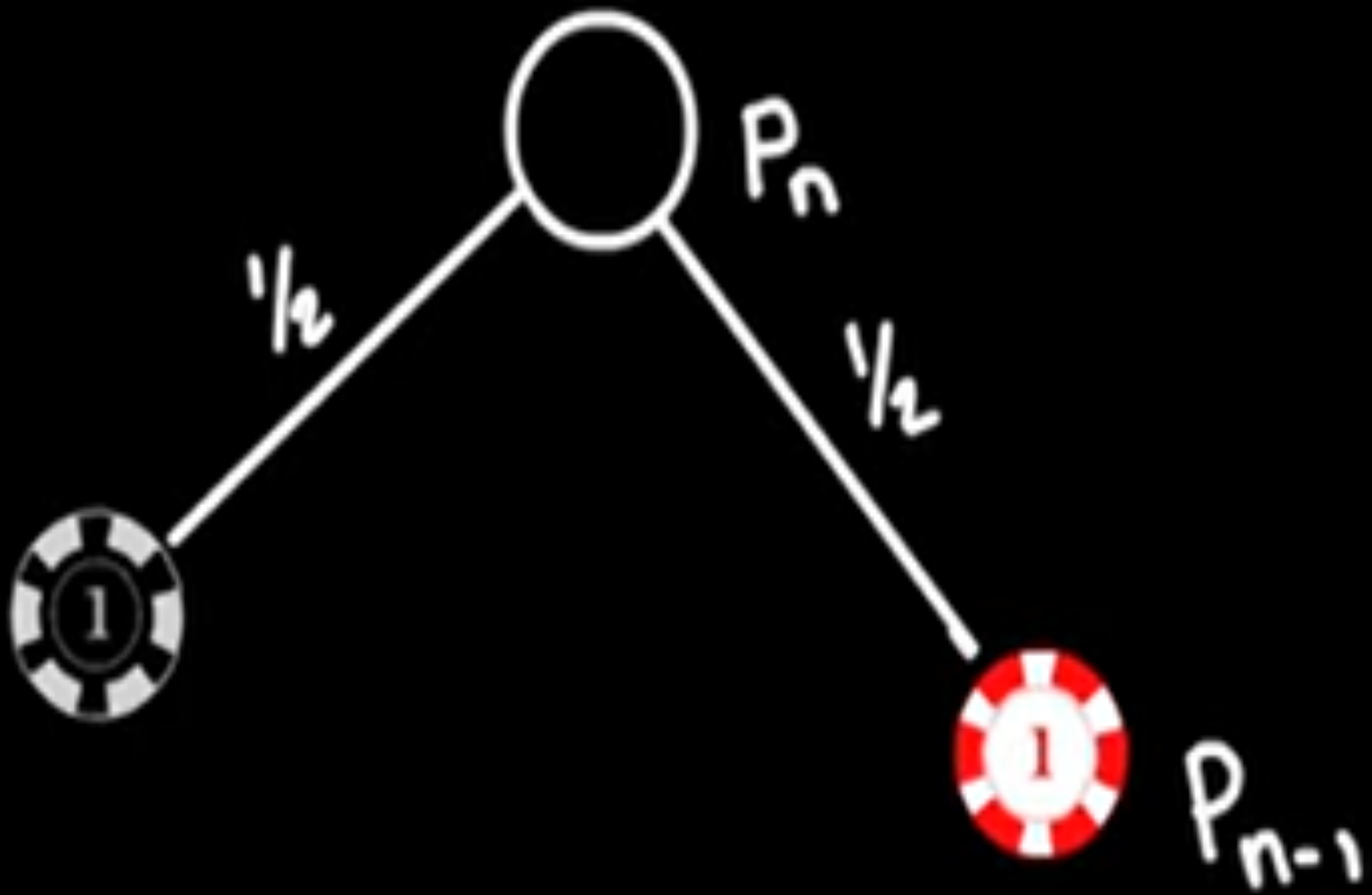
10	$1024
11	$2048
12	$4096
13	$8192
14	$16384
15	$32768
16	$65536
17	$131072
18	$262144
19	$524288

Table Limit

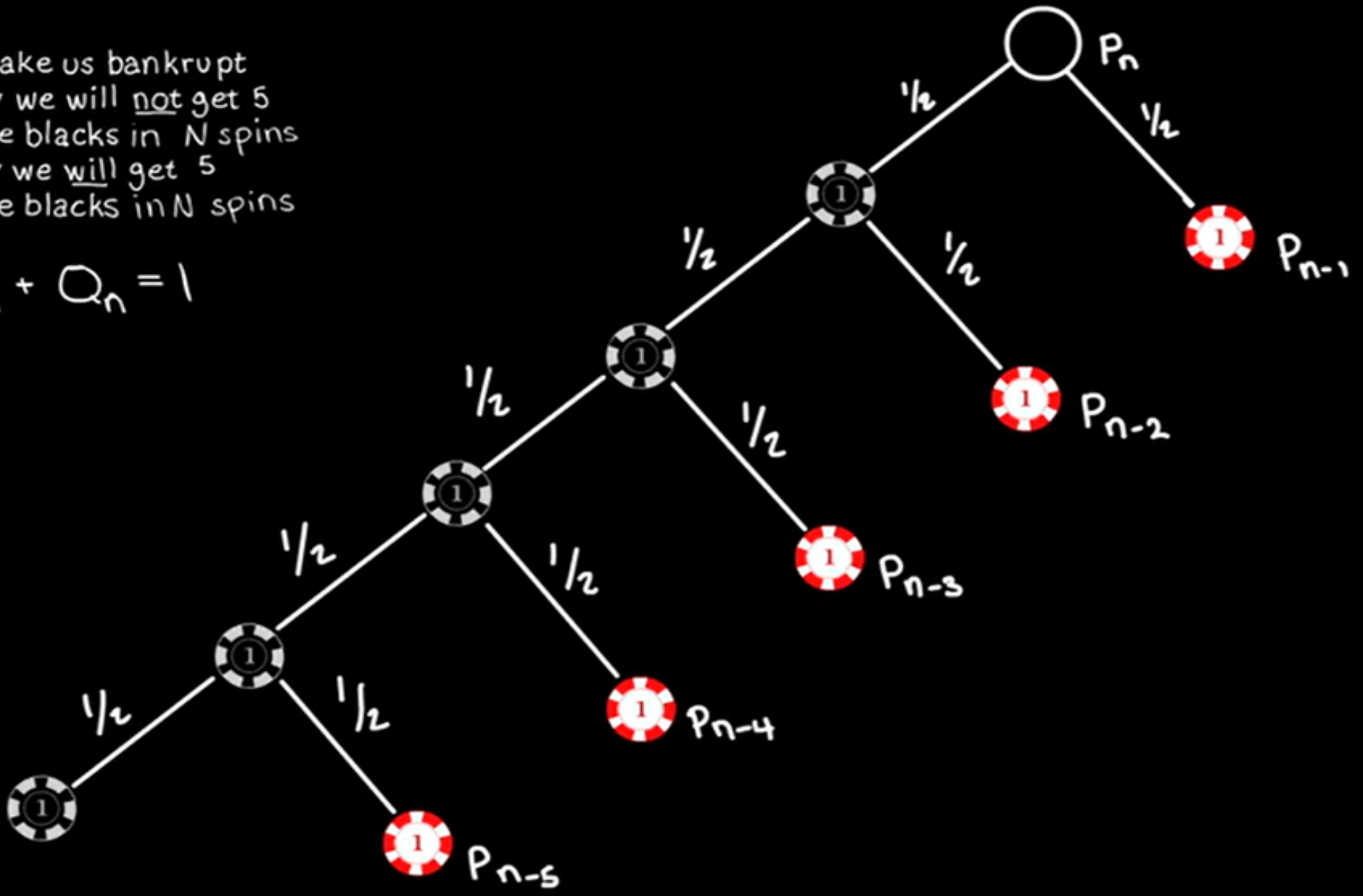
- 5 blacks will make us bankrupt
- P_n = probability we will <u>not</u> get 5 consecutive blacks in N spins
- Q_n = probability we <u>will</u> get 5 consecutive blacks in N spins

$$P_n + Q_n = 1$$


p_n
1/2
1/2
1
1
p_{n-1}

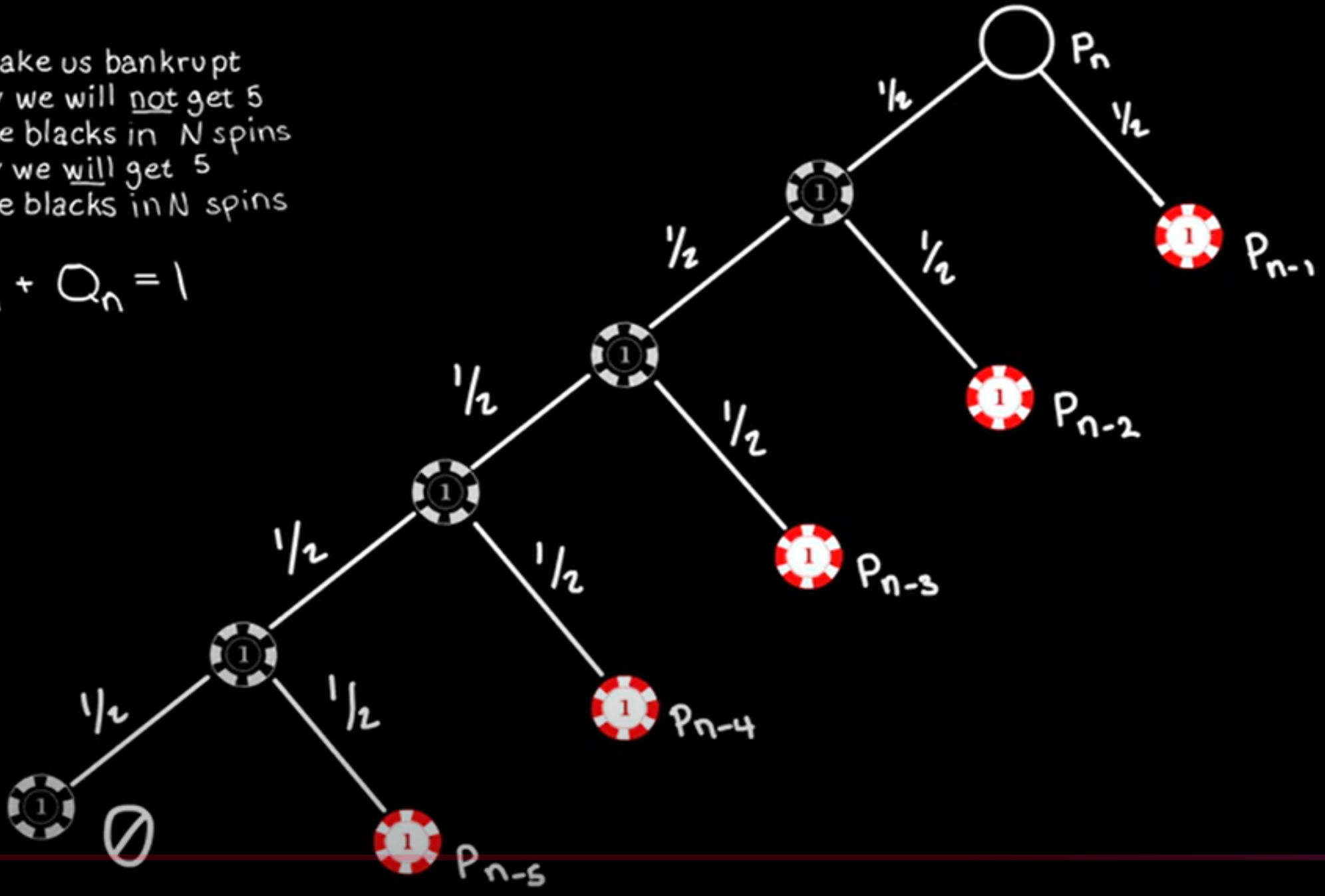
- 5 blacks will make us bankrupt
- P_n = probability we will not get 5 consecutive blacks in N spins
- Q_n = probability we will get 5 consecutive blacks in N spins

$$P_n + Q_n = 1$$

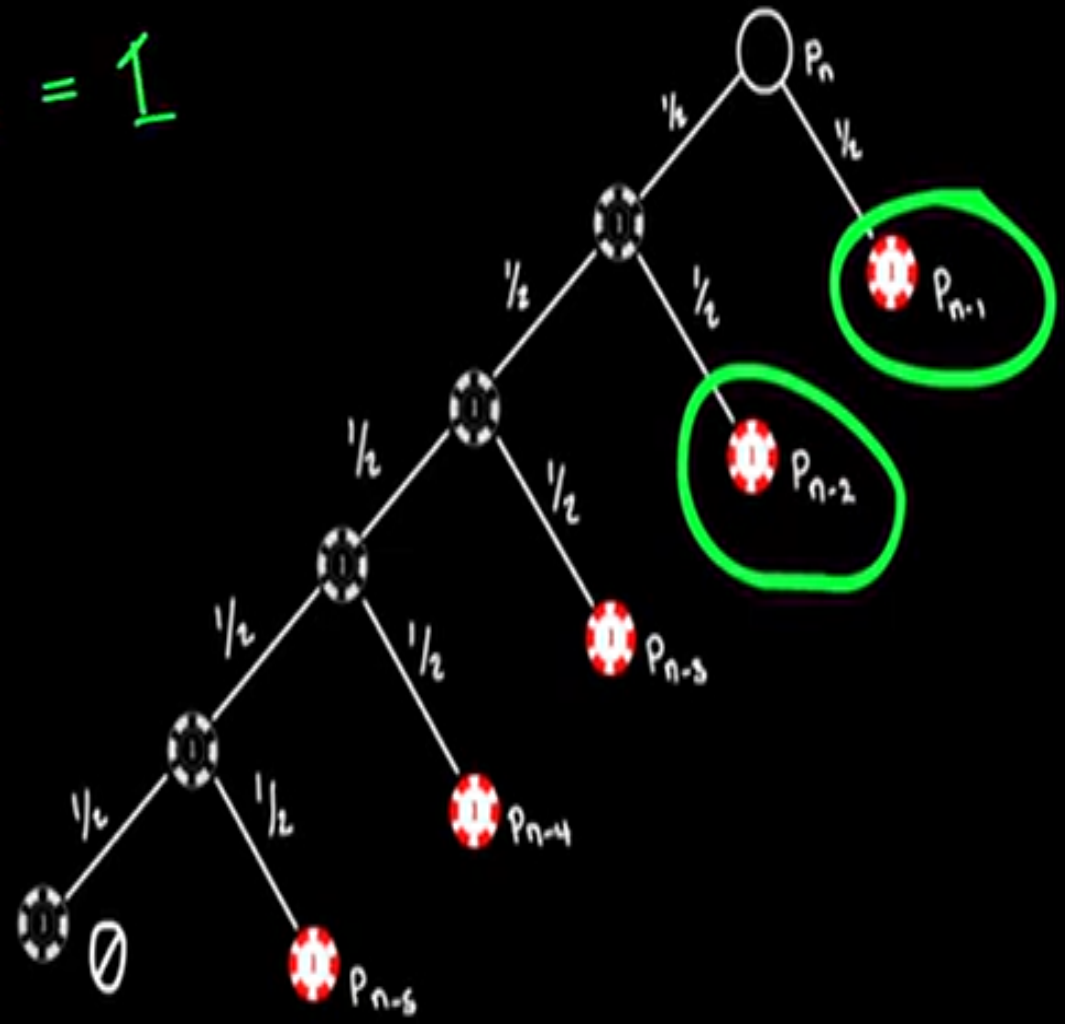


- 5 blacks will make us bankrupt
- P_n = probability we will not get 5 consecutive blacks in N spins
- Q_n = probability we will get 5 consecutive blacks in N spins

$$P_n + Q_n = 1$$



$$P_0 = P_1 = P_2 = P_3 = P_4 = 1$$



$$P_n = \frac{1}{2}(P_{n-1}) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(P_{n-2}) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(P_{n-3}) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(P_{n-4}) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(P_{n-5})$$

Iterate to Solution (target = 5)

Advantage: can do with young students (rational).

n	P(n)	Q(n)	n	P(n)	Q(n)	n	P(n)	Q(n)
0	100.00	0.00	100	18.99	81.01	200	3.41	96.59
1	100.00	0.00	101	18.67	81.33	201	3.35	96.65
2	100.00	0.00	102	18.35	81.65	202	3.29	96.71
3	100.00	0.00	103	18.04	81.96	203	3.24	96.76
4	100.00	0.00	104	17.73	82.27	204	3.18	96.82
5	96.88	3.13	105	17.43	82.57	205	3.13	96.87
6	95.31	4.69	106	17.13	82.87	206	3.08	96.92
7	93.75	6.25	107	16.84	83.16	207	3.02	96.98
8	92.19	7.81	108	16.55	83.45	208	2.97	97.03
9	90.63	9.38	109	16.27	83.73	209	2.92	97.08
10	89.06	10.94	110	15.99	84.01	210	2.87	97.13
11	87.55	12.45	111	15.72	84.28	211	2.82	97.18
12	86.06	13.94	112	15.45	84.55	212	2.77	97.23
13	84.59	15.41	113	15.19	84.81	213	2.73	97.27
14	83.15	16.85	114	14.93	85.07	214	2.68	97.32
15	81.74	18.26	115	14.68	85.32	215	2.64	97.36
16	80.35	19.65	116	14.43	85.57	216	2.59	97.41
17	78.98	21.02	117	14.18	85.82	217	2.55	97.45
18	77.63	22.37	118	13.94	86.06	218	2.50	97.50
19	76.31	23.69	119	13.70	86.30	219	2.46	97.54
20	75.01	24.99	120	13.47	86.53	220	2.42	97.58

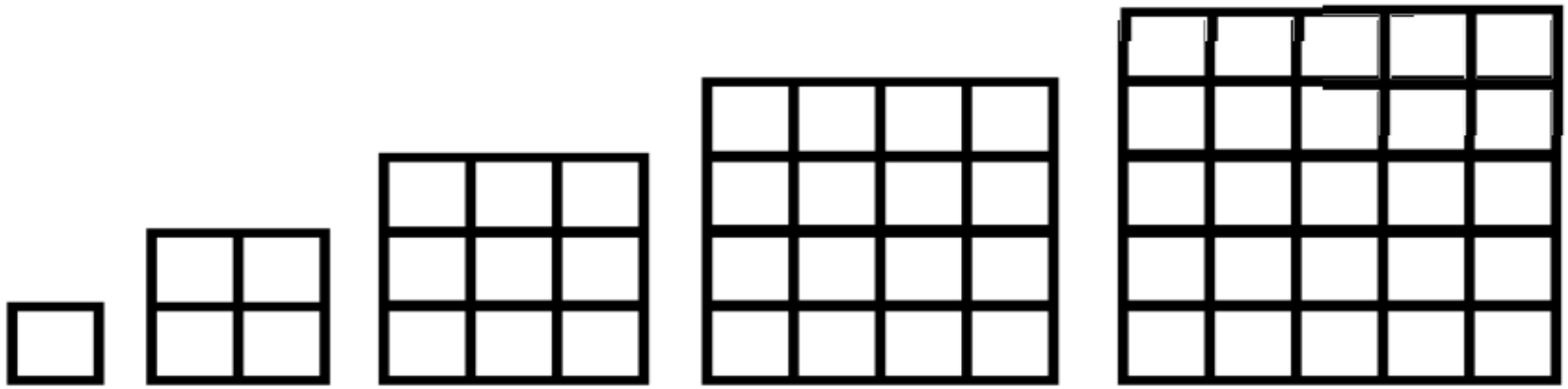
THEORY

- Closed form solution (Generalized Binet Formula).
- Yields large #spins behavior (cannot see from recurrence).
- Lots of ways to solve: linear algebra, characteristic polynomials.
- Generalizing Fibonaccis – what else can model with recurrences? (Predator-Prey, ….)

The I LOVE RECTANGLES Game

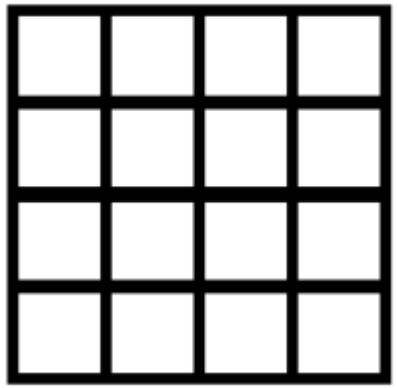
Imagine we have EXACTLY ONE of each size square. We have one 1 by 1 rectangle, one 2 by 2 rectangle, one 3 by 3 rectangle, one 4 by 4 rectangle, and so on.

Here's the rule: we put these squares down ONE AT A TIME, and at EVERY MOMENT IN TIME our shape MUST be a rectangle. Can it be done? Note a square IS a rectangle.

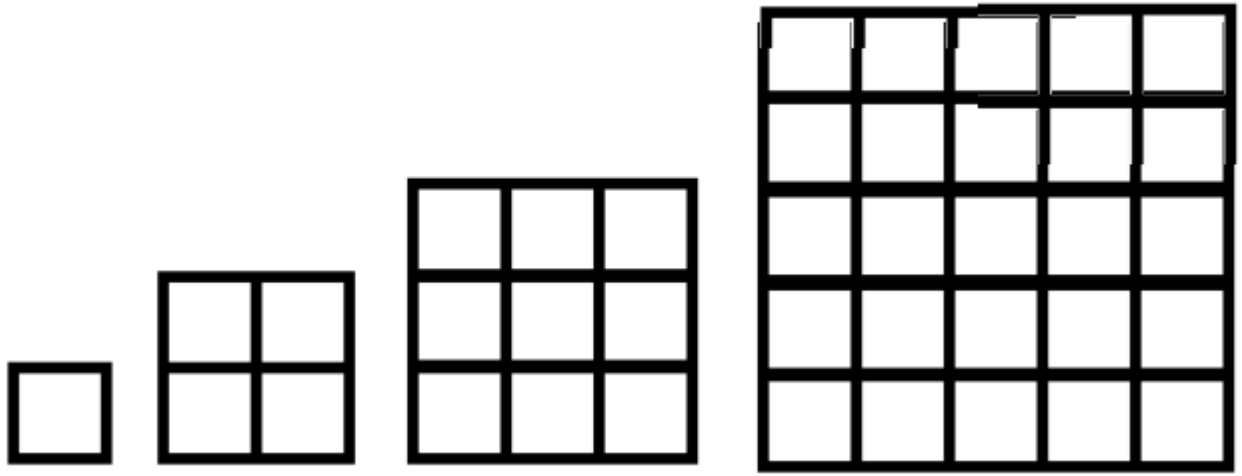


The I LOVE RECTANGLES Game

Imagine we put the 4 by 4 square down. That gives us a rectangle, so far so good. Can we put down anything else next to it and still have a rectangle?



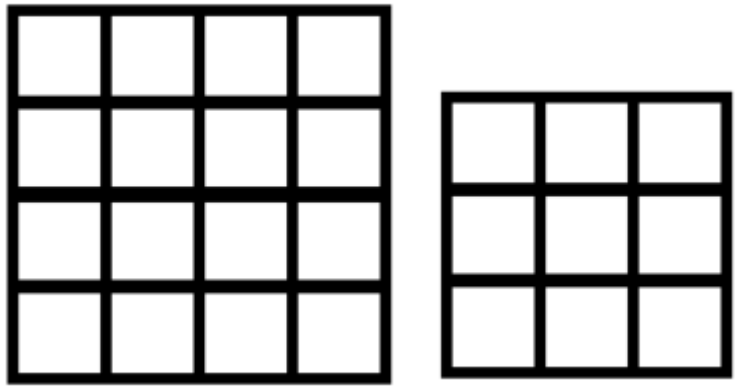
We have placed a 4 by 4 square. This is a rectangle!



These are the squares we have left. We have a 1 by 1, a 2 by 2, a 3 by 3, a 5 by 5, a 6 by 6 (not drawn) and so on. Can we place anything next to the 4 by 4 and still have a rectangle?

