Benford's Law under Zeckendorf expansion

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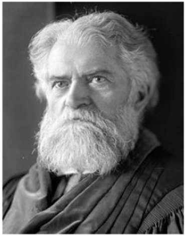
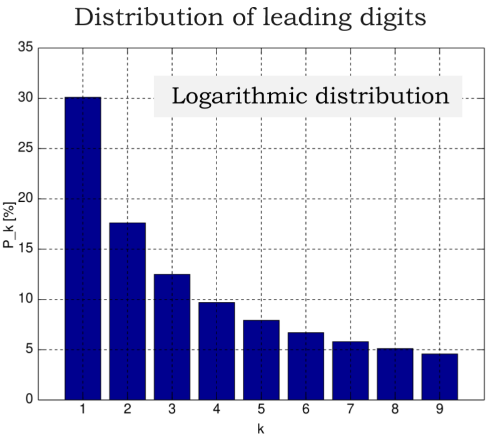
Contents

The page numbers are available down there for later reference.

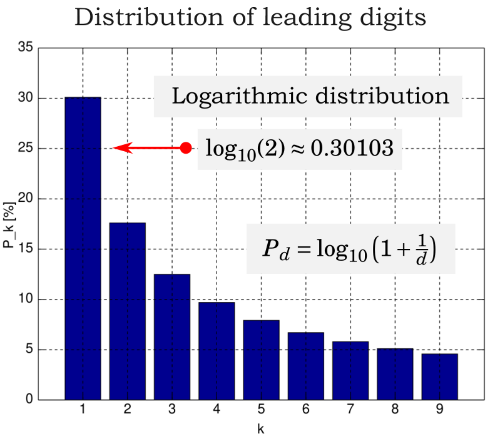
Newcomb-Benford Law

Simon Newcomb	Frank Benford Jr.
	
1835-1909	1883-1948

Newcomb-Benford Law



Newcomb-Benford Law



Notion

Definition: Given a conditional statement $P(n)$ where $n \in \mathbb{N}$, let us define

$$\mathrm{Prob}\left\{ n \in \mathbb{N} : P(n) \right\} := \lim_{n\to\infty} \frac{\#\{k \in \mathbb{N} : P(k),\ k \leq n\}}{n}.$$

Definition:

- Let $\mathrm{LB}_s(n)$ be the leading s digits (or blocks) of n in given expansion.
- Let $\mathrm{TB}_s(n)$ be the last (or terminal) s digits (or blocks) of n in given expansion.

Example: Let $a \in \mathbb{N}$ and $d \in \{1, 2, \ldots, 9\}$ for dec. expansion.

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$$P_a(d) := \text{Prob}\left\{ n \in \mathbb{N} : \text{TB}_1(n^a) = d \right\} \text{ exists.}$$

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$$P_a(d) := \text{Prob}\left\{ n \in \mathbb{N} : \text{TB}_1(n^a) = d \right\} \text{ exists.}$$

```
In[•]:= Table[Mod[n^2, 10], {n, 1, 30}]

Out[•]= {1, 4, 9, 6, 5, 6, 9, 4, 1, 0, 1, 4, 9,
  6, 5, 6, 9, 4, 1, 0, 1, 4, 9, 6, 5, 6, 9, 4, 1, 0}
```

$$P_1(d) = 1/10,$$

$$P_2(0) = 1/10,\ P_2(1) = 1/5,\ P_2(4) = 1/5,\ P_2(5) = 1/10,$$

$$P_2(6) = 1/5,\ P_2(9) = 1/5,\ P_2(d) = 0,\ d \in \{2, 3, 7, 8\}$$

Example: Let $a \in \mathbb{N}$ and $d \in \{1, 2, \ldots, 9\}$ for dec. expansion.

$$P_a(d) := \text{Prob}\left\{ n \in \mathbb{N} : \text{LB}_1(n^a) = d \right\} \text{ does not exist.}$$

$$\text{Demo, } a = 1 : \quad 10000, \ldots, 20000, \ldots, 99999$$

$$\text{Prob}\left\{ n \in [1, 2 \cdot 10^M) : \text{LB}_1(n) = 1 \right\} = \frac{5}{9} + o(1),$$

$$\text{Prob}\left\{ n \in [1, 10^{M+1}) : \text{LB}_1(n) = 1 \right\} = \frac{1}{9} + o(1)$$

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$$\text{Prob}\left\{ n \in [1, 10^{M+1}) : \text{LB}_1(n) = 1 \right\} = \frac{1}{9} + o(1)$$

The existence of the leading digit probability is a special property a sequence can have.

