Modeling the Decrease in Repulsion

- Can the change in zero statistics going from interaction (for small values of the parameter T) to independence (as $T \to \infty$) be modeled using random matrices?
  - Consider orthogonal matrices having a certain number ρ of eigenvalues exactly lying at the central point $+1$.
  - ρ plays the role of a "repulsion parameter" closely related to the rank.

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- Can the change in zero statistics going from interaction (for small values of the parameter T) to independence (as $T \to \infty$) be modeled using random matrices?
  - Consider orthogonal matrices having a certain number ρ of eigenvalues exactly lying at the central point $+1$.
  - ρ plays the role of a "repulsion parameter" closely related to the rank.
- The joint PDF of N pairs of eigenvalues $\{e^{i\theta_j}\}_{1 \le j \le N}$, taken from random orthogonal matrices having other ρ fixed eigenvalues at $+1$ is

$$d\varepsilon_\rho(\theta_1, \ldots, \theta_N) = C_{N,\rho} \prod_{j<k} (\cos\theta_k - \cos\theta_j)^2 \prod_j (1 - \cos\theta_j)^\rho \, d\theta_j.$$

  - This probability measure is well defined for $\rho \in (-\frac{1}{2}, \infty)$.

The Repulsion Parameter ρ

For simplicity, assume that $\mathcal{E}$ is an even orthogonal family depending on a parameter $T \to \infty$.

- The repulsion parameter $\rho = \rho_{\mathcal{E}}(T)$ will monotonically decrease from an initial maximum value $\rho_{\mathcal{E}}(0)$ to a minimum value $\lim_{T\to\infty} \rho_{\mathcal{E}}(T) = 0$ (resp., $\lim_{T\to\infty} \rho_{\mathcal{E}}(T) = 1$ if $\mathcal{E}$ is an odd orthogonal family.)

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- By making ρ vary with T, the statistics of eigenvalues in this model match several of the theoretical and experimental features observed in the critical zeros of $\mathcal{E}$:
  - Repulsion of eigenvalues away from central point when $\rho > 0$. (The larger ρ, the more repulsion.)
  - Independent model statistics when $\rho = 0$.
  - Basically unchanged non-central spacings.

1-Level Density as a Function of ρ

- The standard normalization $x = \frac{N\theta}{\pi}$ makes the eigen-angles θ_j into unit-spaced (on average) “levels” x_j.
- In terms of the x-variable, the limiting 1-level density is given by

$$D_1^{(\rho)}(x) = \rho\delta_0(x) + \pi\left(\frac{\pi x}{2}\left[J_{\rho+\frac{1}{2}}(\pi x)^2 + J_{\rho-\frac{1}{2}}(\pi x)^2\right] - \left(\rho - \tfrac{1}{2}\right) J_{\rho+\frac{1}{2}}(\pi x) J_{\rho-\frac{1}{2}}(\pi x)\right).$$

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

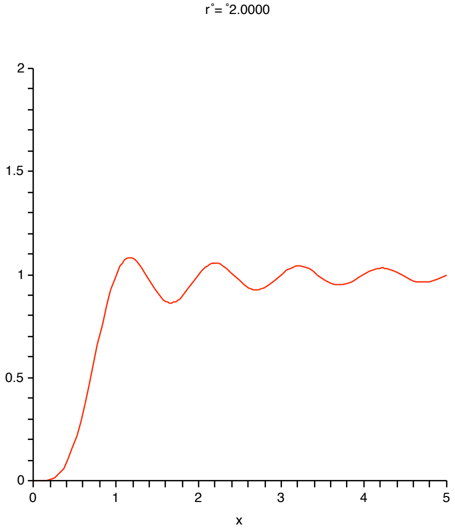

Figure: 1-level density for the ensemble with $\rho = 24/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

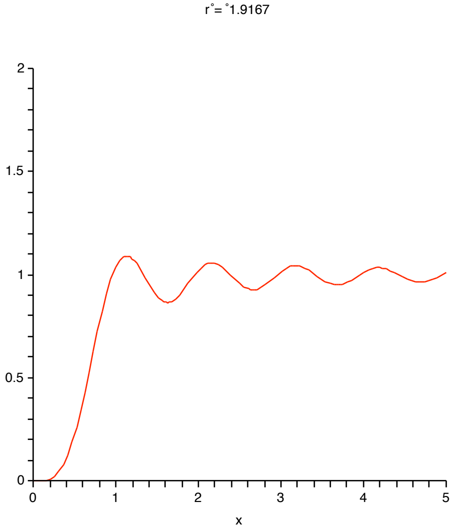

Figure: 1-level density for the ensemble with $\rho = 23/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

