L-FUNCTIONS, RANKS OF ELLIPTIC CURVES AND RANDOM MATRIX THEORY

Finite conductor models for zeros of elliptic curves

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Orthogonal Random Matrix Models

RMT: $2N$ eigenvalues, in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \ \propto \ \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j.$$

Independent Model:

$$\mathcal{A}_{2N,2r} \ = \ \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

Interaction Model:

Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$:

$$d\varepsilon_{2r}(\theta) \ \propto \ \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

with $1 \ \leq \ j,k \ \leq \ N-r$.

1-Level Density

L-function $L(s,f)$: by RH non-trivial zeros $\frac{1}{2}+i\gamma_{f,j}$.

C_f: analytic conductor; $\varphi(x)$: compactly supported even Schwartz function.

$$D_{1,f}(\varphi) \;=\; \sum_j \varphi\left(\frac{\log C_f}{2\pi}\gamma_{f,j}\right)$$

- individual zeros contribute in limit
- most of contribution is from low zeros

Katz-Sarnak Conjecture:

$$\begin{aligned} D_{1,\mathcal{F}}(\varphi) \;&=\; \lim_{N\to\infty}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{1,f}(\varphi) \;=\; \int \varphi(x)\rho_{G(\mathcal{F})}(x)dx \\ &=\; \int \widehat{\varphi}(u)\widehat{\rho}_{G(\mathcal{F})}(u)du. \end{aligned}$$

Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Independent):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Comparing the RMT Models

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) \ = \ \int \varphi(x)\rho_{\mathcal{G}}(x)dx + \ r\varphi(0$$

where

$$\mathcal{G} \ = \ \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Supports Independent and not Interaction model in the limit.

Interesting Families

Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank r over $\mathbb{Q}(T)$.

Natural sub-families:

- Curves of rank r.
- Curves of rank $r + 2$.

Question: Does the sub-family of rank $r+2$ curves in a rank r family behave like the sub-family of rank $r+2$ curves in a rank $r + 2$ family?

Equivalently, does it matter how one conditions on a curve being rank $r + 2$?

Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r+2$.

2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.

Excess Rank

One-parameter family, rank r over $\mathbb{Q}(T)$.

Density Conjecture (Generic Family) $\Longrightarrow$ 50% rank r, r+1.

For many families, observe

Percent with rank r	$\approx$	32%
Percent with rank r+1	$\approx$	48%
Percent with rank r+2	$\approx$	18%
Percent with rank r+3	$\approx$	2%

Problem: small data sets, sub-families, convergence rate $\log(\text{conductor})$.

Data on Excess Rank

$$y^2 + y = x^3 + Tx$$

Each data set 2000 curves from start.

t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

Last set has conductors of size 10^{17}, but on logarithmic scale still small.

RMT: Theoretical Results ($N \to \infty$, Mean $\to 0.321$)

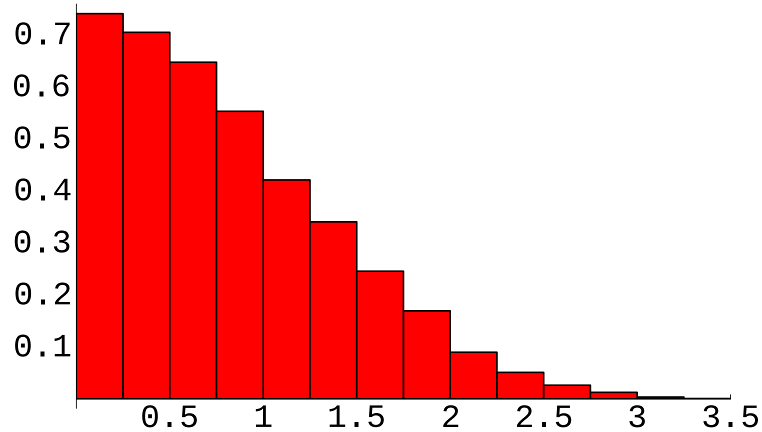

Figure 1a: 1st norm. evalue above 1: 23,040 SO(4) matrices

Mean $= .709$, Std Dev of the Mean $= .601$, Median $= .709$

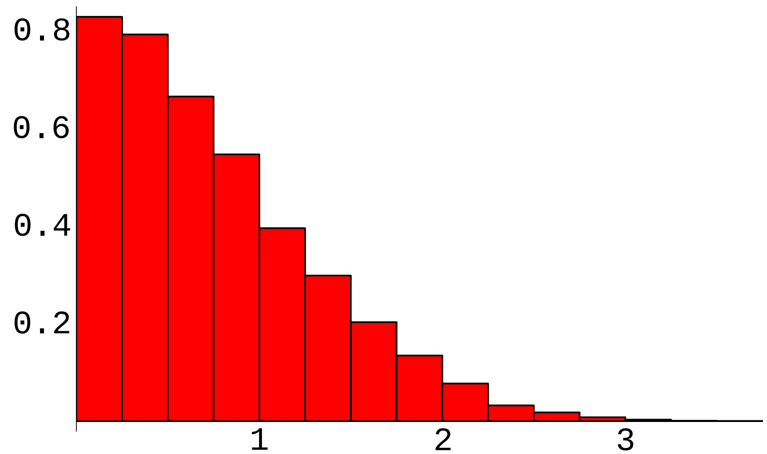

Figure 1b: 1st norm. evalue above 1: 23,040 SO(6) matrices

Mean $= .635$, Std Dev of the Mean $= .574$, Median $= .635$

RMT: Theoretical Results ($N \to \infty$)