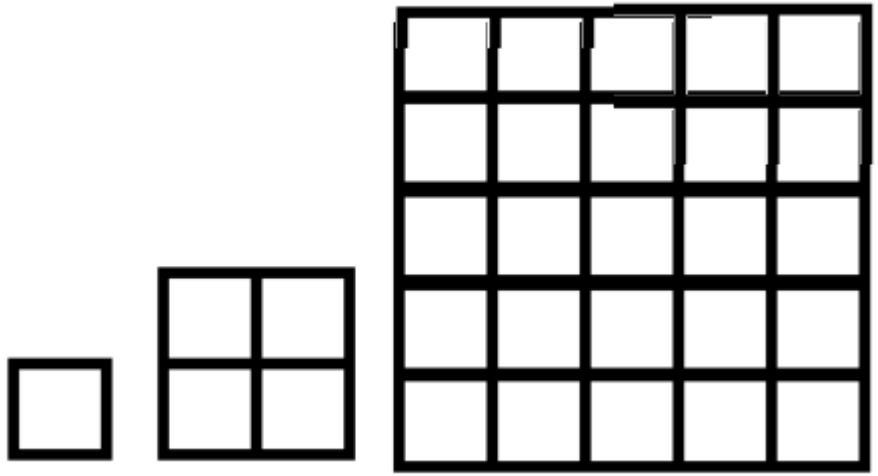
The I LOVE RECTANGLES Game

Imagine we put the 4 by 4 square down. That gives us a rectangle, so far so good. Can we put down anything else? Let's try putting down the 3 by 3.



We have placed a 4 by 4 square. This is a rectangle!

We see the 3 by 3 will not fit next to the 4 by 4 and still give a rectangle!

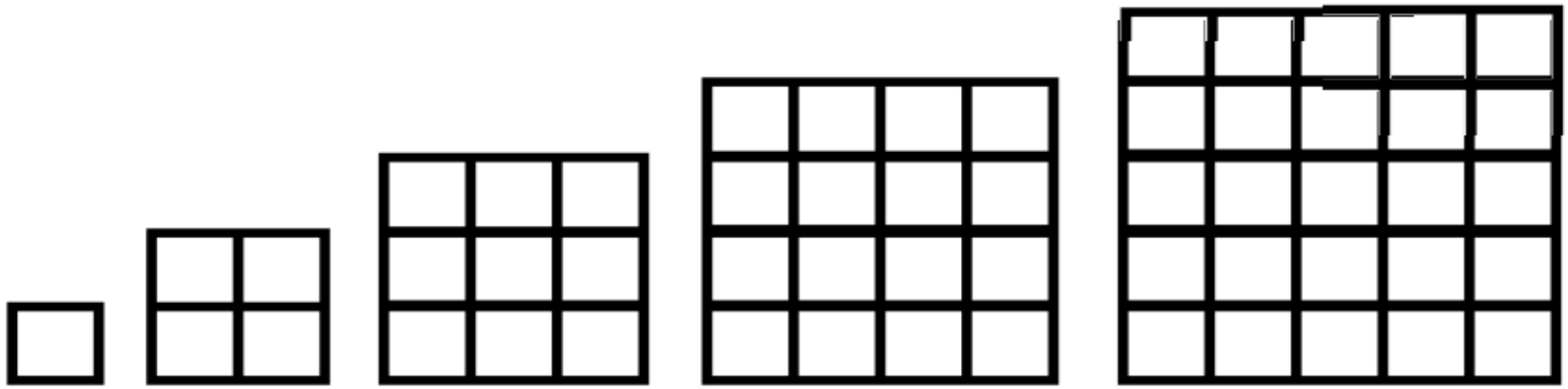


These are the squares we would have left if we try to use a 3 by 3. We would have a 1 by 1, a 2 by 2, a 5 by 5, a 6 by 6 (not drawn) and so on.

The I LOVE RECTANGLES Game

In fact, no matter WHAT square we put down first, we cannot put any more down! If we put down a 5 by 5, to keep it a rectangle we would need something that has a side of length 5, but we only have ONE of each square!

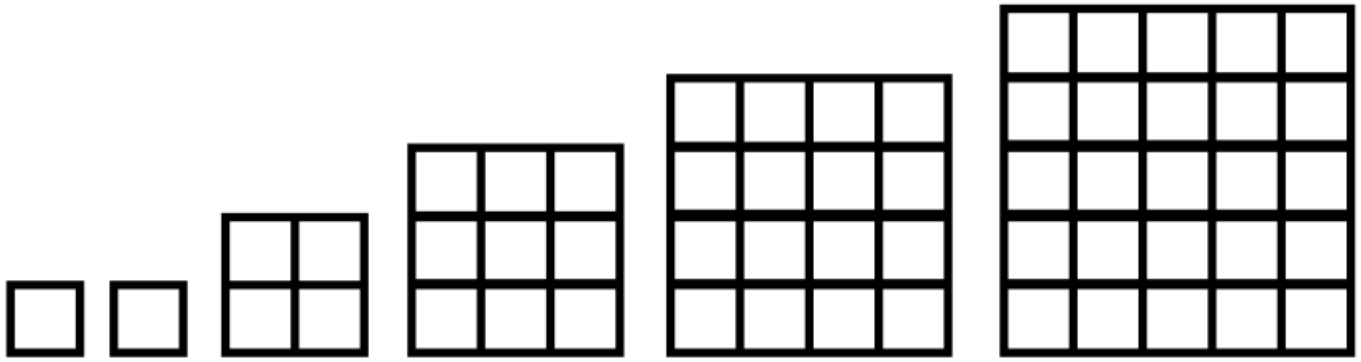
We have to modify the game. We need to give at least ONE more square. What is the smallest square we can give?



The I LOVE RECTANGLES Game

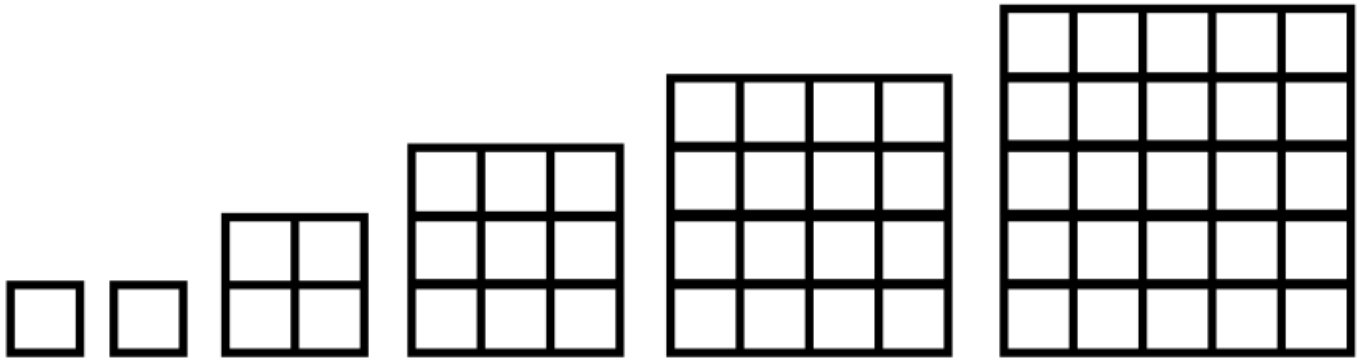
In fact, no matter WHAT square we put down first, we cannot put any more down! If we put down a 5 by 5, to keep it a rectangle we would need something that has a side of length 5, but we only have ONE of each square!

We have to modify the game. We need to give at least ONE more square. What is the smallest square we can give? Answer: a 1 by 1 square! Can we do it now?



The I LOVE RECTANGLES Game

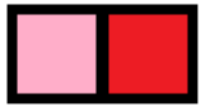
OK, we want to put the squares down one at a time so that we always have a rectangle. We cannot put a square on top of a square. Which should we put down first? Which should we put down second?

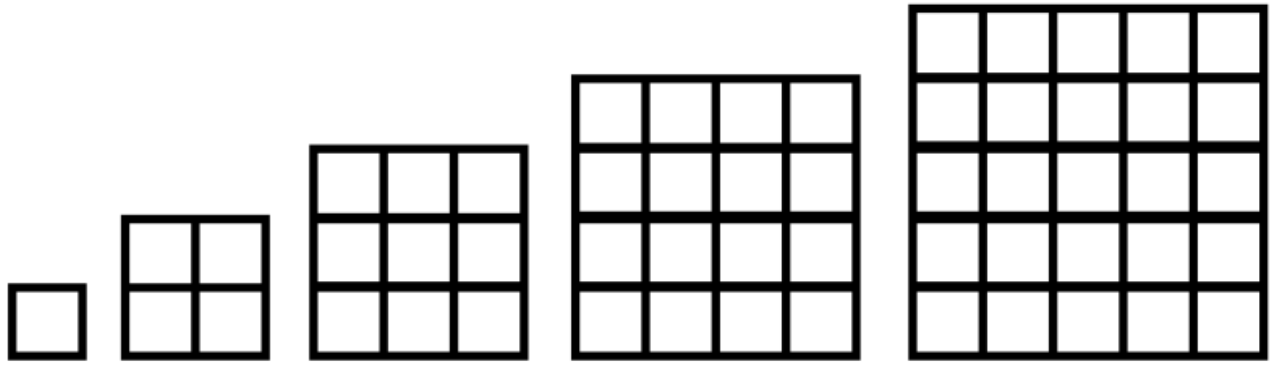


The I LOVE RECTANGLES Game

OK, we want to put the squares down one at a time so that we always have a rectangle. We cannot put a square on top of a square. Which should we put down first? Which should we put down second?

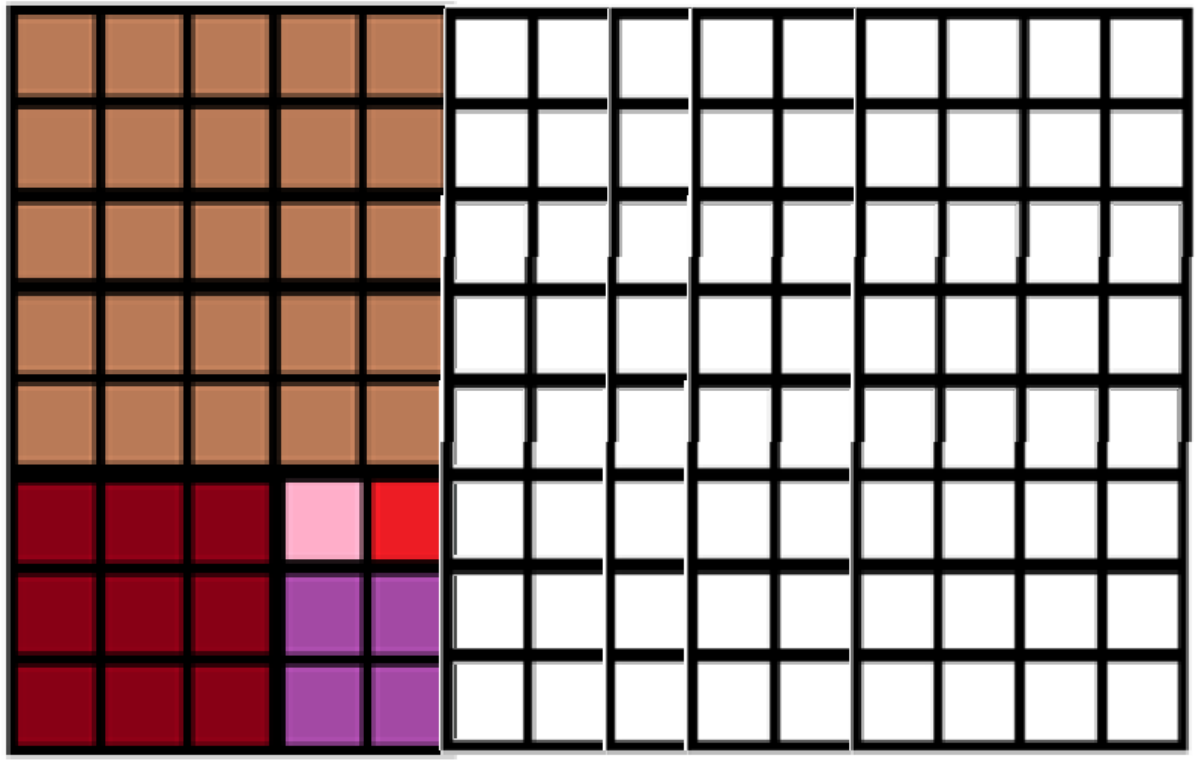
Makes sense to start with the two 1 by 1 squares, as they fit! Here is placing the second 1 by 1 next to the first 1 by 1.





The I LOVE RECTANGLES Game

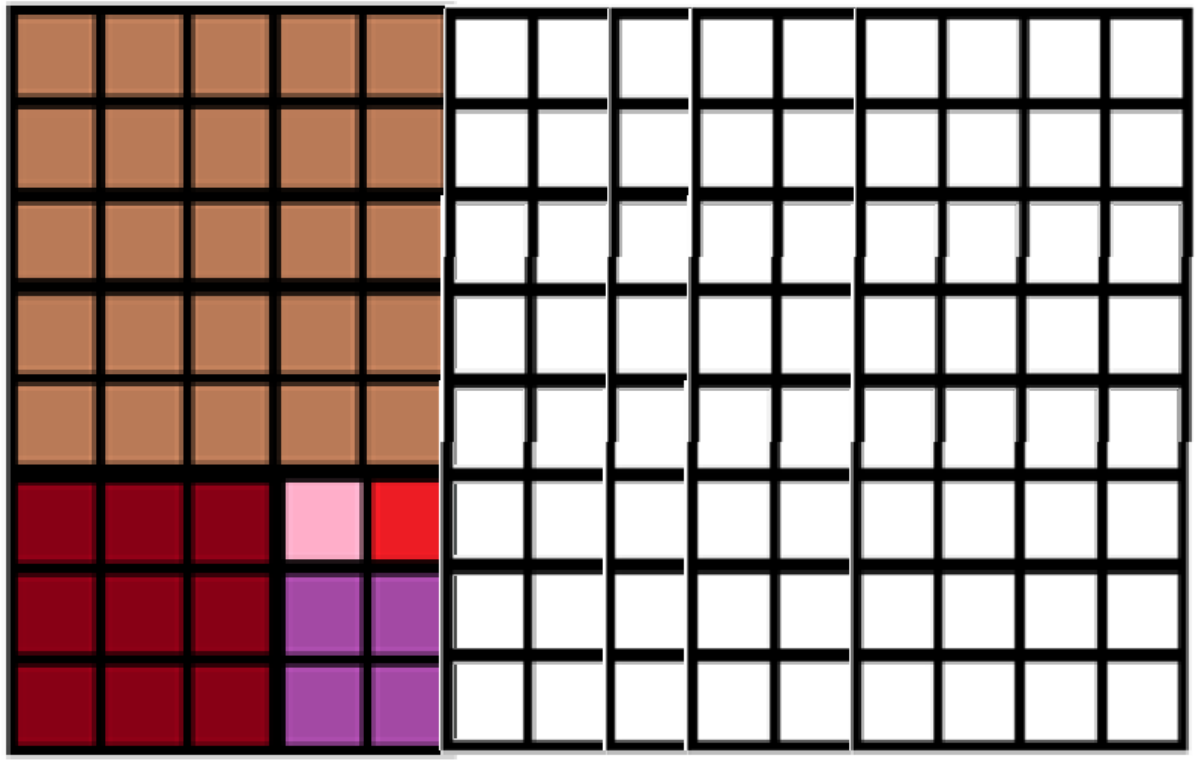
We had a 5 by 8 rectangle. We need to add something with a side of length 5 or 8. Thus we won't use the 4 by 4, the 6 by 6 or the 7 by 7, but we will use the 8 by 8……



The I LOVE RECTANGLES Game

We write down the squares used in the order used:

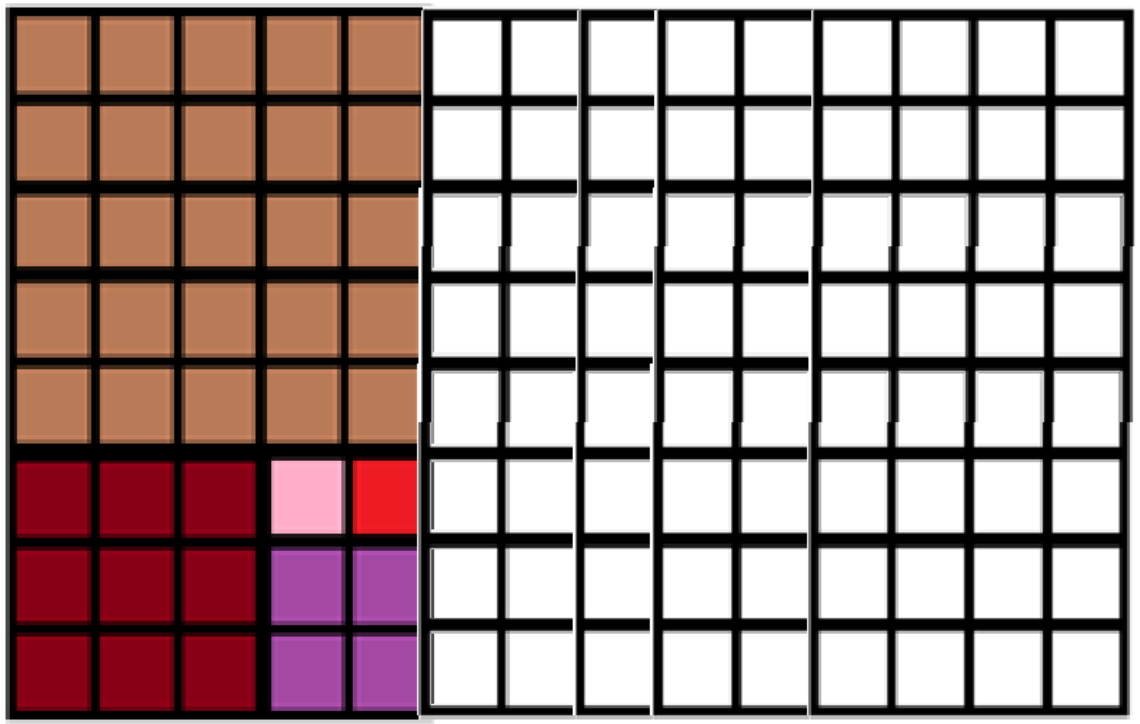
1 by 1, 1 by 1, 2 by 2, 3 by 3, 5 by 5, 8 by 8, …..



The I LOVE RECTANGLES Game

Let's just write down the side lengths of the squares in the order used:

1, 1, 2, 3, 5, 8, …. DO YOU NOTICE A PATTERN?



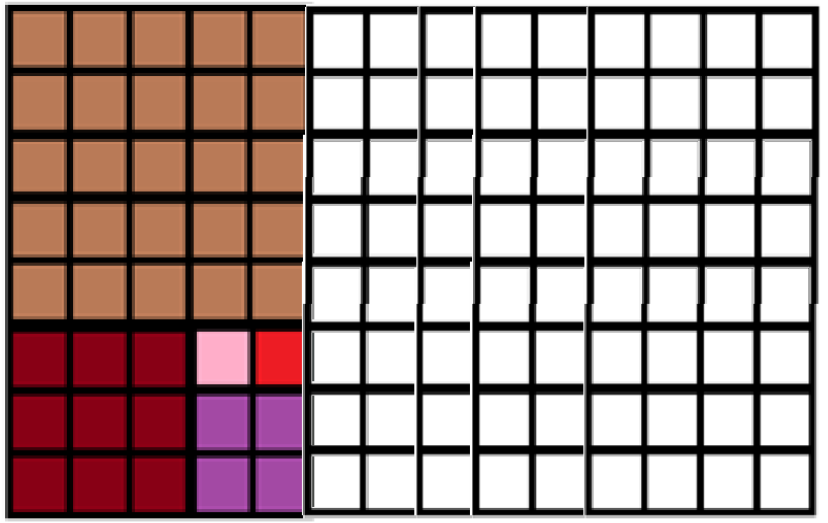
$F_{n+1} = F_n + F_{n-1}$

The I LOVE RECTANGLES Game

Let's just write down the side lengths of the squares in the order used:

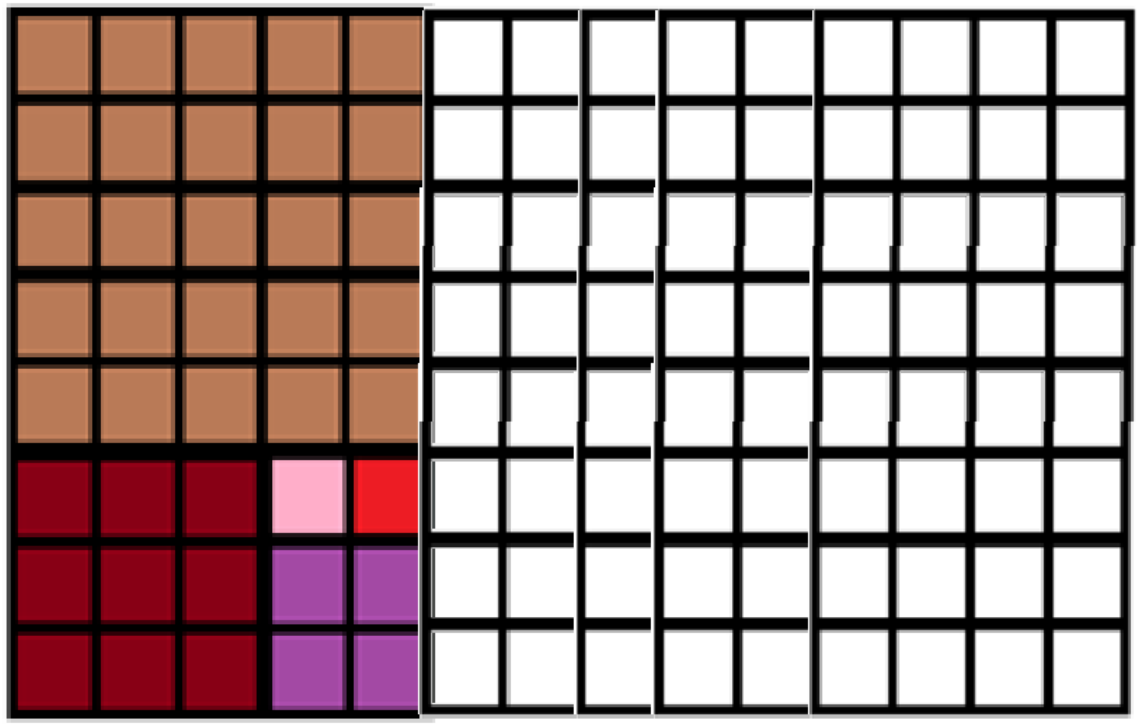
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ….

We start 1, 1, and then after that each term is the sum of the previous two terms! 2 = 1 + 1, 3 = 2 + 1, 5 = 3 + 2, 8 = 5 + 3, and so on. Can you continue the pattern?

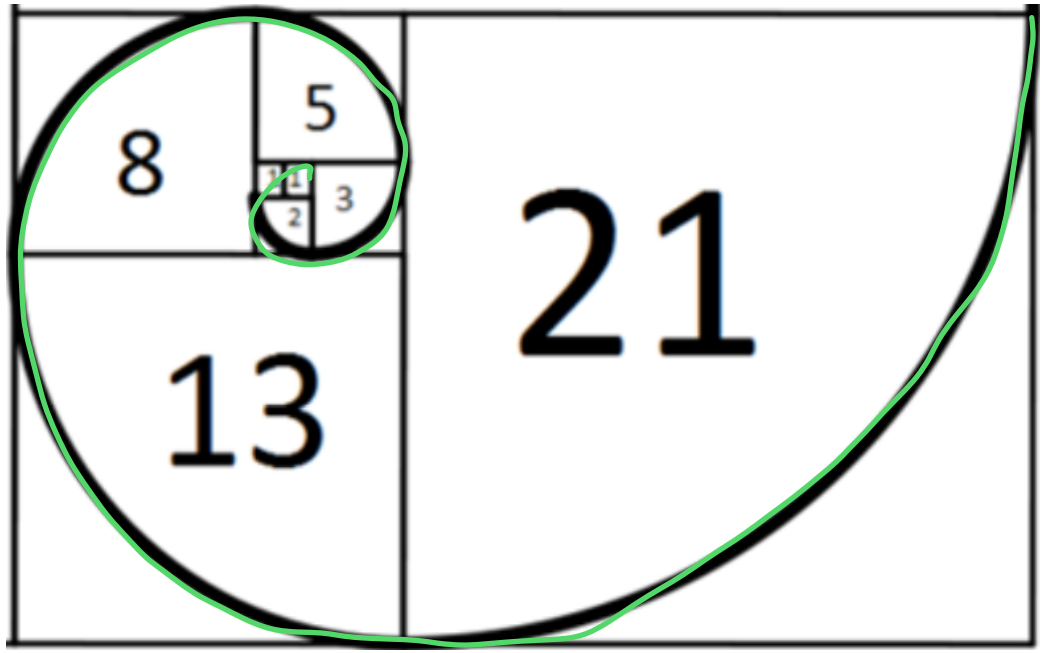


The Fibonacci Sequence

The numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …. are called the Fibonacci numbers, and have many wondrous properties. See for example https://www.youtube.com/watch?v=me6Dnl2DOtM .