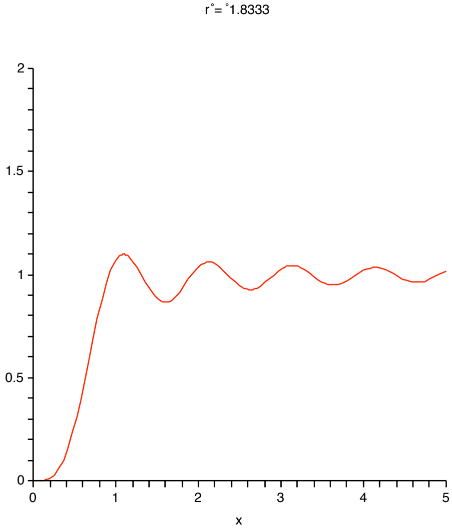

Figure: 1-level density for the ensemble with $\rho = 22/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

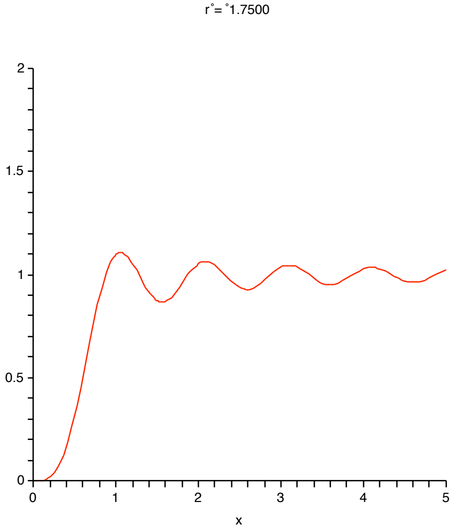

Figure: 1-level density for the ensemble with $\rho = 21/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

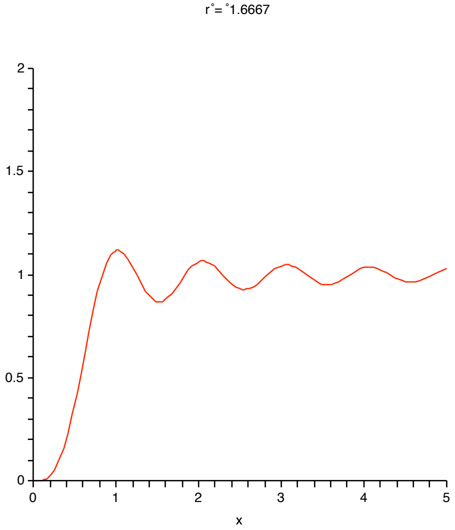

Figure: 1-level density for the ensemble with $\rho = 20/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

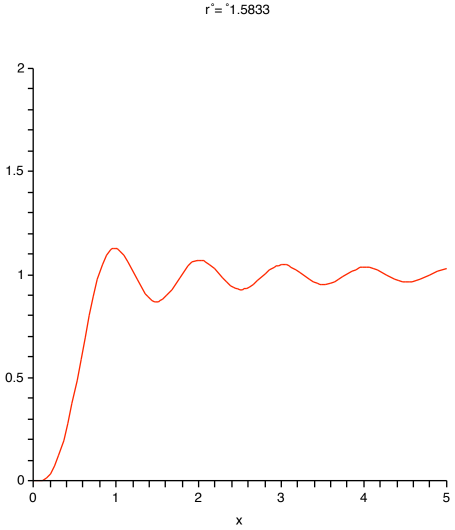

Figure: 1-level density for the ensemble with $\rho = 19/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

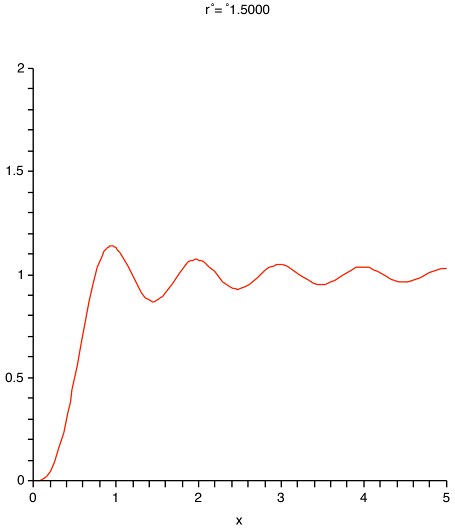

Figure: 1-level density for the ensemble with $\rho = 18/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