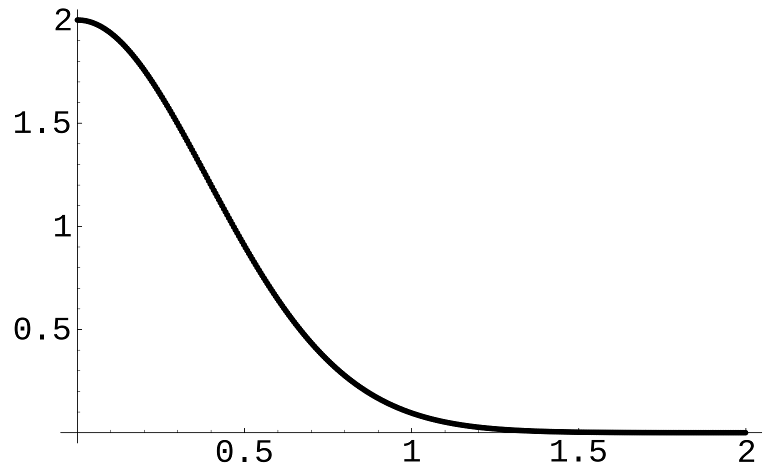

Figure 1c: 1st norm. evalue above 1: SO(even)

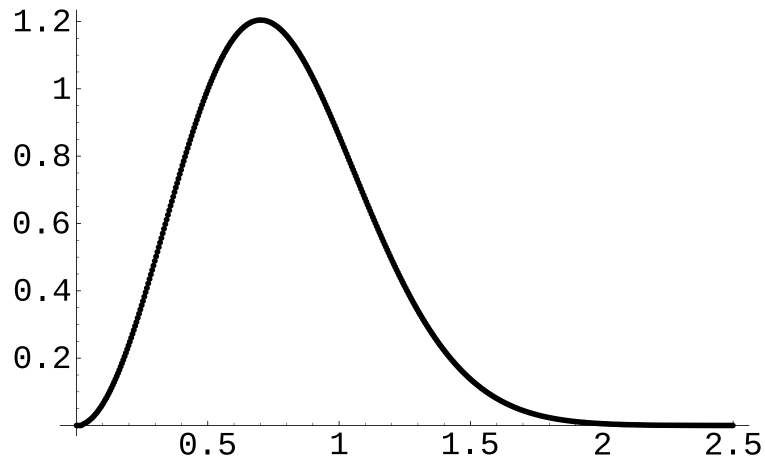

Figure 1d: 1st norm. evalue above 1: SO(odd)

Rank 0 Curves: 1st Normalized Zero above Central Point

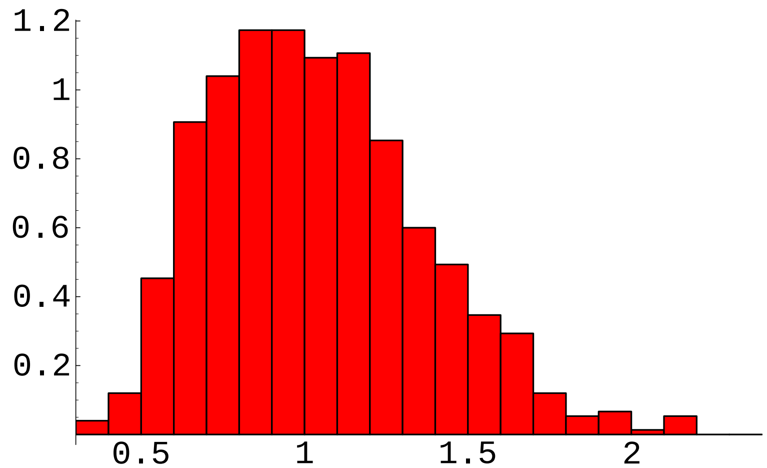

Figure 2a: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\mathrm{cond}) \in [3.2, 12.6]$, median $= 1.00$ mean $= 1.04$, $\sigma_\mu = .32$