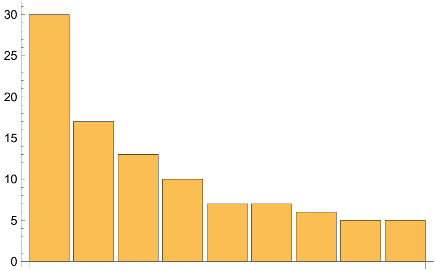
Example: Let $d \in \{1, 2, \ldots, 9\}$ for dec. expansion.

$$\text{Prob}\left\{ n \in \mathbb{N} : \text{LB}_1(2^n) = d \right\} \text{ exists.}$$

```
In[∘]:= list = Table[LD[2^n], {n, 1, 100}]
        FrqList = Table[Count[list, d], {d, 1, 9}];
        BarChart[%]
```

```
Out[∘]= {2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1,
         2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1,
         2, 4, 8, 1, 3, 7, 1, 2, 5, 1, 2, 4, 9, 1, 3, 7, 1, 2, 5, 1,
         2, 4, 9, 1, 3, 7, 1, 2, 5, 1, 2, 4, 9, 1, 3, 7, 1, 3, 6, 1,
         2, 4, 9, 1, 3, 7, 1, 3, 6, 1, 2, 4, 9, 1, 3, 7, 1, 3, 6, 1}
```

Out[∘]=



Example: Let $d \in \{1, 2, \ldots, 9\}$ for dec. expansion.

$$\text{Prob}\left\{ n \in \mathbb{N} : \text{LB}_1(2^n) = d \right\} = \log_{10} \frac{d+1}{d}.$$

$$\sum_{d=1}^{9} \log_{10} \frac{d+1}{d} = 1.$$

$$\text{Prob}\left\{ n \in \mathbb{N} : \text{LB}_1(2^n) = d \right\} = \log_{10} \frac{d+1}{d}$$

Proof: Suppose 2^n has the leading digit d.

$$d10^\ell \le 2^n < (d+1)10^\ell, \quad \ell \in \mathbb{N}$$

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$$d10^\ell \le 2^n < (d+1)10^\ell, \quad \ell \in \mathbb{N}$$

$$\Rightarrow \log_{10} d10^\ell \le n \log_{10} 2 < \log_{10}(d+1)10^\ell$$

$$\Rightarrow \ell + \log_{10} d \le n \log_{10} 2 < \ell + \log_{10}(d+1).$$

$$\Rightarrow$$

$$\mathrm{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_1(2^n) = d \right\} = \log_{10} \frac{d+1}{d}$$

Proof: Suppose 2^n has the leading digit d.

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$$\mathrm{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_1(2^n) = d \right\} = \log_{10} \frac{d+1}{d}$$

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$$\Rightarrow \log_{10} d \le \mathrm{frc}\left(n \log_{10} 2\right) < \log_{10}(d+1)$$

$$\mathrm{Prob}\left\{ n \in \mathbb{N} : a \le \mathrm{frc}\left(n \log_{10} 2\right) < b \right\} = ?$$

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$$\mathrm{Prob}\left\{ n \in \mathbb{N} : a \le \mathrm{frc}\left(n \log_{10} 2\right) < b \right\} = ?$$

Weyl's Equidistribution Theorem: *If α is irrational, then*

$$\mathrm{Prob}\left\{ n \in \mathbb{N} : \mathrm{frc}(n\alpha) < \beta \right\} = \beta.$$

Proof: Suppose 2^n has the leading digit d.

$$d10^\ell \leq 2^n < (d+1)10^\ell, \quad \ell \in \mathbb{N}$$

$$\Rightarrow \log_{10} d \leq \operatorname{frc}\left(n \log_{10} 2\right) < \log_{10}(d+1)$$

Proof: Suppose 2^n has the leading digit d.

$$d10^\ell \le 2^n < (d+1)10^\ell, \quad \ell \in \mathbb{N}$$

$$\Rightarrow \log_{10} d \le \operatorname{frc}\left(n \log_{10} 2\right) < \log_{10}(d+1)$$

$$\operatorname{Prob}\left\{ n \in \mathbb{N} : a \le \operatorname{frc}\left(n \log_{10} 2\right) < b \right\}$$

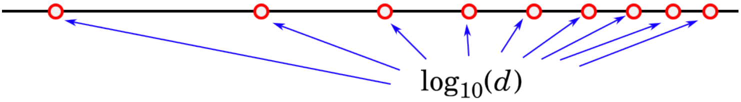
$$= \log_{10}(d+1) - \log_{10} d.$$

Let K_n be a sequence of positive integers.

