- Katz/Sarnak philosophy:
  In the limit as the conductor tends to ∞

  $$\text{zero statistics for families of L-functions}$$
  $$\text{``}=\text{''}$$
  $$\text{eigenvalue statistics of one of the classical compact groups.}$$

- However for finite conductor we observe repulsion of low lying zeros from the critical point. This behaviour is not captured in the models coming from the classical compact groups. We believe that a study of lower order terms will help to understand this phenomenon.

- We investigate the family quadratic twists coming from an elliptic curve L-function (and for simplicity we consider those with even functional equation) because
  - Experimental data from Mike Rubinstein's lcalc.
  - Lower order terms for this family from the ratios conjectures.

  So we can compare theory and data.

- The ratios conjectures (Conrey, Farmer, Zirnbauer) give good estimates for quantities like

$$\sum_{0<d\leq X} \frac{\prod_{k=1}^{K} L(1/2+\alpha_k, \chi_d)}{\prod_{q=1}^{Q} L(1/2+\gamma_q, \chi_d)}.$$

  Simpliest case when $K = Q = 1$.

- For our family we are interested in the following ratio with $\Re(\alpha), \Re(\gamma) > 0$

$$\sum_{d \le X} \frac{L_E(1/2+\alpha, \chi_d)}{L_E(1/2+\gamma, \chi_d)} =: R_E(\alpha, \gamma)$$

and observe that

$$\sum_{d \le X} \frac{L'_E(1/2+r, \chi_d)}{L_E(1/2+r, \chi_d)} = \frac{d}{d\alpha} R_E(\alpha, \gamma)\Big|_{\alpha=\gamma=r}.$$

- For the rest of the talk we focus on the 1-level-density:

$$D_1(\varphi) = \frac{1}{X^*} \sum_{d \le X} \sum_{\gamma_d} \varphi(\gamma_d)$$

where φ is a suitable test function, say an even Schwartz-function and γ_d the ordinate of a generic zero of $L_E(s, \chi_d)$ on the critical line.

By the argument principle we can write

$$D_1(\varphi) = \frac{1}{X^*} \sum_{d \le X} \frac{1}{2\pi i} \left(\int_{(c)} - \int_{(1-c)} \right) \frac{L'(s,\chi_d)}{L(s,\chi_d)} \varphi(-i(s-1/2))ds$$

where $3/4 > c > 1/2 + 1/\log X$.

Hence: If we have a conjecture for

$$\sum_{d \le X} \frac{L_E'(s,\chi_d)}{L_E(s,\chi_d)} \tag{1}$$

we can also give a conjectural answer for $D_1(\varphi)$.

Using the ratios conjectures we get an estimate for (1).

- From the ratios conjecture we get for the 1-level-density

$$D_1(\varphi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t) \frac{1}{X^*}\sum_{d\le X}\Bigg(2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right)$$