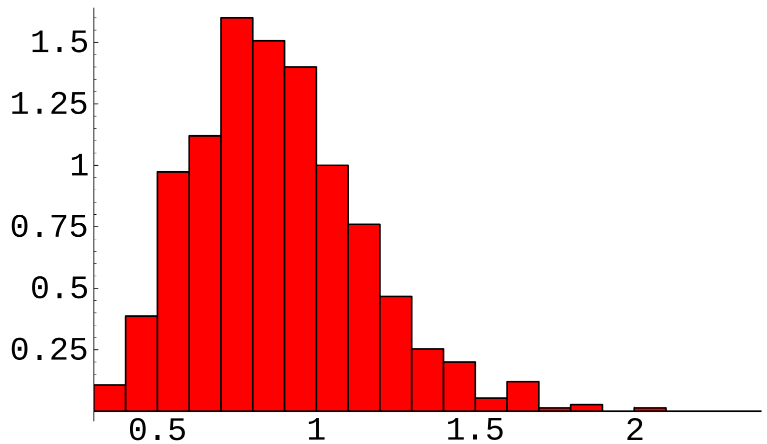

Figure 2b: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\mathrm{cond}) \in [12.6, 14.9]$, median $= .85$, mean $= .88$, $\sigma_\mu = .27$

Rank 2 Curves: 1st Norm. Zero above the Central Point

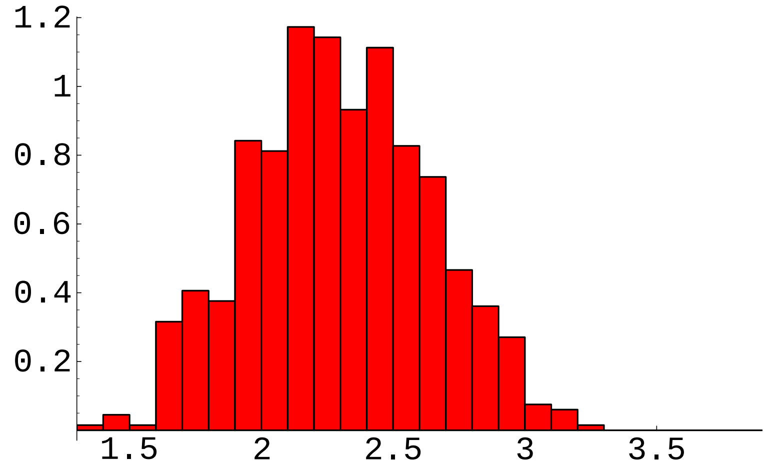

Figure 3a: 665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

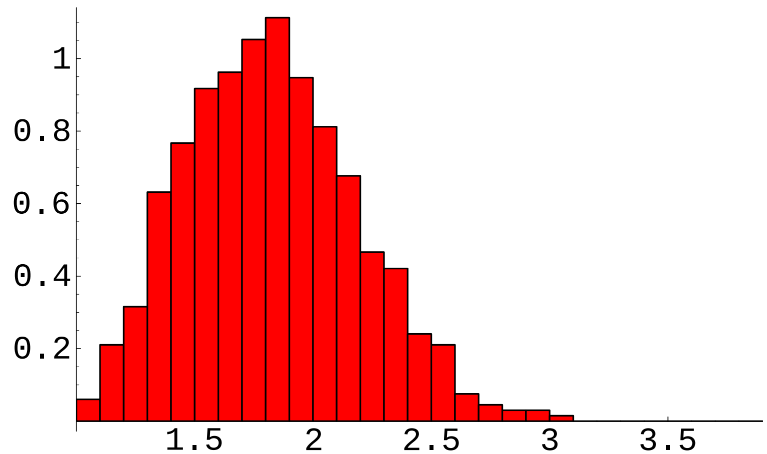

Figure 3b: 665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [16, 16.5]$, median $= 1.81$, mean $= 1.82$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

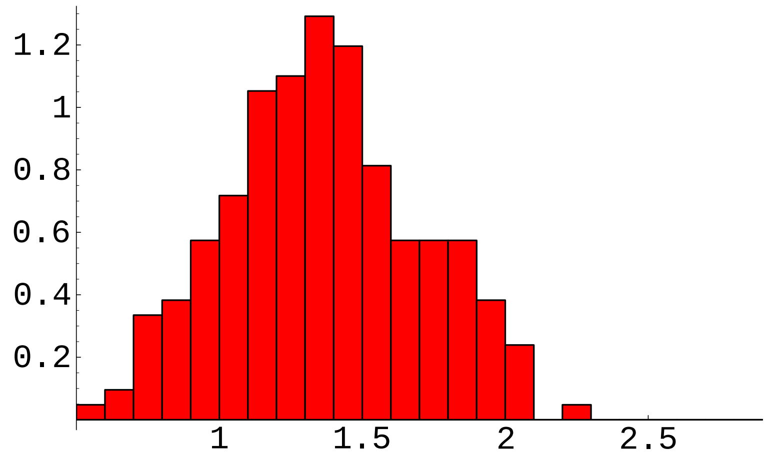

Figure 4a: 209 rank 0 curves from 14 rank 0 families,
$\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

