Distribution of Gaps in Zeckendorf Decompositions from d-dimensional Lattices

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Motivation

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$; $F_1 = 1$, $F_2 = 2$, $F_3 = 3$.

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Alternate Definition of Fibonacci Numbers

Fibonacci numbers form the only sequence such that every positive integer can be written uniquely as sum of non-adjacent terms.

Geometric Variation

We generalize the alternative definition of Fibonacci numbers to a 2-dimensional lattice, where a legal decomposition is called a **simple jump path**. We then construct the lattice grid:

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1	2

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3	5	
1	2	4

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84	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
50	82	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
28	48	74	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
14	24	40	66	$\cdots$	$\cdots$	$\cdots$	$\cdots$
7	12	20	33	59	$\cdots$	$\cdots$	$\cdots$
3	5	9	17	30	56	$\cdots$	$\cdots$
1	2	4	8	16	29	54	$\cdots$

Construction of 2-dimensional Lattice Sequence $y_{i,j}$

We define a legal movement (referred to as **step**) as one that moves at least one unit downward and one unit to the left.

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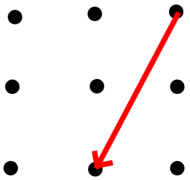

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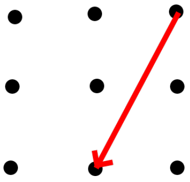

Definition (Simple Jump Path)

A **simple jump path** is a path on the lattice grid where each movement on the lattice grid consists of at least one unit movement to the left and one unit movement downward.

Counting Simple Jump Paths

Number of Simple Jump Paths with Length k

Let $t_d(k;(a_1, a_2, \ldots, a_d))$ denote the number of simple jump paths from $(a_1, a_2, \ldots, a_d)$ to the origin with length k. Then

$$t_d(k; a_1, a_2, \ldots, a_d) \ = \ \binom{a_1 - 1}{k-1}\binom{a_2 - 1}{k-1}\cdots\binom{a_d - 1}{k-1}.$$

Proof: Stars-and-bars problem

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Number of All Simple Jump Paths

Let $s_d(a_1, a_2, \ldots, a_d)$ denote the number of simple jump paths from $(a_1, a_2, \ldots, a_d)$ to the origin. Then

$$s_d(a_1, a_2, \ldots, a_d) \;=\; \sum_{k=1}^{\min(a_1, a_2, \ldots, a_d)} t_d(k; a_1, a_2, \ldots, a_d).$$

Counting Simple Jump Paths

Vandermonde's Identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}.$$

Theorem 1

Recall that $s_d((v))$ denotes the number of simple jump paths from v to $(0, \ldots, 0)$. Applying Vandermonde's Identity, we directly derive

$$s_2(a_1, a_2) \ = \ \binom{a_1 + a_2 - 2}{a_1 - 1}.$$

When $a_1 = a_2 = n$, we have $s_2(n) = \binom{2n-2}{n-1}$.

Counting Simple Jump Paths

It's important to note that combinatorial identities like

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

generally do not have analogues in higher dimension, therefore our research will focus on 2-dimensional case.

Gap Definitions

In the 1-dimensional case, the distribution of gap converges to geometric decay.

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- A **gap vector** of a step from $(x_{m,1}, x_{m,2})$ to $(x_{m+1,1}, x_{m+1,2})$ is the difference vector $(x_{m,1} - x_{m+1,1}, x_{m,2} - x_{m+1,2})$.
For example, for a path from 66 $(4, 4)$ to 9 $(3, 2)$ to 1 $(1, 1)$, the gap vectors are $(4 - 3, 4 - 2) = (1, 2)$ and $(3 - 1, 2 - 1) = (2, 1)$.

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- A **gap sum** is the sum of components of a gap vector.
For example, the gap sum of $(1, 1)$ is 2.

Counting Gap Vectors

Number of Gap Vectors of All Simple Jump Paths

Recall that $t_2(k, n)$ denotes the number of simple jump paths from (n, n) to the origin of length k. Let $g_2(n)$ denote the number of gap vectors of all simple jump paths from (n, n) to the origin, then

$$g_2(n) \;=\; \sum_{k=1}^{n} k \;\, t_2(k, n).$$

Generalization of Gaps

Due to the presence of $n-1$ in the formula below, we work with $n+1$ instead to simplify some of the algebra.

