Bulk and Blip Distributions of Various Random Matrix Ensembles Under Anticommutator Operator

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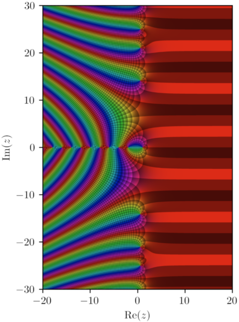
Why care about Large Random Matrices



Complex plot of an L-function

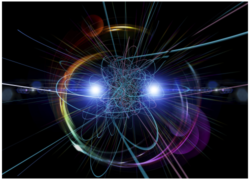
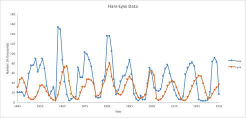


Figure of two particles before collision



Population evolution of a predator-prey population

Random Matrix Ensembles: the GOE

Definition (Gaussian Orthogonal Ensemble)

The GOE X_N is constructed by assigning a random variable a_{ij} to each entry of a square matrix by the following rules:

$$\begin{cases} a_{ij} \; = \; a_{ji} \sim \mathcal{N}(0,1) & i \neq j \\ a_{ii} \sim \mathcal{N}(0,2) & \end{cases}.$$

$$X_N \; = \; \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}.$$

For any instance A_N[1] of X_N, $A_N \; = \; A_N^{\top}$, so eigenvalues of A_N are purely real.

[1] A_N is a fixed matrix w/o randomness

Deriving Spectral distributions

Definition (Spectral Measure of Eigenvalues)

$$\nu_{A_N,N}(x) \ = \ \frac{1}{N}\sum_{i=1}^{N}\delta\left(x-\frac{\lambda_i}{\sqrt{N}}\right),$$

where $\{\lambda_i\}_{i=1}^N$ are the eigenvalues of X_N. Size of eigenvalues are typically $\Theta(\sqrt{N})$.

Moments of the distribution

Lemma (Computing Moments by Trace)

Let $M_{N,k}$ be the k^{th} moment of the spectral distribution $\nu_{A_N,N}$. The moment can be computed by the trace of A_N. i.e.

$$M_{N,k}(A_N) \;=\; \frac{\lambda_1^k + \cdots + \lambda_N^k}{N^{\frac{k}{2}+1}} \;=\; \frac{\mathrm{Tr}(A_N^k)}{N^{\frac{k}{2}+1}}.$$

Remark

The power of N on the numerator depends on the normalization adopted for the definition of spectral density.

Matrix to RMT Ensembles

Corollary

For a random matrix ensemble X_N, the moment of the spectral density can be computed by the expected trace:

$$M_{N,k}(X_N) \;=\; \frac{\mathbb{E}[\mathrm{Tr}(X_N^k)]}{N^{\frac{k}{2}+1}}.$$

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$$M_{N,k}(X_N) \;=\; \frac{\mathbb{E}[\mathrm{Tr}(X_N^k)]}{N^{\frac{k}{2}+1}}.$$

Definition (Limiting Spectral Density of a RMT Ensemble)

The LSD is defined as the spectral density of the RMT ensenble as N approaches infinity. Also, we define the moment of the LSM as follows.

$$\nu_X(x) \;=\; \lim_{N\to\infty} \nu_{X_N,N}(x).$$

LSD's are well-defined for most matrix ensembles.

A Useful Tool

Lemma (Wick's Formula)

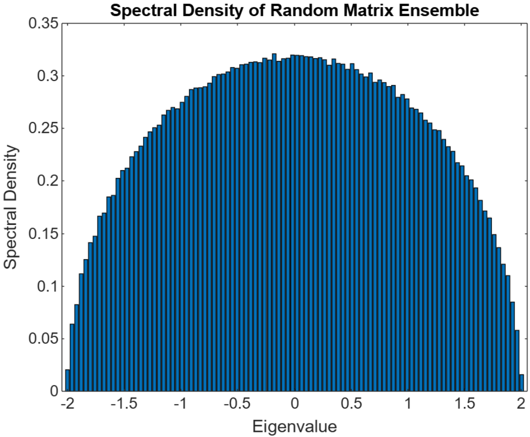
Let $(x_1, \ldots, x_n)$ *be a real Gaussian random vector, and* $\mathcal{P}_2(k)$ *be the set of all pairings of* $[k]$*. Then*

$$\mathbb{E}(x_{i_1} \cdots x_{i_k}) = \sum_{\pi \in \mathcal{P}_2(k)} \mathbb{E}_\pi(x_{i_1}, \ldots, x_{i_k}) \quad \textit{for any } i_1, \ldots, i_k \in [n],$$

where $\mathbb{E}_\pi$ *is the paired expectation, i.e.*

$$\mathbb{E}_{(12)(34)}[x_1 x_2 x_3 x_4] = \mathbb{E}[x_1 x_2]\mathbb{E}[x_3 x_4].$$