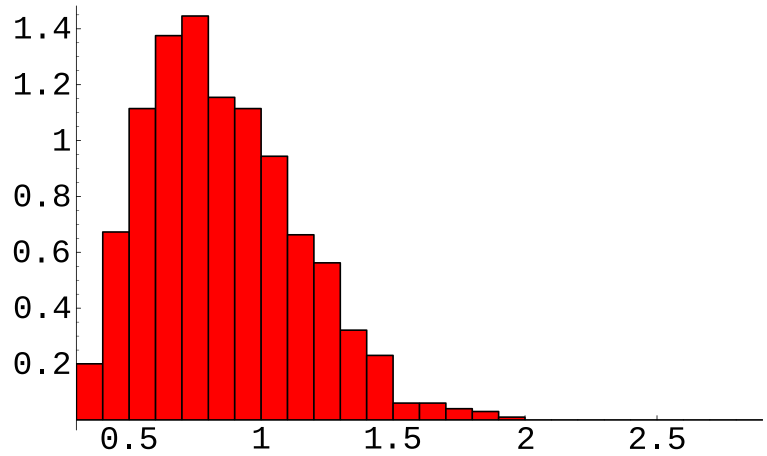

Figure 4b: 996 rank 0 curves from 14 rank 0 families,
$\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	Median $\widetilde{\mu}$	Mean μ	StDev σ_μ	log(conductor)	Number
1: [0,1,1,1,T]	1.28	1.33	0.26	[3.93, 9.66]	7
2: [1,0,0,1,T]	1.39	1.40	0.29	[4.66, 9.94]	11
3: [1,0,0,2,T]	1.40	1.41	0.33	[5.37, 9.97]	11
4: [1,0,0,-1,T]	1.50	1.42	0.37	[4.70, 9.98]	20
5: [1,0,0,-2,T]	1.40	1.48	0.32	[4.95, 9.85]	11
6: [1,0,0,T,0]	1.35	1.37	0.30	[4.74, 9.97]	44
7: [1,0,1,-2,T]	1.25	1.34	0.42	[4.04, 9.46]	10
8: [1,0,2,1,T]	1.40	1.41	0.33	[5.37, 9.97]	11
9: [1,0,-1,1,T]	1.39	1.32	0.25	[7.45, 9.96]	9
10: [1,0,-2,1,T]	1.34	1.34	0.42	[3.26, 9.56]	9
11: [1,1,-2,1,T]	1.21	1.19	0.41	[5.73, 9.92]	6
12: [1,1,-3,1,T]	1.32	1.32	0.32	[5.04, 9.98]	11
13: [1,-2,0,T,0]	1.31	1.29	0.37	[4.73, 9.91]	39
14: [-1,1,-3,1,T]	1.45	1.45	0.31	[5.76, 9.92]	10
All Curves	1.35	1.36	0.33	[3.26, 9.98]	209
Distinct Curves	1.35	1.36	0.33	[3.26, 9.98]	196

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	**Median $\widetilde{\mu}$**	**Mean μ**	**StDev σ_μ**	**log(conductor)**	**Number**
1: [0,1,1,1,T]	0.80	0.86	0.23	[15.02, 15.97]	49
2: [1,0,0,1,T]	0.91	0.93	0.29	[15.00, 15.99]	58
3: [1,0,0,2,T]	0.90	0.94	0.30	[15.00, 16.00]	55
4: [1,0,0,-1,T]	0.80	0.90	0.29	[15.02, 16.00]	59
5: [1,0,0,-2,T]	0.75	0.77	0.25	[15.04, 15.98]	53
6: [1,0,0,T,0]	0.75	0.82	0.27	[15.00, 16.00]	130
7: [1,0,1,-2,T]	0.84	0.84	0.25	[15.04, 15.99]	63
8: [1,0,2,1,T]	0.90	0.94	0.30	[15.00, 16.00]	55
9: [1,0,-1,1,T]	0.86	0.89	0.27	[15.02, 15.98]	57
10: [1,0,-2,1,T]	0.86	0.91	0.30	[15.03, 15.97]	59
11: [1,1,-2,1,T]	0.73	0.79	0.27	[15.00, 16.00]	124
12: [1,1,-3,1,T]	0.98	0.99	0.36	[15.01, 16.00]	66
13: [1,-2,0,T,0]	0.72	0.76	0.27	[15.00, 16.00]	120
14: [-1,1,-3,1,T]	0.90	0.91	0.24	[15.00, 15.99]	48
All Curves	0.81	0.86	0.29	[15.00,16.00]	996
Distinct Curves	0.81	0.86	0.28	[15.00,16.00]	863

Rank 2 Curves: 1st Norm Zero: one-param of rank 0 over $\mathbb{Q}(T)$

first set $\log(\text{cond}) \in [15, 15.5)$; second set $\log(\text{cond}) \in [15.5, 16]$. Median $\widetilde{\mu}$, Mean μ, Std Dev (of Mean) σ_μ.