ADVANCED TOPIC!



$Area = 34 \times 21$

$= 1^2 + 1^2 + 2^2 + 3^2 + \ldots + 21^2$

Proved:

$F_1^2 + F_2^2 + \ldots + F_n^2 = F_n * F_{n+1}$

Advanced: you can calculate area two ways. It is length times width, which here is 21 by 34. It is also the sum of the areas of each square, which is $1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 + 21^2$. These are equal! You can thus prove the sum of the squares of the first n Fibonacci numbers is the nth Fibonacci number times the $(n+1)^{st}$ Fibonacci number!

Rules for Triangle Game

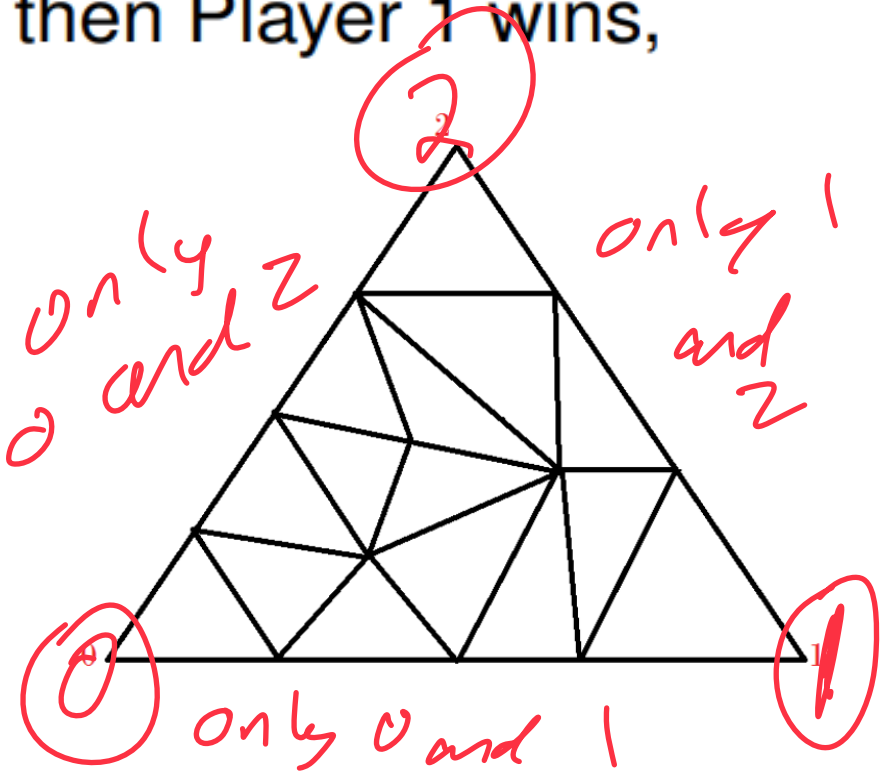
Take an equilateral triangle, label corners 0, 1 and 2.

Subdivide however you wish into triangles.

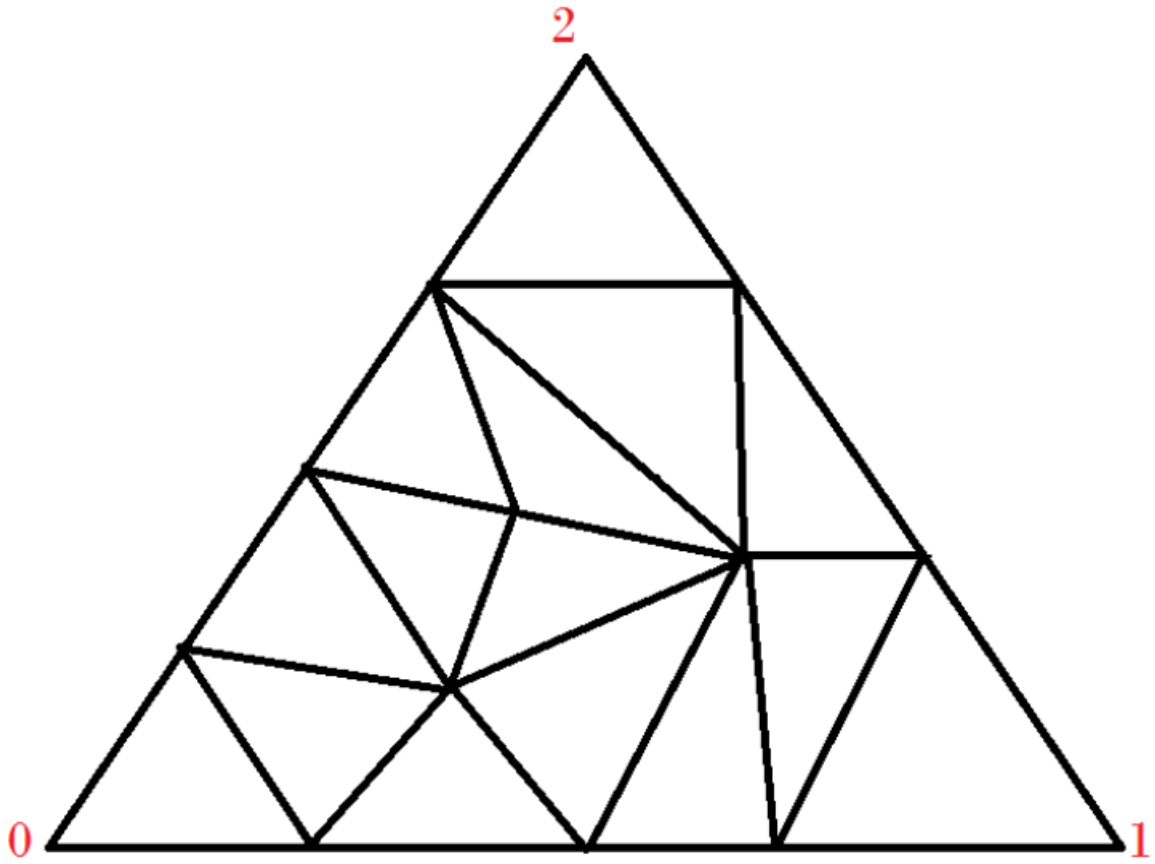
Add labels, if a sub-triangle labeled 0–1–2 then Player 1 wins, else Player 2.

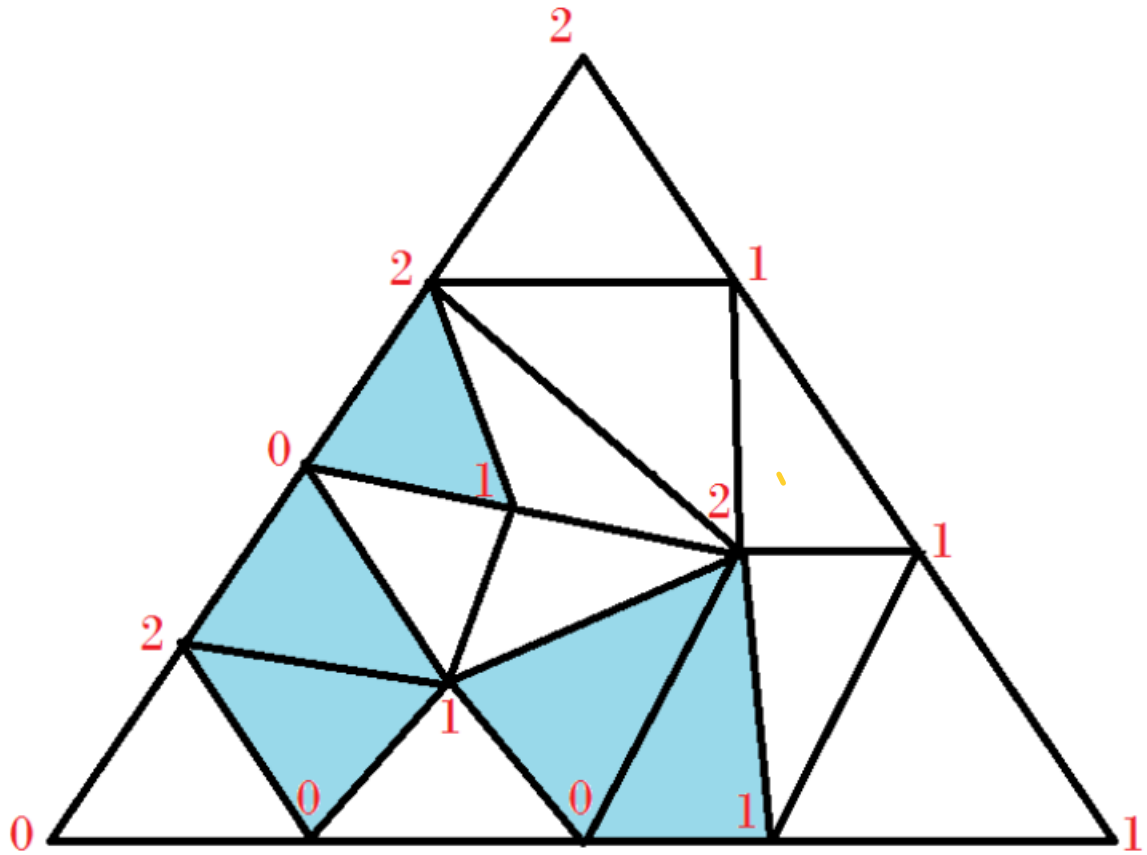
Take turns adding labels, subject to:

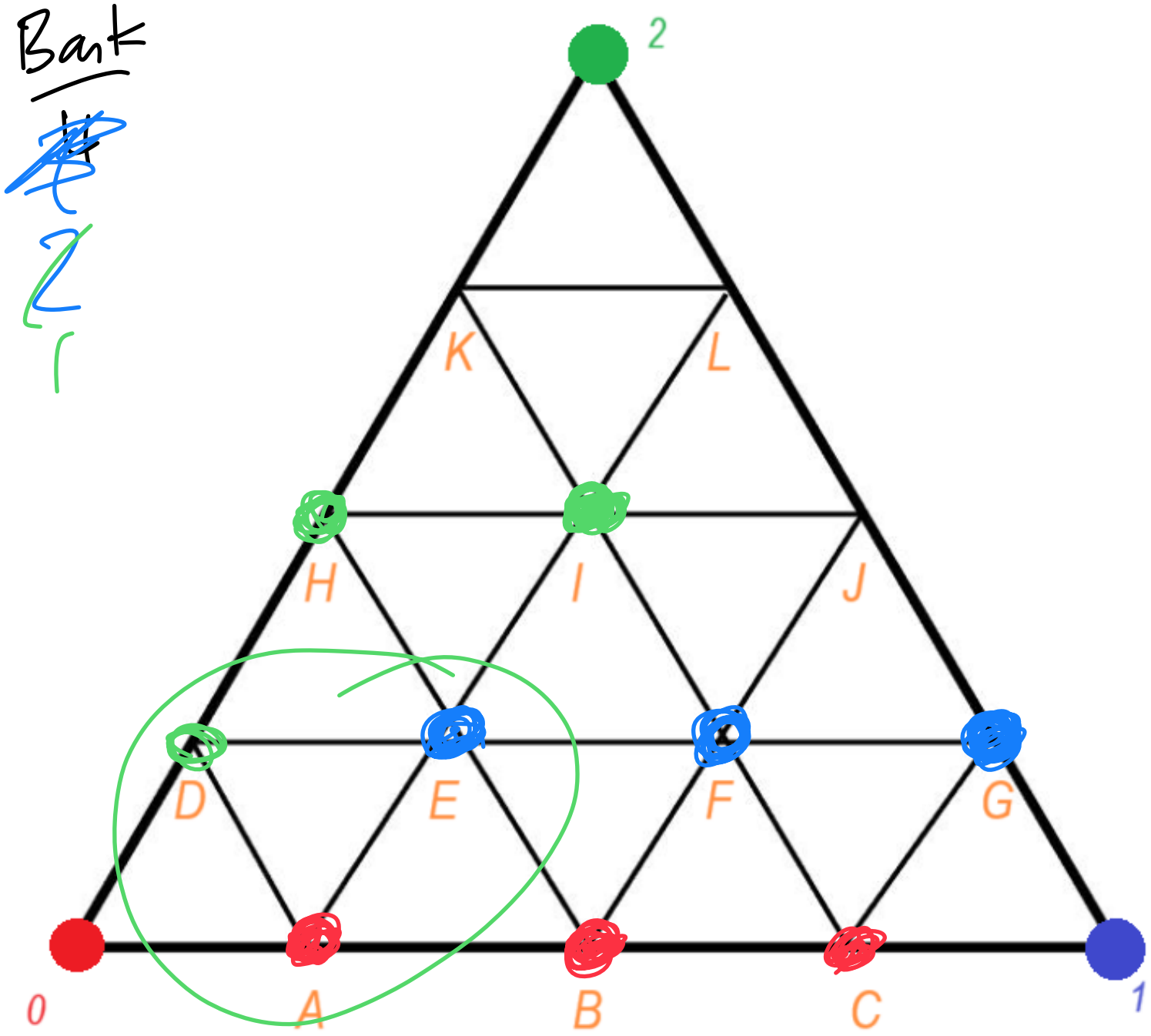
- On 0–1 boundary must use 0 or 1
- On 1–2 boundary must use 1 or 2
- On 0–2 boundary must use 0 or 2



Who has the winning strategy? What is it?


2
0
1


2
2
1
0
1
2
1
2
1
0
0
1
0
1



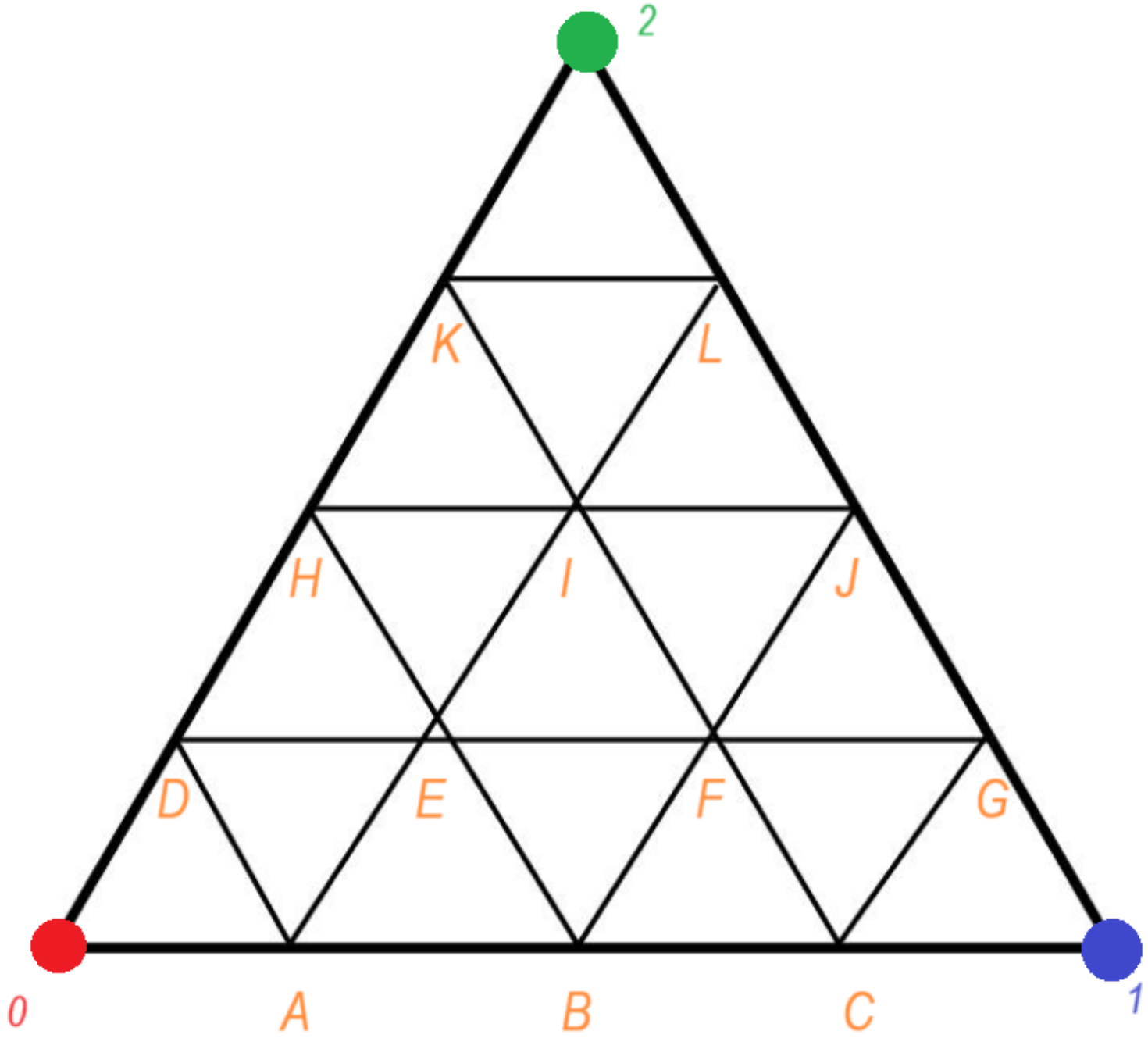
$40: 1 to 1

$20: 2 to 1

$10: 3 to 1

$5: 4 to 1

$1: 5 to 1



$40: 1 to 1

$20: 2 to 1

$10: 3 to 1

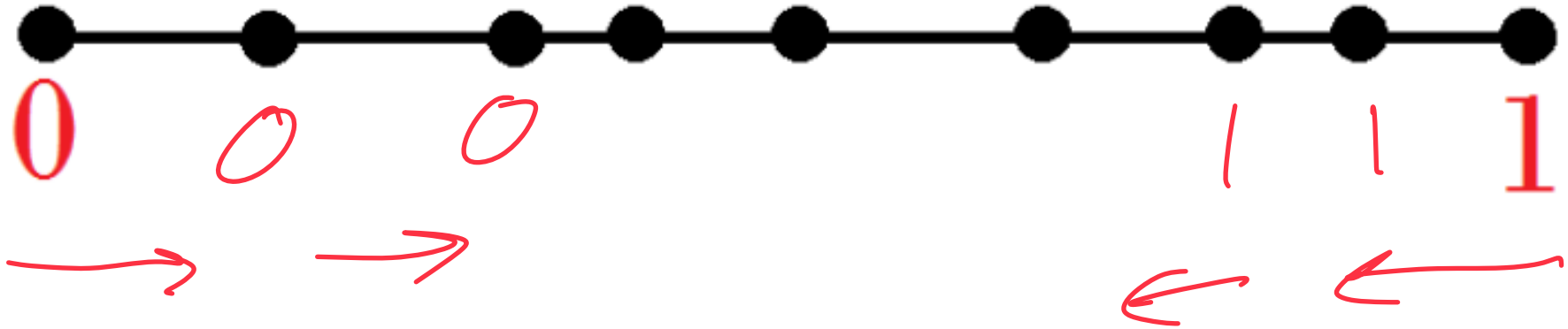
$5: 4 to 1

$1: 5 to 1

$.01 : 99 to 1

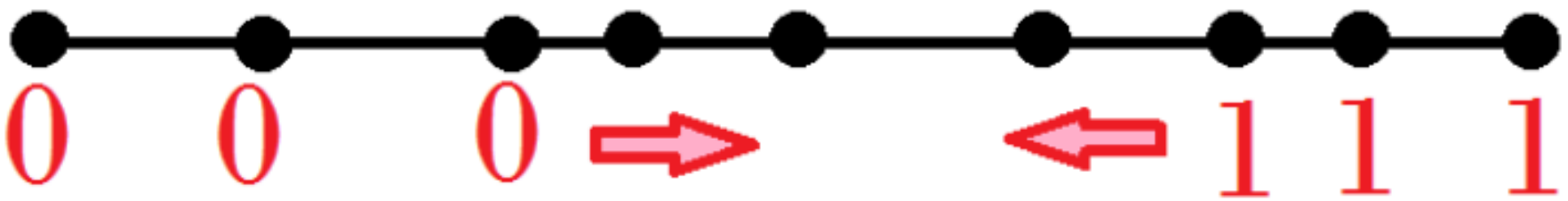
The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.



The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.



Cannot prevent at least one 0–1 segment.

The Line Game (cont)

Mono-variant: as add labels, number of 0–1 segments stays the same or increases by 2.

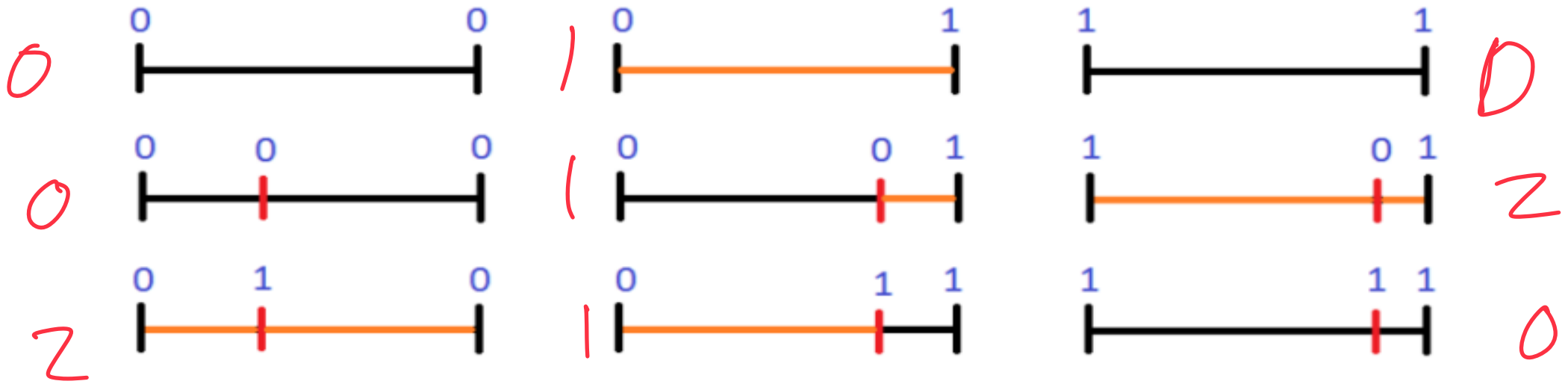


Figure 3. The various cases from the 1-dimensional Sperner proof. Notice in each of the six cases, the change in the number of 0–1 segments is even.

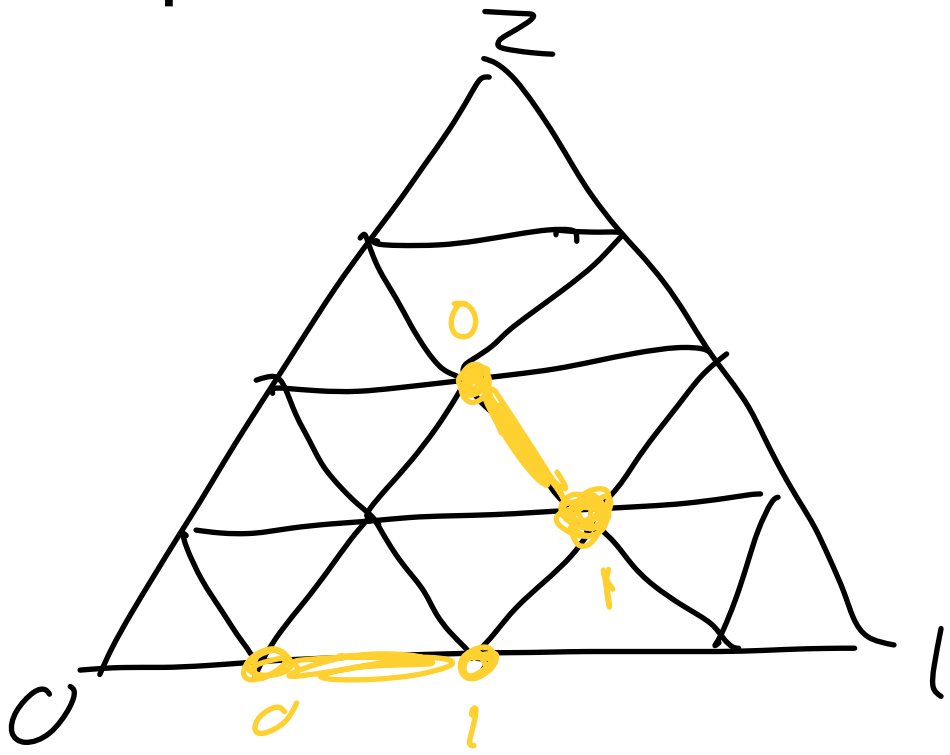
Change # of 0-1 segments by an even #

Can also view this as a parity argument.