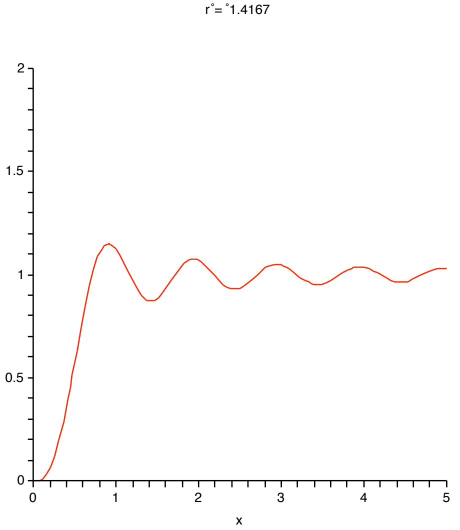

Figure: 1-level density for the ensemble with $\rho = 17/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

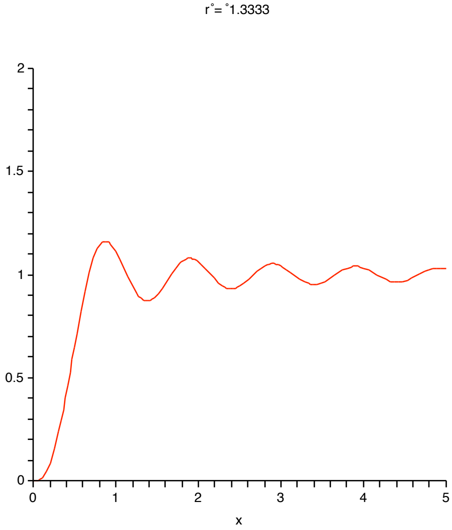

Figure: 1-level density for the ensemble with $\rho = 16/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

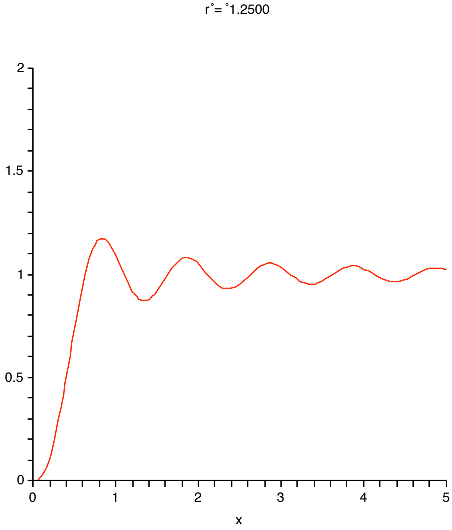

Figure: 1-level density for the ensemble with $\rho = 15/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

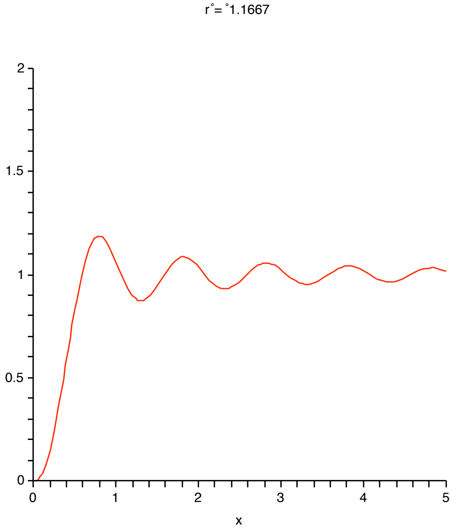

Figure: 1-level density for the ensemble with $\rho = 14/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