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Lemma 1

Consider all the simple paths from $(n+1, n+1)$ to $(0,0)$ in 2-dimensional lattice. Let $G((x,y),(x+v_1,y+v_2))$ denote the number of gap vectors (v_1,v_2) from $(x+v_1,y+v_2)$ to (x,y) within the simple jump paths, then

$$G\left((x,y),(x+v_1,y+v_2)\right) = \binom{x+y-2}{x-1}\binom{2n-v_1-v_2-x-y}{n-v_1-x}$$

Generalization of Gaps

Lemma 2

Let $g\,(n+1;(v_1,v_2))$ denote the number of gap vectors (v_1,v_2) in all the simple jump paths from $(n+1,n+1)$ to $(0,0)$, then

$$g\,(n+1;(v_1,v_2)) = (2n-v_1-v_2-1)\binom{2n-v_1-v_2-2}{n-v_1-1}+2\binom{2n-v_1-v_2}{n-v_1}$$

Sketch of Proof

We study the three different locations of (x, y):
(1) $1 \leq x \leq n - v_1$ and $1 \leq y \leq n - v_2$,
(2) $x = 0$ and $y = 0$,
(3) $x = n - v_1 + 1$ and $y = n - v_2 + 1$.

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Let $p = n - v_1 - 1$ and $q = n - v_2 - 1$, then for case (1),

$$\sum_{x=1}^{n-v_1} \sum_{y=1}^{n-v_2} G\left((x, y), (x + v_1, y + v_2)\right) = \sum_{x=0}^{p} \sum_{y=0}^{q} \binom{x+y}{x} \binom{p+q-(x+y)}{p-x}.$$

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It follows from Theorem 1 that

$$\sum_{x=0}^{p} \binom{x+y}{x} \binom{p+q-(x+y)}{p-x} = \binom{p+q}{p}.$$

Sketch of Proof

Since there are $p+q+1$ values of $x+y \rightarrow$ the number of gap vectors (v_1, v_2) in Case (1) is

$$(p+q+1)\binom{p+q}{p} = (2n-v_1-v_2-1)\binom{2n-v_1-v_2-2}{n-v_1-1}.$$

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For Cases (2) and (3), the number of gap vectors (v_1, v_2) are both

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Adding up all the cases, $g\left(n+1; (v_1, v_2)\right) =$

$$(2n - v_1 - v_2 - 1)\binom{2n - v_1 - v_2 - 2}{n - v_1 - 1} + 2\binom{2n - v_1 - v_2}{n - v_1}.$$

Generalization of Gaps

Lemma 3

Recall that $g_2(n+1) = \sum_{k=1}^{n+1} k \; t_2(k, n+1)$ denotes the number of gap vectors of all simple jump paths from $(n+1, n+1)$ to the origin, then

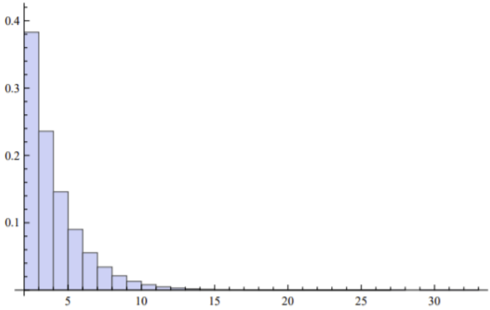
$$g_2(n+1) = \left(\frac{n}{2} + 1\right)\binom{2n}{n}.$$

Note that $t_2(k, n+1)$ is the number of simple jump paths from $(n+1, n+1)$ to $(0, 0)$ of length k.

Gaps in 1-dimensional Case

Theorem (Zeckendorf Gap Distribution)

Consider the distribution of gaps among the decompositions of all the integers $m \in [F_n, F_{n+1})$. For fixed positive integer k, the probability that a gap equals k converges to $1/\phi^k$ for $k \geq 2$.



Distribution of gaps in $[F_{1000}, F_{1001})$

Generalization of Gaps

Theorem 2

Let n be a positive integer. Consider the distribution of gap vectors among all simple jump paths of dimension 2 with starting point $(n+1, n+1)$. For fixed positive integers v_1, v_2, the probability that a gap vector equals (v_1, v_2) converges point-wise to $1/2^{v_1+v_2}$ as $n \to \infty$.