Family	$\widetilde{\mu}$	μ	σ_μ	Number	$\widetilde{\mu}$	μ	σ_μ	Number
1: [0,1,3,1,T]	1.59	1.83	0.49	8	1.71	1.81	0.40	19
2: [1,0,0,1,T]	1.84	1.99	0.44	11	1.81	1.83	0.43	14
3: [1,0,0,2,T]	2.05	2.03	0.26	16	2.08	1.94	0.48	19
4: [1,0,0,-1,T]	2.02	1.98	0.47	13	1.87	1.94	0.32	10
5: [1,0,0,T,0]	2.05	2.02	0.31	23	1.85	1.99	0.46	23
6: [1,0,1,1,T]	1.74	1.85	0.37	15	1.69	1.77	0.38	23
7: [1,0,1,2,T]	1.92	1.95	0.37	16	1.82	1.81	0.33	14
8: [1,0,1,-1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
9: [1,0,1,-2,T]	1.74	1.74	0.43	14	1.82	1.90	0.40	14
10: [1,0,-1,1,T]	2.00	2.00	0.32	22	1.81	1.94	0.42	18
11: [1,0,-2,1,T]	1.97	1.99	0.39	14	2.17	2.14	0.40	18
12: [1,0,-3,1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
13: [1,1,0,T,0]	1.89	1.88	0.31	20	1.82	1.88	0.39	26
14: [1,1,1,1,T]	2.31	2.21	0.41	16	1.75	1.86	0.44	15
15: [1,1,-1,1,T]	2.02	2.01	0.30	11	1.87	1.91	0.32	19
16: [1,1,-2,1,T]	1.95	1.91	0.33	26	1.98	1.97	0.26	18
17: [1,1,-3,1,T]	1.79	1.78	0.25	13	2.00	2.06	0.44	16
18: [1,-2,0,T,0]	1.97	2.05	0.33	24	1.91	1.92	0.44	24
19: [-1,1,0,1,T]	2.11	2.12	0.40	21	1.71	1.88	0.43	17
20: [-1,1,-2,1,T]	1.86	1.92	0.28	23	1.95	1.90	0.36	18
21: [-1,1,-3,1,T]	2.07	2.12	0.57	14	1.81	1.81	0.41	19
All Curves	1.95	1.97	0.37	350	1.85	1.90	0.40	388
Distinct Curves	1.95	1.97	0.37	335	1.85	1.91	0.40	366

Rank 2 Curves: 1st Norm Zero: 21 One-Param of Rank 0 over $\mathbb{Q}(T)$

- Observe the medians and means of the small conductor set to be larger than those from the large conductor set.

- For all curves the Pooled and Unpooled Two-Sample t-Procedure give t-statistics of 2.5 with over 600 degrees of freedom.

- For distinct curves the t-statistics is 2.16 (respectively 2.17) with over 600 degrees of freedom (about a 3% chance).

- Provides evidence against the null hypothesis (that the means are equal) at the $.05$ confidence level (though not at the $.01$ confidence level).

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)
1st Normalized Zero above Central Point

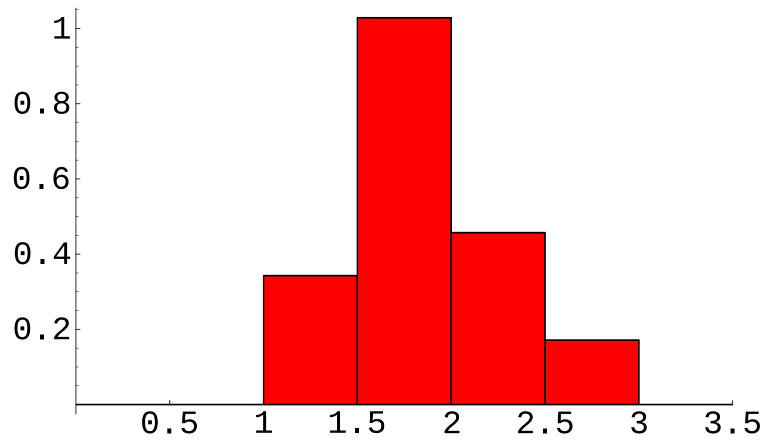

Figure $5a$: 35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

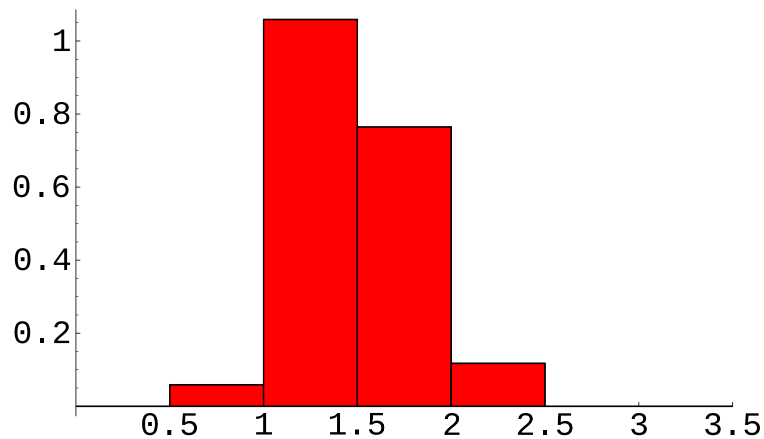

Figure $5b$: 34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Rank 2 Curves: 1st Norm Zero: rank 2 one-param over $\mathbb{Q}(T)$

$\log(\mathrm{cond}) \in [15, 16]$, $t \in [0, 120]$, median is 1.64.