$$+\frac{\Gamma'}{\Gamma}(1/2+it) + \frac{\Gamma'}{\Gamma}(1/2-it)$$

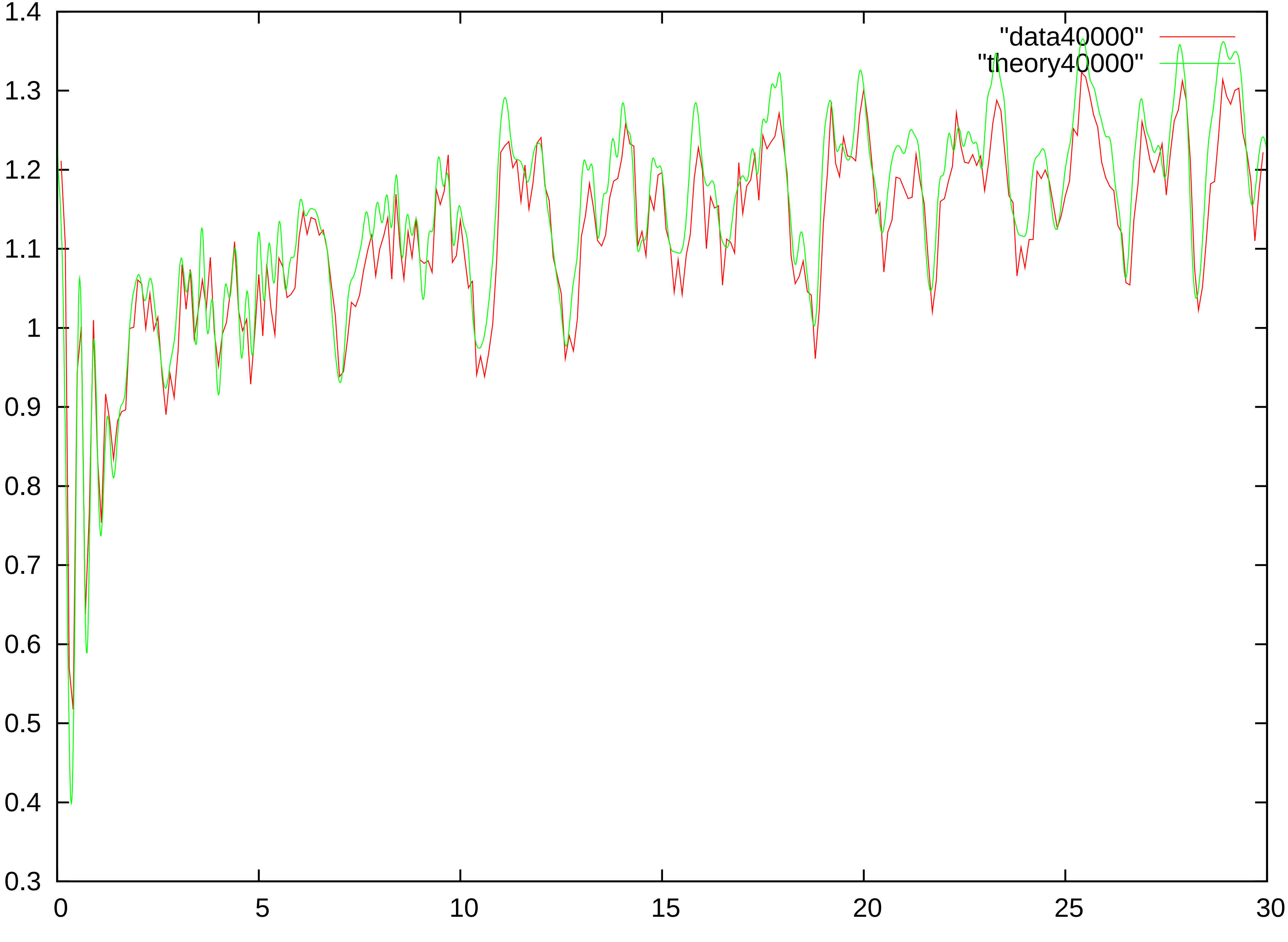
$$+2\Big[-\frac{\zeta'(1+2it)}{\zeta(1+2it)} + \frac{L'_E(\mathrm{sym}^2, 1+2it)}{L_E(\mathrm{sym}^2, 1+2it)} + A'_E(it,it)$$

$$-\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it}\frac{\Gamma(1/2-it)}{\Gamma(1/2+it)}\frac{\zeta(1+2it)L_E(\mathrm{sym}^2,1-2it)}{L_E(\mathrm{sym}^2,1)}A_E(-it,it)\Big]$$

$$+O(X^{-1/2+\varepsilon})$$

where M is the conductor of the elliptic curve E and A_E is a product over primes. We note that