Experimental Spectral Density: $N = 1000$



$$2/2\pi \approx 0.318$$

LSD of GOE: Semicircular Law

Theorem (Semicircular Law)

Let $\{X_N\}_{N=1}^{\infty}$ be a sequence of $N \times N$ GOE random matrices with spectral measure $\{\nu_{X_N,N}\}_{N=1}^{\infty}$. Then, $\{\nu_{X_N,N}\}_{N=1}^{\infty}$ converges weakly almost surely to semicircle distribution

$$\lim_{N\to\infty} \nu_{X_N,N} \;=\; \sigma,$$

where $\sigma := \frac{1}{2\pi}\sqrt{4-t^2}$, in the sense that

$$\lim_{N\to\infty} \mathbb{P}\left(|M_{N,k}(X_N) - M_k(\sigma)| > \epsilon\right) \;=\; 0.$$

Proof Sketch of the semicircle law

Lemma (Moments of GOE)

$$M_k(\sigma) \;=\; \frac{1}{2\pi}\int_{-2}^{2} t^k\sqrt{4-t^2}\,dt \;=\; \begin{cases} C_{k/2} & (k = 0 \mod 2) \\ 0 & (k = 1 \mod 2) \end{cases},$$

where C_k is the k^{th} Catalan number.

Combining two RMT Ensembles

Question

Is there a natural way to combine two random matrix ensembles such that

1. *All the eigenvalues are real;*
2. *The combination is symmetric.*

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Definition

Consider the Anticommutator product, namely

$$\{A, B\} \ := \ AB + BA.$$

Anticommutator of two RMT Ensembles

1. GOE;
2. Palindromic Toeplitz;
3. k-checkerboard.

Spectral Density of the Anticommutator

Definition (Spectral Density of the Anticommutator)

$$\mu_N(x) := \lim_{N\to\infty} \frac{1}{N} \sum_{\lambda\in\Lambda} \delta\left(x - \frac{\lambda}{N}\right).$$

Theorem (Moments of Spectral Density)

$$M_{N,k}(X_N Z_N + Z_N X_N) = \mathbb{E}\left[\mathrm{Tr}\left(\left[\frac{1}{N}(X_N Z_N + Z_N X_N)\right]^k\right)\right].$$

Definition of PTE

Definition (Palindromic Toeplitz)

An $N \times N$ real symmetric palindromic Toeplitz matrix (where N is assumed to be even for simplicity) is a matrix A_N whose entries are paramatrized by $b_0, b_1, \dots, b_{N/2-1}$, where the b_i's are i.i.d. random variables with mean 0 and variance 1:

$$a_{ij} = \begin{cases} b_{|i-j|}, & \text{if } 0 \leq |i-j| \leq \frac{N}{2} - 1 \\ b_{N-1-|i-j|}, & \text{if } \frac{N}{2} \leq |i-j| \leq N-1. \end{cases}$$

This matrix is in the form

$$\begin{pmatrix} b_0 & b_1 & \cdots & b_1 & b_0 \\ b_1 & b_0 & \cdots & b_2 & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1 & b_2 & \cdots & b_0 & b_1 \\ b_0 & b_1 & \cdots & b_1 & b_0 \end{pmatrix}.$$

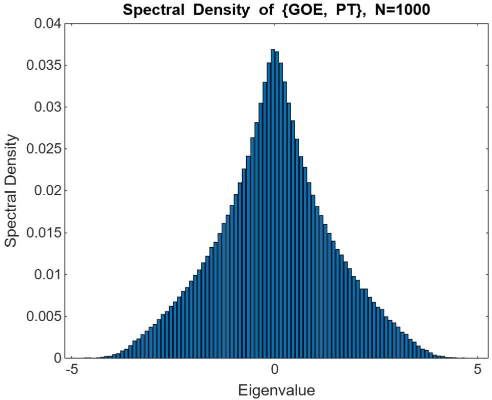
Remark (Kologlu, Kopp, Miller 2011)