Triangle Game:

Triangles with a 0-1 segment

Can prove number of 0-1-2 triangles is odd: thus at least one!



T_{012} = # 0-1-2 triangles : gives one

T_{011} : gives two

T_{001} : gives two

How many 0-1 segments?

odd even even

$1 \cdot T_{012} + 2 \cdot T_{011} + 2 \cdot T_{001}$

$= 2$ (# interior 0-1) $+ 1$ (# borders)

even odd

Triangle Game:

Can prove number of 0-1-2 triangles is odd: thus at least one!

Related to fixed points:

- x is a fixed point of f if f(x) = x.

Triangle Game:

Can prove number of 0-1-2 triangles is odd: thus at least one!

Related to fixed points:

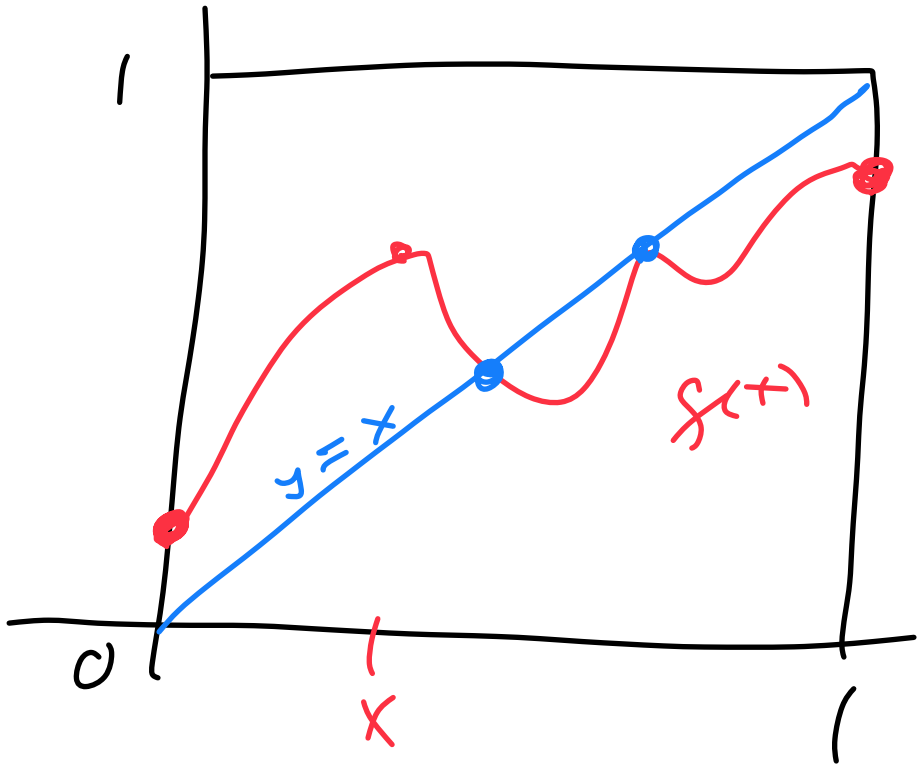
- x is a fixed point of f if f(x) = x.

Two questions:

- What functions have fixed points?
- Why do we care?

Fixed points exist:

Continuous f from [0,1] to [0,1] has a fixed point.



$f(0) \neq 0$

$f(1) \neq 1$

Fixed points exist:

Solving equations equivalent to finding a fixed point!

Imagine have a machine that can give fixed point:

- **Can use this to solve g(x) = a for any g and any a!**

$f(x) := g(x) - a + x$

If x is a fixed point for f Then $f(x) = x$

or $g(x) - a + x = x$ or $g(x) - a = 0$ or

$g(x) = a$

Nash Equilibrium:

Important in game theory: often existence proof that in multi-player games have strategies that lead to no one having an incentive to unilaterally change, not necessarily optimal!

Prisoner's Dilemna:

https://www.shutterstock.com/image-vector/cartoon-stickman-stick-figure-police-officer-2165944019

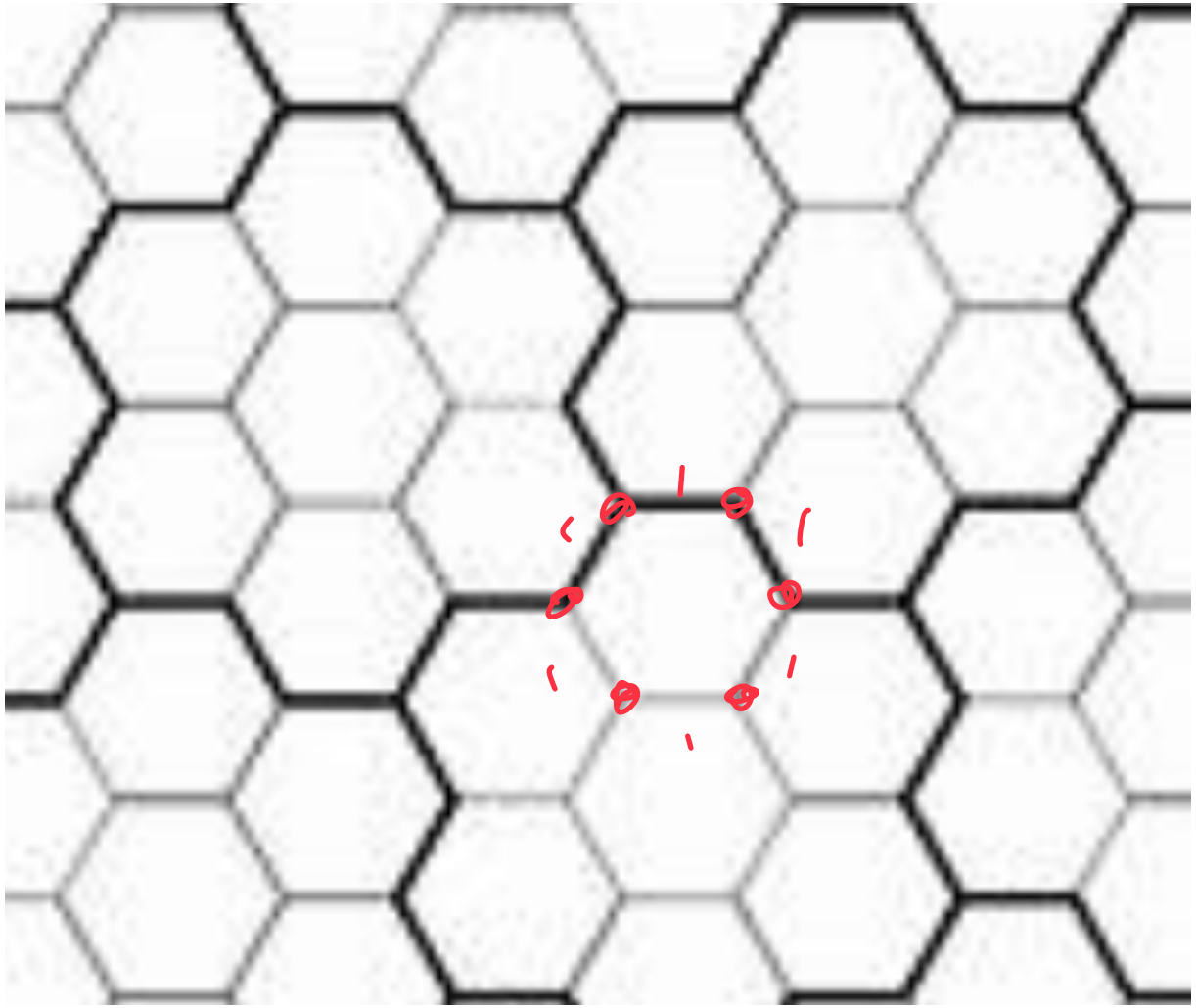
		PRISONER B	
		Confess	Doesn't confess
PRISONER A	Confess	5 years / 5 years Total -10	9 years / Goes free Total -9
	Doesn't confess	Goes free / 9 years Total -9	2 years / 2 years Total -4

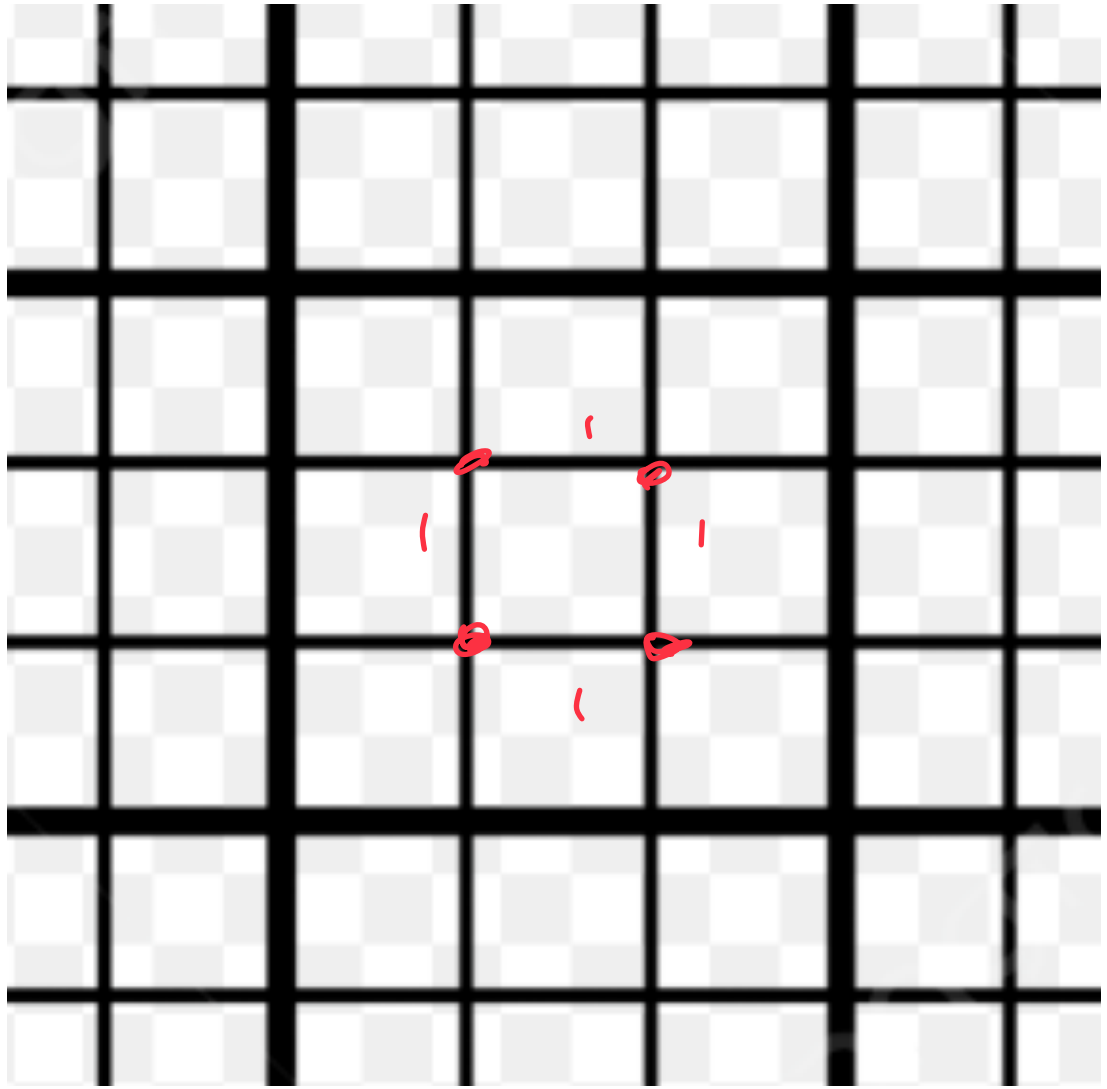
https://image.slideserve.com/1273100/prisoner-s-dilemma1-l.jpg

Sperner and Rental Harmony: Special Guest Francis Su: https://youtu.be/y3f0img9pRE

Geometry Applications:

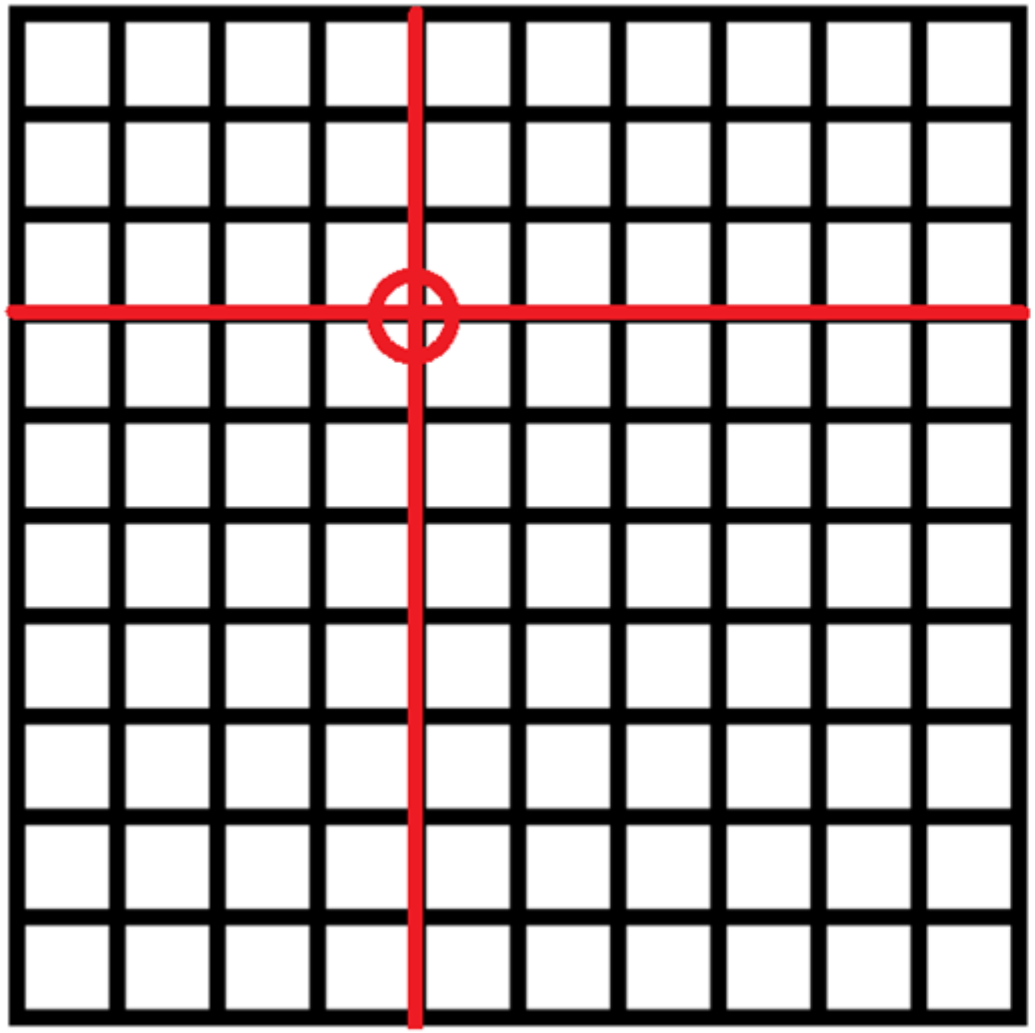
Seating, Pooled Testing, Tomography



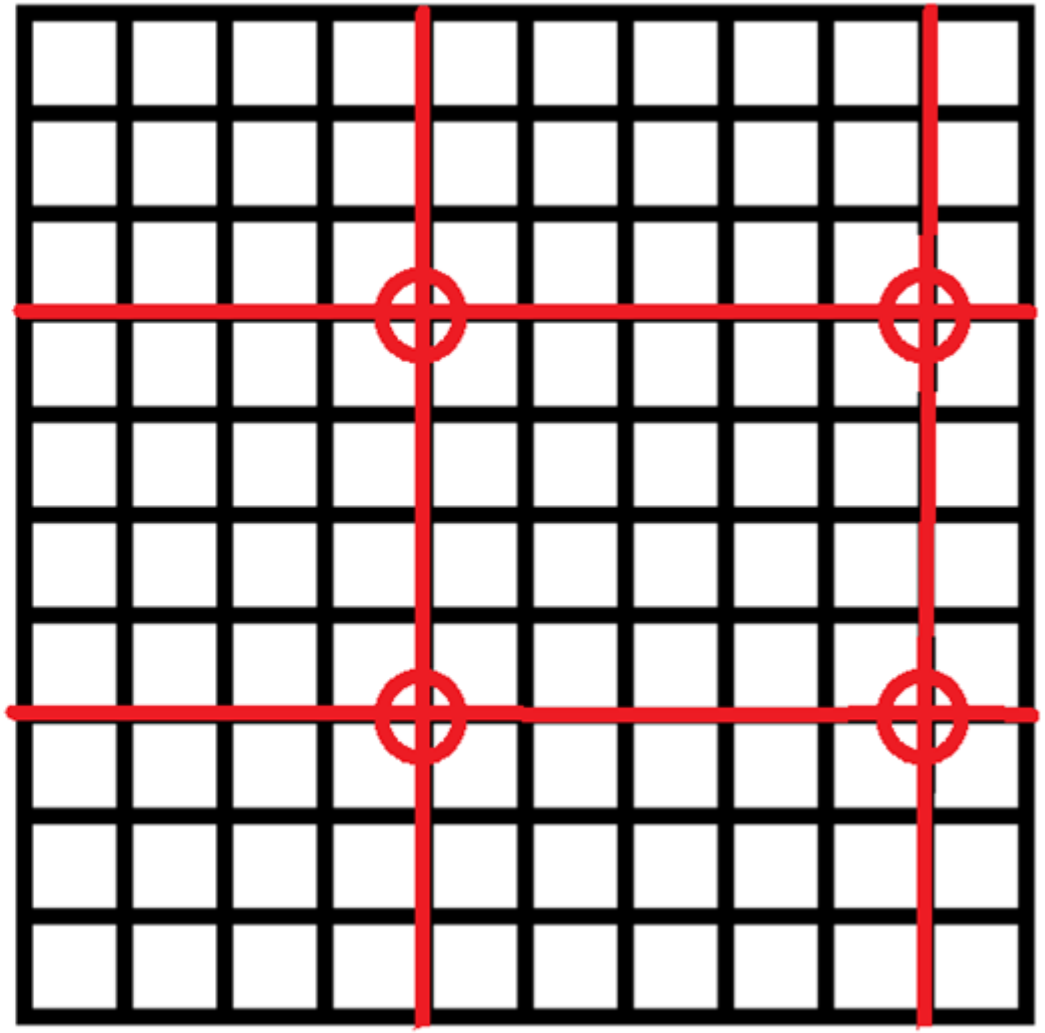


QWERTY

Pooled Testing: Reduces from order n to order $\sqrt{n}$ tests.



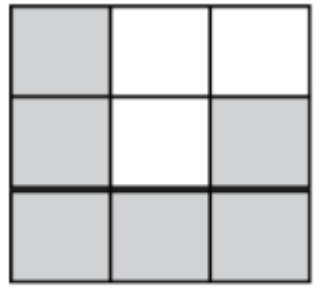
n^2
vs
21



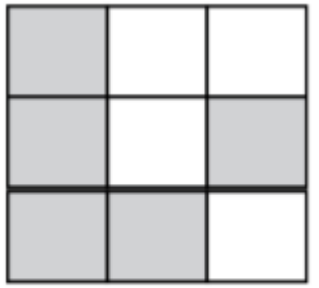
Tomography: A Geometric and Computational Approach

Tomography is the science of examining internal structure with external measurements. Most people think of tomography in the context of medical testing, such as CT scans, but tomography can be used any time it is impossible to directly look inside something. Students tackle activities in which they are challenged to determine what is inside some object. They study how CT scan images of an object are created using 3-D reconstruction of 2-D slices of the object using shadows, pin prints, graphs, and more. The main questions of the module are: How can 3-D images be created from 2-D images (i.e. slices) of it?; How much computational power and skill are required to do these reconstructions; and what do they depend on? This module is appropriate for high school classes in computer science, mathematics, biology, environmental

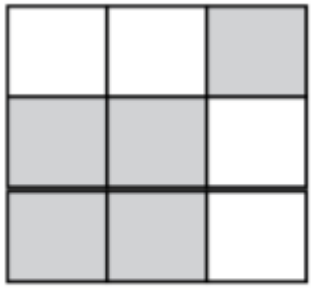
Consider the example below, which shows the top, front, and right views. Each shaded square indicates a Styrofoam ball.



top

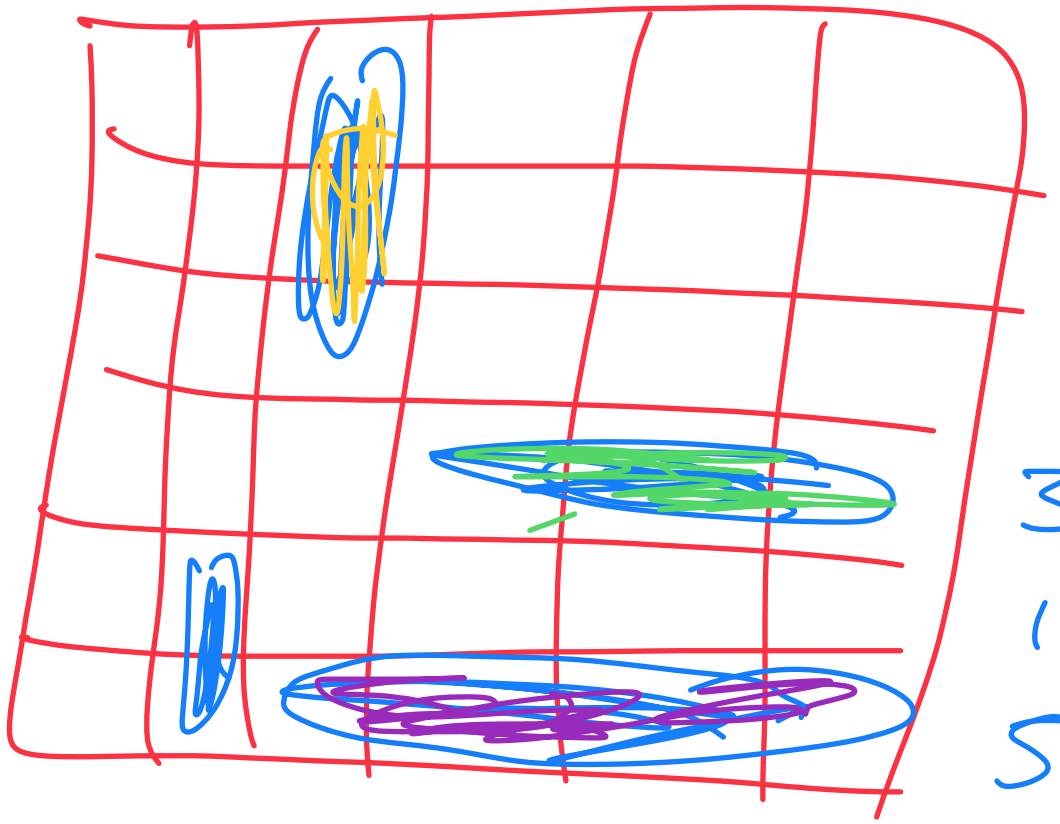


front



right

Battleship



vs

WHAT DO YOU MEAN?!?

Steven Miller, Williams College (sjm1@Williams.edu)

Video 3: What do you MEAN? https://youtu.be/jBKZaCxpgSE (word file here, pdf here) (3/19/2020): Comfort with Algebra sufficient: 40 minutes

Definitions

Means and averages

- Given x and y, the average or mean is the number in between
- ArithmeticMean(x,y) = (x + y) / 2.
- There is more than one mean that can be defined!
- What properties should a mean have? Assume $0 < x \leq y$.

Desired Properties

We want:

- $x \le \text{mean}(x,y) \le y$. Should be "in between"
- $\text{mean}(x,x) = x$.

Does ArithmeticMean(x,y) = (x+y)/2 satisfy these properties?

Scaling property:

$$\text{Mean}(ax, ay) = a\ \text{mean}(x,y)$$

Desired Properties

We want:

1. $x \le \text{mean}(x,y) \le y$. Should be “in between”
2. $\text{mean}(x,x) = x$.

Does ArithmeticMean(x,y) = (x+y)/2 satisfy these properties?