Last Digits: $K_n \pmod{10}$

Leading Digits: $\log_{10}(K_n) \pmod{1}$

Def: For real numbers x, y, z where $z \neq 0$, $x \equiv y \pmod{z}$ if $x - y = zm$ for $m \in \mathbb{Z}$.



$$\mathrm{frc}\left(\log_{10}(20240712)\right) = \log_{10}(\qquad\qquad)$$

Let K_n be a sequence of positive integers.

Last Digits: $K_n \pmod{10}$

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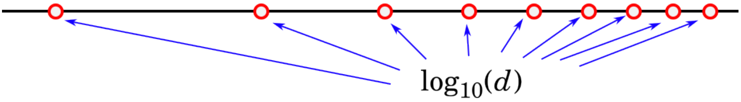
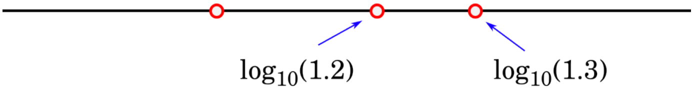


$$\operatorname{frc}\left(\log_{10}(20240712)\right) = \log_{10}(2.0240712)$$

Example: $\mathrm{LB}_2(2^n) = (1,2)$.

$$\log_{10}(1.2) \le \mathrm{frc}\left(\log_{10} 2^n\right) < \log_{10}(1.3)$$



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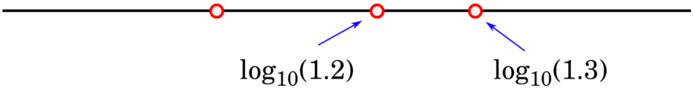

$$\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_2(2^n) = (1,2) \right\} = \log_{10}(1.3) - \log_{10}(1.2)$$

$$= \log_{10} \frac{1.3}{1.2} = \log_{10} \frac{13}{12} \approx 0.0347$$

Example: $\mathrm{LB}_3(2^n) = (2,5,9)$.

$$\mathrm{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_2(2^n) = (2,5,9) \right\} = \log_{10} \frac{2.60}{2.59} \approx 0.00167358$$

Strong Benford's Law

A sequence K_n *satisfies strong Benford's Law* if

$$\text{Prob}\left\{ n \in \mathbb{N} : \text{LB}_s(K_n) = (d_1, \ldots, d_s) \right\} = \log_{10} \frac{\sum_{k=1}^{s} d_k 10^{s-k} + 1}{\sum_{k=1}^{s} d_k 10^{s-k}}.$$

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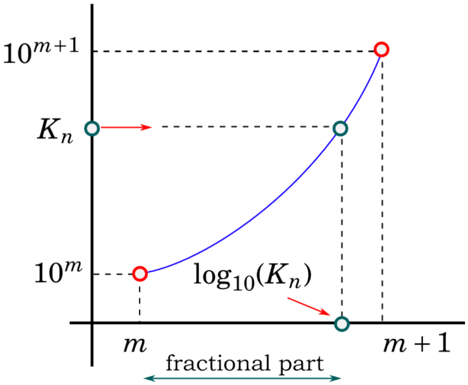
The sequence a^n where $\log_{10} a$ is irrational satisfies strong Benford's Law.

Strong Benford's Law



Zeckendorf expansions

- By Zeckendorf's theorem, each positive integer n has a unique Zeckendorf expansion, i.e.,

$$n = \sum_{k=1}^{m} a_k F_{m-k+1} = a_1 F_m + a_2 F_{m-1} + \cdots + a_m F_1$$

where $a_k \in A := \{0,1\}$, $a_1 = 1$, and $a_k a_{k+1} = 0$ for all $k \in \mathbb{N}$.

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- The following collection is lexicographically ordered:

$$\mathscr{F}_\circ := \bigcup_{m=1}^{\infty} \{(a_1,\dots,a_m) \in A^m : a_1 = 1,\ a_k a_{k+1} = 0\ \forall k\}.$$

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- The following collection is lexicographically ordered:

$$\mathscr{F}_\circ := \bigcup_{m=1}^{\infty} \{(a_1,\ldots,a_m) \in A^m : a_1 = 1,\ a_k a_{k+1} = 0\ \forall k\}.$$

- The function $\mathrm{eval} : \mathscr{F}_\circ \to \mathbb{N}$ given below is an increasing bijection:

$$(a_1,\ldots,a_m) \mapsto \sum_{k=1}^{m} a_k F_{m-k+1}.$$

Zeckendorf expansions

- Let $(a_1, \ldots, a_m) = \mathrm{eval}^{-1}(n)$. If $m \geq s$, define $\mathrm{LB}_s(n) := (a_1, \ldots, a_s)$, called *the leading block of n with length s under Zeckendorf expansion*.