Family	Mean	Standard Deviation	log(conductor)	Number
1: [1,T,0,-3-2T,1]	1.91	0.25	[15.74,16.00]	2
2: [1,T,-19,-T-1,0]	1.57	0.36	[15.17,15.63]	4
3: [1,T,2,-T-1,0]	1.29		[15.47, 15.47]	1
4: [1,T,-16,-T-1,0]	1.75	0.19	[15.07,15.86]	4
5: [1,T,13,-T-1,0]	1.53	0.25	[15.08,15.91]	3
6: [1,T,-14,-T-1,0]	1.69	0.32	[15.06,15.22]	3
7: [1,T,10,-T-1,0]	1.62	0.28	[15.70,15.89]	3
8: [0,T,11,-T-1,0]	1.98		[15.87,15.87]	1
9: [1,T,-11,-T-1,0]				
10: [0,T,7,-T-1,0]	1.54	0.17	[15.08,15.90]	7
11: [1,T,-8,-T-1,0]	1.58	0.18	[15.23,25.95]	6
12: [1,T,19,-T-1,0]				
13: [0,T,3,-T-1,0]	1.96	0.25	[15.23, 15.66]	3
14: [0,T,19,-T-1,0]				
15: [1,T,17,-T-1,0]	1.64	0.23	[15.09, 15.98]	4
16: [0,T,9,-T-1,0]	1.59	0.29	[15.01, 15.85]	5
17: [0,T,1,-T-1,0]	1.51		[15.99, 15.99]	1
18: [1,T,-7,-T-1,0]	1.45	0.23	[15.14, 15.43]	4
19: [1,T,8,-T-1,0]	1.53	0.24	[15.02, 15.89]	10
20: [1,T,-2,-T-1,0]	1.60		[15.98, 15.98]	1
21: [0,T,13,-T-1,0]	1.67	0.01	[15.01, 15.92]	2
All Curves	1.61	0.25	[15.01, 16.00]	64

Function Field Example (with Sal Butt, Chris Hall)

$y^2 = x^3 + (t^5 + a_1 t^4 + a_0)x + (t^3 + b_2 t^2 + b_1 t + b_0)$, $a_i, b_i \in \mathbb{F}_5$

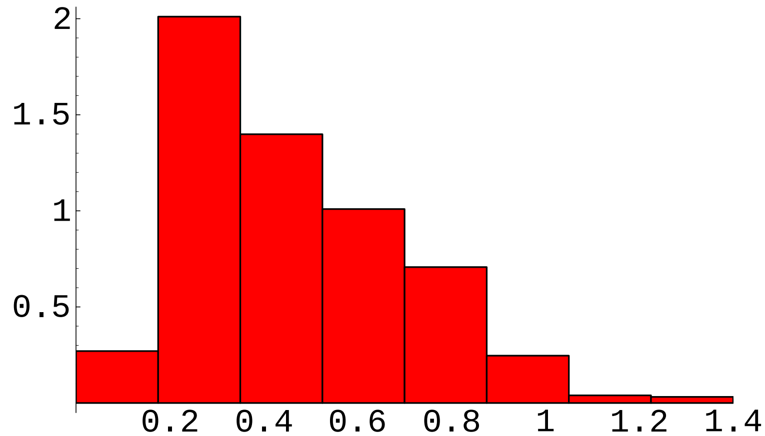

Figure $6a$: Normalized first eigenangle: 719 rank 0 curves.

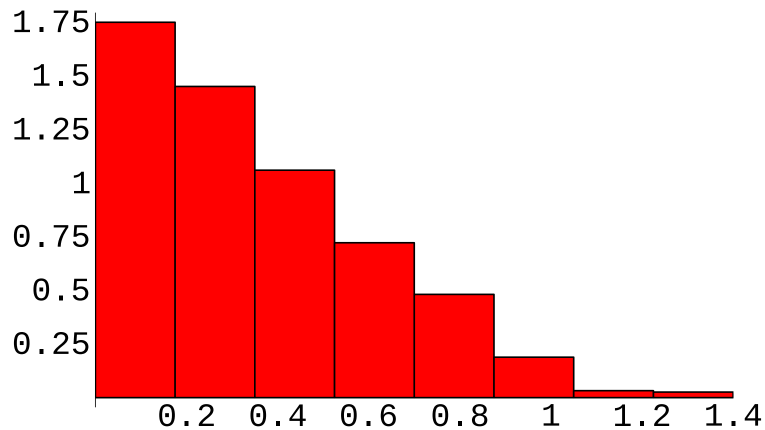

Figure $6b$: Normalized first eigenangle: 978 curves
(719 rank 0 curve, 254 rank 2 curves, 5 rank 4 curves).

Repulsion or Attraction?

Conductors in $[15, 16]$; first set is rank 0 curves from 14 one-parameter families of rank 0 over $\mathbb{Q}$; second set rank 2 curves from 21 one-parameter families of rank 0 over $\mathbb{Q}$. The t-statistics exceed 6.

Family	**2nd vs 1st Zero**	**3rd vs 2nd Zero**	**Number**
Rank 0 Curves	2.16	3.41	863
Rank 2 Curves	1.93	3.27	701

The repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.