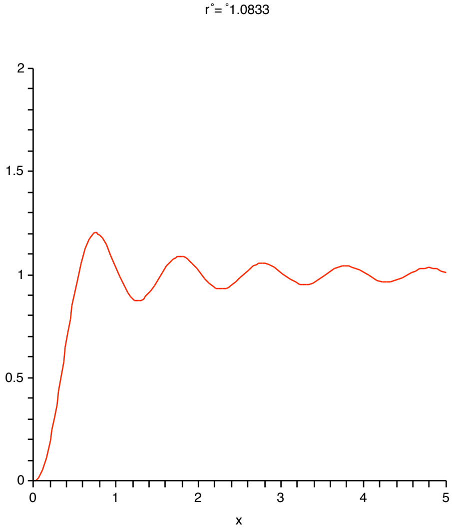

Figure: 1-level density for the ensemble with $\rho = 13/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

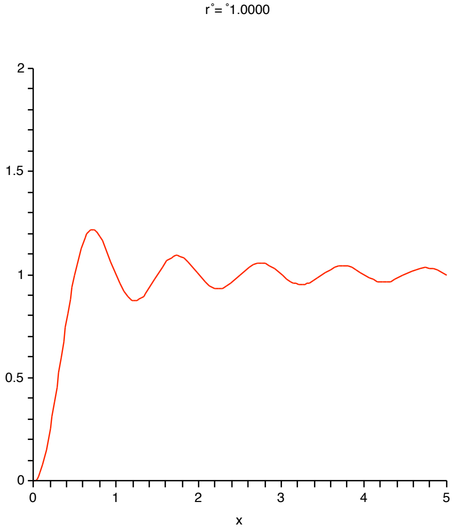

Figure: 1-level density for the ensemble with $\rho = 12/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

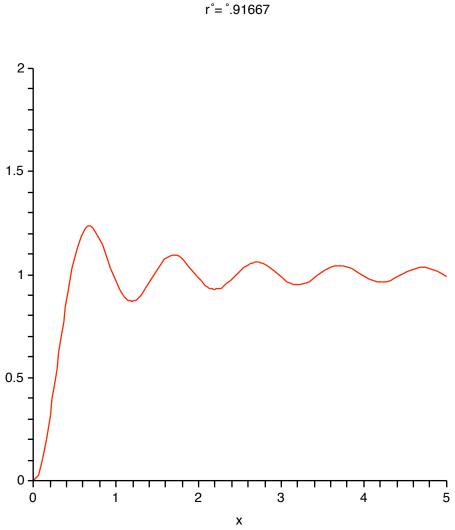

Figure: 1-level density for the ensemble with $\rho = 11/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

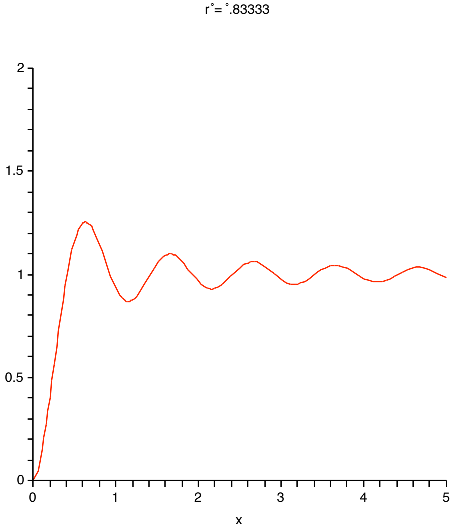

Figure: 1-level density for the ensemble with $\rho = 10/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

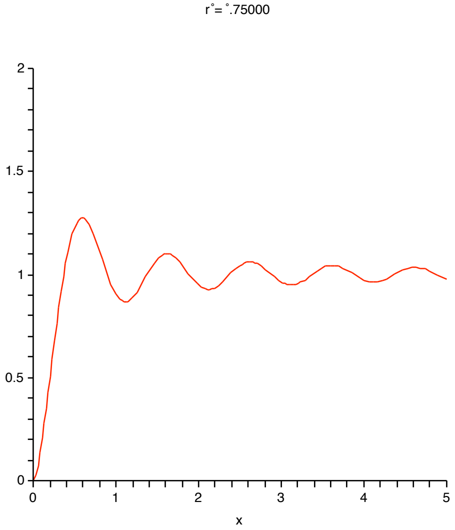

Figure: 1-level density for the ensemble with $\rho = 9/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