Zeckendorf expansions

- Let $(a_1,\ldots,a_m)=\mathrm{eval}^{-1}(n)$. If $m\geq s$, define $\mathrm{LB}_s(n):=(a_1,\ldots,a_s)$, called *the leading block of n with length s under Zeckendorf expansion*.
- If the following probabilities are common for all integers $a\geq 2$:

$$\mathrm{Prob}\left\{n\in\mathbb{N}:\mathrm{LB}_3(a^n)=(1,0,0)\right\},\ \mathrm{Prob}\left\{n\in\mathbb{N}:\mathrm{LB}_3(a^n)=(1,0,1)\right\},$$

then we declare it to be *Benford's Law under Zeckendorf expansion*.

Theorem: *Let $a > 1$ be an integer.*

$$\text{Prob}\left\{ n \in \mathbb{N} : \text{LB}_3(a^n) = (1,0,0) \right\} = \log_\phi(1+\omega^2) \approx .672,$$

$$\text{Prob}\left\{ n \in \mathbb{N} : \text{LB}_3(a^n) = (1,0,1) \right\} = \log_\phi \frac{\phi}{1+\omega^2} \approx .328$$

where $\omega := 1/\phi$.

Strong Benford's Law under Zeckendorf expansion

Theorem: *Let $a > 1$ be an integer.*

$$\begin{aligned} \mathrm{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_6(a^n) = (1,0,0,0,1,0) \right\} &= \log_\phi \frac{1+\omega^3}{1+\omega^4} \approx 0.157 \\ \mathrm{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_6(a^n) = (1,0,1,0,1,0) \right\} &= \log_\phi \frac{\phi}{1+\omega^2+\omega^4} \approx 0.119. \end{aligned}$$

- $(1,0,0,0,1,0) \mapsto 1 + 0\omega^1 + 0\omega^2 + 0\omega^3 + 0\omega^4,$

  $\mathrm{lub}_{\mathscr{F}_\circ}(1,0,0,0,1,0) = (1,0,0,1,0,0),$

- $(1,0,1,0,1,0) \mapsto 1 + 0\omega^1 + \omega^2 + 0\omega^3 + \omega^4,$

  $\mathrm{lub}_{\mathscr{F}_\circ}(1,0,1,0,1,0) = (1,0,0,0,0,0,0)$

Strong Benford's Law under Zeckendorf expansion

Theorem:

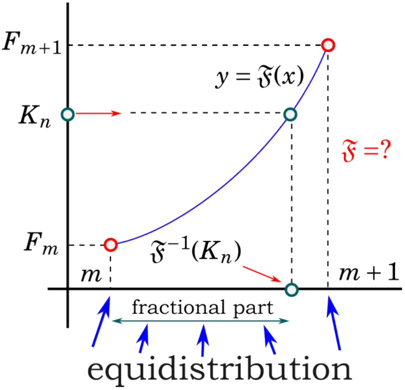
Let $a > 1$ *be an integer,* $\mathbf{b} = (b_1, \dots, b_s) \in \mathscr{F}_\circ$*, and* $\mathrm{lub}_{\mathscr{F}_\circ}(b_1, \dots, b_s) = (\widetilde{b}_1, \dots, \widetilde{b}_s)$*, or* $= (1, 0, \dots, 0) \in \mathbb{Z}^{s+1}$*. Then,*

$$\mathrm{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(a^n) = \mathbf{b} \right\} = \log_\phi \frac{\sum_{k=1}^{s} \widetilde{b}_k \omega^{k-1}}{\sum_{k=1}^{s} b_k \omega^{k-1}}, \quad \text{or } \log_\phi \frac{\phi}{\sum_{k=1}^{s} b_k \omega^{k-1}}.$$

We declare this property to be

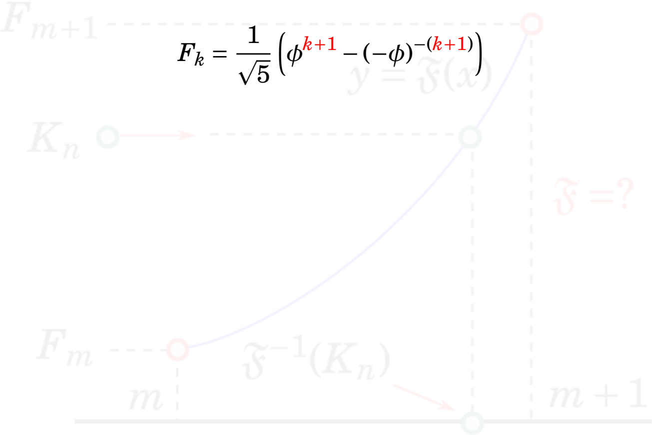
strong Benford's Law under Zeckendorf expansion.