The structure of the PTE creates Diophantine Obstructions which make certain terms of the expected trace vanish in the $N \to \infty$. In particular, terms that have a ***Crossing Pairing*** *vanish.*

Moments of $\{\mathrm{GOE}, \mathrm{PT}\}$

Question

Can we compute the moments of the Spectral distribution of the anticommutator of two ensembles GOE, PT?




```
2th moment: 2.005185
3th moment: -0.000116
4th moment: 12.220592
5th moment: -0.059222
6th moment: 110.056541
7th moment: 2.953869
8th moment: 1177.779577
```

Figure: Moments of $\{GOE, PTE\}$

Computing Normalized Spectral Density

Remark

Foiling trace expansion and invoking Wick's formula, we verify that GOE's have crossings do not contribute to the moments as $N \to \infty$.

Example

$$N^3\mu(2) = \mathbb{E}[\mathrm{Tr}(XZXZ)] + \mathbb{E}[\mathrm{Tr}(XZZX)] + \mathbb{E}[\mathrm{Tr}(ZXXZ)] + \mathbb{E}[\mathrm{Tr}(ZXZX)]$$
$$= 2\left(\mathbb{E}[\mathrm{Tr}(XXZZ)] + \mathbb{E}[\mathrm{Tr}(XZXZ)]\right).$$

The following computations give motivation for the following definitions.

Special Words

Definition (Special Words)

A special word of length $2k$ is composed of k blocks of $\{XX, ZX, XZ\}$. The characteristic of a special word w, $\chi(w)$, is the number XX blocks.

Example

When $k = 3$,

$$XX\ ZX\ XZ$$

has length 6, characteristic 1.

Set of Special Words

Definition (Set of Special Words)

$H_{n,k}$ is the set of special words of length $2n$, characteristic k.

Example

For $n = 2$ and $k = 1$:

$$H_{2,1} = \{XX\,ZX, XX\,XZ, ZX\,XX, XZ\,XX\}.$$

Valid Pairings

Definition (Valid Pairings)

A pairing is valid if each paired letter are the same.

Example

For the word $XX\,ZZ$, a valid pairing is

$$(12)(34)$$

and an invalid pairing is

$$(13)(24).$$

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Remark (Motivation for Validness)

Assuming x, z are independent r.v. with mean zero, then

$$\mathbb{E}[xz] \;=\; 0.$$

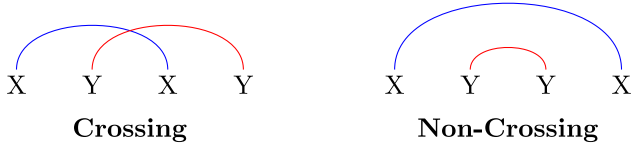
Non-Crossing Pairings

Definition (Non-Crossing Pairings)

A non-crossing pairing is a valid pairing where for any two pairs $\{i, k\}$ and $\{j, l\}$, it is not the case that $i < j < k < l$.

Example

For the word $XXZZ$, the pairing (12)(34) is non-crossing, while the pairing (13)(24) is crossing because $1 < 2 < 3 < 4$.



The crossing pairs vanish due to Diophantine Obstructions.

Pairing Number

Definition (Pairing Number)

$\nu_{n,k}$ is the number of valid, non-crossing pairings for all words in $H_{n,k}$, i.e.

$$\nu_{n,k} = \sum_{w \in H_{n,k}} \varphi(w),$$

where $\varphi(w)$ counts valid, non-crossing pairings of w.

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Definition (Auxiliary Sequence)

Define

$$\sigma_{n,s,k} \ := \sum_{w \in H_{n,s,k}} \varphi(w).$$

$H_{n,s,k}$ is the set of all words with n total blocks, at least s blocks of XX in the beginning, k blocks of XZ, ZX.

