Can the change in zero statistics going from interaction (for small values of the parameter $T$) to independence (as $T \to \infty$) be modeled using random matrices?

Consider orthogonal matrices having a certain number $\rho$ of eigenvalues exactly lying at the central point $+1$.

$\rho$ plays the role of a "repulsion parameter" closely related to the rank.
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The joint PDF of $N$ pairs of eigenvalues $\{e^{i\theta_j}\}_{1 \leq j \leq N}$, taken from random orthogonal matrices having other $\rho$ fixed eigenvalues at +1 is

$$d\varepsilon_\rho(\theta_1, \ldots, \theta_N) = C_{N,\rho} \prod_{j<k} (\cos \theta_k - \cos \theta_j)^2 \prod_j (1 - \cos \theta_j)^\rho \ d\theta_j.$$ 

This probability measure is well defined for $\rho \in (-\frac{1}{2}, \infty)$. 
The Repulsion Parameter $\rho$

For simplicity, assume that $\mathcal{E}$ is an even orthogonal family depending on a parameter $T \to \infty$.

- The repulsion parameter $\rho = \rho_\mathcal{E}(T)$ will monotonically decrease from an initial maximum value $\rho_\mathcal{E}(0)$ to a minimum value $\lim_{T \to \infty} \rho_\mathcal{E}(T) = 0$ (resp., $\lim_{T \to \infty} \rho_\mathcal{E}(T) = 1$ if $\mathcal{E}$ is an odd orthogonal family.)
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- By making $\rho$ vary with $T$, the statistics of eigenvalues in this model match several of the theoretical and experimental features observed in the critical zeros of $\mathcal{E}$:
  - Repulsion of eigenvalues away from central point when $\rho > 0$. (The larger $\rho$, the more repulsion.)
  - Independent model statistics when $\rho = 0$.
  - Basically unchanged non-central spacings.
The standard normalization $x = \frac{N\theta}{\pi}$ makes the eigen-angles $\theta_j$ into unit-spaced (on average) “levels” $x_j$.

In terms of the $x$-variable, the limiting 1-level density is given by

$$
D_1^{(\rho)}(x) = \rho \delta_0(x) + \pi \left( \frac{\pi x}{2} \left[ J_{\rho+\frac{1}{2}}(\pi x)^2 + J_{\rho-\frac{1}{2}}(\pi x)^2 \right] - (\rho - \frac{1}{2}) J_{\rho+\frac{1}{2}}(\pi x) J_{\rho-\frac{1}{2}}(\pi x) \right).
$$
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

Figure: 1-level density for the ensemble with $\rho = 24/12$. 
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

Figure: 1-level density for the ensemble with $\rho = \frac{23}{12}$. 

$\rho = 1.9167$
Figure: 1-level density for the ensemble with $\rho = 22/12$. 

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

$$r^* = 1.7500$$

**Figure:** 1-level density for the ensemble with $\rho = \frac{21}{12}$.
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

$r^* = 1.6667$

Figure: 1-level density for the ensemble with $\rho = 20/12$. 

E. Dueñez, D. K. Huynh, S. J. Miller (UTSA, Modeling Decreasing Repulsion in Families)
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

$$r^* = 1.5833$$

Figure: 1-level density for the ensemble with $\rho = \frac{19}{12}$. 

E. Dueñez, D. K. Huynh, S. J. Miller (UTSA, Modeling Decreasing Repulsion in Families)
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

Figure: 1-level density for the ensemble with $\rho = 18/12$. E. Dueñez, D. K. Huynh, S. J. Miller (UTSA, Modeling Decreasing Repulsion in Families)
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

$r^* = 1.4167$

Figure: 1-level density for the ensemble with $\rho = \frac{17}{12}$. E. Dueñez, D. K. Huynh, S. J. Miller (UTSA, Modeling Decreasing Repulsion in Families)
Animated Example of Decreasing repulsion: \( 2 \geq \rho \geq 0 \)

Figure: 1-level density for the ensemble with \( \rho \approx 16/12 \).
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

Figure: 1-level density for the ensemble with $\rho = 15/12$. 

E. Dueñez, D. K. Huynh, S. J. Miller (UTSA, Modeling Decreasing Repulsion in Families)
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

Figure: 1-level density for the ensemble with $\rho = \frac{14}{12}$. E. Dueñez, D. K. Huynh, S. J. Miller (UTSA, Modeling Decreasing Repulsion in Families)
Figure: 1-level density for the ensemble with $\rho = \frac{13}{12}$.
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$
Figure: 1-level density for the ensemble with $\rho = 11/12$. 

E. Dueñez, D. K. Huynh, S. J. Miller (UTSA, Modeling Decreasing Repulsion in Families)
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

Figure: 1-level density for the ensemble with $\rho = \frac{10}{12}$.
Figure: 1-level density for the ensemble with $\rho = 9/12$. 

$\rho \geq 0$
Figure: 1-level density for the ensemble with $\rho = 8/12$. 

$\rho \geq 0$
Figure: 1-level density for the ensemble with $\rho = \frac{7}{12}$. 

$\rho \geq 0$
Animated Example of Decreasing repulsion: \( 2 \geq \rho \geq 0 \)

\[ r^* = 0.50000 \]

**Figure:** 1-level density for the ensemble with \( \rho = 6/12 \).
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

Figure: 1-level density for the ensemble with $\rho = 5/12$. 

$r^* = 0.41667$
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

Figure: 1-level density for the ensemble with $\rho = 4/12$. 
Figure: 1-level density for the ensemble with $\rho^\ast = 3/12$. 

$\rho \geq r^\ast \geq 0$
Figure: 1-level density for the ensemble with $\rho = 2/12$. 

$\rho \geq 0$
An animated example of decreasing repulsion: \( 2 \geq \rho \geq 0 \)

Figure: 1-level density for the ensemble with \( \rho = \frac{1}{12} \).
Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

Figure: 1-level density for the ensemble with $\rho = 0/12$. 

E. Dueñez, D. K. Huynh, S. J. Miller  (UTSA, Modeling Decreasing Repulsion in Families)
The Effect of the Parameter $\rho$

- As $\rho$ varies from $\rho(0)$ to 0 the “central repulsion” decreases and, at $r = 0$, it disappears completely.
- Any $\rho > 0$ merely tends to shift all the eigenvalues to the right: they are pushed away, but the relative spacings between them are basically unchanged.
What is the Function $\rho_{\mathcal{E}}(T)$?

First issue: What should $\rho_{\mathcal{E}}(0)$ be?
- Choice #1: Take $\rho_{\mathcal{E}}(0)$ equal to the geometric rank of a family $\mathcal{E}$ over $\mathbb{Q}(T)$.

Second issue: How does $\rho_{\mathcal{E}}(T)$ vary with $T$?
$r$ should probably go to zero inversely with the log conductors of curves in $\mathcal{E}$. Best bet so far: $\rho_{\mathcal{E}}(T) = \langle r \rangle_{\mathcal{E}}(T) \log T + 1$ if odd family.
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  - Best bet so far:

$$
\rho_\mathcal{E}(T) = \frac{\langle r \rangle_{\mathcal{E}(T)}}{\log T} \quad (+1 \text{ if odd family}).
$$