Sketch of Proof

Recall that $g\left(n+1;\left(v_1,v_2\right)\right)$ denotes the number of gap vectors $\left(v_1,v_2\right)$ and $g_2(n+1)$ denotes the number of all gap vectors in all simple jump paths from $(n+1,n+1)$ to the origin. Let $P(n+1;v_1,v_2)$ denote the probability that a given gap vector is (v_1,v_2), then

$$P(n+1;v_1,v_2)=\frac{g\left(n+1;\left(v_1,v_2\right)\right)}{g_2\left(n+1\right)}.$$

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Recall that $g\left(n+1;\left(v_1,v_2\right)\right)$ denotes the number of gap vectors $\left(v_1,v_2\right)$ and $g_2(n+1)$ denotes the number of all gap vectors in all simple jump paths from $(n+1,n+1)$ to the origin. Let $P(n+1;v_1,v_2)$ denote the probability that a given gap vector is (v_1,v_2), then

$$P(n+1;v_1,v_2)=\frac{g\left(n+1;\left(v_1,v_2\right)\right)}{g_2\left(n+1\right)}.$$

Applying Lemma 2 and Lemma 3 and simplifying the result, we obtain

$$\lim_{n\to\infty} P(n+1;v_1,v_2) \;=\; \lim_{n\to\infty} \frac{\frac{(2n-v_1-v_2-1)!}{(n-v_1-1)!(n-v_2-1)!}}{\left(\frac{n}{2}+1\right)\frac{(2n)!}{n!n!}}.$$

Sketch of Proof

Using Stirling's approximation, we obtain

$$\lim_{n\to\infty} P(n+1; v_1, v_2) = \lim_{n\to\infty} \frac{2n - v_1 - v_2 - 1}{\frac{n}{2} + 1} \frac{e^{-1}}{2^{v_1+v_2+2}} \times \left(1 + \frac{\frac{v_1 - v_2 + 1}{2}}{n - v_1 - 1}\right)^{n-v_1-1} \left(1 + \frac{\frac{v_2 - v_1 + 1}{2}}{n - v_2 - 1}\right)^{n-v_2-1}.$$

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Finally, applying the equation $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$, we simplify the result into

$$\lim_{n\to\infty} P(n+1; v_1, v_2) = \frac{1}{2^{v_1+v_2}}.$$

Generalization of Gaps

Theorem 3

Let n be a positive integer. Consider the distribution of gap sums among all simple jump paths of dimension 2 with starting point $(n+1, n+1)$. The probability that a gap sum equals an integer $v \geq 2$ converges to $(v-1)/2^v$ as $n \to \infty$ (the probability of a gap sum of 0 or 1 is zero).

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Sketch of Proof:

Let $P(v)$ denote the probability that a given gap sum equals $v \geq 2$. Since for each v, there are $v-1$ pairs of (v_1, v_2) such that $v_1 + v_2 = v$. Thus by Theorem 2,

$$\lim_{n\to\infty} P(v) \;=\; (v-1)\left(\frac{1}{2}\right)^v.$$

Future Work

Conjecture (d-dimensional Gap Vector)

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We proved that

$$P(v_1, \ldots, v_d) \;=\; 2\,\frac{g_{n-v_1,\ldots,n-v_d}}{g_{n,\ldots,n}} + \sum_{i} \frac{g_{i_1,\ldots,i_d}\, g_{n-i_1-v_1,\ldots,n-i_d-v_d}}{g_{n,\ldots,n}},$$

where $g_{a_1,\ldots,a_d} = \sum_{k=1}^{\infty} k\binom{a_1-1}{k-1}\cdots\binom{a_d-1}{k-1}$, and the first term goes to zero as $n \to \infty$. It remains to prove convergence for the second term.

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Conjecture (d-dimensional Gap Sum)

As $n \to \infty$, the distribution of the gap sums in the Zeckendorf decompositions from d-dimensional lattice grid approaches geometric decay.

Future Work

Generalization to Euclidean Distances

Our method can potentially be generalized to study the distribution of the Euclidean distances between summands. The analysis involves counting the number of diophantine equations that have solutions within the range of $\{1, 2, \ldots, n\}$.

Thank you!