Moment Computation as a Combinatorial Problem

Theorem (Moments of $\{PTE, GOE\}$)

The n^{th} moment of the LSD of the anticommutator ensemble of $\{GOE, PTE\}$ can be computed by the pairing number, i.e.

$$M_n \;=\; \nu_{n,n} \;=\; \sigma_{n,s=0,k=-n}.$$

Initial Conditions for $\sigma_{n,s,k}$

Theorem (Initial Conditions for $\sigma_{n,s,k}$)

For $n, s, k \in \mathbb{Z}_{\text{pos}}$,

- $\sigma_{n,s,k} = 0$ *if* $s + k > n$,
- $\sigma_{n,s,2k+1} = 0$,
- $\sigma_{n,s,-k} = 0$,
- $\sigma_{n,s,0} = (2n - 1)!!$.

Proof.

Conditions $s + k > n$ and $k < 0$ will not generate a valid word. k odd does not generate valid pairings.
If $k = 0$, then the word is comprised solely of X's, which reduces to the *GOE* case. □

Theorem 3: Recurrence Relation for $\sigma_{n,s,2k}$

Theorem (Recurrence Relation for $\sigma_{n,s,2k}$)

The recurrence relation for $\sigma_{n,s,2k}$ is given by:

$$\sigma_{n,s,2k} = \sum_{p=s+1}^{n} \sum_{q=p+1}^{n} \sum_{r=0}^{2k} \left[\sigma_{n-q+p,p,r} \cdot \sigma_{q-p-1,0,2k-2-r}\right]$$

$$+ \sum_{p=s+1}^{n} \sum_{q=p+1}^{n} \sum_{r=0}^{2k} \left[\sigma_{n-q+p-1,p-1,r} \cdot \sigma_{q-p,1,2k-2-r}\right].$$

Recurrence Relation: Demo

Proposition

The pairing number of the word is given by the sum of the product of pairing numbers generated by the Y-slicings.

$$\varphi(W) \;=\; \sum_{\substack{W_1, V_1 \\ \textit{Type 1}}} \varphi(W_1)\varphi(V_1) + \sum_{\substack{W_2, V_2 \\ \textit{Type 2}}} \varphi(W_2)p(V_2).$$

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Corollary (Informal justification of the recurrence relation)

$$\sigma_{n,s,2k} \;=$$

$$\sum_{p=s+1}^{n}\sum_{q=p+1}^{n}\sum_{r=0}^{2k}\left[\sigma_{n-q+p,p,r}\cdot\sigma_{q-p-1,0,2k-2-r}+\sigma_{n-q+p-1,p-1,r}\cdot\sigma_{q-p,1,2k-2-r}\right].$$

Numerical computations of theoretical moments

```
Table of \sigma_{n, k} for N = 15:
    k:     0    1    2    3    4    5
n
  0:     1
  1:     0    1
  2:     2    0    3
  3:     0   12    0   15
  4:    12    0   84    0  105
  5:     0  160    0  720    0  945
  6:   104    0 1908    0 7470    010395
```

Figure: Numerical values of $\sigma_{n,s=0,k}$

Recap: k-Checkerboard

Definition ((k,w)-Checkerboard)

An $N \times N$ (k,w)-**checkerboard matrix** $M = (m_{ij})$ is a matrix whose entries are defined as

$$m_{i,j} = \begin{cases} a_{i,j} & \text{if } i \not\equiv j \mod k \\ w & \text{if } i \equiv j \mod k \end{cases},$$

where $a_{ij} = a_{ji}$ with a_{ij} i.i.d. random variables with mean 0 and variance 1, and $w \in \mathbb{R}$. For example, $(2,w)$-checkerboard matrices look like the following:

$$M = \begin{pmatrix} w & a_{0,1} & w & a_{0,1} & w & \cdots & a_{0,N-1} \\ a_{0,1} & w & a_{1,2} & w & a_{1,4} & \cdots & w \\ w & a_{1,2} & w & a_{2,3} & w & \cdots & a_{2,N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

We refer to the $(k,1)$-checkerboard ensemble as the k-checkerboard ensemble.

Anticommutator of Checkerboards

Question

What is the limiting spectral distribution of the anticommutator of k-checkerboard and j-checkerboard?

Multiple Regimes

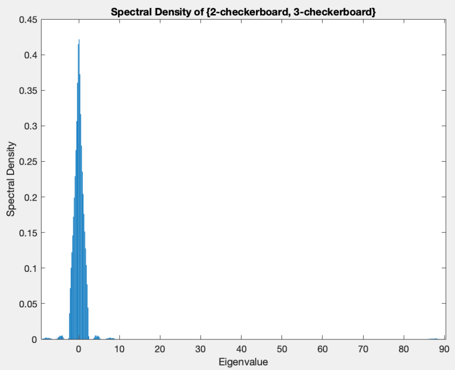

Figure: Multiple Regimes

There is one bulk regime and five other smaller regimes (blip regimes).