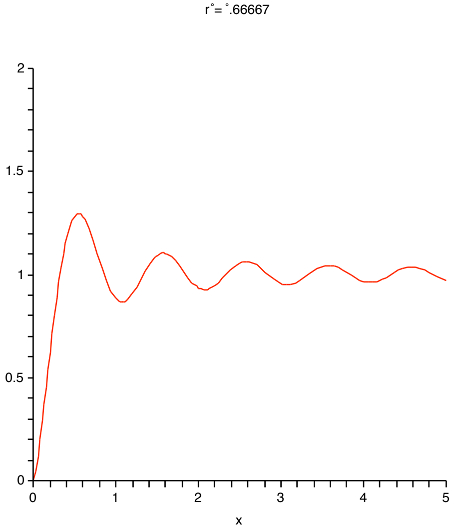

Figure: 1-level density for the ensemble with $\rho = 8/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

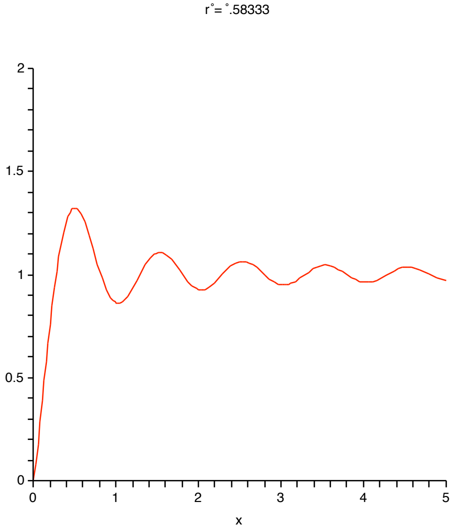

Figure: 1-level density for the ensemble with $\rho = 7/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

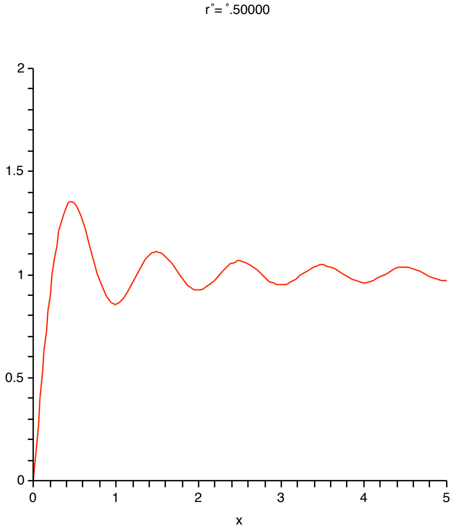

Figure: 1-level density for the ensemble with $\rho = 6/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

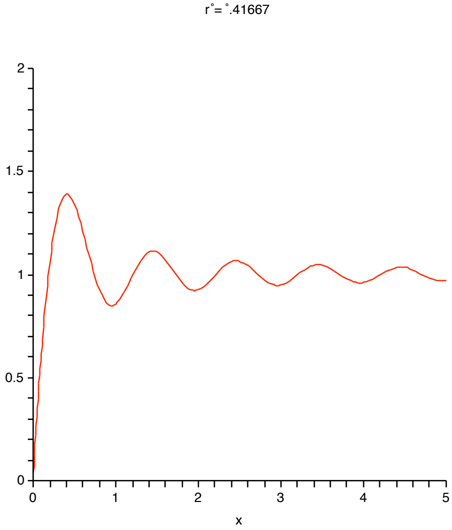

Figure: 1-level density for the ensemble with $\rho = 5/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

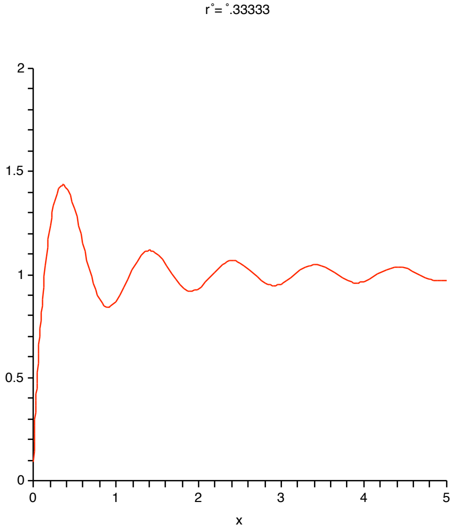

Figure: 1-level density for the ensemble with $\rho = 4/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