Proof of (1): Since $0 < x \le y$, we have $x + x \le x + y \le y + y$.
So we know $2x \le x + y \le 2y$. Divide everything by 2 and we get
$x \le (x+y)/2 \le y$ or $x \le \text{ArithmeticMean}(x,y) \le y$.
We proved the first result!

Desired Properties

We want:

1. $x \leq \text{mean}(x,y) \leq y$. Should be “in between”
2. $\text{mean}(x,x) = x$.

Does ArithmeticMean(x,y) = (x+y)/2 satisfy these properties?

Proof of (2):Does ArithmeticMean(x,x) equal x?
Yes! ArithmeticMean(x,x) = (x+x)/2 = 2x / 2 = x.
So the ArithmeticMean(x,y) = (x+y)/2 satisfies our two properties.

We write AM(x,y) = ArithmeticMean(x,y) = (x+y)/2 to save space.

Question

Is there another choice of mean that satisfies the two properties we wish?

We want:

1. $x \leq \text{mean}(x,y) \leq y$. Should be “in between”
2. $\text{mean}(x,x) = x$.

Thoughts?

Question

Is there another choice of mean that satisfies the two properties we wish?

We want:

1. $x \le \text{mean}(x,y) \le y$. Should be “in between”
2. $\text{mean}(x,x) = x$.

Try $\text{mean}(x,y) = \text{Sqrt}(x\ y)$.

- Check: $\text{Sqrt}(2 * 8) = \text{Sqrt}(16) = 4$ and that IS between 2 and 8.
- Check: $\text{Sqrt}(1 * 100) = \text{Sqrt}(100) = 10$ and that is between 1 and 100.

So maybe this is another choice of mean. Maybe it also satisfies the two properties….

Question

Try mean(x,y) = Sqrt(x,y). Must show

1. $x \leq \text{mean}(x,y) \leq y$. Should be “in between”
2. mean(x,x) = x.

First property: Show if $0 < x \leq y$ then $x \leq \text{Sqrt}(x\ y) \leq y$.

We know $x \leq y$ so $x\ x \leq x\ y \leq y\ y$

But $x^2 \leq x\ y \leq y^2$. Now take the square-root!

$\text{Sqrt}(x^2) = x$ and $\text{Sqrt}(y^2) = y$, so get $x \leq \text{Sqrt}(x\ y) \leq y$, as claimed!

Question

Try mean(x,y) = Sqrt(x,y). Must show

1. $x \leq \text{mean}(x,y) \leq y$. Should be “in between”
2. mean(x,x) = x.

Second is easier!

We have Sqrt(x x) = Sqrt(x^2) = x. We are done!

We call this the GEOMETRIC MEAN. We write GM(x,y) = Sqrt(x y)

Two Means

$0 < x \le y$

So we have two choices of mean:

- AM(x, y) = (x + y) / 2
- GM(x, y) = Sqrt(x y)

BOTH have two good properties:

- For $0 < x \le y$ both satisfy $x \le \text{mean}(x,y) \le y$ and $\text{mean}(x,x) = x$.

More used to the first.

Try x = 2 and y = 8:

- Get AM(2,8) = (2 + 8) / 2 = 10 / 2 = 5
- Get GM(2,8) = Sqrt(2 * 8) = Sqrt(16) = 4

WLOG Either x or y is 1

$\frac{x+1}{2}$ vs $\sqrt{x \cdot 1}$

$\frac{x^2+2x+1}{4}$ vs x

x^2+2x+1 vs $4x$

$x^2-2x+1 \ge 0$

$(x-1)^2 \ge 0$

More used to the first.

Try x = 3 and y = 12

- Then AM(3, 12) = 15/2 = 7.5
- And GM(3,12) = Sqrt(36) = 6.

Two Means

So we have two choices of mean:

- AM(x, y) = (x + y) / 2
- GM(x, y) = Sqrt(x y)

BOTH have two good properties:

- For $0 < x \le y$ both satisfy $x \le \text{mean}(x,y) \le y$ and mean(x,x) = x.

Try x = 1 and y is VERY large….

- Then AM(1,y) = (1 + y)/2 which is APPROXIMATELY y/2
- But GM(1,y) = Sqrt(y) which is MUCH smaller if y is large.
- Note if y is small we would say (1 + y)/2 is approximately .5

CONJECTURE: GM(x,y) ??? AM(x,y)

Two Means

So we have two choices of mean:

- AM(x, y) = (x + y) / 2
- GM(x, y) = Sqrt(x y)

BOTH have two good properties:

- For $0 < x \le y$ both satisfy $x \le \text{mean}(x,y) \le y$ and mean(x,x) = x.

Try x = 1 and y is VERY large….

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- But GM(1,y) = Sqrt(y) which is MUCH smaller if y is large.
- Note if y is small we would say (1 + y)/2 is approximately .5

CONJECTURE: $GM(x,y) \le AM(x,y)$

CONJECTURE: GM(x,y) ≤ AM(x,y)

PROOF: Consider: $0 < x \leq y$, what is true about $(\text{Sqrt}(x) - \text{Sqrt}(y))^2$? It must be positive…

- So $0 \leq (\text{Sqrt}(x) - \text{Sqrt}(y))^2$.

Remember FOIL: $(a - b)^2 = (a - b)(a - b) = a\ a - a\ b - b\ a + b\ b$: First Outside Inside Last

- So $(a-b)^2 = a^2 - 2\ a\ b + b^2$

We are looking at $(\text{Sqrt}(x) - \text{Sqrt}(y))^2$.

- $0 \leq (\text{Sqrt}(x) - \text{Sqrt}(y))^2 = \text{Sqrt}(x)^2 - 2\ \text{Sqrt}(x)\ \text{Sqrt}(y) + \text{Sqrt}(y)^2$.
- $0 \leq x - 2\ \text{Sqrt}(x\ y) + y$

Trying to get $AM(x,y) = (x+y)/2$ and $GM(x,y) = \text{Sqrt}(x,y)$

- $2\ \text{Sqrt}(x,y) \leq x + y$
- $\text{Sqrt}(x,y) \leq (x+y)/2$
- $GM(x,y) \leq AM(x,y)$.

We proved it!

Extensions

What if we had three objects: $0 < x \le y \le z$?

- $AM(x,y,z) = (x+y+z) / 3$
- $GM(x,y,z) = (x\ y\ z)^{1/3}$.

$x = GM(x,x,x) = (x^3)^{1/3}$

Is there another combination?

- $((x\ y\ +\ y\ z\ +\ x\ z) / ???\ )^{???}$

$x = \left(\frac{x^2+x^2+x^2}{a}\right)^b = \left(\frac{3x^2}{a}\right)^b$

Food for thought: can you find a choice of a and b such that

- $((xy + yz + zx) / a)^b$ is a mean, so it would satisfy
- $x \le TripleMean(x,y,z) \le z$ and $TripleMean(x,x,x) = x$

$a = 3$

$b = 1/2$

What would be a good choice of x, y and z to build some intuition?

Extensions

What if we had three objects: $0 < x \le y \le z$?

- $AM(x,y,z) = (x+y+z) / 3$
- $GM(x,y,z) = (x\ y\ z)^{1/3}$.

Is there another combination?

- $((x\ y\ +\ y\ z\ +\ x\ z) / ???\ )^{???}$

Food for thought: can you find a choice of a and b such that

- $((xy + yz + zx) / a)^b$ is a mean, so it would satisfy
- $x \le TripleMean(x,y,z) \le z$ and $TripleMean(x,x,x) = x$

If $x = y = z$ then $((xx + xx + xx) / a)^b = (3\ x^2 / a)^b = x$ for ALL x.

- SO b = ??? and a = ???

Extensions

What if we had three objects: $0 < x \le y \le z$?

- $AM(x,y,z) = (x+y+z) / 3$
- $GM(x,y,z) = (x\ y\ z)^{1/3}$.

Is there another combination? YES

- $((x\ y\ +\ y\ z\ +\ x\ z) / 3\ )^{1/2}$

If $x = y = z$ then $((xx + xx + xx) / a)^b = (3\ x^2 / a)^b = x$ for ALL x.

- SO $b = ½$ and $a = 3$

SO this is our guess….

- Try $x = 3$ and $y = 4$ and $z = 5$
- $TripleMean(3,4,5) = (\ (12 + 20 + 15) / 3\ )^{1/2} = (47/3)^{1/2}$ is approximately 3.958
- This IS a reasonable answer! It is more than 3, less than 5!

Final Thoughts

AM(x,y) = (x+y)/2 GM(x,y) = Sqrt(x y)

Test 1 Get 1 and on Test 2 get 100

- AM(1, 100) = (1 + 100)/2 = 50.5
- GM(1,100) = Sqrt(1 100) = 10

Recall

- Log(x y) = Log(x) + Log(y)
- So there is a relation between logarithms, AM and GM

$p\,x + (1-p)\,y$ with $0 \le p \le 1$

$x, y \Rightarrow z_x, z_y$ get $p z_x + (1-p) z_y$

$= z(p x + (1-p) y)$

These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.

Video 13: The Three Hat Problem and Error Correcting Codes: https://youtu.be/oMeKf7AhAa4 (powerpoint here, pdf here) (4/6/2020): Aimed for 3rd Grade and up: 26 minutes

The Hat Game

Steven and Kayla Miller, Williams College, sjm1@williams.edu

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The Hat Game

- There are three people.
- Each person gets a RED or YELLOW hat (randomly assigned) independent of what the other two people get.
- You can’t see your hat, but you can see the other two hats.

The Hat Game

- There are three people.
- Each person gets a RED or YELLOW hat (randomly assigned) independent of what the other two people get.
- You can't see your hat, but you can see the other two hats.
- You can strategize with each other, but once the game starts you can't talk to anyone or give signs.
- At the count of three, each person chooses to either say their hat color or stay silent.
- If everyone who speaks is right, you all win; if one or more is wrong, you all lose.
- What is your best strategy, and how often will you win?

The Hat Game

The Hat game seems hard.

Hint: What are odds that someone is correct, if they randomly guess a color?

Can you use this information to help you get a strategy that wins half the time?

You can only say RED or YELLOW, and if even one person who speaks is wrong the entire team loses; thus if two people are right and one is wrong, YOU LOSE!

Can you come up with a strategy where exactly half the time your team wins? You must decide when each person speaks and what they will say.

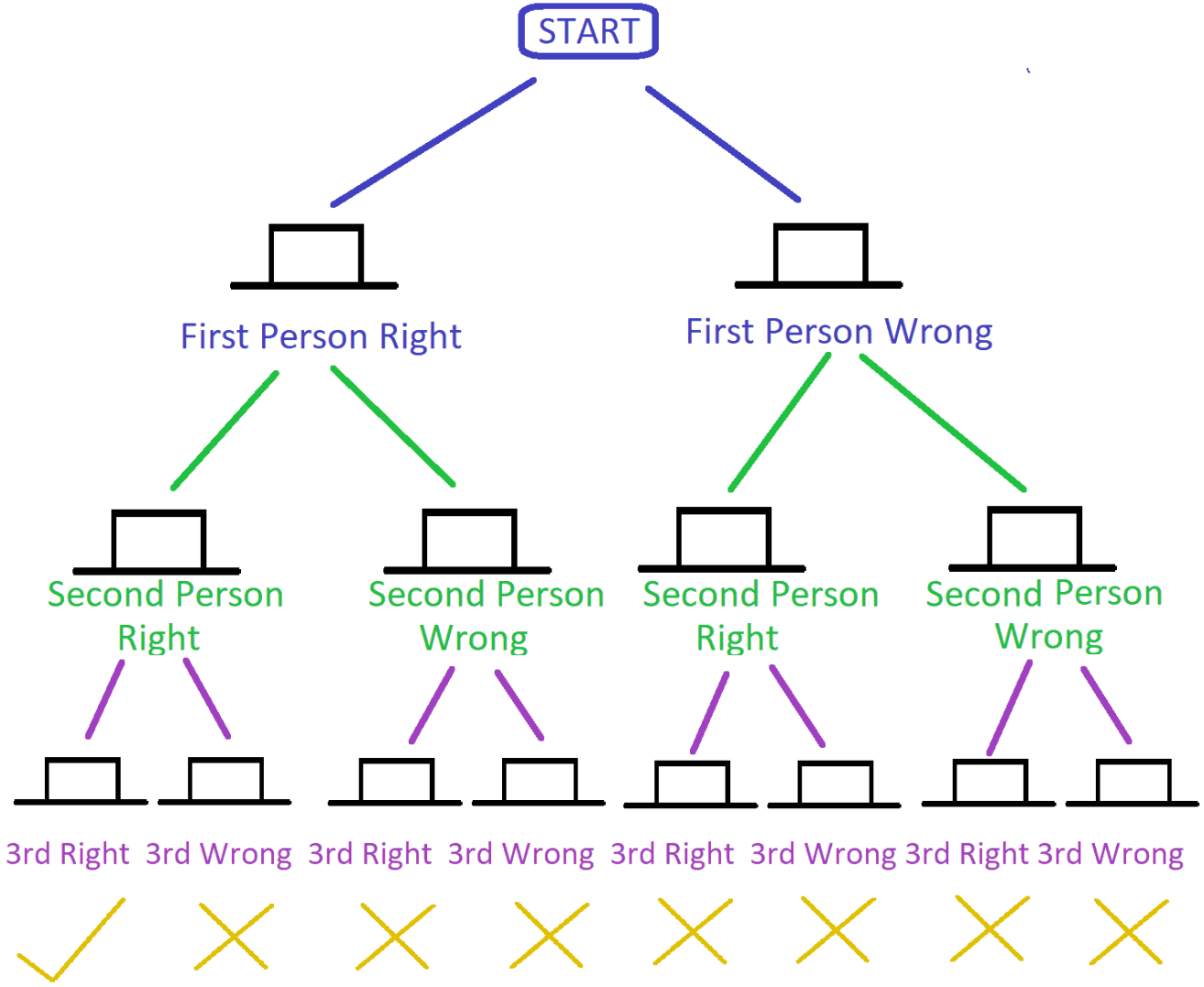
The Hat Game

If each person were to speak randomly, they would have a 50% chance of being right and a 50% chance of being wrong. So 1 out of 2 times correct.

Thus the odds that two who speak randomly are both right is 25%, or 1 out of 4 times. Why?

What if three people speak randomly?
What are the odds all three are right?

It is 12.5% or 1 out of 8.



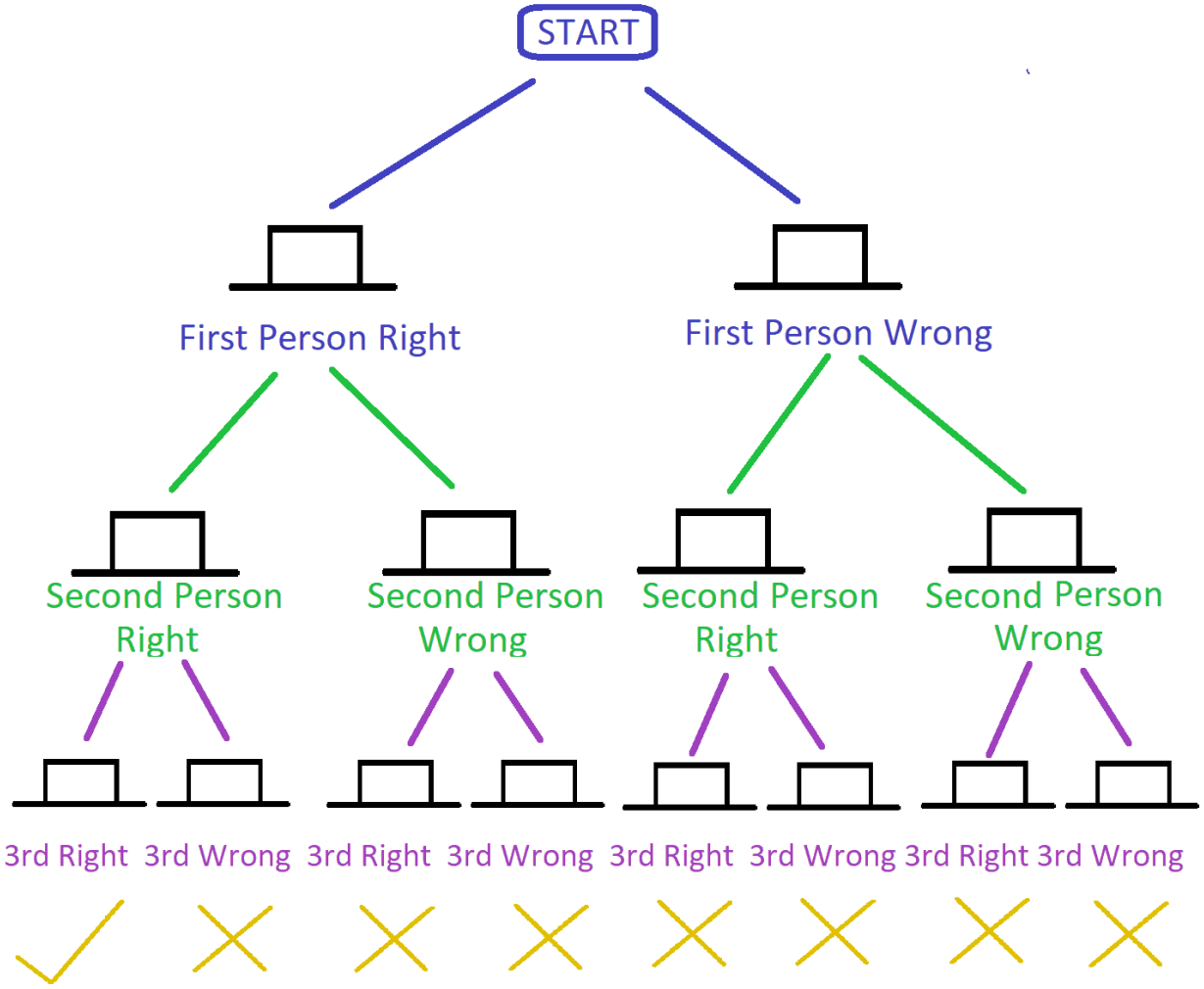
The Hat Game

Remember our FIRST goal is is to find a strategy that will win half the time. We see if one person speaks randomly that works!

If two speak randomly it is worse, and works just 1 out of 4 times.

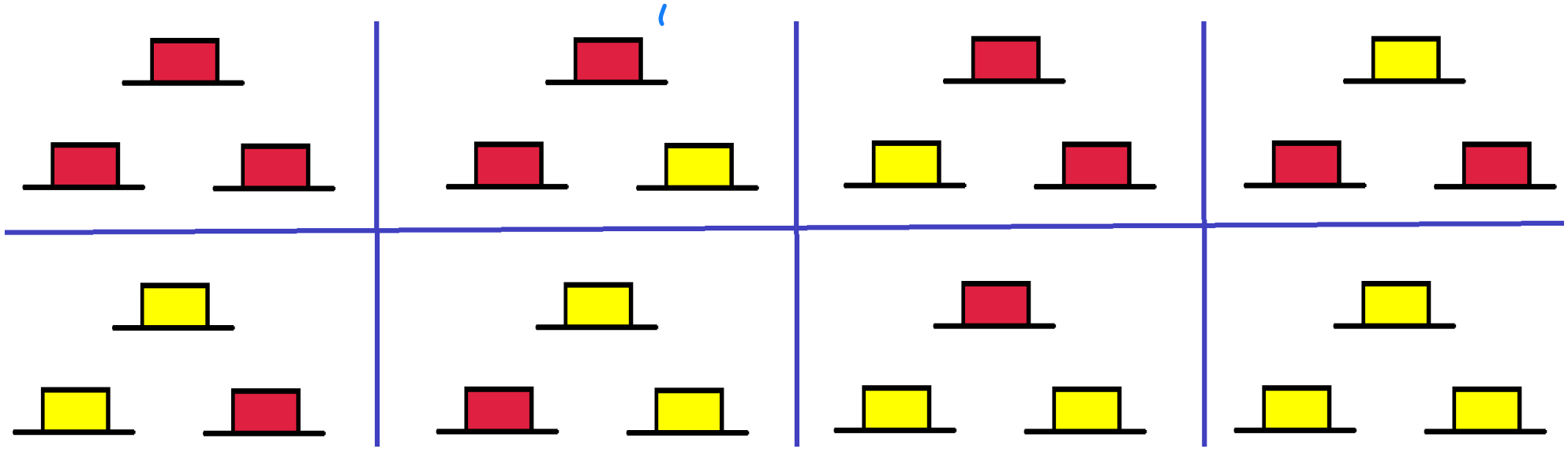
If three speak randomly it is even worse, working just 1 out of 8 times.

Can we do better than winning half the time?



The Hat Game

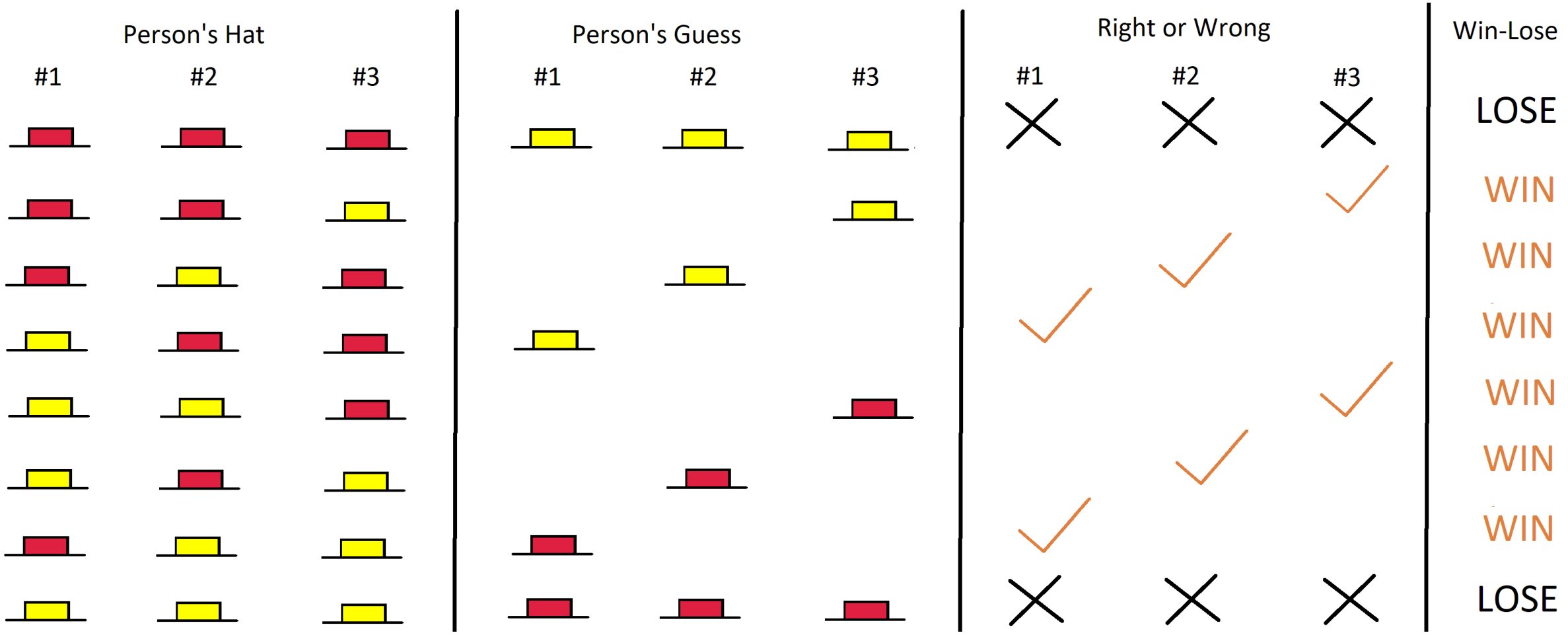
Think about what each person sees in each case….



What are the possible hat combinations each person can see?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Hat Game: Analysis



There is a lot of interesting behavior above – what do you see in the analysis? Are you surprised? Should anyone be able to be right more than half the time? Is anyone?

The Hat Game: Analysis

Notice each person speaks four times and is silent four times; they are right twice and wrong twice.

We have six wrong answers and six right ones; we get to 75% because when we are wrong, we are WRONG, while when we are right we just scrape by.

The Hat Game

We saw that we can win the hat game 75% of the time.

The strategy was to speak only if you see two hats that are the same, and say the opposite color.

Why do we care about this problem?

This has applications to error detection and error correction codes!

Sending Messages

Imagine we want to send a message.

We could send letters one at a time, but there is a danger.

What if one of the letters is mis-received?

Maybe we are trying to charge something on Amazon to our credit card, but the wrong digit is sent.