A Closer Look

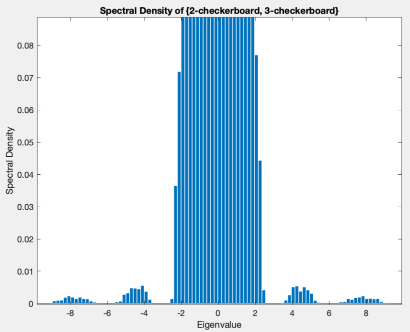

Figure: Intermediary Blips

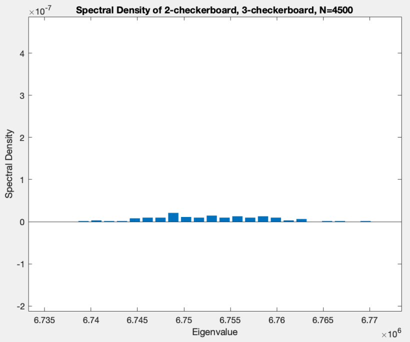

Figure: Largest Blip

Limiting Spectral Distribution

Observation

Numerical simulation tells us location of 5 blip regimes:

1. $\frac{N^2}{kj} + \Theta(N)$ *(1 blip eigenvalue);*
2. $\pm\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2} + \Theta(N)$ *($k-1$ blip eigevalues);*
3. $\pm\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2} + \Theta(N)$ *($j-1$ blip eigenvalues).*

Remark

Standard techniques fail to find centered distribution → construction of weight functions.

Definitions

We focus on the spectral distribution of the largest blip.

Definition

The **empirical largest blip spectral measure** of $\{A_N, B_N\}$:

$$\mu_{\{A_N,B_N\}}(x) = \sum_{\lambda \text{ eigenvalues}} g_0^{2n}\left(\frac{jk\lambda}{2N^2}\right)\delta\left(x - \left(\frac{\lambda - \frac{2}{jk}N^2}{N}\right)\right),$$

where $g_0^{2n}(x) = x^{2n}(2-x)^{2n}$, $n(N) = \log\log(N)$.

Weight Function for Largest Blip Regime

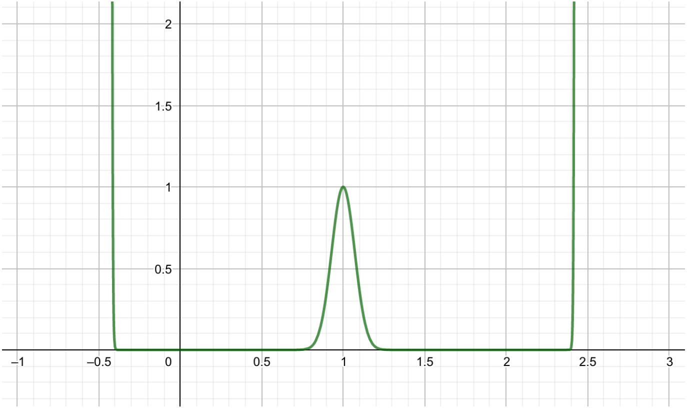

Figure: $g_0(x)^{100} = x^{100}(2-x)^{100}$

Moments of the Empirical Largest Blip Spectral Measure

Theorem

The m^{th} moment of the largest blip spectral measure is

$$\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\}}\right] = \sum_{\substack{m_{1a}+m_{1b}+m_{2a}+m_{2b}=m;\\ m_{1a},m_{1b} \text{ even}}} C(m,m_{1a},m_{2a},m_{1b},m_{2b})$$

$$\left(k\sqrt{1-\frac{1}{k}}\right)^{m_{1a}+2m_{2a}} \left(j\sqrt{1-\frac{1}{j}}\right)^{m_{1b}+2m_{2b}},$$

where $C(m,m_{1a},m_{2a},m_{1b},m_{2b}) := m!\left(\frac{2}{jk}\right)^m \frac{2^{\frac{m_{1a}+m_{1b}}{2}-2(m_{2a}+m_{2b})}m_{1a}!!m_{1b}!!}{m_{1a}!m_{1b}!m_{2a}!m_{2b}!}.$

Thanks

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