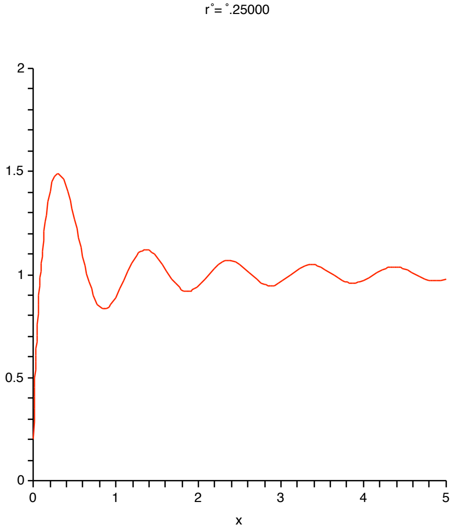

Figure: 1-level density for the ensemble with $\rho = 3/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

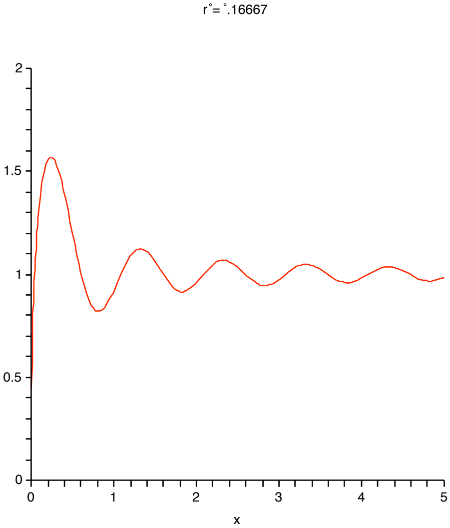

Figure: 1-level density for the ensemble with $\rho = 2/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

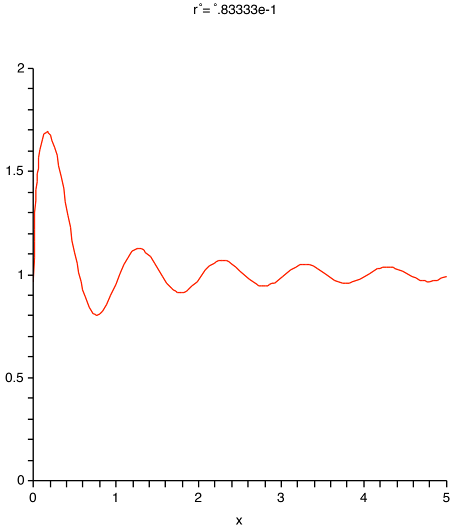

Figure: 1-level density for the ensemble with $\rho = 1/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

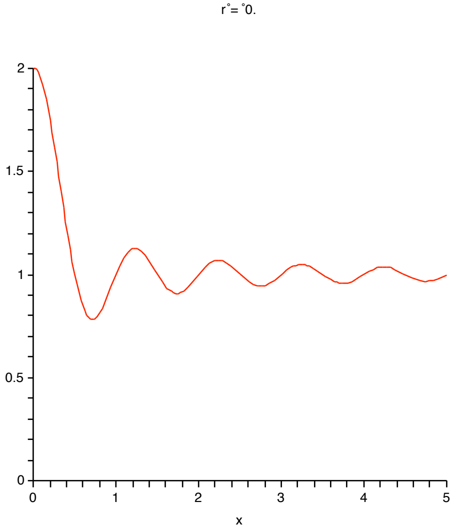

Figure: 1-level density for the ensemble with $\rho = 0/12$.

The Effect of the Parameter ρ

- As ρ varies from $\rho(0)$ to 0 the "central repulsion" decreases and, at $r = 0$, it disappears completely.
- Any $\rho > 0$ merely tends to shift all the eigenvalues to the right: they are pushed away, but the relative spacings between them are basically unchanged.

What is the Function $\rho_{\mathcal{E}}(T)$?

- First issue: What should $\rho_{\mathcal{E}}(0)$ be?
  - Choice #1: Take $\rho_{\mathcal{E}}(0)$ equal to the geometric rank of a family $\mathcal{E}$ over $\mathbb{Q}(T)$.

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- Second issue: How does $\rho_{\mathcal{E}}(T)$ vary with T?
  - r should probably go to zero inversely with the log conductors of curves in $\mathcal{E}$.
    - Best bet so far:

$$\rho_{\mathcal{E}}(T) = \frac{\langle r \rangle_{\mathcal{E}(T)}}{\log T} \qquad (+1 \text{ if odd family.})$$