Can also interpret as attraction.

Comparison b/w One-Param Families of Different Rank

First normalized zero above the central point.

- The first family is the 701 rank 2 curves from the 21 one-parameter families of rank 0 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$;
- the second family is the 64 rank 2 curves from the 21 one-parameter families of rank 2 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$.

Family	**Median**	**Mean**	**Std. Dev.**	**Number**
Rank 2 Curves (Rank 0 Families)	1.926	1.936	0.388	701
Rank 2 Curves (Rank 2 Families)	1.642	1.610	0.247	64

- t-statistic is 6.60, indicating the means differ.
- The mean of the first normalized zero of rank 2 curves in a family above the central point (for conductors in this range) depends on *how* we choose the curves.

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	**701 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	**23 Rank 4 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	**64 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

Appendices

The first appendix list various standard conjectures. The second appendix gives the formula to numerically approximate the analytic rank of an elliptic curve. For a curve of conductor C_E, one needs about $\sqrt{C_E}\log C_E$ Fourier coefficients. The third is the statement (with assumptions) of the main theoretical result for the one-level density of one-parameter families of Elliptic curves over $\mathbb{Q}(T)$.

Appendix I: Standard Conjectures

Generalized Riemann Hypothesis (for Elliptic Curves) *Let $L(s, E)$ be the (normalized) L-function of the elliptic curve E. Then the non-trivial zeros of $L(s, E)$ satisfy Re$(s) = \frac{1}{2}$.*

Birch and Swinnerton-Dyer Conjecture [BSD1], [BSD2] *Let E be an elliptic curve of geometric rank r over $\mathbb{Q}$ (the Mordell-Weil group is $\mathbb{Z}^r \oplus T$, T is the subset of torsion points). Then the analytic rank (the order of vanishing of the L-function at the central point) is also r.*

Tate's Conjecture for Elliptic Surfaces [Ta] *Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the L-series attached to $H^2_{\acute{e}t}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to* **C** *and satisfies $-$ord$_{s=2} L_2(\mathcal{E}, s) =$ rank $NS(\mathcal{E}/\mathbb{Q})$, where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line Re$(s) = 2$.*

Most of the 1-param families we investigate are rational surfaces, where Tate's conjecture is known. See [RSi].

Appendix II: Numerically Approximating Ranks: Preliminaries

Cusp form f, level N, weight 2:

$$\begin{aligned} f(-1/Nz) &= -\epsilon N z^2 f(z) \\ f(i/y\sqrt{N}) &= \epsilon y^2 f(iy/\sqrt{N}). \end{aligned}$$

Define

$$\begin{aligned} L(f,s) &= (2\pi)^s \Gamma(s)^{-1} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z} \\ \Lambda(f,s) &= (2\pi)^{-s} N^{s/2} \Gamma(s) L(f,s) = \int_0^{\infty} f(iy/\sqrt{N}) y^{s-1} dy. \end{aligned}$$

Get

$$\Lambda(f,s) = \epsilon \Lambda(f, 2-s), \quad \epsilon = \pm 1.$$

To each E corresponds an f, write $\int_0^\infty = \int_0^1 + \int_1^\infty$ and use transformations.

Algorithm for $L^r(s, E)$: I

$$\begin{aligned}
\Lambda(E,s) &= \int_0^\infty f(iy/\sqrt{N})y^{s-1}dy \\
&= \int_0^1 f(iy/\sqrt{N})y^{s-1}dy + \int_1^\infty f(iy/\sqrt{N})y^{s-1}dy \\
&= \int_1^\infty f(iy/\sqrt{N})(y^{s-1} + \epsilon y^{1-s})dy.
\end{aligned}$$

Differentiate k times with respect to s:

$$\Lambda^{(k)}(E,s) = \int_1^\infty f(iy/\sqrt{N})(\log y)^k(y^{s-1} + \epsilon(-1)^k y^{1-s})dy.$$

At $s = 1$,

$$\Lambda^{(k)}(E,1) = (1 + \epsilon(-1)^k)\int_1^\infty f(iy/\sqrt{N})(\log\ y)^k dy.$$

Trivially zero for half of k; let r be analytic rank.

Algorithm for $L^r(s,E)$: II

$$\begin{aligned}\Lambda^{(r)}(E,1) &= 2\int_1^\infty f(iy/\sqrt{N})(\log\ y)^r dy \\ &= 2\sum_{n=1}^\infty a_n \int_1^\infty e^{-2\pi ny/\sqrt{N}}(\log\ y)^r dy.\end{aligned}$$

Integrating by parts

$$\Lambda^{(r)}(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^\infty \frac{a_n}{n}\int_1^\infty e^{-2\pi ny/\sqrt{N}}(\log\ y)^{r-1}\frac{dy}{y}.$$

We obtain

$$L^{(r)}(E,1) = 2r!\sum_{n=1}^\infty \frac{a_n}{n} G_r\left(\frac{2\pi n}{\sqrt{N}}\right),$$

where

$$G_r(x) = \frac{1}{(r-1)!}\int_1^\infty e^{-xy}(\log y)^{r-1}\frac{dy}{y}.$$

Expansion of $G_r(x)$

$$G_r(x) = P_r\left(\log\frac{1}{x}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-r}}{n^r\cdot n!}x^n$$