Strong Benford's Law



What is $\mathfrak{F}(x) = ?$

$$F_k = \frac{1}{\sqrt{5}}\left(\phi^{k+1} - (-\phi)^{-(k+1)}\right)$$

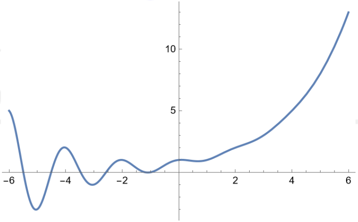


What is $\mathfrak{F}(x) = ?$

$$F_k = \frac{1}{\sqrt{5}}\left(\phi^{k+1} - (-\phi)^{-(k+1)}\right)$$

$$\text{Consider} \quad \Re\,\frac{1}{\sqrt{5}}\left(\phi^{z+1} - e^{-(z+1)\mathrm{Log}(-\phi)}\right)$$

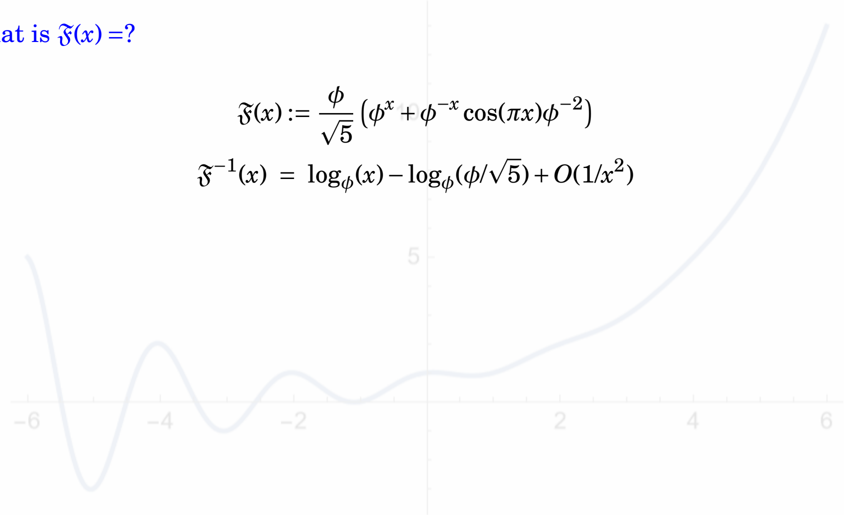
$$\mathfrak{F}(x) := \frac{\phi}{\sqrt{5}}\left(\phi^{x} + \phi^{-x}\cos(\pi x)\phi^{-2}\right) \text{ worked!}$$



What is $\mathfrak{F}(x) = ?$

$$\mathfrak{F}(x) := \frac{\phi}{\sqrt{5}} \left(\phi^{x} + \phi^{-x} \cos(\pi x) \phi^{-2} \right)$$

$$\mathfrak{F}^{-1}(x) \; = \; \log_{\phi}(x) - \log_{\phi}(\phi/\sqrt{5}) + O(1/x^{2})$$



What is $\mathfrak{F}(x) = ?$

$$\mathfrak{F}(x) := \frac{\phi}{\sqrt{5}}\left(\phi^{x} + \phi^{-x}\cos(\pi x)\phi^{-2}\right)$$

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Theorem: *Let $K_n = a^n$ where $a > 1$ is an integer. Then, $\operatorname{frc}\left(\mathfrak{F}^{-1}(K_n)\right)$ is equidistributed.*

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Theorem: *Let* $K_n = a^n$ *where* $a > 1$ *is an integer. Then,* $\operatorname{frc}\left(\mathfrak{F}^{-1}(K_n)\right)$ *is equidistributed.*

$$\text{Example:} \quad \operatorname{frc}\left(\mathfrak{F}^{-1}(F_m + F_{m-2})\right) = \mathfrak{F}^{-1}(F_m + F_{m-2}) - m$$

$$= \log_{\phi}\frac{F_m + F_{m-2}}{\phi^{m}} - \log_{\phi}(\phi/\sqrt{5}) + O(1/\phi^{m})$$

$$= \log_{\phi}(1 + \omega^{2}) + O(\omega^{m})$$

Strong Benford's Law under Zeckendorf expansion

Theorem:

