Alternating More Sum Than Difference (MSTD) Sets

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We say A is sum-dominated or a **more sums than differences** (MSTD) set if $|A + A| > |A - A|$.

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- If there exists an $a^*\in\mathbb{Z}$ such that $A=\{a^*\}-A$, then A is *symmetric* with respect to a^* and A is *sum-difference balanced* ($|A+A| = |A-A|$).

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- Several methods for constructing MSTD sets exist.

Problem Statement

At the recent CANT (Combinatorial and Additive Number Theory Conference), Samuel Alexander posed the following: **Find a sequence of sets with $A_{i-1} \subset A_i$ that alternate being sum and difference dominated.**

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- Let $A_1 \subset [0, m]$ be a MSTD set where $m \in \mathbb{N}$ and let $p > m + 1$, $p \in \mathbb{N}$. Then

$$A_2 := [0, m] \cup \{p\}$$

is difference-dominated.

- To get a MSTD A_3, we apply Theorem 4 from Nathanson (2007) and obtain

$$\begin{aligned} B &= [0, n-1] \setminus \{r\} \\ A_3^* &= B \cup \{2n-r, 3n-r, \ldots, (k+1)n-r\} \cup (a^* - B) \\ A_3 &= A_3^* \cup \{n\} \end{aligned}$$

  where $n = p + 2$ if p is odd, $n = p + 5$ if p is even, $2 \leq r \leq n - 3$, $k \geq 2$, and $a^* = (k+3)n - 2r$.
- We have $A_1 \subset A_2 \subset A_3$, and we can extend this sequence infinitely by setting $m = \max(A_3)$ and repeating the steps used to generate A_2 and A_3.

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- Let A_1 be MSTD with $0 \in A_1$, and $n > \max(A_1)$ satisfying:
  - $|(A_1 + A_1) \mod n| = |(A_1 - A_1) \mod n|$
  - $2y - x - 1 > |A_1 + A_1| - |A_1 - A_1|$ where:
    - $x = |a \in A_1 : n + a \notin A_1 + A_1|$
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    - $x = |a \in A_1 : n + a \notin A_1 + A_1|$
    - $y = |b \in A_1 : n - b \notin A_1 - A_1|$
- Then, $A_2 = A_1 \cup \{n\}$ is difference-dominated, and $A_3 = A_1 + \{0, n\}$ is sum-dominated.

Non-filling Method 1

- We have $A_1 \subset A_2 \subset A_3$
- For $l \geq 2$, let

$$\begin{aligned} A_{2l} &= A_{2l-1} \cup \{ln\} \\ A_{2l+1} &= A_{2l-1} \cup (A + ln) \end{aligned}$$

- Clearly, $A_{2l} \subset A_{2l+1}$. We proved A_{2l} and A_{2l+1} alternate being sum- and difference-dominated.
- Using these constructions, we are able to generate the desired infinite sequence.

Non-filling Method 2

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- $A_1 = A \cup \{-1, 15\}$ is MSTD. $A_2 = A_1 \cup \{19\}$ is difference-dominated. $A_3 = A_2 \cup \{-5\}$ is MSTD.

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- By extending the arithmetic progression $\{3, 7, 11\}$, we are able to generate the desired sequence.
- $A_1 = A \cup \{-1, 15\}$ is MSTD. $A_2 = A_1 \cup \{19\}$ is difference-dominated. $A_3 = A_2 \cup \{-5\}$ is MSTD.
- For $l \geq 2$, $A_{2l} = A_{2l-1} \cup \{4l + 15\}$ is difference-dominated, and $A_{2l+1} = A_{2l} \cup \{-4l - 1\}$ is MSTD.

Table: Comparison of MSTD set growth characteristics

Method	$\lvert A_1 \rvert$	A_1 Diam.	Growth
Filling 1	≥ 8	≥ 14	Exponential
Non-Filling 1	≥ 8	≥ 14	Linear
Non-Filling 2	≥ 8	≥ 14	Linear
Filling 2	≥ 10	≥ 17	Linear
Non-Filling 3	≥ 11	≥ 18	Linear
Non-Filling 4	≥ 13	≥ 33	Linear

not presented

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