Or maybe their system records the wrong value for our order and sends us the wrong item!

We want to tell if a mistake is made, and even better correct such a mistake!

Detecting and Correcting Errors

This is a very rich subject; I have longer lectures on it, this is a short introduction.

A popular choice is the TELL ME THREE TIMES method, though better methods exist.

We assume the probability of making a mistake is low (otherwise it would be hard to send anything), and thus in any short message we will almost always have zero or at most one error.

If we want to send H E L L O we would send HHH EEE LLL LLL OOO.

If we ever get a block where the three digits are not the same we switch the different digit to the repeated one. What do you think RRR EIE DDD SSS AOO XXP is?

It could be RID SAP, but it is almost surely RED SOX! We are using the idea of the three hat problem!

Binary

You might object to sending letters, as it is possible we could have two errors and get three letters in a message!

The hat code method works because there are just two options – if there are just two options and one is wrong, the other must be right! This is illustrated brilliantly in the Seinfeld episode “The Opposite”: https://www.youtube.com/watch?v=1Y_6fZGSOQI

One solution is to send 0’s and 1’s and use these to build up letters: 00000 is A, 00001 is B, 00010 is C, 00011 is D, 00100 is E, 00101 is F, …, 11010 is Z. This is Binary (so instead of base 10 we have base 2).

Morse Code

You might object to sending letters, as it is possible we could have two errors and get three letters in a message!

Another solution is to use Morse Code to transmit letters. We represent each letter with dots and dashes….

A ●–	J ●–––	S ●●●
B –●●●	K –●–	T –
C –●–●	L ●–●●	U ●●–
D –●●	M ––	V ●●●–
E ●	N –●	W ●––
F ●●–●	O –––	X –●●–
G ––●	P ●––●	Y –●––
H ●●●●	Q ––●–	Z ––●●
I ●●	R ●–●	

What stays with us….

Learned a lot of good things today.

- Attacked a big problem by breaking into simpler one first to explore.
- Saw math can have applications in the real world.
- Sometimes choices made a long time ago are still with us today. For example, what does QWERTY mean?

What stays with us....

Sometimes choices made a long time ag
are still with us today. For example, what
does QWERTY mean?

Pirates and Math:
Dividing the Spoils and Game Theory

Steven J Miller, Williams College (sjm1@williams.edu) https://web.williams.edu/Mathematics/sjmiller/public_html/

https://www.shutterstock.com/image-vector/cartoon-pirate-walking-plank-260nw-715720567.jpg

shutterstock.com 715720567

Pirate Riddle

- Five pirates, named 1, 2, 3, 4 and 5.
- Must split 100 gold coins.
- The lowest named pirate alive proposes a division.
- If 50% or more vote for the plan it passes and is done.
- If 50% or more vote against the plan it fails, the proposer walks the plank and dies, and the lowest named surviving pirate proposes a plan.
- Repeat till a plan is accepted.
- Assume the pirates are intelligent – what is the final division?

Questions

- Is the game play clear? Can it be interpreted more than one way?
- How should you try to analyze the game?
- Have you seen anything like this before?
- Anything special about 5 pirates and 100 coins?

Game Rules (again)

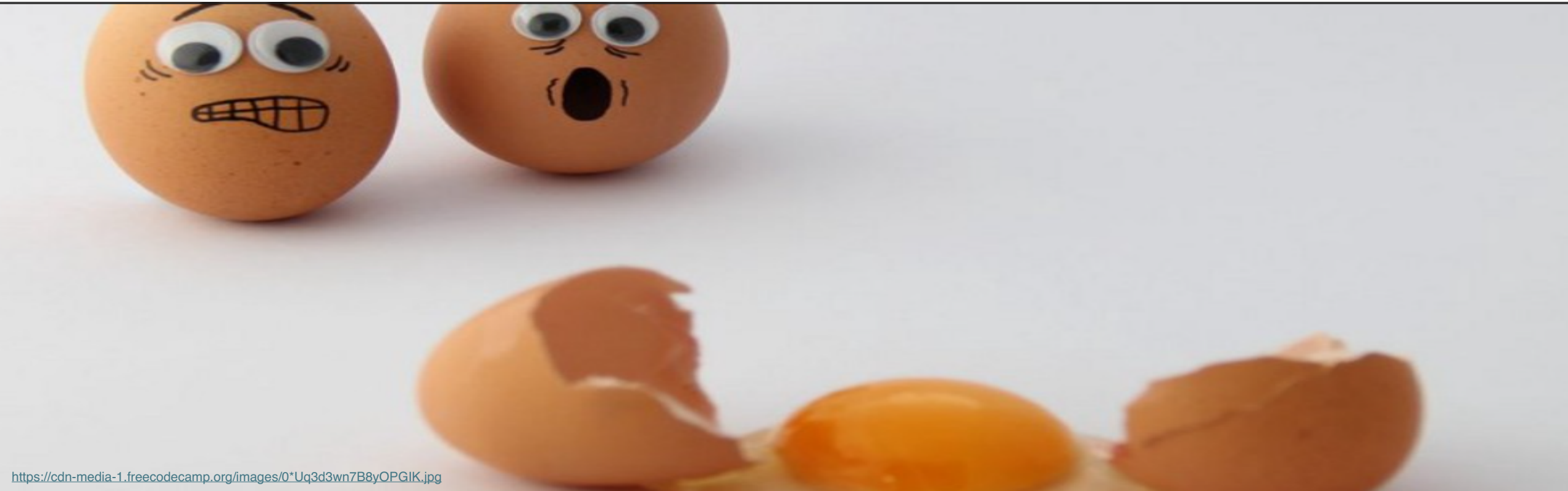
- Five pirates, named 1, 2, 3, 4 and 5.
- Must split 100 gold coins.
- The lowest named pirate alive proposes a division.
- If 50% or more vote for the plan it passes and is done.
- If 50% or more vote against the plan it fails, the proposer walks the plank and dies, and the lowest named surviving pirate proposes a plan.
- Repeat till a plan is accepted.
- Assume the pirates are intelligent – what is the final division?

Play smaller game and work up….

Pirates and Math: Dividing the Spoils and Game Theory, TCNJ Math Camp, July 13, 2025. July 27, 2025. pdf (blank) pdf (filled in) Hampshire College, July 21, 2025. pdf (talk online: https://youtu.be/7waPYdokENM)

Egg Drop Mathematics:
It IS all its cracked up to be!

Steven J Miller, Williams College (sjm1@williams.edu) https://web.williams.edu/Mathematics/sjmiller/public_html/



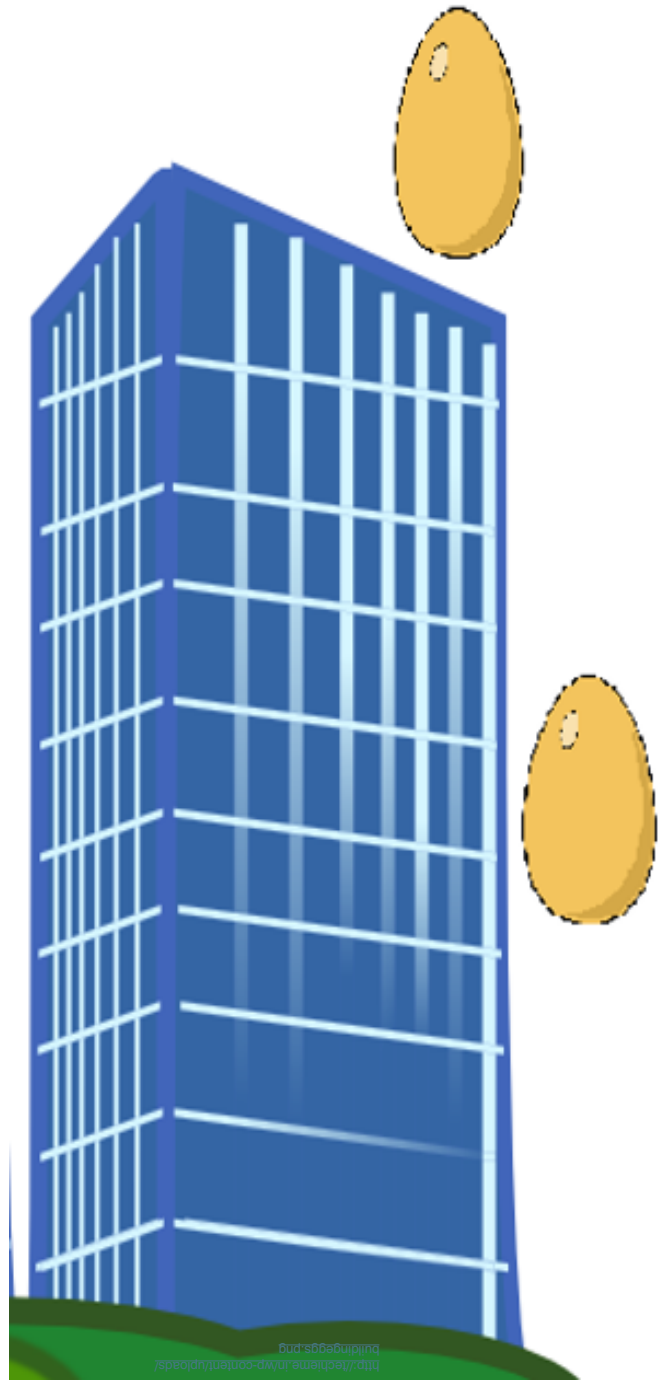
https://cdn-media-1.freecodecamp.org/images/0*Uq3d3wn7B8yOPGIK.jpg

Building with N floors, have 2 golden eggs.

Special eggs: some floor **n** such that if you drop from below **n** no damage; can drop as many times as wish.

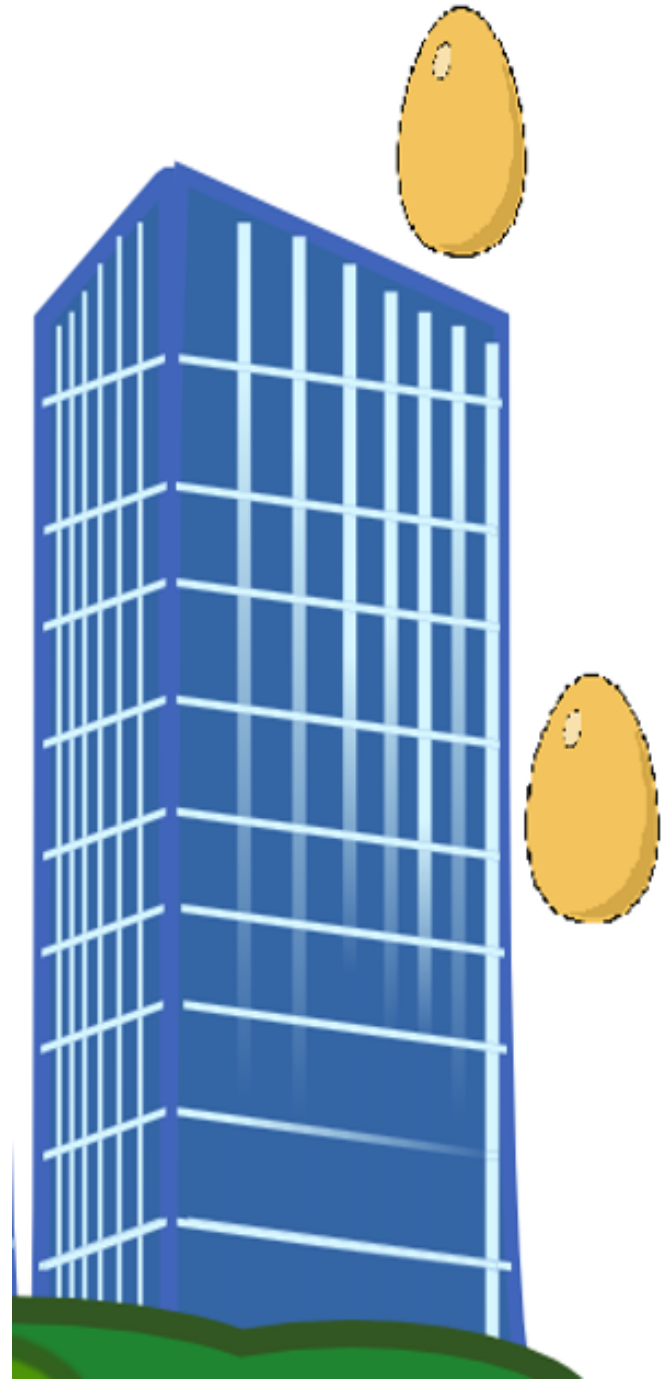
If drop even once from floor **n** or higher immediately break.

Find in as few drops as you can what **n** is (the lowest floor where if you drop from there it breaks). Doesn't matter if have any of the golden eggs at the end - just want to know **n**.



Interpretation:

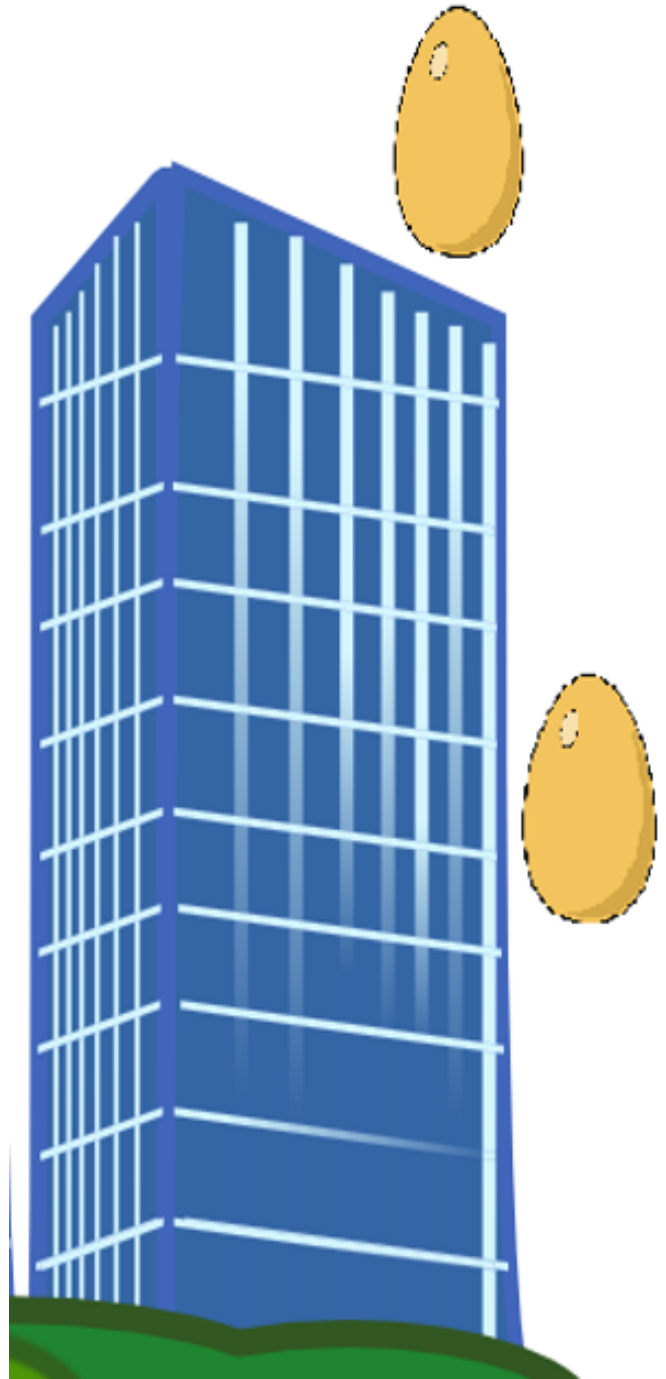
How do you interpret finding n in as few drops as possible?



Interpretation:

How do you interpret finding n in as few drops as possible?

- Minimize worse case.
- Minimize average case.

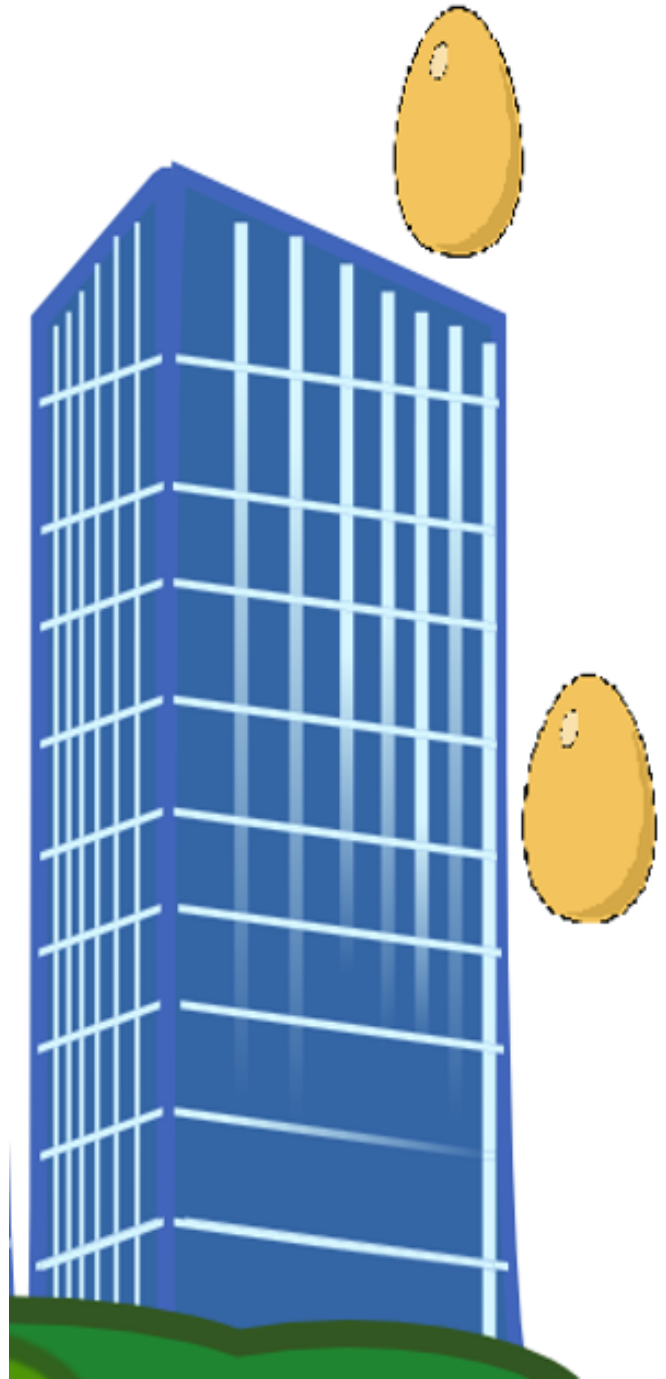


General Advice:

When given a hard problem:

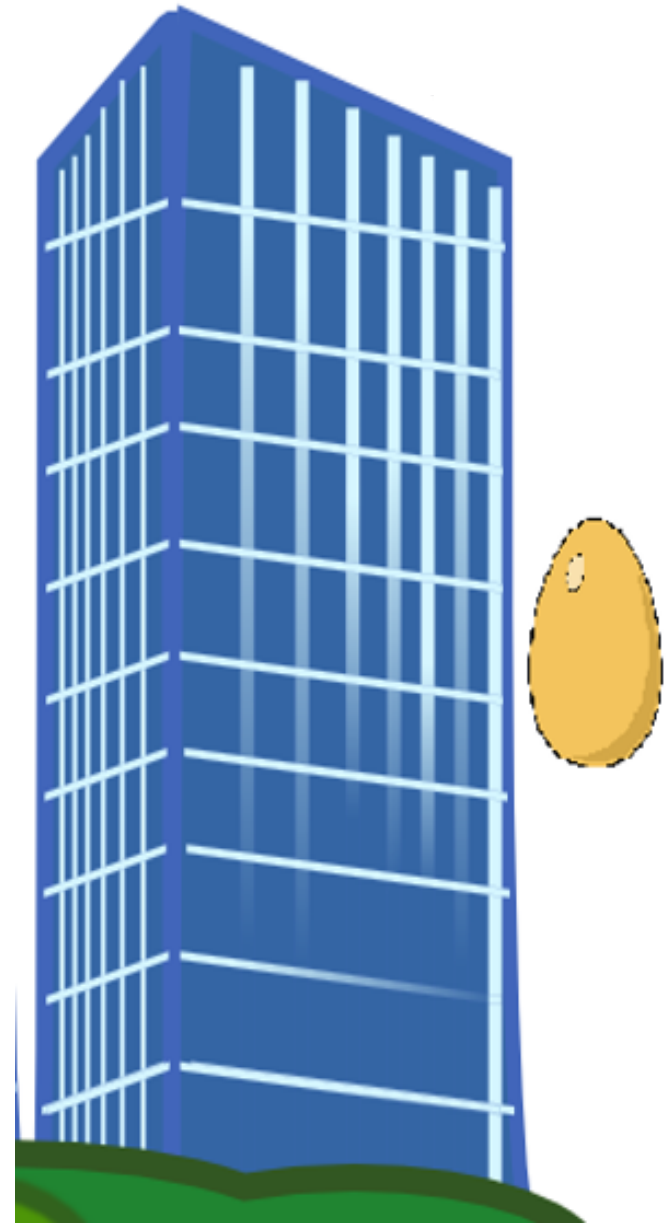
- try to do an easier version first, and
- try to do specific values of parameters.

What is an easier problem?



Simple Case: 1 Egg

What is the solution?

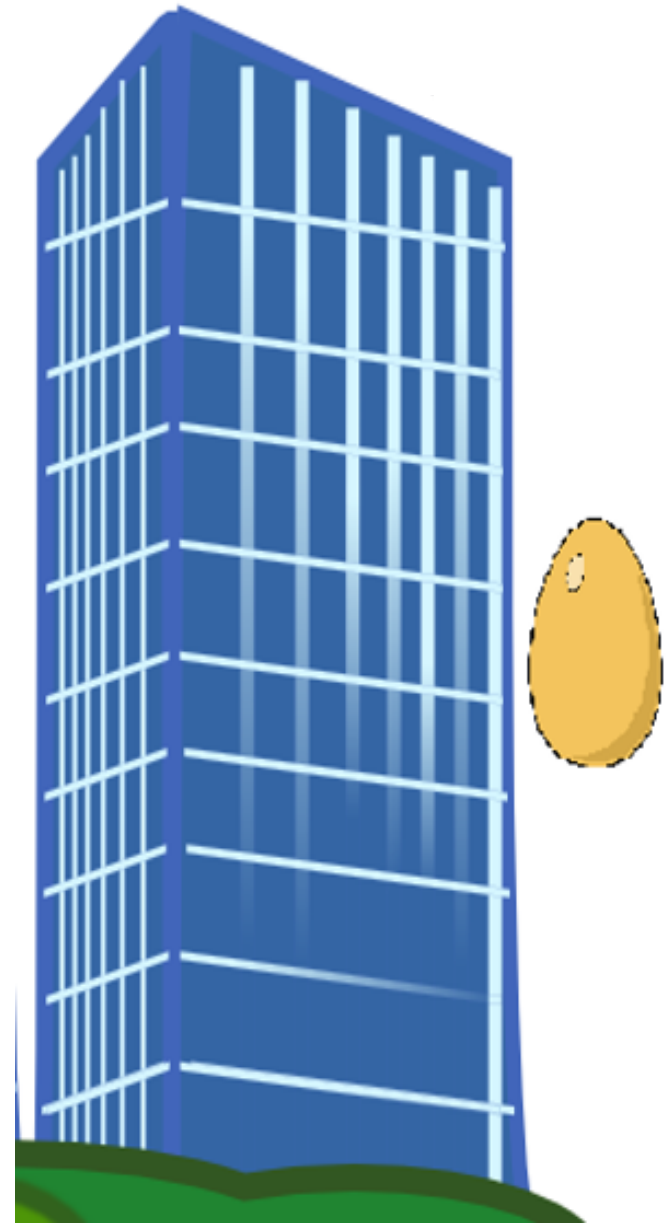


Simple Case: 1 Egg

What is the solution?

Only possibility is go 1, 2, 3, … till break.