Let $a > 1$ *be an integer,* $(b_1, \ldots, b_s) \in \mathscr{F}_\circ$*, and* $\operatorname{lub}_{\mathscr{F}_\circ}(b_1, \ldots, b_s) = (\widetilde{b}_1, \ldots, \widetilde{b}_s)$*, or* $= (1, 0, \ldots, 0) \in \mathbb{Z}^{s+1}$*. Then,*

$$\operatorname{Prob}\left\{ n \in \mathbb{N} : \operatorname{LB}_s(a^n) = \mathbf{b} \right\} = \log_\phi \frac{\sum_{k=1}^{s} \widetilde{b}_k \phi^{-(k-1)}}{\sum_{k=1}^{s} b_k \phi^{-(k-1)}},$$

$$\text{or } \log_\phi \frac{\phi}{\sum_{k=1}^{s} b_k \phi^{-(k-1)}}.$$

We declare this property to be

strong Benford's Law under Zeckendorf expansion.

Strong Benford's Law under Zeckendorf expansion

Theorem: *Let K_n be an increasing sequence of positive integers. Then, $\operatorname{frc}\left(\mathfrak{F}^{-1}(K_n)\right)$ is equidistributed if and only if K_n satisfies strong Benford's Law under Zeckendorf expansion.*

$$\mathfrak{F}(x) := \frac{\phi}{\sqrt{5}}\left(\phi^x + \phi^{-x}\cos(\pi x)\phi^{-2}\right)$$

$$\mathfrak{F}^{-1}(x) \;=\; \log_\phi(x) - \log_\phi(\phi/\sqrt{5}) + O(1/x^2)$$

Strong Benford's Law under Zeckendorf expansion

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Example: (Non-Benford Law)

Let L_n be the Lucas sequence $(L_1, L_2) = (1,3)$ and $L_{n+2} = L_{n+1} + L_n$.

Strong Benford's Law under Zeckendorf expansion

$$\mathfrak{F}(x) := \frac{\phi}{\sqrt{5}}\left(\phi^{x} + \phi^{-x}\cos(\pi x)\phi^{-2}\right)$$

$$\mathfrak{F}^{-1}(x) \;=\; \log_{\phi}(x) - \log_{\phi}(\phi/\sqrt{5}) + O(1/x^{2})$$

Example: (Non-Benford Law)

Let L_n be the Lucas sequence $(L_1, L_2) = (1, 3)$ and $L_{n+2} = L_{n+1} + L_n$.

$$\text{Prob}\left\{ n \in \mathbb{N} : \text{LB}_5(L_n) = (1,0,0,0,1) \right\} = 1,$$

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Strong Benford's Law under Zeckendorf expansion

$$\mathfrak{F}(x) := \frac{\phi}{\sqrt{5}} \left(\phi^{x} + \phi^{-x}\cos(\pi x)\phi^{-2}\right)$$

$$\mathfrak{F}^{-1}(x) \;=\; \log_{\phi}(x) - \log_{\phi}(\phi/\sqrt{5}) + O(1/x^2)$$

Example: (Non-Benford Law)

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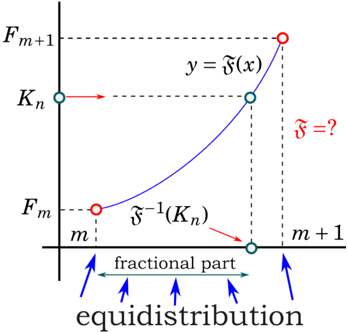
$$\mathfrak{F}(x) := \frac{\phi}{\sqrt{5}}\left(\phi^x + \phi^{-x}\cos(\pi x)\phi^{-2}\right)$$

$\mathfrak{F}(x)$ is called *the Benford continuation* of F_m.

Strong Benford's Law



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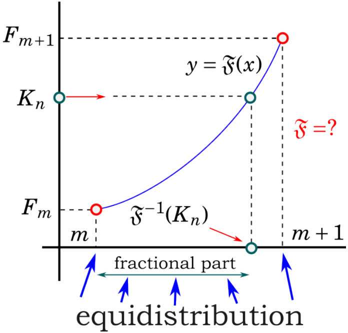

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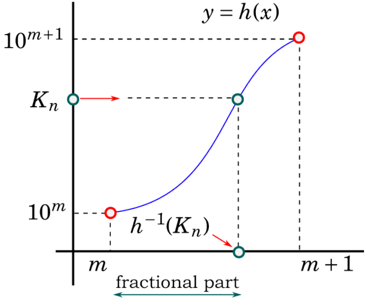