– the ratios conjecture give all terms down to $O(X^{-1/2+\varepsilon})$ which is a very precise prediction.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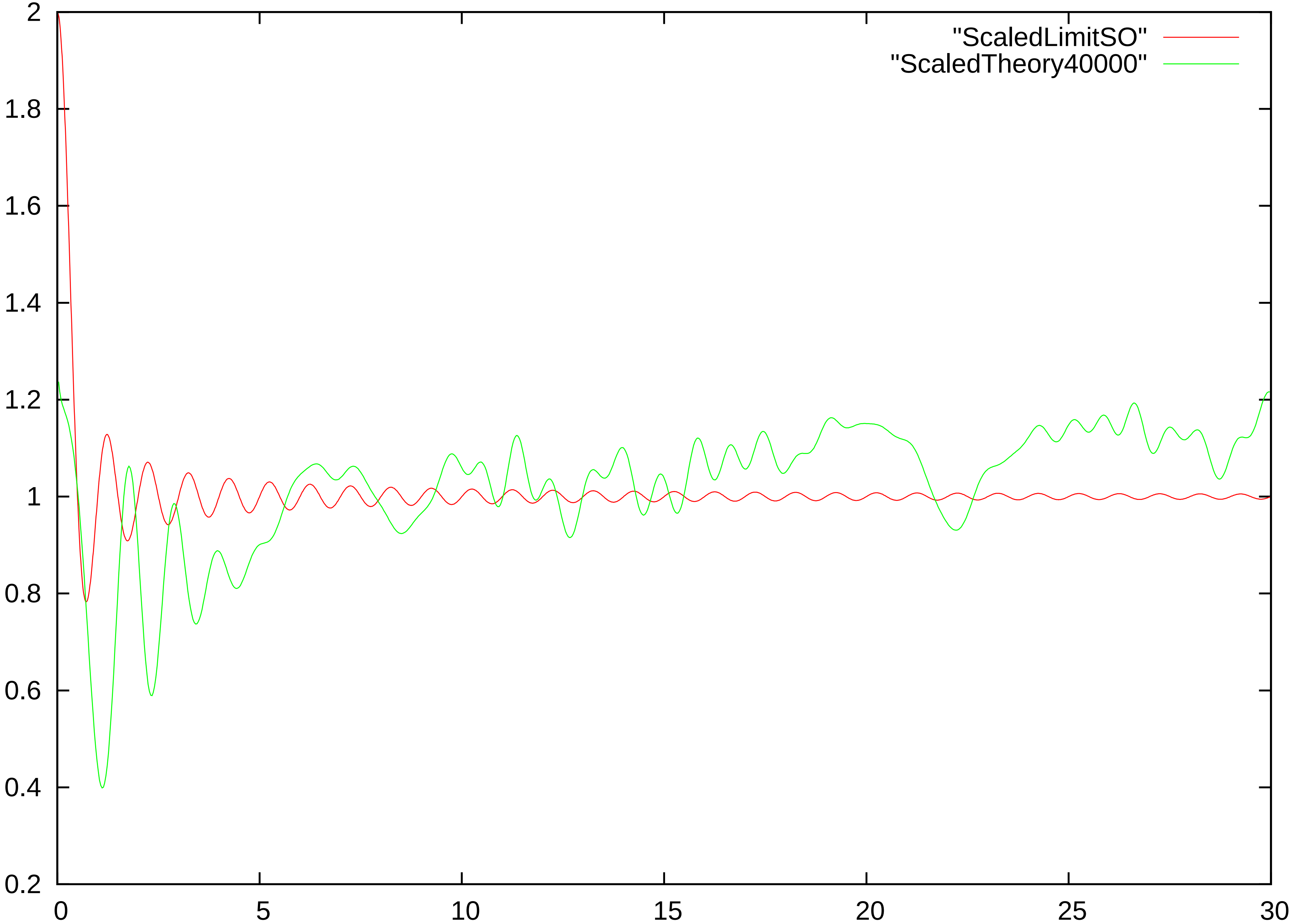

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