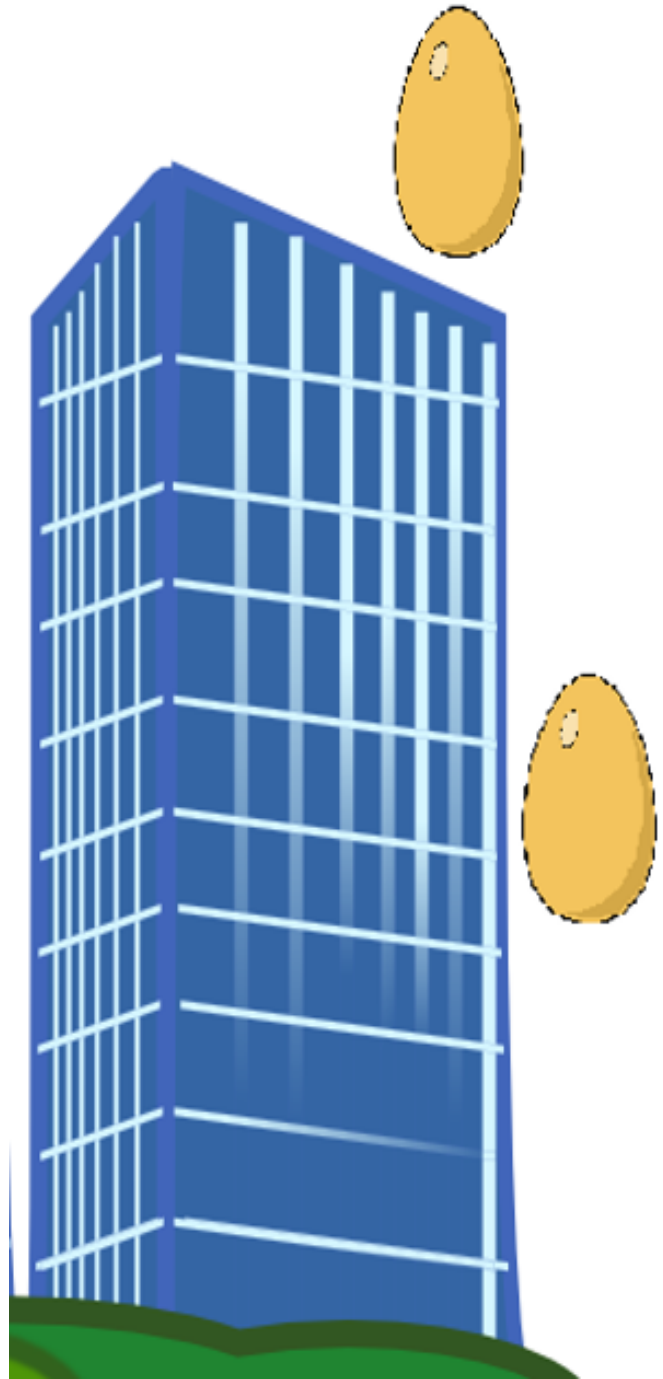
Worse case is order N drops.



Next Case: 2 Eggs

Once one cracks, reduced to 1 egg case.

What are possible strategies?



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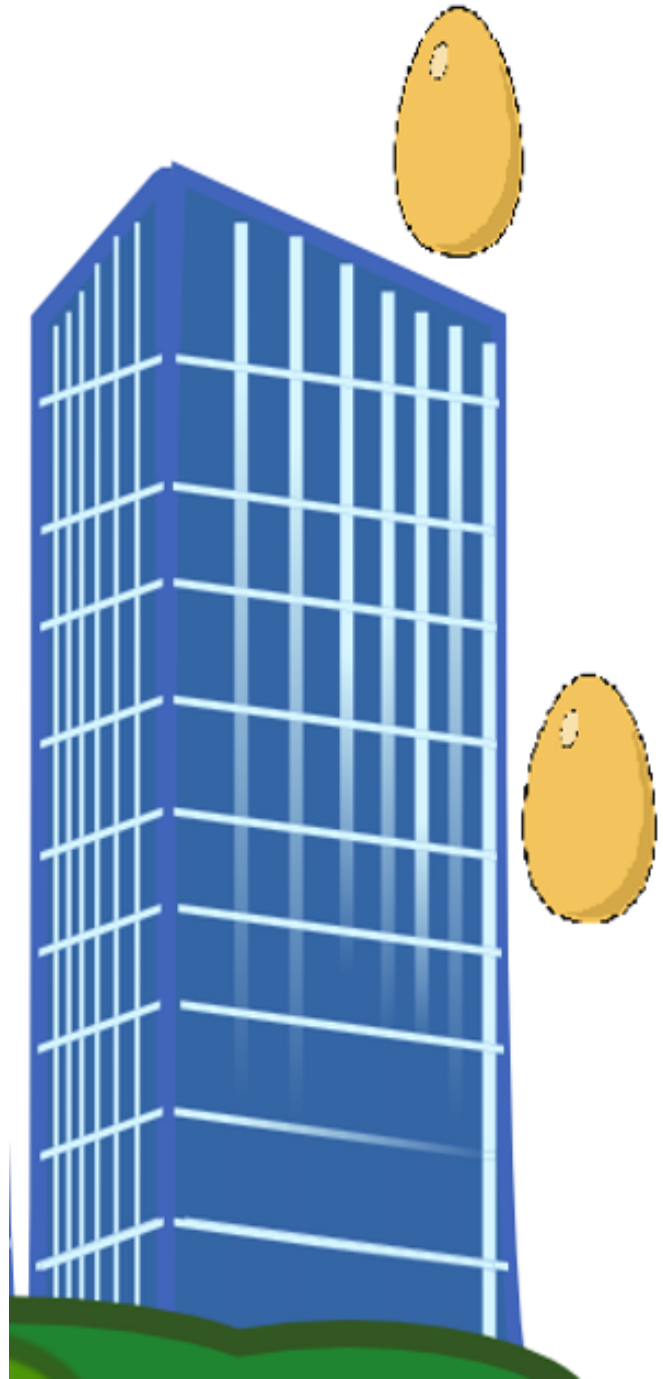
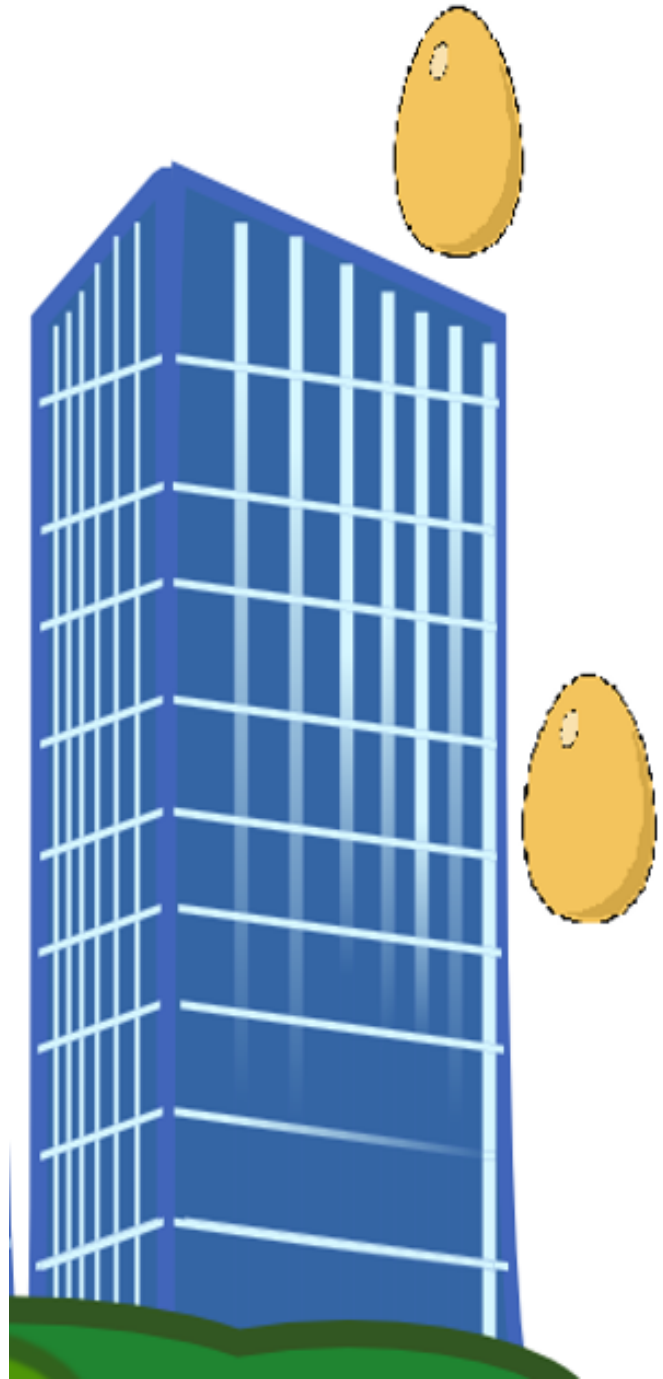


Next Case: 2 Eggs

Once one cracks, reduced to 1 egg case.

What are possible strategies?
Extreme cases:

- Drop every 2nd floor.
- Drop at N/2.
- (more generally drop every x)



Competing Influences

Drop every 2nd floor.

- Once first breaks fast, but could take many drops.
- #Drops = N/2 + 1

Drop at N/2

- If doesn't crack eliminate a lot, when crack lot to check.
- #Drops = 1 + (N/2 – 1).

Both basically on the order of N/2 drops….

Competing Influences: Balance

Drop every x floors.

Competing Influences: Balance

Reduced to choosing x to minimize

$$\frac{N}{x} + x\,.$$

Competing Influences: Balance

Reduced to choosing x to minimize

$$\frac{N}{x} + x\,.$$

Set two terms equal to each other to balance:

$$\frac{N}{x} = x \quad so \quad N = x2 \quad or \quad x = N^{1/2}\,.$$

Gives #Drops = $\frac{N}{N^{1/2}} + N^{1/2}$ - 1 or about 2 $N^{1/2}$.

```
Plot[100 / x + x, {x, 1, 100}]
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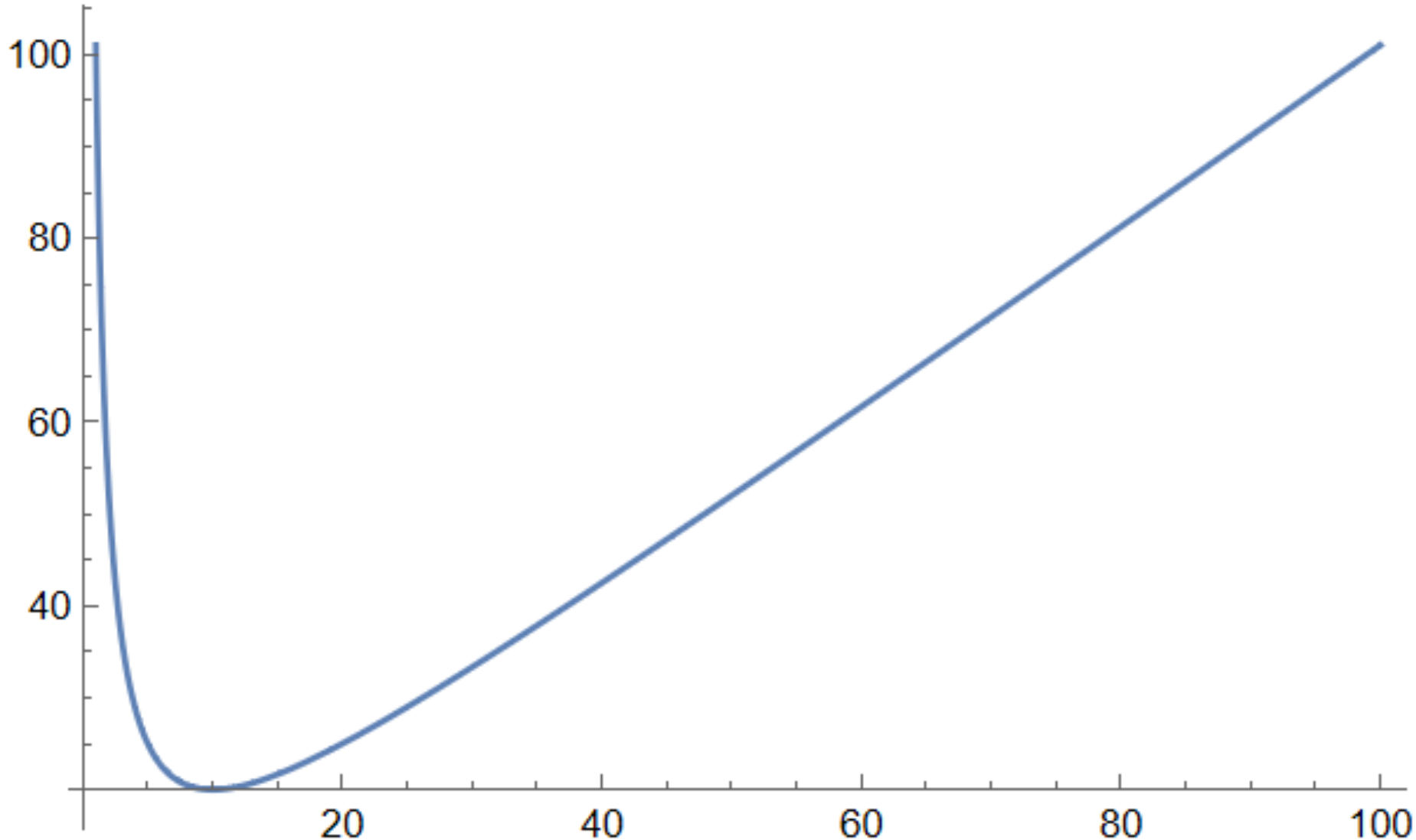
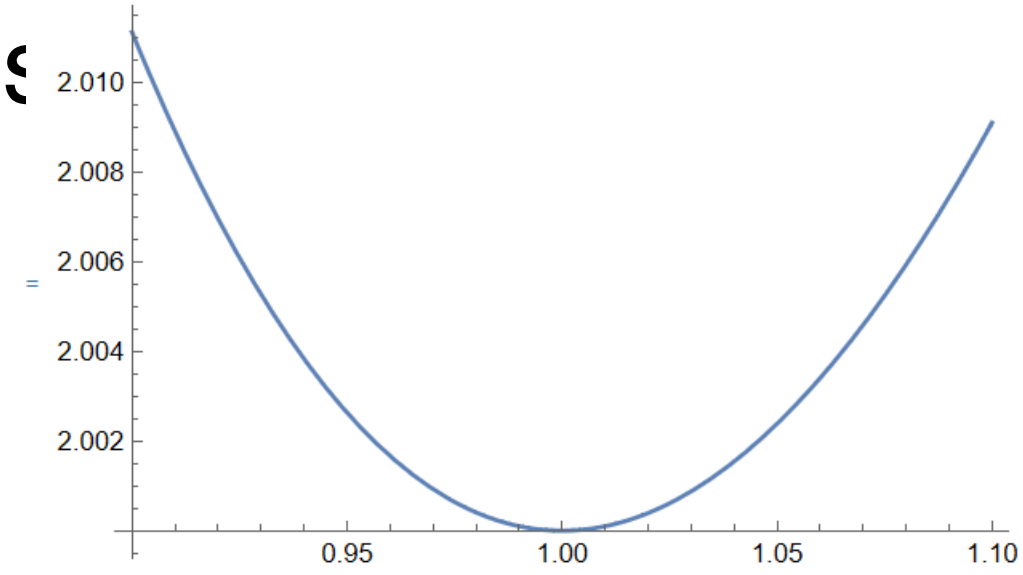


Write x = t $N^{1/2}$ in #Drops = $\frac{N}{x} + x - 1$.

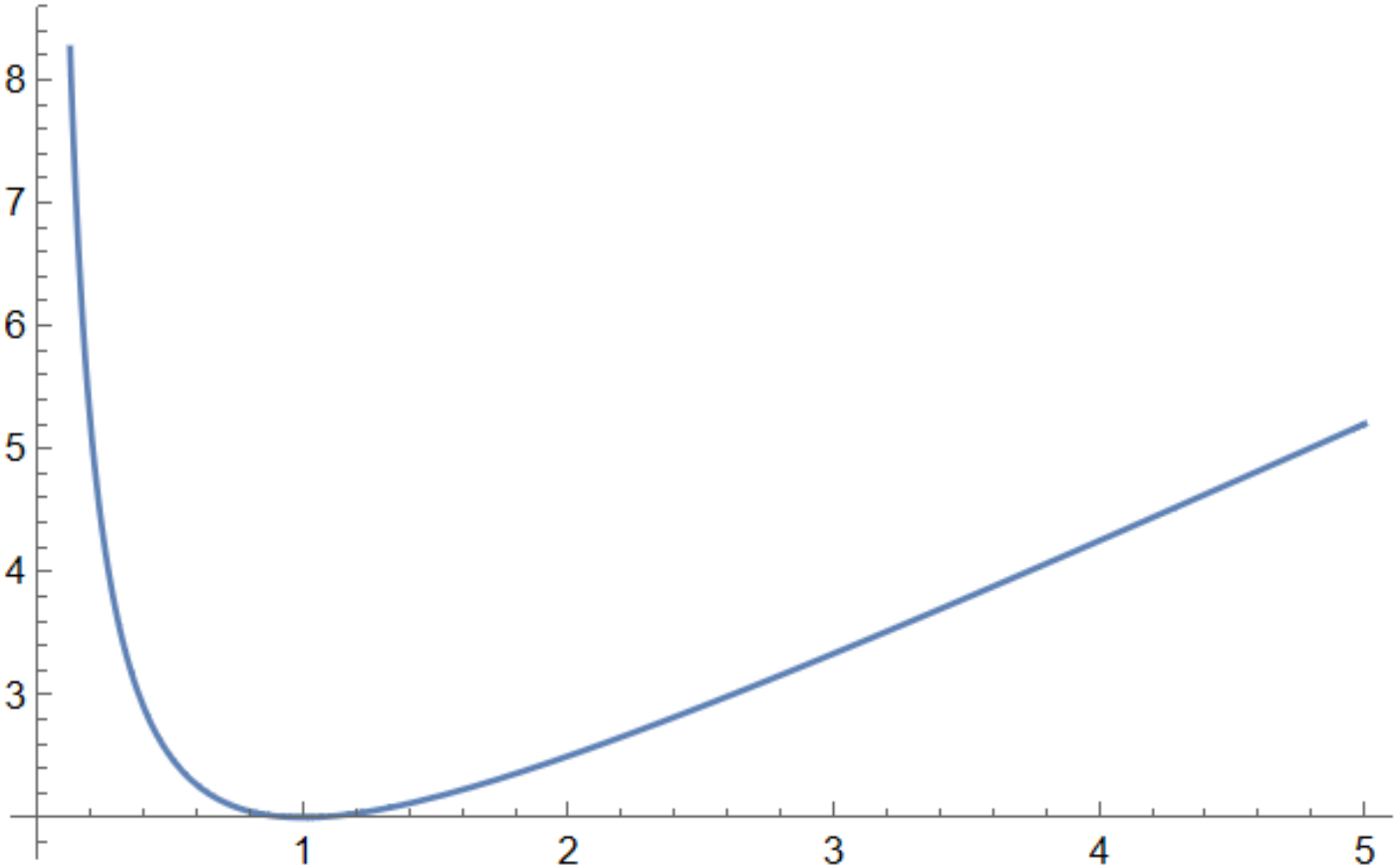
Gives #Drops = $\frac{N}{t\ N^{1/2}} + t\ N^{1/2}$ - 1.

This is just $N^{1/2}$ $(\frac{1}{t} + t)$,

s ²!