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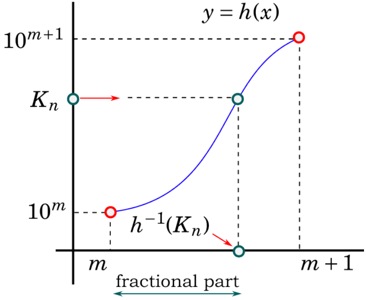

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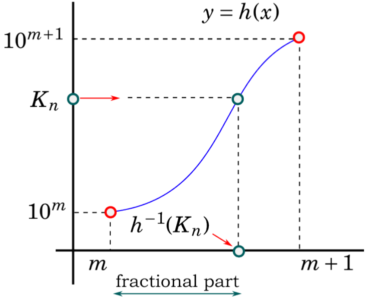

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The Benford continuation:
$h(x) = 10^{x-1}$

- Normalize the intervals to $[0, 1]$.
- Impose convergence of $h(x)$.

Definition: A continuous function $h : [1,\infty) \to \mathbb{R}$ is called a *uniform continuation of* 10^{n-1} if $h(n) = 10^{n-1}$ for all $n \in \mathbb{N}$, and the following sequence of functions $h_n : [0,1] \to [0,1]$ uniformly converges to an increasing (continuous) function:

$$h_n(p) = \frac{h(n+p) - h(n)}{h(n+1) - h(n)}.$$

If h is a uniform continuation of 10^{n-1}, let $h_\infty : [0,1] \to [0,1]$ denote the increasing continuous function given by $h_\infty(p) = \lim_{n\to\infty} h_n(p)$.

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Theorem: *Let h be a uniform continuation of 10^{n-1}, and suppose $\operatorname{frc}\left(h^{-1}(K_n)\right)$ is equidistributed. Then,*

$$\operatorname{Prob}\left\{\, n\in\mathbb{N}:\mathrm{LB}_s(K_n)=d\,\right\}={h_\infty}^{-1}\left(\frac{(d+1)-10^{s-1}}{9\cdot 10^{s-1}}\right)-{h_\infty}^{-1}\left(\frac{d-10^{s-1}}{9\cdot 10^{s-1}}\right).$$

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$$h_\infty(p) = \lim_{n\to\infty} \frac{10^{n+p} - 10^n}{10^{n+1} - 10^n} = \tfrac{1}{9}\left(10^p - 1\right)$$

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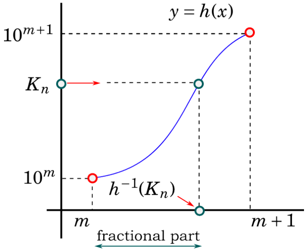
The continuation: $h_\infty(x) = x$

Equidistribution of $\mathrm{frc}\left(h^{-1}(K_n)\right)$

Strong Non-Benford Law



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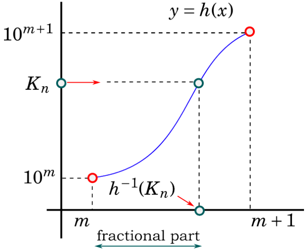

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$$K_n = \left\lfloor 10^{n+\log_{10}(9\operatorname{frc}(n\pi)+1)} \right\rfloor$$

22, 354, 4823, 60973, 737166, 8646003,

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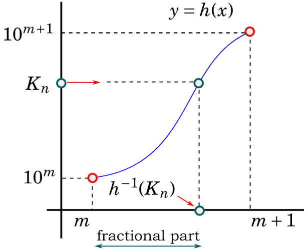

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Any leading block of length s has probability $\frac{1}{9\cdot 10^{s-1}}$.

Review

Theorem: *Let h be a uniform continuation of 10^{n-1}, and suppose $\operatorname{frc}\left(h^{-1}(K_n)\right)$ is equidistributed. Let d be a positive integer of s decimal digits. Then, the probability of the s leading decimal digits of K_n being d is equal to*

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Uniform continuations $\rightsquigarrow$ Leading digits distributions

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$$\left(\operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(K_n) \le \left\lfloor 10^{s-1}\beta \right\rfloor \right\}, \quad s \in \mathbb{N} \right)$$

$$h_K^*\left(\tfrac{1}{9}(\beta-1)\right) := \lim_{s\to\infty} \operatorname{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(K_n) \le \left\lfloor 10^{s-1}\beta \right\rfloor \right\}$$

where h_K^* is continuous. The sequence K_n is said to *have continuous leading block distribution under base*-10 *expansion*.