$P_r(t)$ is a polynomial of degree r, $P_r(t) = Q_r(t-\gamma)$.

$$\begin{aligned}
Q_1(t) &= t;\\
Q_2(t) &= \frac{1}{2}t^2 + \frac{\pi^2}{12};\\
Q_3(t) &= \frac{1}{6}t^3 + \frac{\pi^2}{12}t - \frac{\zeta(3)}{3};\\
Q_4(t) &= \frac{1}{24}t^4 + \frac{\pi^2}{24}t^2 - \frac{\zeta(3)}{3}t + \frac{\pi^4}{160};\\
Q_5(t) &= \frac{1}{120}t^5 + \frac{\pi^2}{72}t^3 - \frac{\zeta(3)}{6}t^2 + \frac{\pi^4}{160}t - \frac{\zeta(5)}{5} - \frac{\zeta(3)\pi^2}{36}.
\end{aligned}$$

For $r = 0$,

$$\Lambda(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}\,e^{-2\pi ny/\sqrt{N}}.$$

Need about $\sqrt{N}$ or $\sqrt{N}\log N$ terms.

Appendix III: 1-Level Density

Definitions:

$$D_{n,\mathcal{F}}(\varphi) = \frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \varphi_i \left(\frac{\log C_E}{2\pi} \gamma_E^{(j_i)} \right)$$

$D_{n,\mathcal{F}}^{(r)}(\varphi)$: n-level density with contribution of r zeros at central point removed.

$\mathcal{F}_N$: Rational one-parameter family, $t \in [N, 2N]$, conductors monotone.

ASSUMPTIONS

1-parameter family of Ell Curves, rank r over $\mathbb{Q}(T)$, rational surface. Assume

- GRH;
- $j(t)$ non-constant;
- Sq-Free Sieve if $\Delta(t)$ has irr poly factor of deg ≥ 4.

Pass to positive percent sub-seq where conductors polynomial of degree m.

φ_i even Schwartz, support σ_i:

- $\sigma_1 < \min\left(\frac{1}{2}, \frac{2}{3m}\right)$ for 1-level
- $\sigma_1 + \sigma_2 < \frac{1}{3m}$ for 2-level.

MAIN RESULT

Theorem (Miller 2004): Under previous conditions, as $N \to \infty$, $n = 1, 2$:

$$D^{(r)}_{n,\mathcal{F}_N}(\varphi) \longrightarrow \int \varphi(x) W_{\mathcal{G}}(x) dx,$$

where

$$\mathcal{G} \ = \ \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

1 and 2-level densities confirm Katz-Sarnak, B-SD predictions for small support.

Examples

Constant-Sign Families:

1. $y^2 = x^3 + 2^4(-3)^3(9t+1)^2$,
   $9t+1$ Square-Free: all even.

2. $y^2 = x^3 \pm 4(4t+2)x$,
   $4t+2$ Square-Free:
   $+$ all odd, $-$ all even.

3. $y^2 = x^3 + tx^2 - (t+3)x + 1$,
   t^2+3t+9 Square-Free: all odd.

First two rank 0 over $\mathbb{Q}(T)$, third is rank 1.

Without 2-Level Density, couldn't say *which* orthogonal group.

Examples (cont)

Rational Surface of Rank 6 over $\mathbf{Q}(t)$:

$$\begin{aligned} y^2 \ = \ & x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x \\ & +(2ct - D)(t^2 + 2t - A + 1)^2 \end{aligned}$$

$$\begin{aligned} A &= 8,916,100,448,256,000,000 \\ B &= -811,365,140,824,616,222,208 \\ C &= 26,497,490,347,321,493,520,384 \\ D &= -343,107,594,345,448,813,363,200 \\ a &= 16,660,111,104 \\ b &= -1,603,174,809,600 \\ c &= 2,149,908,480,000 \end{aligned}$$

Need GRH, Sq-Free Sieve to handle sieving.

Appendix IV: t-Statistics

The Pooled Two-Sample t-Procedure is

$$t \;=\; (\overline{X_1} - \overline{X_2}) \Big/ s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},$$

where $\overline{X_i}$ is the sample mean of n_i observations of population i, s_i is the sample standard deviation and

$$s_p \;=\; \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

is the pooled variance; t has a t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

The Unpooled Two-Sample t-Procedure is

$$t \;=\; (\overline{X_1} - \overline{X_2}) \Big/ \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}};$$

this is approximately a t distribution with

$$\frac{(n_1 - 1)\;(n_2 - 1)\;(n_2 s_1^2 + n_1 s_2^2)^2}{(n_2 - 1)n_2^2 s_1^4 + (n_1 - 1)n_1^2 s_2^4}$$

degrees of freedom

Bibliography Caveat

Warning: this bibliography hasn't been updated for a few years, and could be a little out of date. It is meant to serve as a first reference. Please email additional references to sjmiller@math.brown.edu.

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