```
Plot[1/t + t, {t, 0, 5}]
```



If know calculus: want to minimize $f(x) = N/x + x$:

- Endpoints: $f(1)$ and $f(N)$ are of order N.
- $f'(x) = -N/x^2 + 1$, critical point $f'(x) = 0$ or $x = N^{1/2}$.
- Easily see minimum, or note $f''(x) = 2N/x^3 > 0$.

Balancing Application

Imagine have two algorithms:

- One always takes 1000 seconds.
- One takes 1 second except one in a million inputs take 1,000,000,000 seconds.

Both take on average approximately 1000 seconds….

Balancing Application

Imagine have two algorithms:

- One always takes 1000 seconds.
- One takes 1 second except one in a million inputs take 1,000,000,000 seconds.

Both take on average approximately 1000 seconds….

…but what if run algorithm 1 and if takes more than 2 seconds on an input switch to first?

Average of about 1 second!

Improving Strategy with 2 Eggs

Consider triangular numbers and dynamic rescaling.

- Do not move in constant steps of x floors.
- Do x, then x-1 if doesn't crack, then x-2….
  - Advantage is always same number of drops!
  - Basically if doesn't crack doing 2 egg problem but now with N-x floors (after first drop).

Improving Strategy with 2 Eggs

Consider triangular numbers and dynamic rescaling.

- Do not move in constant steps of x floors.
- Do x, then x-1 if doesn't crack, then x-2….
  - Advantage is always same number of drops!
  - Basically if doesn't crack doing 2 egg problem but now with N-x floors (after first drop).

Example: N = 105 = 14 + 13 + 12 + … + 1:

(1 + 13) or (2 + 12) or (3 + 11)

What if we have 3 Eggs? Or k eggs?

What if we have 3 Eggs? Or k eggs?

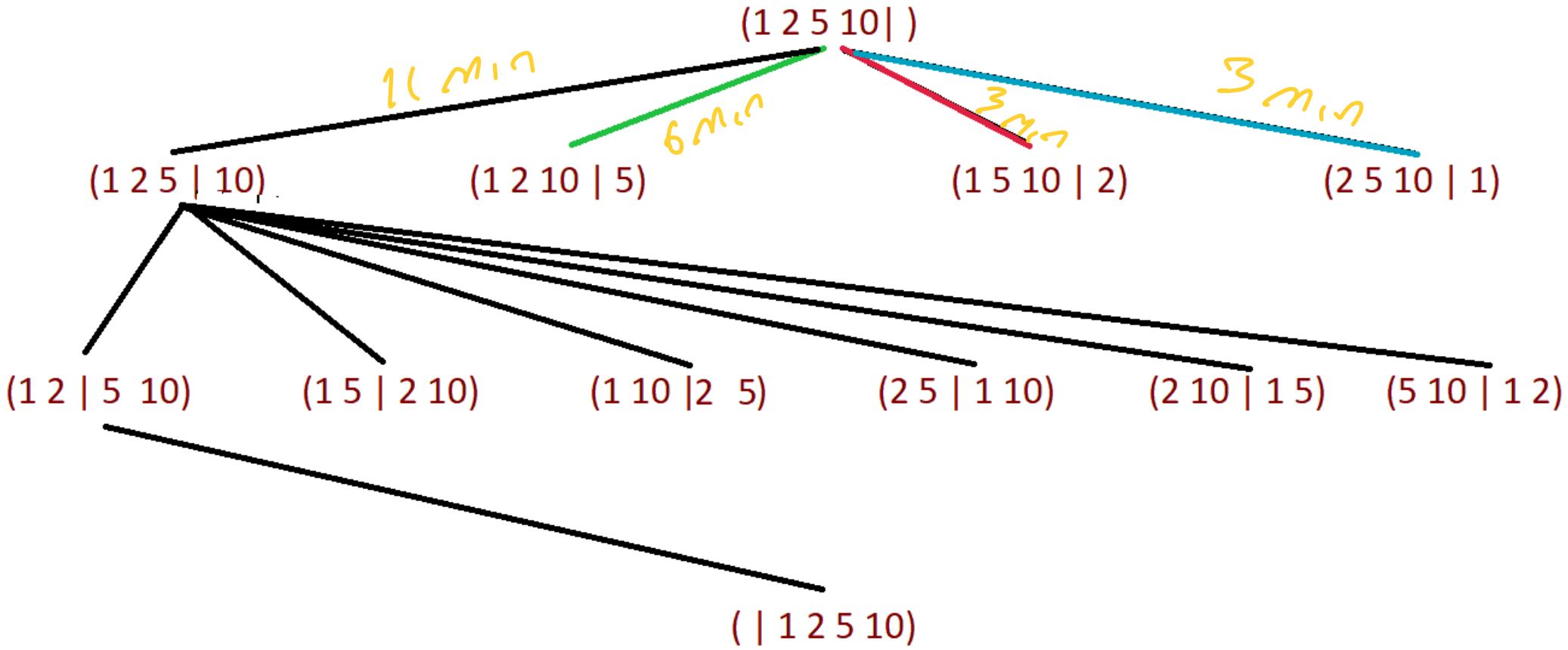
For 3 eggs: once one cracks, 2 egg problem.
If do every x it would be, worse case:

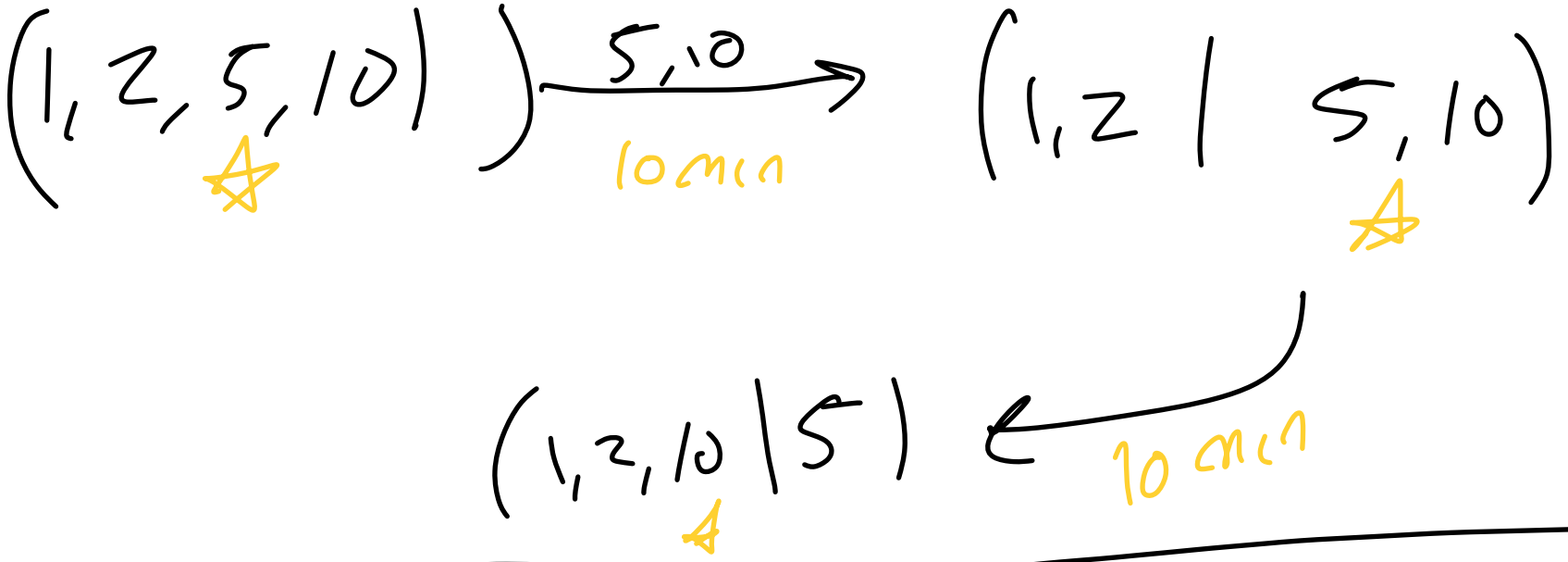
Last Time: *We'll Cross That Bridge* Proble

- We have a rickety old bridge, it's nighttime and dark but we have one flashlight, and we are being chased by zombies (or coronavirus victims). They are 17.5 minutes from us.
- Only two people can cross the bridge at a time, and those crossing must have the flashlight. Thus two can go across with the flashlight and then one returns with the flashlight.
- We must get all people across before the attackers arrive. The four people are named 1, 2, 5 and 10; these names are how long it takes each of them to cross individually; if two go over together the time it takes is the larger of the two times. We cannot have three cross together or the bridge collapses.

Show it is possible to get the four people over in 17 minutes; if everyone is not over by 17.5 minutes the bridge is infected / destroyed and we lose….

Key Idea: Good way to list possibilities: Is this missing anything?



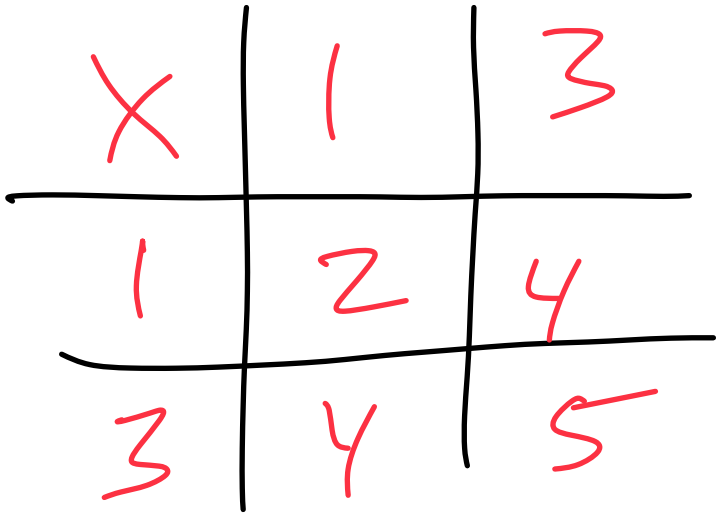


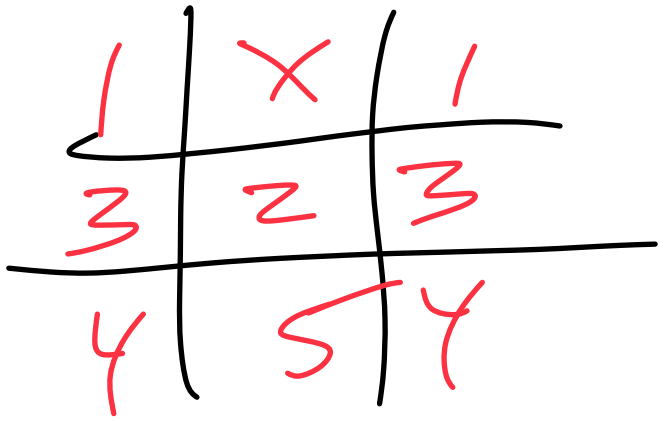
(1, 2, 5, 10 |) $\xrightarrow[10\text{ min}]{1,10}$ (2, 5 | 1, 10) $\xrightarrow[1\text{ min}]{1}$ (1, 2, 5 | 10) $\xrightarrow[5\text{ min}]{1,5}$ (2 | 1, 5, 10

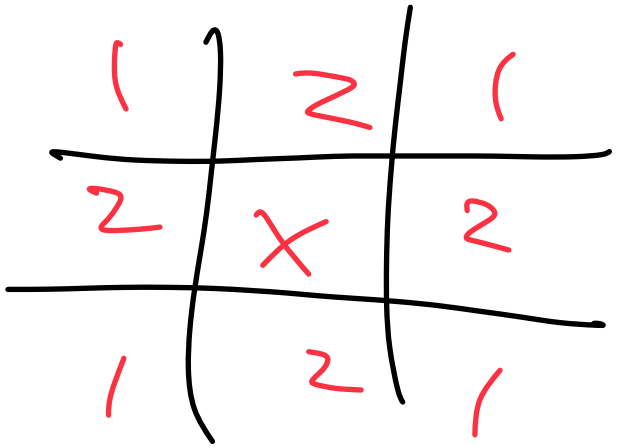
5 and 10 go over, 5 comes back: 15 min : fail

Tic-Tac-Toe

$9 - 8 = 72$

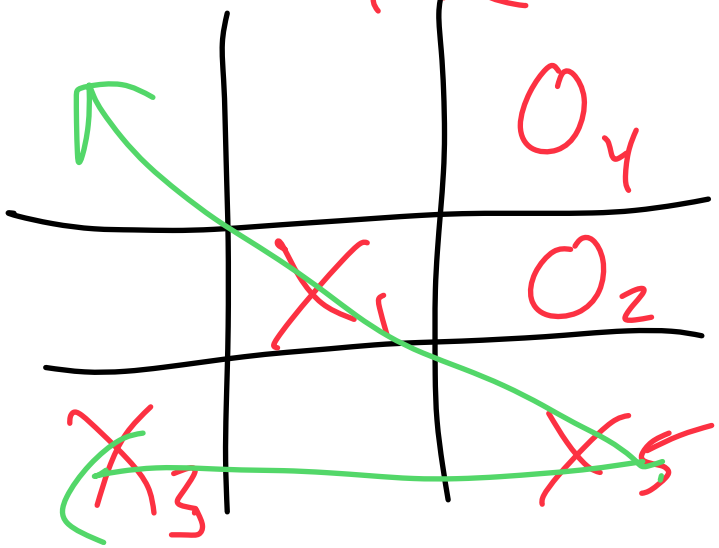


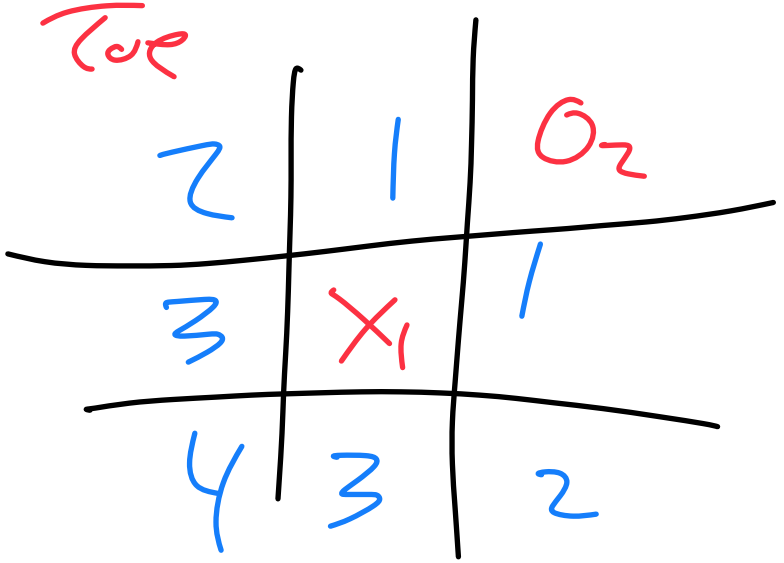




$5 + 5 + 2 = 12$

Tic - Tac - Toe





Toe - Tac - Tic or CoT - CaT - CiT

Reverse Tic Tac Toe

1, 2, 3, 4, 5, 6, 7, 8, 9

$$\frac{\square}{\square\square} + \frac{\square}{\square\square} + \frac{\square}{\square\square} = 1$$

$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

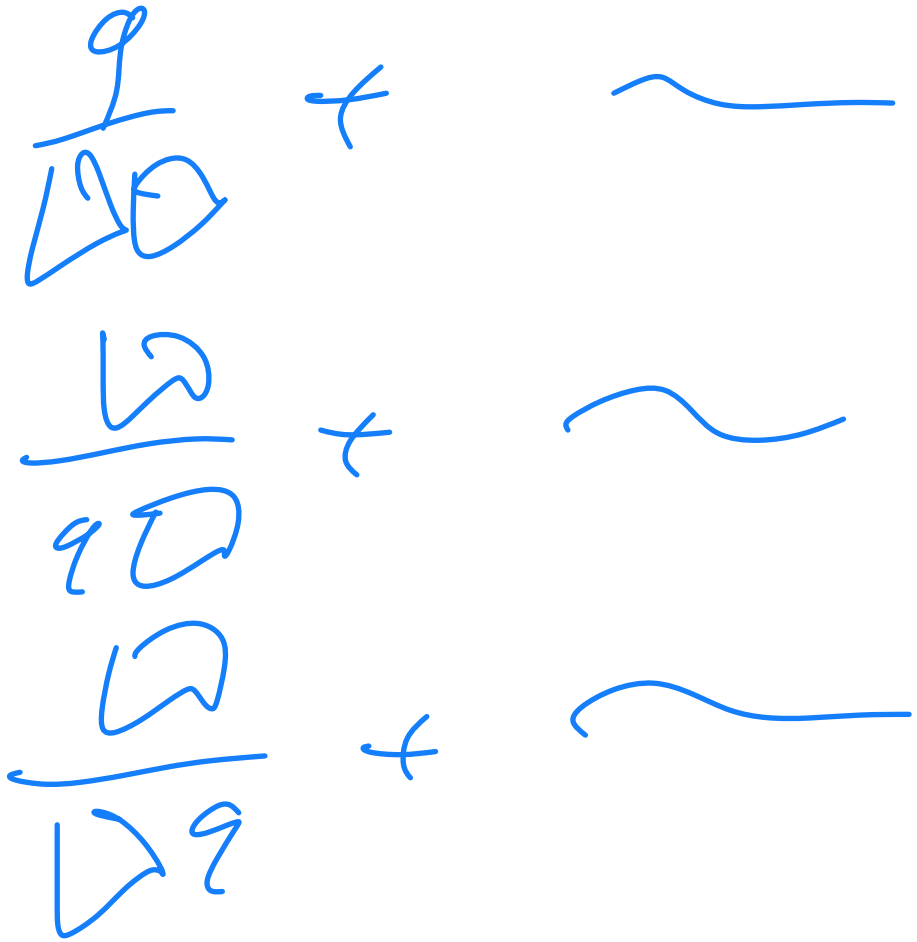
$9 \cdot 8 \cdot 7 \approx \frac{1000}{2}$

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

$= 1000 \cdot 360$ or $360{,}000$

3600 sec/hr

Wlog, one of the first 3 is the 9



$3 \cdot 8!$

Instead of

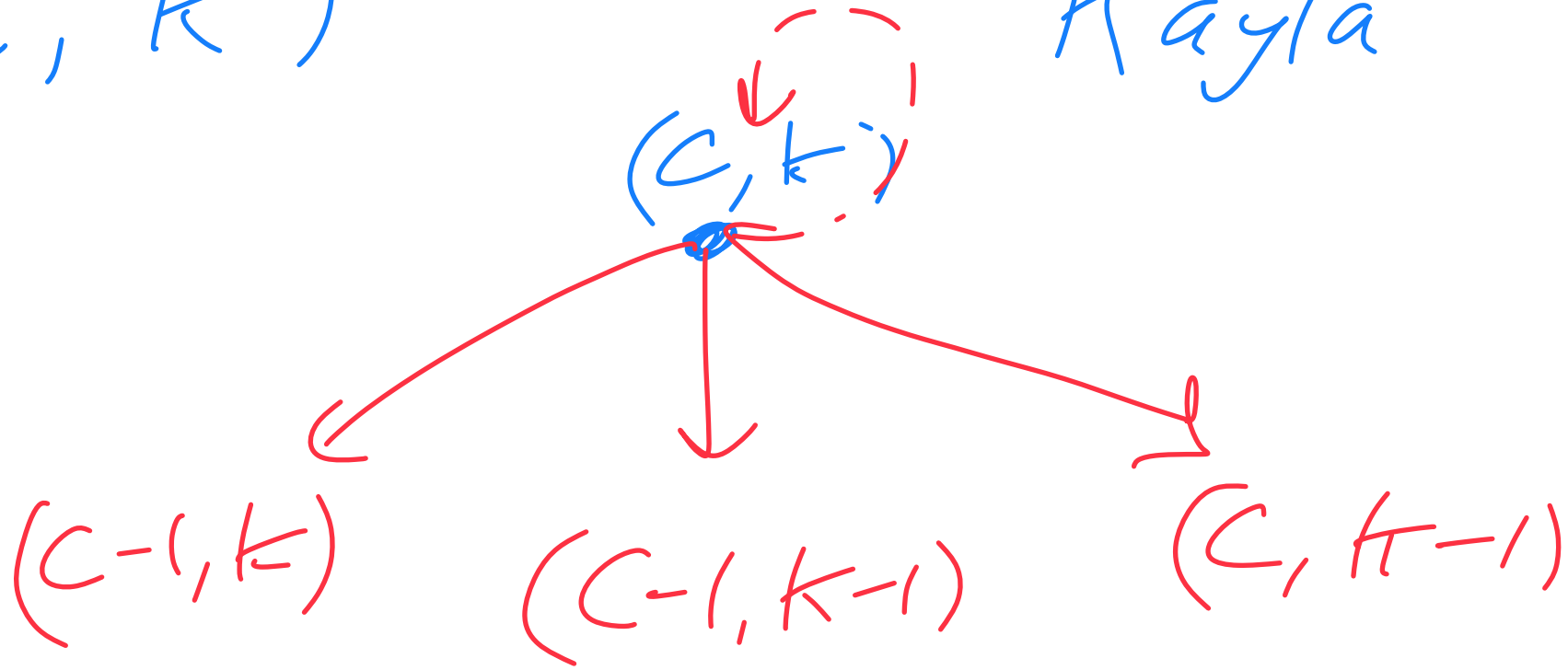
$9 \cdot 8!$

M&M game

(c, k)

Cameron

Kayla



Multiplication

x^{100} : cost is 99 multiplications

$x^1 x^1 = x^2$

$x^2 x^2 = x^4$

$x^4 x^4 = x^8$

$x^8 x^8 = x^{16}$

$x^{16} x^{16} = x^{32}$

$x^{32} x^{32} = x^{64}$

6 mul

$100 = 64 + 32 + 4$

$100 = 64 + 36$

$= 64 + 32 + 4$

$x^{100} = x^{64} x^{32} x^4$

$(X, 1)$

↓

$(X^2, 1)$

(X^4, X^4)

(X^8, X^4)

(X^{16}, X^4)

(X^{32}, X^{36})

(X^{64}, X^{100})

$$X^{100} = X^{64} X^{32} X^4$$

X^n: $n-1$ multiplications (brute force)

Repeated Squaring: at most $2 \log_2(n)$

10^{200}

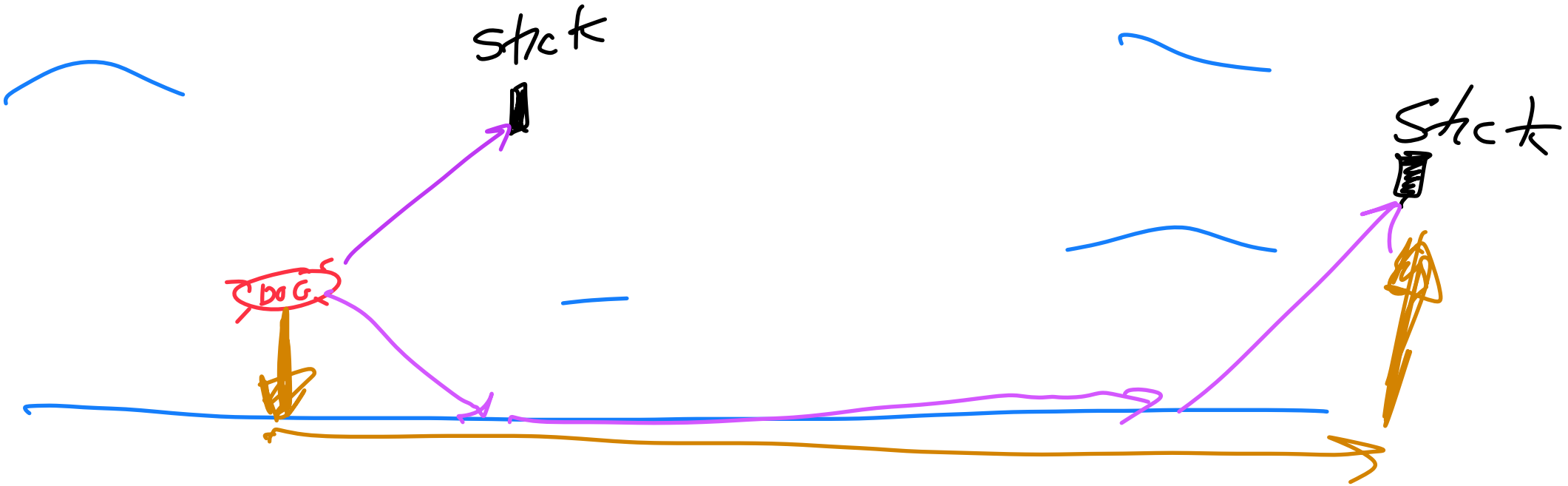
$\log_2(10^{200}) = 200 \log_2(10) \leq 800$

$\log_2(2) = 1$

$\log_2(4) = 2$

$\log_2(8) = 3$

$\log_2(16) = 4$



$$\frac{x+y}{2} \geq \sqrt{xy}$$

New Problem: Legal 21

Young Saul, a budding mathematician and printer, is making himself a fake ID. He needs it to say he's 21. The problem is he's not using a computer, but rather he has some symbols he's bought from the store, and that's it. He has one 1, one 5, one 6, one 7, and an unlimited supply of + – * / (the operations addition, subtraction, multiplication and division). Using each number exactly once (but you can use any number of +, any number of -, …) how can he get 21 from 1, 5, 6, 7?

- Note: You can't do things like 15+6 = 21. You have to use the four operations as 'binary' operations: ((1+5)*6)+7.
- Note: We strongly oppose creating fake IDs…..

Legal 21: How to attack it?

Have the numbers 1, 5, 6, 7; can use any number of +, -, *, / and want to combine and get 21.

1+5+6+7 = 19

5*6-1-7 = 22

(7*6)/(5-1) = 21/2

Legal 21: How to attack it?

We need a way to go through all the possibilities and make sure we don't miss anything.

Note that +,-,* and / are all BINARY operations – they take two inputs and give one output.

Thus if we look at (6*7) / (5-1) we can view this as the following:

$b_{/}(b_{*}(6,7), b_{-}(5,1))$. This means we do a multiplication with two numbers, a subtraction with another two, and then divide the results.

Legal 21: How to attack it?

Thus if we look at (6*7) / (5-1) we can view this as the following:

$b_/(b_*(6,7), b_-(5,1))$. This means we do a multiplication with two numbers, a subtraction with another two, and then divide the results.

More generally, let b_i denote a binary operation. Then we can really view (6*7) / (5-1) as a specific realization of the structure

$$b_1(b_2(w,x), b_3(y,z))$$

where w, x, y, z are the numbers 1, 5, 6, 7 in some order, and the b_i's are different binary operations.

- How many ways can we order 1, 5, 6, 7 to substitute for w, x, y, z?
- How many ways can we choose binary operations for b_1, b_2, b_3?
- How many combinations are there to check for this structure?

Legal 21: How to attack it?

Thus if we look at (6*7) / (5-1) we can view this as the following:

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$$b_1(b_2(w,x), b_3(y,z))$$

where w, x, y, z are the numbers 1, 5, 6, 7 in some order, and the b_i's are different binary operations.

- How many ways can we order 1, 5, 6, 7 to substitute for w, x, y, z? 4! = 4*3*2*1 = 24
- How many ways can we choose binary operations for b_1, b_2, b_3? 4 * 4 * 4 = 64
- How many combinations are there to check for this structure? 24 * 64 = 1536

Legal 21: How to attack it?

More generally, let b_i denote a binary operation. Then we can really view (6*7) / (5-1) as a specific realization of the structure

$$b_1(b_2(w,x), b_3(y,z))$$

We just showed there are 1536 ways to substitute for this!

That is a lot for us to do with pen and paper, but nothing for a computer!

Are there other structures we can have other than something like (6*7)/(5-1), i.e., $b_1(b_2(w,x), b_3(y,z))$? What are they?

Legal 21: Possible Structures

Here are all the possible structures

(1) $b_1(b_2(w,x), b_3(y,z))$. Example: (w+x) + (y+z).

(2) $b_1(b_2(b_3(w,x), y), z)$. Example: ((w+x)+y) + z.

(3) $b_1(z, b_2(y, b_3(w,x)))$. Example: z + (y + (w+x)).

(4) $b_1(b_2(y, b_3(w,x)), z)$. Example: (y + (w+x)) + z.

(5) $b_1(z, b_2(b_3(w,x), y))$. Example: z + ((w+x) + y).

Legal 21: Possible Structures

The hardest part is making sure you don't miss any sentence structures. For me, I found it very helpful to think about adding four numbers and all the different ways I could group it (hence the examples listed above).

We now loop through all 4!=24 ways of assigning 1,5,6,7 to w,x,y,z, and we loop through all ways of assigning binary operations (there are 4*4*4 = 4^3 = 64 ways to do this).

Note that for some sentences, different assignments lead to the same output; if all the binary operations are addition then all 4! = 24 arrangements of the four numbers lead to the same output. It's faster to write simple code and execute it then to spend a lot of time telling the computer not to do certain calculations that give the same output as other cases. In programming, it's often good advice to not worry about being too clever unless you run into issues with how fast the code runs.

Thus, there are 5 * 4^3* 4! = 7680 candidates. While this is too large for most humans to do by hand, a computer can output the result of checking all of these almost instantaneously. The solution involves two divisions, which might explain why most people are unable to find it:

$\frac{6}{1 - \frac{5}{7}}$. This gives us 21 from 42/2, and is of the sentence (3) type.

Related Problems

Same rules, try to get the target number using each input number once and only once; you can only use +, -, * and /.

- 21 from 1, 5, 6, 7
- 21 from 2, 3, 5, 7 (two different “sentence structures” work)
- 24 from 1, 5, 5, 5 (so you have exactly three 5’s and one 1).
- 24 from 5, 9, 13, 22
- 25 from 2, 4, 6, 8