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Example: Let $\beta = 3.141592\cdots \in [1,10)$.

$$s = 1 : \lfloor 10^{s-1}\beta \rfloor = 3, \quad \mathbf{b} \le 3 \Rightarrow (1),(2),(3).$$
$$s = 2 : \lfloor 10^{s-1}\beta \rfloor = 31, \quad \mathbf{b} \le 31 \Rightarrow (1,0),(1,1),\ldots,(3,1).$$
$$s = 3 : \lfloor 10^{s-1}\beta \rfloor = 314, \quad \mathbf{b} \le 314 \Rightarrow (1,0,0),(1,0,1),\ldots,(3,1,4).$$

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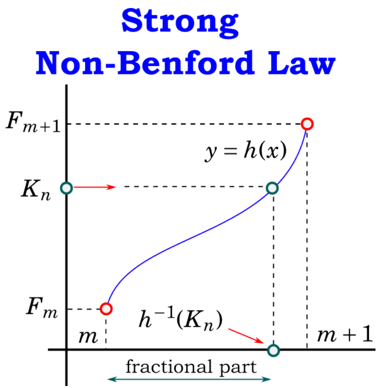
Cor: *If K_n satisfies strong Benford's Law under base-*10 *expansion, then* $\text{frc}\left(\log_{10}(K_n)\right)$ *is equidistributed.*

Under Zeckendorf expansion



Uniform continuation:

$$\mathfrak{F}_n : [0,1] \to [0,1]$$

$$\mathfrak{F}_n(p) := \frac{\mathfrak{F}(n+p) - \mathfrak{F}(n)}{\mathfrak{F}(n+1) - \mathfrak{F}(n)}$$

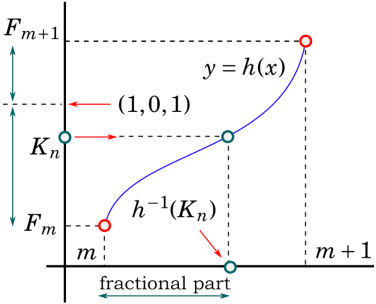
$$= \frac{\mathfrak{F}(n+p) - \mathfrak{F}(n)}{F_{n-1}}$$

$$= \phi(\phi^p - 1) + o(1)$$

$$\Rightarrow \mathfrak{F}_\infty(p) = \phi(\phi^p - 1)$$

Under Zeckendorf expansion

Strong Non-Benford Law



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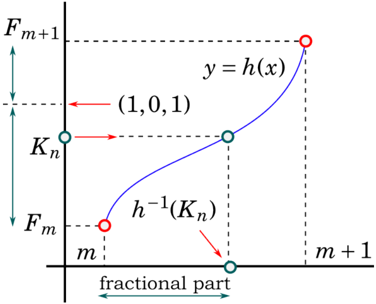

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$$= \lim_{n\to\infty} \frac{(F_n + F_{n-2}) - F_n}{F_{n+1} - F_n} = \omega := \frac{1}{\phi}$$

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Theorem: *Let* $K_n \to \infty$ *such that* $\mathrm{frc}\big(h^{-1}(K_n)\big)$ *is equidistributed where* h *is a uniform continuation of* F_m. *Let* $\mathbf{b} \in \mathscr{F}_\circ \cap \mathbb{Z}^s$ *where* $s \geq 3$. *Then,*

$$\mathrm{Prob}\left\{ n \in \mathbb{N} : \mathrm{LB}_s(K_n) = \mathbf{b} \right\} = {h_\infty}^{-1}\big(\phi(\widetilde{\mathbf{b}} \cdot \widehat{F} - 1)\big) - {h_\infty}^{-1}\big(\phi(\mathbf{b} \cdot \widehat{F} - 1)\big).$$

where $(c_1, \ldots, c_s) \cdot \widehat{F} := \sum_{k=1}^{s} c_k \omega^{k-1} = c_1 + c_2\omega^1 + c_3\omega^2 + \cdots$ *or* $:= \phi$.

Which leading digits distributions? $\rightsquigarrow$ Uniform continuations

Definition: (Zeckendorf expansion of real numbers)

Let $\mathscr{F}_{\circ}^{\star}$ be the subset of $(c_1, c_2, \dots) \in \{0,1\}^{\infty}$ such that $c_1 = 1$, $c_k \in \{0,1\}$ and $c_k c_{k+1} = 0$ for all $k \in \mathbb{N}$.

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Proposition: *The function* $\text{eval}_\star : \mathscr{F}_\circ^\star \to [1, 1+\omega]$ *given by the following is surjective and "almost injective:"*

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Benford's Law and other distribution of leading digits
under generalized Zeckendorf expansions;

See `arXiv:` Benford's Law under Zeckendorf Expansion

Thank you